3.8 Branch conjectures

Martin and Steel, [7], proposed two hypotheses with regard to iteration trees on V.

(UBH) *The Unique Branch Hypothesis*:

Suppose $\mathcal{T}$ is an iteration tree on a premouse (V_Θ, δ). Then $\mathcal{T}$ does not have two distinct cofinal wellfounded branches. □

(CBH) *The Cofinal Branch Hypothesis*:

Suppose $\mathcal{T}$ is an iteration tree on a premouse (V_Θ, δ). Then:

(1) If $\mathcal{T}$ has limit length then $\mathcal{T}$ has a cofinal wellfounded branch;

(2) If $\mathcal{T}$ has successor length, $\eta + 1$, then $\mathcal{T}$ can be freely extended to an iteration tree of length $\eta + 2$. □

Unfortunately if there is a supercompact cardinal then these hypotheses are each false in essentially the simplest cases. Define an iteration tree on V to be *short* if no extender occurring in the iteration tree is a long extender. Both UBH and CBH refer only to iteration trees which are short.

An iteration tree, $\mathcal{T}$, is *non-overlapping* if

$$\mathrm{LTH}(E_\alpha) \leq \mathrm{CRT}(E_\beta)$$

for all $\alpha + 1 <_{\mathcal{T}} \beta + 1$. The iteration tree, $\mathcal{T}$, is *totally non-overlapping* if

$$j_{E_\alpha}(\mathrm{SPT}(E_\alpha)) < \mathrm{CRT}(E_\beta)$$

for all $\alpha + 1 <_{\mathcal{T}} \beta + 1$.

In [10], Neeman and Steel give a much simpler construction for counterexamples to both UBH and CBH than the construction given here in the proof of Theorem 97. Their construction requires much weaker large cardinal hypotheses and the counterexamples produced have the same underlying tree orders as the examples constructed here, but their counterexamples are not iteration trees which are non-overlapping. For the special case of non-overlapping iteration trees, Steel has shown that hypotheses below the level of superstrong are probably not sufficient, [18].

Theorem 97. *Suppose that there is a supercompact cardinal.*

(1) *There is a short, totally non-overlapping,* (+2)*-iteration tree on V of length ω with only two cofinal branches and each is wellfounded.*

(2) *There is a short, totally non-overlapping,* (+2)*-iteration tree on V of length $\omega \cdot \omega$ with only one cofinal branch and this branch is illfounded.*

Proof. We sketch the proof which involves some material which is a little outside the scope of this paper.

Fix $\delta_0 < \delta$ and an elementary embedding,

$$j : V \to M$$

with $\mathrm{CRT}(j) = \delta_0$ such that $j(\delta_0) = \delta$ and such that $V_{\delta+\omega} \subseteq M$.

Since δ is supercompact, (δ_0, j) exists. It is the existence of (δ_0, j) which is all that we require. In fact we only require that $V_{\delta+2} \subseteq M$ and even this can be weakened.

It is useful to introduce some notation. Suppose that $\kappa_0 \leq \kappa_1$ are measurable cardinals. Suppose that $X_0 \subseteq V_{\kappa_0+2}$ is a set of κ_0-complete ultrafilters on V_{κ_0} and $X_1 \subseteq V_{\kappa_1+2}$ is a set of κ_1-complete ultrafilters on V_{κ_1} and that $|X_0| = |X_1|$.

A bijection

$$\pi : X_0 \to X_1$$

is a *tower isomorphism* if the following hold for all sequences

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from X_0.

(1.1) $\langle U_i : i < \omega \rangle$ is a tower if and only if $\langle \pi(U_i) : i < \omega \rangle$ is a tower,

(1.2) If $\langle U_i : i < \omega \rangle$ is a tower then the tower $\langle U_i : i < \omega \rangle$ is wellfounded if and only if the tower, $\langle \pi(U_i) : i < \omega \rangle$, is wellfounded.

The sets X_0 and X_1 are *tower isomorphic* if there exists a tower isomorphism,

$$\pi : X_0 \to X_1.$$

Suppose that there exists a Woodin cardinal γ such that

$$|X_0| < \gamma < \kappa_0$$

and that

$$\pi : X_0 \to X_1$$

is a tower isomorphism. Suppose $g \subseteq \mathbb{P}$ is V-generic for a partial order $\mathbb{P} \in V_\gamma$. Then in $V[g]$, π is a tower isomorphism where we identify the elements $U \in X_0 \cup X_1$ with the ultrafilters they generate in $V[g]$. The verification uses Lemma 145 (see page 162) and the generic elementary embeddings associated to the stationary tower.

Another preliminary fact we shall need is the following. Again suppose $X_0 \subseteq V_{\kappa_0+2}$ is a set of κ_0-complete ultrafilters on V_{κ_0}. Suppose that κ_0 is a limit of Woodin cardinals and that $|X_0| < \kappa_0$. Then there is a set $Y_0 \subseteq V_{\kappa_0+2}$ of κ_0-complete ultrafilters on V_{κ_0} and a function,

$$e : X_0 \to Y_0$$

such that all sequences

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from X_0:

(2.1) $\langle U_i : i < \omega\rangle$ is a tower if and only if $\langle e(U_i) : i < \omega\rangle$ is a tower,

(2.2) If $\langle U_i : i < \omega\rangle$ is a tower then the tower $\langle U_i : i < \omega\rangle$ is wellfounded if and only if the tower, $\langle e(U_i) : i < \omega\rangle$, is illfounded.

As above this property of e persists to all generic extensions of V given by partial orders $\mathbb{P} \in V_{\kappa_0}$.

We fix some more notation and isolate what is really the key to the proof. For this we fix

$$G \subset \mathrm{Coll}(\omega, V_{\delta_0+2})$$

such that G is V-generic.

Using G which we regard as a surjection,

$$G : \omega \to V_{\delta_0+2},$$

we define a reduction map

$$\mathcal{R}_G : V_\delta \to V_{\delta_0+2}$$

as follows. Suppose that $\langle U_i : i < \omega\rangle$ is a tower of δ_0^+-complete ultrafilters from V_δ. To simplify notation and with no essential loss of generality we can suppose that for some $\kappa < \delta$, each ultrafilter U_i concentrates on $(V_\kappa)^i$. In any case necessarily (by our conventions on towers), U_0 is the principal ultrafilter concentrating on $\{\emptyset\} = (V_\kappa)^0$.

Set $\mathcal{R}_G(U_0) = U_0$. Fix $i + 1 < \omega$ and suppose

$$s = \langle G(0), a_0, G(1), a_1, \ldots, G(i), a_i\rangle$$

is such that for all $n \le i$, $a_n \in V_\kappa$. Let

$$U_s = \{A \subseteq V_{\delta_0} \mid s \in j(A)\}.$$

Thus U_s is a δ_0-complete ultrafilter on V_{δ_0}. Since U_{i+1} is δ_0^+-complete there must exist a set $A \in U_{i+1}$ and $U \in V_{\delta_0+2}$ such that for all

$$\langle a_0, \ldots, a_i\rangle \in A,$$

if $s = \langle G(0), a_0, G(1), a_1, \ldots, G(i), a_i\rangle$ then $U_s = U$. Define for $B \subseteq (V_{\delta_0})^i$,

$$B \in \mathcal{R}_G(U_i)$$

if $\left\{t \in (V_{\delta_0})^{2i} \mid t|i \in B\right\} \in U$. Thus $\mathcal{R}_G(U_i)$ is simply the projection of U to $(V_{\delta_0})^i$ (and the only reason for not setting $\mathcal{R}_G(U_i) = U$ is in order to conform with our conventions on towers). Note that since $\langle U_i : i < \omega\rangle$ is a tower, $\langle \mathcal{R}_G(U_i) : i < \omega\rangle$ is a tower.

Let

$$\mathcal{X}_G = \{j(f)(G|i) \mid f : V_{\delta_0} \to V, i < \omega\} \prec M.$$

Thus

$$\mathcal{X}_G = \{j(f)(a) \mid f : V_{\delta_0} \to V, a \in V_{\delta_0+2}\}.$$

We claim, and this claim follows easily from the definitions, that for all towers, $\langle U_i : i < \omega\rangle$, from $V_\delta \cap \mathcal{X}_G$ consisting of δ_0^+-complete ultrafilters, the following are equivalent where N is the transitive collapse of $\mathcal{X}_G$ and where for each $i < \omega$, U_i^N is the image of U_i under the collapsing map.

(3.1) The tower $\langle \mathcal{R}_G(U_i) : i < \omega\rangle$ is wellfounded.

(3.2) The direct limit of $\langle \mathrm{Ult}(N, U_i^N) : i < \omega\rangle$ is wellfounded.

The definition of the reduction map, $\mathcal{R}_G$, only requires that $V_\delta \subseteq M$. The key consequence of $V_{\delta+2} \subseteq M$ is that there exist cofinally many $\kappa < \delta$ such that κ is measurable and such that there exists a set $Y \subset V_{\kappa+2}$ of κ-complete ultrafilters such that Y is tower isomorphic to X where X is the set of all δ_0-complete ultrafilters on V_{δ_0}. This follows by reflection in M since

$$\{j(U) \mid U \in X\} \in M$$

and since δ is superstrong in M. This is all (beyond $V_\delta \subseteq M$) that is required for the construction.

Let X be the set of all δ_0-complete ultrafilters on V_{δ_0} and let $\kappa < \delta$ be least such that

(4.1) $\delta_0 < \kappa$,

(4.2) $V_\kappa \prec V_\delta$,

(4.3) κ measurable and there exists a set $Y \subset V_{\kappa+2}$ of κ-complete ultrafilters on V_κ such that Y is tower isomorphic to X.

By the definability of κ, $\kappa \in \mathcal{X}_G$. Fix

$$Y \subseteq V_{\kappa+2}$$

such that Y is a set of κ-complete ultrafilters on V_κ such that Y is tower isomorphic to X and such that $Y \in \mathcal{X}_G$. Fix a tower isomorphism,

$$\pi : X \to Y,$$

such that $\pi \in \mathcal{X}_G$.

Since $V_\kappa \prec V_\delta$,

$$V_\kappa[G] \vDash \mathrm{ZFC} + \text{“There is a proper class of Woodin cardinals”}.$$

Therefore in $V[G]$, if $A \subseteq \mathbb{R}^{V[G]}$ is $(<\kappa)$-weakly homogeneously Suslin then $(A, \mathbb{R}^{V[G]})^\#$ is $(<\kappa)$-weakly homogeneously Suslin.

For each partial function,

$$F : V_{\delta_0+2} \to V_{\kappa+2}$$

there is a canonical set $A_F^G \subseteq \mathbb{R}^{V[G]}$ such that A_F^G is $(<\kappa)$-weakly homogeneously Suslin in $V[G]$. The set A_F^G is the set of all $x \in \omega^\omega$ such that

$$\langle G(x(i)) : i < \omega\rangle$$

is a wellfounded tower of κ complete ultrafilters on V_κ from the range of F and here we identify G with the corresponding surjection,

$$G : \omega \to V_{\delta_0+2},$$

and as above we identify each κ-complete ultrafilter in V with the ultrafilter it generates in $V[G]$. Notice that

(5.1) if $G^* \subseteq \mathrm{Coll}(\omega, V_{\delta_0+2})$ is V-generic and

$$V[G] = V[G^*]$$

then A_F^G and $A_F^{G^*}$ are continuously reducible to each other.

Since κ is a measurable limit of Woodin cardinals, for each set $A \subseteq \mathbb{R}^{V[G]}$ such that A is $(<\kappa)$-weakly homogeneously Suslin in $V[G]$, there exists an injective function

$$F : V_{\delta_0+2} \to V_{\kappa+2}$$

in V such that A is continuously reducible to A_F^G.

Fix a function

$$F : V_{\delta_0+2} \to V_{\kappa+2}$$

such that $F \in \mathcal{X}_G$ and such that $\left((A_\pi^G, \mathbb{R}^{V[G]})^\#\right)^\#$ is continuously reducible to A_F^G and let Y_F be the set of κ-complete ultrafilters U on V_κ such that U is in the range of F.

Fix a set $Z \subseteq V_{\kappa+2}$ of κ-complete ultrafilters on V_κ and a surjection

$$e : Y_F \to Z$$

such that $(Z, e) \in V$ and such that in V, for all sequences

$$\langle U_i : i < \omega\rangle$$

of ultrafilters from Y_F:

(6.1) $\langle U_i : i < \omega\rangle$ is a tower if and only if $\langle e(U_i) : i < \omega\rangle$ is a tower,

(6.2) If $\langle U_i : i < \omega\rangle$ is a tower then the tower $\langle U_i : i < \omega\rangle$ is wellfounded if and only if the tower, $\langle e(U_i) : i < \omega\rangle$, is illfounded.

As indicated above, this property of e must hold in $V[G]$. Again we can and do choose $(e, Z) \in \mathcal{X}_G$.

We now come to the key claim.

(7.1) There exists in $V[G]$ a tower,

$$\langle U_i : i < \omega\rangle$$

of ultrafilters from Y_F such that both the towers,

$$\langle j(\mathcal{R}_G(U_i)) : i < \omega\rangle \text{ and } \langle j(\mathcal{R}_G(e(U_i))) : i < \omega\rangle,$$

are wellfounded in $V[G]$.

To verify this assume toward a contradiction that in $V[G]$ no such tower exists. We claim that in V, there exists a closed unbounded set

$$C \subset \mathcal{P}_{\omega_1}(V_{\delta_0+2})$$

such that for each function

$$\hat{G} : \omega \to V_{\delta_0+2},$$

if $\hat{G}[\omega] \in C$ then the following hold where we are defining $\mathcal{R}_G(U)$ in the obvious fashion.

(8.1) $(F, e, \pi, Z) \in \mathcal{X}_{\hat{G}}$,

(8.2) $\left((A^{\hat{G}}_{\pi}, \mathbb{R})^{\#}\right)^{\#}$ is continuously reducible to $A^{\hat{G}}_{F}$,

(8.3) There is no tower

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from $F \circ \hat{G}[\omega]$ such that both the towers,

$$\langle j(\mathcal{R}_{\hat{G}}(U_i)) : i < \omega \rangle \text{ and } \langle j(\mathcal{R}_{\hat{G}}(e(U_i))) : i < \omega \rangle,$$

are wellfounded in V.

This follows by Lemma 145 (see page 162) using the stationary tower at least Woodin cardinal above δ_0.

Fix

$$\hat{G} : \omega \to V_{\delta_0+2}$$

such that $\hat{G}[\omega] \in C$. Thus for all towers

$$\langle U_i : i < \omega \rangle$$

of ultrafilters from $F \circ \hat{G}[\omega]$, the following must be equivalent.

(9.1) $\langle U_i : i < \omega \rangle$ is wellfounded.

(9.2) $\langle R_{\hat{G}}(i) : i < \omega \rangle$ is wellfounded.

(9.3) $\langle j(R_{\hat{G}}(i)) : i < \omega \rangle$ is wellfounded.

But then

$$(A^{\hat{G}}_{\pi}, \mathbb{R})^{\#} \in L(A^{\hat{G}}_{\pi}, \mathbb{R})$$

and this is a contradiction.

Therefore (7.1) must hold and we will use the witness to give the counterexample for part (1) of the theorem. Before giving the details we establish a variant of (7.1) which will give the counterexample for part (2) of the theorem.

We require a definition. Suppose that

$$\langle \langle U^n_i : i < \omega \rangle : n < \omega \rangle$$

is a sequence of towers of κ-complete ultrafilters on V_κ. For each $0 < i < \omega$ let U_i^* be the κ-complete ultrafilter on V_κ given by the product,

$$U_i^0 \times \cdots \times U_i^{i-1}.$$

Thus $\langle U_i^* : i < \omega \rangle$ is naturally regarded as a tower of ultrafilters using the natural projection maps:

$$p_{i_2,i_1} : V_\kappa^{i_2 \cdot i_2} \to V_\kappa^{i_1 \cdot i_1}$$

where for $0 < i_1 < i_2 < \omega$,

$$p_{i_2,i_1}(s_1 + s_2 + \ldots + s_{i_2}) = (s_1|i_1) + (s_2|i_1) + \ldots + (s_{i_2}|i_2),$$

and so here we are violating our notational convention on towers.

It is straightforward to show that the tower $\langle U_i^* : i < \omega \rangle$ is wellfounded if and only if each of the towers, $\langle U_i^n : i < \omega \rangle$, is wellfounded. But we caution that this equivalence requires that each of the towers, $\langle U_i^n : i < \omega \rangle$, belongs to V.

The second key claim is the following.

(10.1) There exists in $V[G]$ a sequence,

$$\langle \langle U_i^n : i < \omega \rangle : n < \omega \rangle$$

of towers of ultrafilters from Y_F such that

a) For all $m < \omega$, the towers

$$\langle j(\mathcal{R}_G(U_i^n)) : i < \omega \rangle$$

for $n \leq m$ are jointly wellfounded in $V[G]$,

b) The tower,

$$\langle j(\mathcal{R}_G(U_i^*)) : i < \omega \rangle,$$

is not wellfounded in $V[G]$.

We work in $V[G]$. Let $B \subseteq \mathbb{R}^{V[G]}$ be the set of all $x \in \mathbb{R}^{V[G]}$ such that x codes a countable elementary substructure,

$$\sigma \prec \left(V[G]_{\omega+1}, (A_\pi^G, \mathbb{R}^{V[G]})^\# \right).$$

Since $\left((A_\pi^G, \mathbb{R}^{V[G]})^\# \right)^\#$ is continuously reducible to A_F^G, there exists a function,

$$h : \omega^{<\omega} \to Y_F,$$

which witnesses that B is homogeneously Suslin, therefore for all $x \in \mathbb{R}^{V[G]}$, $x \in B$ if and only if

$$\langle U_i : i < \omega \rangle$$

is a wellfounded tower where for each $i < \omega$, $U_i = h(x|i)$. Since $(\pi, F) \in \mathcal{X}_G$ we can choose $h \in \mathcal{X}_G[G]$. Let $T \in \mathcal{X}_G[G]$ be a tree such that h witnesses the homogeneity of T. Thus $p[T] = B$.

Let $\hat{M}$ be the transitive collapse of $\mathcal{X}_G$ and let

$$\hat{j} : V \to \hat{M}$$

be the induced elementary embedding. Let

$$\hat{k} : \hat{M} \to M$$

invert the collapsing map. Thus $\hat{k}$ is an elementary embedding, $j = \hat{k} \circ \hat{j}$, and $\mathrm{CRT}(\hat{k}) > \delta_0$ and $V_{\delta_0+2} \subseteq \hat{M}$.

For each $U \in Y_F$ let $\hat{U}$ be the image of U under the transitive collapse of $\mathcal{X}_G$ (since $F \in \mathcal{X}_G$, $Y_F \subseteq \mathcal{X}_G$ and so this makes sense).

Suppose that

$$\langle U_i : i < \omega \rangle \in V[G]$$

is a tower of ultrafilters from Y_F.

(11.1) If the tower $\langle U_i : i < \omega \rangle$ is wellfounded then so is the tower, $\langle \mathcal{R}_G(U_i) : i < \omega \rangle$.

(11.2) If the tower, $\langle \mathcal{R}_G(U_i) : i < \omega \rangle$, is wellfounded then the direct limit of

$$\langle \mathrm{Ult}(\hat{M}, \hat{U}_i) : i < \omega \rangle$$

under the natural embeddings is wellfounded.

Similarly, suppose that

$$\langle \langle U_i^n : i < \omega \rangle : n < \omega \rangle \in V[G]$$

is a sequence of towers of ultrafilters from Y_F.

(12.1) If the tower $\langle U_i^* : i < \omega \rangle$ is wellfounded then so is the tower,

$$\langle \mathcal{R}_G(U_i^*) : i < \omega \rangle.$$

(12.2) If the tower, $\langle \mathcal{R}_G(U_i^*) : i < \omega \rangle$, is wellfounded then the direct limit of

$$\left\langle \mathrm{Ult}\left(\hat{M}, \widehat{U_i^*}\right) : i < \omega \right\rangle$$

under the natural embeddings is wellfounded.

We come to the key point. Suppose $x \in \mathbb{R}^{V[G]}$ and that the direct limit of

$$\langle \mathrm{Ult}(\hat{M}, \hat{U}_i) : i < \omega \rangle$$

under the natural embeddings is wellfounded, where for each $i < \omega$, $U_i = \hat{U}$ and $U = h(x|i)$. Then x codes the sharp of a countable set $a \subseteq \mathbb{R}^{V[G]}$ such that $a = \mathbb{R}^{V[G]} \cap L(a)$. This follows by absoluteness since h witnesses that B is homogeneously Suslin and since $h \in \mathcal{X}_G$. The point is that $\hat{k}$ lifts to an elementary embedding

$$\hat{k}_G : \hat{M}[G] \to M[G]$$

and there exists $\hat{h} \in \hat{M}[G]$ such that $\hat{k}_G(\hat{h}) = h$.

Let B^* be the set of all finite sequences,

$$\langle x_0, \ldots, x_n \rangle \in (\mathbb{R}^{V[G]})^{<\omega}$$

such that for all $k \leq n$, the tower

$$\langle j(\mathcal{R}_G(h(x_k|i))) : i < \omega \rangle$$

is wellfounded. This implies that the towers,

$$\langle \hat{h}(x_k|i) : i < \omega \rangle = \left\langle \widehat{h(x_k|i)} : i < \omega \right\rangle,$$

for $k \leq n$, are jointly wellfounded over $\hat{M}$ (in the obvious sense).

The set B^* is definable from parameters in

$$\langle V[G]_{\omega+1}, A^G_\pi, \in \rangle.$$

We claim there must exist an infinite sequence,

$$\langle x_k : k < \omega \rangle \in V[G]$$

such that

(13.1) for all $n < \omega$, $\langle x_k : k \leq n \rangle \in B^*$,

(13.2) for all $k_1 < k_2 < \omega$, $(a_{k_1})^\# \subseteq (a_{k_2})^\#$

(13.3) $\cup \left\{ (a_k)^\# \mid k < \omega \right\} \neq (\cup \{a_k \mid k < \omega\})^\#$,

where for each $k < \omega$, $a_k \subseteq \mathbb{R}^{V[G]}$ is the countable set such that x_k codes $(a_k)^\#$.

If no such sequence $\langle x_k : k < \omega \rangle$ exists then there is a wellfounded relation which definable in

$$\langle V[G]_{\omega+1}, B^*, \in \rangle,$$

and which has rank greater that $\Theta^{L(A^G_\pi)}$ and this contradicts that B^* is projective in A^G_π. Here the relevant point is that if $\langle \sigma_k : k < \omega \rangle \in V[G]$ is an increasing sequence of countable elementary substructures of

$$\left(V[G]_{\omega+1}, (A^G_\pi, \mathbb{R}^{V[G]})^\# \right)$$

then there exists a sequence $\langle x_k : k < \omega \rangle$ such that for each $k < \omega$, x_k codes σ_k and such that for all $n < \omega$, $\langle x_k : k \leq n \rangle \in B^*$.

Therefore the sequence $\langle x_k : k < \omega \rangle$ exists as specified. For each $n < \omega$ let

$$\langle U^n_i : i < \omega \rangle = \langle h(x_n|i) : i < \omega \rangle.$$

Let $\langle U^*_i : i < \omega \rangle$ be the associated tower as defined above. We claim that the tower,

$$\langle j(\mathcal{R}_G(U^*_i)) : i < \omega \rangle$$

is not wellfounded. If not then the direct limit of the iteration of $\hat{M}$ given by the sequence of towers,

$$\left\langle \langle \widehat{U^n_i} : i < \omega \rangle : n < \omega \right\rangle$$

is wellfounded and this yields an elementary embedding,

$$j^* : \hat{M}[G] \to M^*[G] \subseteq V[G]$$

such that for all $k < \omega$, $x_k \in p[j^*(\hat{T})]$, where $\hat{T}$ is the image of T under the transitive collapse of $\mathcal{X}_G[G]$. But then by the elementary of j^* and the wellfoundedness of $M^*[G]$ there must exist a sequence $\langle y_k : k < \omega\rangle \in V[G]$ such that

(14.1) for all $k_1 < k_2 < \omega$, $(a_{k_1})^\# \subseteq (a_{k_2})^\#$

(14.2) $\cup\left\{(a_k)^\# \mid k < \omega\right\} \neq (\cup\{a_k \mid k < \omega\})^\#$,

where for each $k < \omega$, $a_k \subseteq \mathbb{R}^{V[G]}$ is the countable set such that y_k codes $(a_k)^\#$.

For each $k < \omega$, $y_k \in B$ and so $a_k \subseteq (A^G_\pi, \mathbb{R}^{V[G]})^\#$ and so

$$\cup\left\{(a_k)^\# \mid k < \omega\right\} \subseteq (A^G_\pi, \mathbb{R}^{V[G]})^\#,$$

and this is a contradiction. Therefore the tower,

$$\langle j(\mathcal{R}_G(U^*_i)) : i < \omega\rangle,$$

is not wellfounded and so the sequence of towers,

$$\langle\langle U^n_i : i < \omega\rangle : n < \omega\rangle,$$

witnesses that (10.1) holds.

We finish by outlining how (7.1) and (10.1) yield the counterexamples for the theorem. In V, for each function

$$g : \omega \to V_{\delta_0+2}$$

there is a reduction map

$$\mathcal{R}_g : V_\delta \to V_{\delta_0+2}$$

which is defined from g exactly as $\mathcal{R}_G$ is defined in $V[G]$ from G. For each such function g, let

$$\mathcal{X}_g = \{j(f)(g|i) \mid f : V_{\delta_0} \to V, i < \omega\} \prec M$$

and let M_g be the transitive collapse of $\mathcal{X}_g$. Let

$$j_g : V \to M_g$$

and

$$k_g : M_g \to M$$

be the associated elementary embeddings.

By absoluteness, using (7.1) and (10.1), there exists in V a function

$$G_0 : \omega \to V_{\delta_0+2}$$

such that

(15.1) $(F, e, Y_F, Z) \in \mathcal{X}_{G_0}$, (15.2) There exists a tower,

$$\langle U_i : i < \omega\rangle$$

of ultrafilters from $Y_F \cap \mathcal{X}_{G_0}$ such that both the towers,

$$\langle j(\mathcal{R}_{G_0}(U_i)) : i < \omega\rangle \text{ and } \langle j(\mathcal{R}_{G_0}(e(U_i))) : i < \omega\rangle,$$

are wellfounded.

(15.3) There exists a sequence,

$$\langle\langle U_i^n : i < \omega\rangle : n < \omega\rangle$$

of towers of ultrafilters from $Y_F \cap \mathcal{X}_{G_0}$ such that

a) For all $m < \omega$, the towers

$$\langle j(\mathcal{R}_{G_0}(U_i^n)) : i < \omega\rangle$$

for $n \leq m$ are jointly wellfounded,

b) The tower $\langle j(\mathcal{R}_{G_0}(U_i^*)) : i < \omega\rangle$ is not wellfounded.

Let $(F_{G_0}, e_{G_0}, Y_F^{G_0}, Z_{G_0})$ be the image of (F, e, Y_F, Z) under the transitive collapse of $\mathcal{X}_{G_0}$. Thus we established the following where for a tower

$$\langle U_i : i < \omega\rangle$$

of ultrafilters from M_{G_0}, we say the tower is wellfounded if the induced direct limit of the ultrapowers, $\text{Ult}(M_{G_0}, U_i)$, is wellfounded. The point of course is that we are not requiring that the sequence $\langle U_i : i < \omega\rangle$ be an element of M_{G_0} (and so the equivalence with countable completeness need not hold).

(16.1) There exists a tower,

$$\langle U_i : i < \omega\rangle$$

of ultrafilters from $Y_F^{G_0}$ such that both the towers,

$$\langle U_i : i < \omega\rangle \text{ and } \langle e_{G_0}(U_i) : i < \omega\rangle,$$

are wellfounded.

(16.2) There exists a sequence,

$$\langle\langle U_i^n : i < \omega\rangle : n < \omega\rangle$$

of towers of ultrafilters from $Y_F^{G_0}$ such that:

a) For all $m < \omega$, the towers $\langle\langle U_i^n : i < \omega\rangle : n \leq m\rangle$ are jointly wellfounded;

b) The tower $\langle U_i^* : i < \omega\rangle$ is not wellfounded.

From this we obtain the counterexamples witnessing the theorem, though first we produce counterexamples where the iteration trees are short and non-overlapping (not totally non-overlapping).

From [7], an iteration tree

$$\mathcal{T} = \langle N_m, E_m, j_{m,n} : m < \eta, m <_{\mathcal{T}} n\rangle.$$

on a premouse (N, δ_N) is an *alternating chain* if $\eta \leq \omega$ and if for all $0 < n < m < \eta$, $n <_{\mathcal{T}} m$ if and only if n and m are either both even or both odd. Thus if $\eta = \omega$ then $\mathcal{T}$ has exactly two cofinal branches, an even branch and an odd branch.

Fix $\delta_0 < \gamma < \Theta < \kappa_{G_0}$ such that

$$(M_{G_0} \cap V_\Theta, \gamma)$$

is a premouse such that γ is a limit of Woodin cardinals in M_{G_0} and where κ_{G_0} is the image of κ under the transitive collapse of $\mathcal{X}_{G_0}$. Note that if $\mathcal{T} \in M_{G_0}$ is a countable iteration tree on the premouse $(M_{G_0} \cap V_\Theta, \gamma)$ then $\mathcal{T}$ defines a iteration tree on M_{G_0}.

Fix a function,

$$H : Y_F^{G_0} \cup Z_{G_0} \to M_{G_0}$$

such that

(17.1) $H \in M_{G_0}$,

(17.2) $H(U) \in U$ for all $U \in Y_F^{G_0} \cup Z_{G_0}$,

(17.3) For all towers $\langle U_i : i < \omega\rangle \in M_{G_0}$ of ultrafilters from $Y_F^{G_0} \cup Z_{G_0}$, the tower is wellfounded if and only if there exists a function

$$f : \omega \to M_{G_0}$$

such that for all $i < \omega$, $f|i \in H(U_i)$.

The existence of H follows from the fact,

$$|Y_F^{G_0} \cup Z_{G_0}|^{M_{G_0}} < \kappa_{G_0},$$

since each ultrafilter $U \in Y_F^{G_0} \cup Z_{G_0}$ is κ_{G_0}-complete in M_{G_0}. One property that H has and which will need need is the following.

(18.1) Suppose that $j^* : M_{G_0} \to N^*$ is an elementary embedding and that

$$\langle\langle U_i^k : i < \omega\rangle : k < \omega\rangle$$

is a sequence of towers of ultrafilters from $j^*\left(Y_F^{G_0} \cup Z_{G_0}\right)$ and that for each $k < \omega$ there is a function

$$f_k^* : \omega \to N^*$$

such that for all $i < \omega$, $f_k^*|i \in j^*(H)(U_i^k)$. Then the tower,

$$\langle U_i^* : i < \omega\rangle,$$

is wellfounded over N^*.

There are Woodin cardinals in M_{G_0} in the interval, (δ_0, κ_{G_0}) and so applying the basic construction of [7] within M_{G_0} to the set of ultrafilters, $Y_F^{G_0}$, one obtains a function

$$I : (Y_F^{G_0} \cup Z_{G_0})^{<\omega} \to M_{G_0}$$

such that $I \in M_{G_0}$ and such that the following hold.

(19.1) For all finite towers $s \in (Y_F^{G_0} \cup Z_{G_0})^{<\omega}$, $I(s)$ is a finite alternating chain on the premouse $(M_{G_0} \cap V_\Theta, \gamma)$ with all associated critical points above δ_0 and which is totally non-overlapping.

(19.2) If s_1 and s_2 are finite towers from $Y_F^{G_0} \cup Z_{G_0}$ and s_2 extends s_1 then $I(s_2)$ extends $I(s_1)$,

(19.3) for all infinite towers $\langle U_i : i < \omega\rangle \in M_{G_0}$ of ultrafilters from $Y_F^{G_0} \cup Z_{G_0}$,

a) the tower is wellfounded if and only if the even branch of the alternating chain of length ω given by $\{I(\langle U_i : i \leq n\rangle) \mid n < \omega\}$ is wellfounded,

b) the tower is not wellfounded if and only if the odd branch of the alternating chain of length ω given by $\{I(\langle U_i : i \leq n\rangle) \mid n < \omega\}$ is wellfounded.

(19.4) for all infinite towers $\langle U_i : i < \omega\rangle$ of ultrafilters from $Y_F^{G_0} \cup Z_{G_0}$, if the even branch of the alternating chain of length ω given by $\{I(\langle U_i : i \leq n\rangle) \mid n < \omega\}$ is wellfounded and

$$j^* : M_{G_0} \to M^*$$

is the induced elementary embedding, then there exists a function

$$f^* : \omega \to M^*$$

such that for all $i < \omega$, $f^*|i \in j^*(H(U_i))$.

We emphasize that (19.4) holds for all towers from $Y_F^{G_0} \cup Z_{G_0}$, even those towers which are not elements of M_{G_0}.

Let $\langle U_i : i < \omega\rangle$ be a tower of ultrafilters from $Y_F^{G_0}$ which witnesses that (16.1) holds. Let $\mathcal{T}_0$ be the alternating chain on $(M_{G_0} \cap V_\Theta, \gamma)$ given by applying I to the initial segments of $\langle U_i : i < \omega\rangle$. We claim that both the even and odd branches of $\mathcal{T}_0$ are wellfounded acting on M_{G_0}. Suppose not and we first suppose that the even branch is not wellfounded acting on M_{G_0}.

Let N_0 be the direct limit of $\mathrm{Ult}(M_{G_0}, U_i)$ and let

$$j_0 : M_{G_0} \to N_0$$

be the associated elementary embedding. Since $\Theta < \kappa_{G_0}$ and since each of the ultrafilters, U_i, is κ_{G_0}-complete, $j_0(\mathcal{T}_0) = \mathcal{T}_0$.

There is a function

$$f : \omega \to N_0$$

such that for all $i < \omega$, $f|i \in j_0(H)(e_{G_0}(U_i))$ and even branch of $j_0(\mathcal{T}_0)$ is not wellfounded acting on N_0. This implies that the tree of attempts to refute the property that $j_0(I)$ must have in N_0, is not wellfounded and this is a contradiction.

We next suppose that the odd branch of $\mathcal{T}_0$ is not wellfounded acting on M_{G_0}. Now let N_0 be the direct limit of $\mathrm{Ult}(M_{G_0}, e_{G_0}(U_i))$ and let

$$j_0 : M_{G_0} \to N_0$$

be the associated elementary embedding. Again we have $j_0(\mathcal{T}_0) = \mathcal{T}_0$ and in this case there is a function

$$f : \omega \to N_0$$

such that for all $i < \omega$, $f|i \in j_0(H)(e_{G_0}(U_i))$. Thus in N_0 the tree of attempts to find a tower, $\langle U'_i : i < \omega\rangle$, of ultrafilters from $j_0(Y_F^{G_0})$ and a function

$$f' : \omega \to N_0,$$

such that

(20.1) the odd branch of the alternating chain given by applying $j_0(I)$ to the initial segments of $\langle U'_i : i < \omega\rangle$ is not wellfounded acting on N_0,

(20.2) for all $i < \omega$, $f'|i \in j_0(e_{G_0}(U'_i))$.

This again contradicts the properties that $j_0(I)$ must have in N_0, since by the properties of $j_0(e_{G_0})$ and $j_0(H)$, the tower $\langle U'_i : i < \omega\rangle$ must be illfounded in N_0

The same argument shows that the odd branch of $\mathcal{T}_0$ must be wellfounded acting on M_{G_0}. The iteration tree given by j_{G_0} followed by $\mathcal{T}_0$ is a non-overlapping tree on V with exactly two cofinal branched each of which is wellfounded.

Let $\langle\langle U_i^n : i < \omega\rangle : n < \omega\rangle$ be a sequence of towers of ultrafilters from $Y_F^{G_0}$ which witnesses (16.2).

Let

$$j_0 : M_{G_0} \to N_0$$

be the elementary embedding given by $\langle U_i^0 : i < \omega\rangle$ and by induction on $k < \omega$, let

$$j_{k+1} : N_k \to N_{k+1}$$

be the elementary embedding given by the tower,

$$\langle (j_k \circ \cdots \circ j_0)(U_i^{k+1}) : i < \omega\rangle.$$

Since for all $n < \omega$, the towers

$$\langle\langle U_i^k : i < \omega\rangle : k \leq n\rangle$$

are jointly wellfounded, for each $k < \omega$, the tower

$$\langle (j_k \circ \cdots \circ j_0)(U_i^{k+1}) : i < \omega\rangle$$

is wellfounded over N_k.

Since the tower $\langle U_i^* : i < \omega\rangle$ is not wellfounded over M_{G_0}, the direct limit of the N_k under the maps, j_{k+1}, is not wellfounded.

For each $k < \omega$, there is a function

$$f_k : \omega \to N_k$$

such that for all $i < \omega$,

$$f|i \in (j_k \circ \cdots \circ j_0)\left(H(U_i^k)\right).$$

Define an iteration tree

$$\mathcal{T} = \langle M_\alpha, E_\beta, j^{\mathcal{T}}_{\gamma,\alpha} : \alpha < \eta, \beta + 1 < \omega \cdot \omega, \gamma <_{\mathcal{T}} \alpha\rangle$$

as follows. We define $\mathcal{T}|(\omega \cdot (k+1) + 1)$ by induction on k.

Let $\mathcal{T}|(\omega + 1)$ be the iteration tree given by applying I to the initial segments of $\langle U_i^0 : i < \omega\rangle$ and then taking the even branch. Then by induction, let

$$\mathcal{T}|(\omega \cdot (k+2) + 1) = \mathcal{T}|(\omega \cdot (k+1) + 1) + \mathcal{S},$$

where $\mathcal{S}$ is the iteration tree given by applying $j^{\mathcal{T}}_{0,\omega\cdot(k+1)}(I)$ to the initial segments of the tower,

$$\langle j^{\mathcal{T}}_{0,\omega\cdot(k+1)}(U_i^{k+1}) : i < \omega\rangle$$

and then taking the even branch.

If at some stage k, the definition of $\mathcal{T}|(\omega\cdot(k+1)+1)$ fails (because of illfoundedness) then for all $n \geq 0$, the construction must fail over N_n using

$$j_n \circ \cdots \circ j_0(I)$$

and the towers, $\langle\langle j_n \circ \cdots \circ j_0(U_i^m) : i < \omega\rangle : m \leq k\rangle$.

But (taking $n = k + 1$) this yields that for some $m \leq k$ there exists an elementary embedding,

$$j^* : N_{k+1} \to N^*,$$

such that even branch of the iteration tree on N^* given by applying

$$j^* \circ j_{k+1} \circ \cdots \circ j_0(I)$$

to the initial segments of the tower,

$$\langle j^* \circ j_{k+1} \circ \cdots \circ j_0(U_i^m) : i < \omega\rangle,$$

is not wellfounded (over N^*). But for all $i < \omega$,

$$j^* \circ j_{k+1} \circ \cdots \circ j_{m+1}(f_m)|i \in j^* \circ j_{k+1} \circ \cdots \circ j_0(H(U_i^m))$$

and this is a contradiction.

Thus the definition of $\mathcal{T}$ succeeds to define an iteration tree on M_{G_0} of length $\omega \cdot \omega$. The tree $\mathcal{T}$ has only one cofinal branch. We must show that this branch is not wellfounded. Assume toward a contradiction that this branch is wellfounded and let

$$j^* : M_{G_0} \to M^*$$

be the associated elementary embedding. By the key property (19.4) of I, applied in $M_{\omega\cdot(k+1)}$ to $j^{\mathcal{T}}_{0,\omega\cdot(k+1)}(I)$, it follows that for each $k<\omega$, there is a function

$$f^*_k : \omega \to M^*$$

such that for all $i<\omega$,

$$f^*_k|i \in j^*(H)(U^k_i).$$

By (18.1) this implies that the tower,

$$\langle j^*(U^*_i) : i<\omega\rangle$$

is wellfounded over M^* and this contradicts that the tower,

$$\langle U^*_i : i<\omega\rangle,$$

is not wellfounded over M_{G_0}.

The trees $\mathcal{T}_0$ and $\mathcal{T}$ are each short totally non-overlapping trees on M_{G_0} with all critical points above δ_0. Further

$$M_{G_0} = \mathrm{Ult}(V, E_0)$$

where E_0 is an extender with $\mathrm{CRT}(E_0)=\delta_0$ and with

$$\mathrm{LTH}(E_0) \le |M_{G_0}\cap V_{\delta_0+2}|^{M_{G_0}}.$$

The difficulty is that $j_{G_0}(\delta_0) > \kappa_{G_0}$ and these trees are each based on the premouse, $(M_{G_0}\cap V_\Theta, \gamma)$ and so while the induced iteration trees on V are necessarily non-overlapping, the induced iteration trees on V are not totally non-overlapping.

There is a function

$$f : \delta_0 \to \delta_0$$

such that for all $\alpha<\delta_0$ and for all $\gamma<\delta_0$, for all $W\subseteq V_{\gamma+2}$ of γ-complete ultrafilters on V_γ, $\gamma<\delta_0$ and for all $W\subseteq V_{\gamma+2}$ of γ-complete ultrafilters on V_γ if $|W|\le|V_{\alpha+2}|$ and if $f(\alpha)<\gamma$ then for cofinally many $\gamma^*<\delta_0$ there exists a set W^* of γ^*-complete ultrafilters on V_{γ^*} such that W is tower isomorphic to W^*. Clearly we can choose f such that f is definable in V_{δ_0} and so by choice of κ,

$$\kappa_{G_0} > j_{G_0}(f)(\delta_0).$$

Note that $j_{G_0}(\delta)=\delta$ and so

$$j_{G_0}(V_{\delta_0}) \prec j_{G_0}(V_\delta) = M_{G_0}\cap V_\delta.$$

This implies that in M_{G_0}, for cofinally many $\gamma<\delta$ there exist sets $W^\gamma_0, W^\gamma_1 \in M_{G_0}$ such that

(21.1) $W^\gamma_0 \subseteq M_{G_0}\cap V_{\gamma+2}$ and W^γ_0 is tower isomorphic in M_{G_0} with $Y^{G_0}_F$,

(21.2) $W^\gamma_1 \subseteq M_{G_0}\cap V_{\gamma+2}$ and W^γ_1 is tower isomorphic in M_{G_0} with Z_{G_0}.

By choosing $\gamma > j_{G_0}(\delta_0)$ sufficiently large so that there are Woodin cardinals in M_{G_0} in the interval, $(j_{G_0}(\delta_0), \gamma)$, and using (W_0^γ, W_1^γ) in place of $(Y_F^{G_0}, Z_{G_0})$ one produces $\mathcal{T}_0, \mathcal{T}$ such that the induced iteration trees on V are each totally non-overlapping. □

If δ is supercompact then the counterexamples of Theorem 97 can easily be constructed (following the proof of Theorem 97) such that for a given set $\mathcal{E} \subseteq V_\delta$ of extenders which witnesses that δ is a Woodin cardinal and which is closed under initial segments, each extender, E, of the iteration tree, except for the first extender E_0, has the following properties in the model from which E is selected.

(1) $E \in \mathcal{E}^*$;

(2) $\text{LTH}(E) = \rho(E)$ and $\text{LTH}(E)$ is strongly inaccessible;

where $\mathcal{E}^*$ is the image of $\mathcal{E}$ in that model. Further (as in the proof of Theorem 97) E_0 can be chosen to be very "short" :

(3) $\text{LTH}(E_0) \leq \left(2^{2^\kappa}\right)^{\text{Ult}(V,E_0)}$ where $\kappa = \text{CRT}(E_0)$.

Let $\mathcal{F}_\mathcal{E}$ be the set of all short extenders $F \in V_\delta$ such that F satisfies (2) and such that if $\gamma = \rho(F)$, then

$$j_F(\mathcal{E}) \cap V_\gamma = \mathcal{E} \cap V_\gamma$$

and $(V_\gamma, \mathcal{E} \cap V_\gamma) \prec (V_\delta, \mathcal{E})$.

In the case of the counterexample to UBH still more can be required and this also follows from the proof of Theorem 97. If $\mathcal{T}$ is the iteration tree on $\text{Ult}(V, E_0)$ with exactly two cofinal branches, b and c, each of which are wellfounded, then there exists an extender $F_0 \in \mathcal{F}_\mathcal{E}$, and elementary embeddings,

$$\pi_b : \text{Ult}(V, E_0) \to \text{Ult}(V, F_0)$$

and

$$\pi_c : \text{Ult}(V, E_0) \to \text{Ult}(V, F_0)$$

each determined by their restrictions to $\text{LTH}(E_0)$ such that

(4) $\mathcal{T}$ copies by π_b to an iteration tree on $\text{Ult}(V, F_0)$ for which c copies to an illfounded branch,

(5) $\mathcal{T}$ copies by π_c to an iteration tree on $\text{Ult}(V, F_0)$ for which b copies to an illfounded branch,

(6) $\mathcal{T}$ copied by π_b yields an iteration tree on V (with first extender given by F_0) which is totally non-overlapping,

(7) $\mathcal{T}$ copied by π_c yields an iteration tree on V (with first extender given by F_0) which is totally non-overlapping.