# Hugh Woodin 2.

def. (F-M)

weak$^2$ $\Box$ holds at $\kappa$ iff

$\exists\ (C_\alpha : \alpha < \kappa^+)$ s.t. $C_\alpha \subset \alpha\ \forall \alpha$

and f.a. lim $\alpha < \kappa^+$,

① $C_\alpha$ is closed and cof. in $\alpha$

② $|C_\alpha| \leq \kappa$ and $cf(\alpha) < \kappa \Rightarrow otp(C_\alpha) < \kappa$.

③ $\forall \xi < \alpha\ \exists \beta < \alpha\ \ C_\alpha \cap \xi = C_\beta \cap \xi$.

lem. if $\kappa$ is $\kappa^{+\omega}$-supercompact,

then weak$^2$ $\Box$ fails at $\kappa^{+\omega}$.

Pf: let $(C_\alpha : \alpha < \kappa^{+\omega+1})$ witness weak$^2$ $\Box$

at $\kappa^{+\omega}$. let

$$j : V \longrightarrow M$$

s.t. $j''\kappa^{+\omega+1} \in M$.

notice $\kappa$ is also $\kappa^{+\omega+1}$ supercompact (pick

$f$ : $\kappa^{+\omega+1} \leftrightarrow (\kappa^{+\omega})^\omega$ ).

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[dr, "j_0"'] & M \\ & N \arrow[u, "k_0"'] \end{tikzcd}$$

$N \sim \mathrm{ult}(V; E_j \restriction j(\kappa^{+\omega}))$

set $\alpha = \mathrm{crit}(k_0)$

$= \sup j'' \kappa^{+\omega+1} < \cancel{\kappa}\ j(\kappa^{+\omega+1})$.

$j(\langle C_\beta : \beta < \kappa^{+\omega+1} \rangle)$

$\parallel$

$\langle C^*_\beta : \beta < j(\kappa^{+\omega+1}) \rangle$

$M \models \mathrm{cf}(\alpha) = \kappa^{+\omega+1}$.

$\mathrm{otp}(C^*_\alpha) < j(\kappa^{+\omega}) = j_0(\kappa^{+\omega})$.

$$(C^*_\beta : \beta < \alpha_0) = j_0(C_\beta : \beta < \kappa^{+\omega+1})$$

$\Rightarrow$ $C^*_\alpha \cap \xi \in N$ for all
$\xi < \alpha$ by the fact that
$(C_\beta : \beta < \kappa^{+\omega+1})$ is a weak$^2$
square sequence.

let $D = \{\beta < \kappa^{+\omega+1} : \underset{\underset{j(\beta)}{\shortparallel}}{j_0(\beta)} \in C^*_{\alpha_0}\}$

$D$ is an $\omega$-club in $\alpha_0$.
hence we may fix $\gamma < \kappa^{+\omega+1}$
s.t. $|D \cap \gamma| \geq \kappa^{+\omega}$.

note $j_0{}'' D \cap \gamma = j'' D \cap \gamma$
is covered by $C^*_\alpha \cap \underset{\underset{j(\gamma)}{\shortparallel}}{j_0(\gamma)}$

~~we correct~~

but $\mathcal{P}(C^*_\alpha \cap \hat{j}_0(\gamma))^M =$

$\mathcal{P}(C^*_\alpha \cap \hat{j}_0(\gamma))^N$ (using GCH)

$\Rightarrow \hat{\not\exists}\ \hat{j}_0''(D \cap \gamma) \in M$. $\lightning$

without GCH:

$j_0'' \kappa^{+\omega} \subset Z$ for some $Z \in N$,

$|Z|^M < j(\kappa^{+\omega})$.

$\Rightarrow \hat{j}_0'' \kappa^{+\omega+1} \subset Y$, where

$|Y| = |Z^\omega| < \hat{j}_0(\kappa^{+\omega})$

$\Rightarrow \hat{j}_0(\kappa^{+\omega+1}) > \sup(\hat{j}_0'' \kappa^{+\omega+1})$

$\lightning$

<u>def.</u> $\mathbb{P} \subset OR \times V$

then $\mathbb{P}$ is $\omega$- <u>weakly amenable</u>

if the following hold

① $\alpha \in \mathrm{dom}(\mathbb{P}) \longrightarrow$ ~~$\mathbb{P}(\alpha)$~~ $\mathbb{P}_\alpha \subset$

$J_\alpha[\mathbb{P}] \models ZFC -$ powerset

② $\alpha \in \mathrm{dom}(\mathbb{P}) \Longrightarrow$

$\mathbb{P}_\alpha \subset \omega \times J_\alpha(\mathbb{P})$ and

$\forall k < \omega \ \exists \theta_k \leq \alpha \ (\mathbb{P}_\alpha)_k \cap J_\beta[\mathbb{P}] \in J_{\theta_k}(\mathbb{P})$

$(\mathbb{P}_\alpha)_k \subset J_{\theta_k}[\mathbb{P}]$.

<u>lem.</u> assume GCH and $V = L[\mathbb{P}]$

for some $\mathbb{P} \subset OR \times V$.

TFAE.

(1) weak$^2$ $\square$ holds at all $\kappa > \omega$.

(2) $\exists\, \mathbb{P}^*$ $\mathbb{P}^*$ is $\Sigma_2$ dfg.ble on $(L(\mathbb{P});\ \mathbb{P})$, $V = L(\mathbb{P}^*)$ and all levels of $\mathbb{P}^*$ are amenable and sound.

(3) $\exists\, \mathbb{P}^*$ $\mathbb{P}^*$ is $\Sigma_2$-dfg. the on $(L(\mathbb{P});\ \mathbb{P})$ s.t. $V = L(\mathbb{P}^*)$ and all the level $\mathbb{P}^*$ are sound and $\omega$-weakly amenable.

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$\mathcal{L}$, $\mathring{\mathbb{P}}$ $\mathring{P}$

$M = (J_\alpha[A],\ \mathbb{P}|\alpha,\ \mathbb{P}_\alpha)$

for $\alpha \in \mathrm{dom}(\mathbb{P})$

set $\mathbb{P}_\alpha = \emptyset$ if $\alpha \notin \mathrm{dom}(\mathbb{P})$.

$\mathcal{L}^+$ is $\mathcal{L}$ expanded by 3-ary predicate $\dot{T}_n$, $1 \leq n < \omega$.

$\theta$ is $r\Sigma_1$ iff $\dot{\theta}(a)$ says

$$(J_\alpha(\mathbb{P}), \mathbb{P}|\alpha, \mathbb{P}_\alpha) \models \phi(a),$$

which is equiv. to,

$$\exists b \subset \mathbb{P}_\alpha \text{ finite}$$

$$(J_\alpha(\mathbb{P}), \mathbb{P}|\alpha) \models \underbrace{\psi(a,b)}_{\Sigma_1}$$

def. $J(\mathbb{P})$ is sound iff f.a. $\alpha$,

$\forall n < \omega$ either

(1) $\rho_n(J(\mathbb{P})|\alpha) = \alpha$ or

(2) $j(\mathbb{P})|\alpha = \mathrm{Hull}_n^{j(\mathbb{P})|\alpha}(\rho \cup \{p\})$

for some $p \in \mathcal{J}_\alpha(\mathbb{P})$, and

$$\rho = \rho_n(\mathcal{J}_\alpha(\mathbb{P})/\alpha).$$

<u>notat.</u>: $\mathcal{J}(\mathbb{P})/\alpha =$

$(\mathcal{J}_\alpha(\mathbb{P}), \mathbb{P}/\alpha, \mathbb{P}_\alpha)$ if $\alpha \in \mathrm{dom}(\mathbb{P})$

$(\mathcal{J}_\alpha(\mathbb{P}), \mathbb{P}/\alpha)$ o.w.

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~~Supp.~~

**lem.** ~~Supp.~~ $\mathbb{P}$ ~~$\in$~~ $OR \times V$ and

~~$\mathcal{J}(\mathbb{P}) \models GCH$.~~ TFAE

proof:

① $\Rightarrow$ ②

supp. $\kappa$ is an unctbl. cardinal, and let $C = (C_\alpha : \alpha < \kappa^+)$ witness weak$^2$ $\square$ at $\kappa$.

let $D = \{\alpha \in (\kappa, \kappa^+) :$

$$(J_\alpha(\mathbb{P}),\ \mathbb{P}|\alpha,\ C|\alpha) \prec (J_{\kappa^+}[\mathbb{P}],\ \mathbb{P}|\kappa^+,\ C)\}.$$

assume w.l.o.g. $\gamma \cdot$ ∀ cardinals $\gamma$,

$$H_\gamma \subset J_\gamma[\mathbb{P}] .$$

define $\mathbb{P}^* | [\kappa, \kappa^+)$ to be the

$(J(\mathbb{P}), \mathbb{P})$ — least set

$$\mathbb{H} \subset [\kappa, \kappa^+) \times \underset{\substack{\shortparallel \\ H_{\kappa^+}}}{J_{\kappa^+}(\mathbb{P})} \ni$$

s.t.

① $J_{\kappa+1}[\mathbb{H}] \supseteq H_\kappa$ .

② if $(J_\eta[\mathbb{H}], \mathbb{H}\restriction\eta) \not\models$ ZFC − power,

then $\mathbb{H}_\eta = \emptyset$ $\forall \eta$

~~③ $\kappa$ is the largest cardinal of~~

$\cancel{J_\eta[\mathbb{H}]}$

and $\forall \eta \in (\kappa, \kappa^+)$ s.t.

$$(J_\eta[\mathbb{H}], \mathbb{H}|\eta) \models ZFC^- \text{ powerset}$$

the following hold.

(a) $\kappa$ is the largest cardinal of $J_\eta[\mathbb{H}]$.

(b) $cf(\eta) < cf(\kappa)$.

then $\mathbb{H}_\eta$ is a closed cof. subset of $\eta$, $otp(\mathbb{H}_\eta) = cf(\eta)$,

and $\mathbb{H}_\eta \cap \zeta \in J_\eta[\mathbb{H}]$ $\forall \zeta < \eta$.

(c) Supp. $cf(\eta) = cf(\kappa)$.

~~$\mathbb{H}^*$~~ let $\eta^* \in D$ be least above $\eta$.

then $\exists E \subset \kappa$ which codes

$$(J_{\eta^*}[\mathbb{P}], \mathbb{P}|\eta^*, C|\eta^*)$$

and then are increasing continuous cof. functions

$$f : cf(\kappa) \longrightarrow \eta$$
$$g : cf(\kappa) \longrightarrow \kappa$$

$$\mathbb{H}_\eta = \{ ( f(\xi), g(\xi) \cap E ) : \xi < cf(\kappa) \}.$$

(d) supp. $cf(\eta) > cf(\kappa)$.

~~supp.~~

then $\eta \in D \wedge J_\eta[\mathbb{P}] \subset J_\eta[\mathbb{H}]$

$\wedge \quad \mathbb{H}_\eta = C_\eta$

arrange by weak$^2$ $\square$ - property.

3 $\Rightarrow$ 1

assume $\mathbb{P} = \mathbb{P}^*$, so $V = L(\mathbb{P})$ + all levels are $\omega$-weakly overlap + sound.

fix $\kappa > \omega$, $\gamma = \mathrm{cf}(\kappa)$.

$X \prec (J_{\kappa^+}[\mathbb{P}] \,;\, \mathbb{P}|\kappa^+)$

$|X| = \kappa$, $\kappa \subset X$.

need to see can singularize $X \cap \kappa^+$ by a club all of whose initial segments are in $X$.

easy: $\mathrm{cf}(X \cap \kappa^+) \overset{\leq}{\cancel{\geq}} \gamma$.

so let's assume $\mathrm{cf}(X \cap \kappa^+) > \gamma$.

let $X \cap \kappa^+ < \alpha < \kappa^+$ be least

s.t. $\rho_n(M) = \kappa$, some $n$,

where $M = (J_\alpha(\mathbb{P}), \mathbb{P}|\alpha, \mathbb{P}_\alpha)$.

& fix $n$ least.

assume $n = 1$, $\alpha \in \mathrm{dom}(\mathbb{P})$.

~~there is a partial~~

$M = \mathrm{Hull}_1^M(\kappa \cup \{p\})$, some $p \in M$.

thus there is a partial surjection

$f : \kappa \longrightarrow X \cap \kappa^+$

s.t. $f$ is generalized $r\Sigma_1$-def.

for $p$.

i.e.

$r\Sigma_1$ with

occurrences of

$r\Sigma_1$ shorter-terms

for each $m < \omega$, let $M_{(m)} =$

$$\left( J_\alpha[\mathbb{P}] ;\ \mathbb{P}|\alpha ,\ \underset{\substack{\parallel \\ \mathbb{P}_\alpha \cap (m \times J_\alpha(\mathbb{P}))}}{\mathbb{P}_\alpha | m} \right).$$

recall $\mathbb{P}_\alpha \subset \omega \times J_\alpha(\mathbb{P})$.

~~for each $m < \omega$, let~~

$f_{(m)}$ is $f$, when everything is interpreted in $M_{(m)}$ rather than $M$.

$f_{(m)}(\xi)\uparrow$ iff $f(\xi)$ makes ref. to a pred. with index $\geq m$

since $cf(\kappa) < cf(X \cap \kappa^+) < \kappa$,

$\exists$ $\xi < \kappa$ and $m < \omega$ such that

range $(f_{(m)} \restriction \xi)$

is cofinal.

let $\delta = cf(X \cap \kappa^+)$. so

$\exists$ cont. cof. function,

$g : \delta \longrightarrow X \cap \kappa^+$

which is gen. $+\Sigma_1$ - dy. in $M_{(m)}$.

assume $m$ is as small as possible.

$m > 0$, o.w. $M_{(m)}$ is amenable.

in fact, assume $m$ is as small as

possible among all

$$( J_\alpha[\mathbb{P}], \mathbb{P}|\alpha, P ),$$

$P \subset \bar{m} \times J_\alpha[\mathbb{P}]$,

some $\bar{m}$.

for each $k < m$, let $\theta_k$ ~~=~~ be the

~~sup~~ least $\theta$ s.t.

$$(\mathbb{P}_\alpha)_k \subset J_\theta(\mathbb{P}),$$

i.e., the place where $(\mathbb{P}_\alpha)_k$ is amenable.

let $I$ be the set of all finite sequences $(\beta_k : k < m)$, $\beta_k < \theta_k$ for all $k < m$.

directed via partial $<$.

for each $a \in I$, let

$$M_a = (J_\alpha(\mathbb{P}), \mathbb{P}|\alpha, \mathbb{P}_a)$$

now interpret $g$ in $M_a$, i.e., if def. of $g(\xi)$ needs ~~st~~ ref. to a pred. not availabl in $\mathbb{P}_a$, $g^{M_a}\uparrow$.

here: f.a. $\eta < \delta$ $\exists\, a \in I$ $\forall b > a$

$$g^{M_b}(\eta) = g(\eta).$$

claim. $\forall k < m \quad cf(\theta_k) = \delta$.

o.w. don't need this $\mathbb{P}_{\alpha,k}$, as i can do with the rest of the $\mathbb{P}_\alpha$'s, $k < m$. $\dashv$

Case 1. $cf(\alpha) \geq \delta$.

for each $a \in I$ and for each $\sup(a) < \beta < \alpha$, let $g_{(\beta,a)}$ be $g$ interpreted in

$$M_{(\beta,a)} = ( J_\beta [\mathbb{P}], \mathbb{P} \restriction \beta, \mathbb{P}_a )$$

$cof(\alpha) \geq \delta$, so

$$\forall \eta < \delta \; \exists a \; \exists \beta \quad g \restriction \eta = g_{(\beta,a)} \restriction \eta$$

so all the restriction of $g$ are in $M$.

~~the~~

( actually shows $cf(\alpha) \leq \delta$ )

<u>Case</u> 2. $cf(\alpha) < \delta$.

but $cf(\theta_k) = \delta \quad \forall k < m$.

so $\sup_{k<m} (\theta_k) < \alpha$.

since $cf(\alpha) < \delta$, there must exist

$X \cap \kappa^+ < \beta < \alpha$

and $h : \delta \longrightarrow X \cap \kappa^+$

increasing, cof. with

$h$ $r\Sigma_1$- def. on $(J_\beta[\mathbb{P}], \mathbb{P}|\beta, \mathbb{P}_{\alpha,m})$

$$M^\beta_a = (J_\beta[\mathbb{P}], \mathbb{P}|\beta, \mathbb{P}_a).$$

for each $\eta < \delta$ $\exists a \in I$

$h^{M_b}(\eta) = h(\eta) \quad \forall b > a$ in $I$

$\Longrightarrow$ for each $\eta < \delta$ there is some $a \in I$

$h|\eta = h^{M_a}|\eta$.

but $M_a^\beta \in J_\alpha[\mathbb{P}]$.

$\dashv$