The higher sharp (Yizheng Zhu)

Th. (Martin - Steel) n Woodins + a measurable above $\Rightarrow$ $\underset{\sim}{\Pi}^1_{n+1}$ det.

Th. (Neeman - Woodin) TFAE.

(1) $\forall x \in \mathbb{R}$ $\exists$ ctbly. iterable $M_n^\#(x)$

(2) $\underset{\sim}{\Pi}^1_{n+1}$ det.

assume PD from now on.

	HF	$L_{\omega_1^{ck}}$	L	$L_{\kappa_3}(T_2)$ (least T_2-admissible) ~ M_1
structure (DST / IMT)	HF	$L_{\omega_1^{ck}}$	L	$L_{\kappa_3}(T_2)$ ~ M_1
mouse set	rec. reals	$\mathbb{R} \cap L_{\omega_1^{ck}}$ $=$ $\{x : x \text{ is } \Delta^1_1\}$	$\mathbb{R} \cap L$ $=$ $\{x : x \text{ is } \Delta^1_2 \text{ in a ctbl. ordinal}\}$	$\mathbb{R} \cap L_{\kappa_3}(T_2) = \mathbb{R} \cap M_1 = \{x : x \text{ is } \Delta^1_3 \text{ in a ctbl. ordinal}\}$
definability	$\Sigma_1^{HF} = \Sigma^0_1$	$\Sigma_1^{L_{\omega_1^{ck}}} = \Pi^1_1$	$\Sigma_1^L = \Sigma^1_2$	$\Sigma_1^{L_{\kappa_3}(T_2)}(\{T_2\}) = \Pi^1_3$
jump operator over mouse set (DST / IMT)	$0'$	Kleene's O	$0^\#$	$0^{2\#}$ ~ $M_1^\#$

$L(T_3)$
$\wr$
M_2

$\mathbb{R} \cap L(T_3) =$

$\mathbb{R} \cap M_2 =$

$\{x : x \text{ is } \Delta^1_4$
in a cth. ordinal $\}$

$\Sigma_1^{L(T_3)} =$
$\Sigma_1^{M_2} = \Sigma^1_4$

$0^{3\#}$
$M_2^\#$

<u>question</u>: what is the fine structure of $L_{\kappa_3}(T_2)$?

$0^{3\#}$ is the theory of $L_{\tilde{\delta}^1_3}[T_3]$ with level 3 indiscernibles.

<u>Th</u> (Steel) Let $M^\#_{2,\infty}$ be the direct limit of ctbl. iterates of $M^\#_2$. Then $\underset{\sim}{\delta}^1_3$ is the least $< \delta^{M^\#_{2,\infty}}$-strong in $M^\#_{2,\infty}$, where $\delta^{M^\#_{2,\infty}}$ is the bottom woodin, and $L_{\underset{\sim}{\delta}^1_3}[T_3] = M^\#_{2,\infty} | \underset{\sim}{\delta}^1_3$.

level 3 indiscernibles:

indiscernibility w.r.t. tuples of ind.

indexing by trees

<u>question:</u> are the following equivalent?

(1) $\underset{\sim}{\Pi}^1_n$ det. + Π^1_{n+1} det.

(2) there is a countably iterable $M^\#_n$.

$n=0$ true by martin-harrington

$n=1$ true by ~~[illegible]~~ woodin

$n>1$ odd. true.

$n>1$ even ?

conjecture :

$$\underset{\sim}{\Delta}^1_{2n+1} \text{ det.} + \Pi^1_{2n+1} \text{ det.} \Rightarrow M^\#_{2n} .$$

e.g., $\underset{\sim}{\Delta}^1_3$ det. $\Rightarrow$ $\forall x\ L(T_3, x)$ is a mouse.

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effective well-ordered cardinals		ω	ω_1	$u_2, u_3, \dots, u_\omega$

last level : $\underset{\sim}{\delta}^1_3$.

<u>def.</u> I^x = the class of $L[x]$ – silver ind.

$$\bigcap_{x \in \mathbb{R}} I^x = \text{uniform ind.}$$
$$= \{ u_\alpha : \alpha \in \mathrm{OR} \}$$

$$u_1 = \omega_1$$
$$u_\alpha \le \aleph_\alpha .$$

<u>fact</u>. if $\xi < u_{n+1}$, then $\exists$ skolem term τ and a real x s.t.

$$\xi = \tau^{L[x]}(x, u_1, \ldots, u_n).$$

a <u>sharp code</u> is a pair $(\ulcorner\tau\urcorner, x^{\#})$ s.t.

$$\tau^{L[x]}(x, u_1, \ldots, u_n) \in OR$$

the set of sharp codes is Π^1_2.

$$|(\ulcorner\tau\urcorner, x^{\#})| = \tau^{L[x]}(x, u_1, \ldots, u_n).$$

"v, w are sharp codes $\wedge$ $|v| = |w|$" is Δ^1_3.

if $\sigma : m+1 \longrightarrow n+1$ is order preserving.

then $j^{\sigma} : u_{m+1} \longrightarrow u_{n+1}$

$$j^{\sigma}(\tau^{L[x]}(x, u_1, \ldots, u_m)) =$$

$$\tau^{L[x]}(x, u_{\sigma(1)}, \ldots, u_{\sigma(m)}).$$

def. let T be a rec. tree on ω s.t.

$z \in [T]$ iff $z \in 2^\omega$ is a remarkable EM blueprint over a real.

fix an enumeration of skolem terms $(\tau_i)_{i<\omega}$. τ_i l_i+1 ary.

$(s, (\alpha_0, \dots, \alpha_n)) \in T_2$ iff

for each $i, k \leq n$, for each order pres. $\sigma : l_i + 1 \to l_k + 1$, if

"$\tau_i(\dot{x}, v_{\sigma(1)}, \dots, v_{\sigma(l_i)}) = \tau_k(\dot{x}, v_1, \dots, v_{l_k})$"

is true in s, then $j^\sigma(\alpha_i) = \alpha_k$.

uth. (martin – solovay)

$$p[T_2] = \{ x^\# : x \in \mathbb{R} \}.$$

$\forall x$ if $(\alpha_i)_{i<\omega}$ is the leftmost branch of $(T_2)_{x^\#}$, then

$$\alpha_i = \tau_i^{L[x]}(x, u_1, \dots, u_{l_i}).$$

<u>thm.</u> (becker – kechris, kechris – martin)

supp. $A \subset \mathbb{R} \times u_\omega$. TFAE.

(1) A is Π^1_3.

(2) there is a Σ_1 fmla s.t.

$$(x, \alpha) \in A \iff L_{u_3^x}(T_2, x) \models \varphi(T_2, x, \alpha)$$

<u>dfn.</u> $A \subset \mathbb{R} \times u_\omega$ is Π^1_3 iff

$$A^* = \{ (x, w) : w \text{ is a sharp code} \wedge (x, |w|) \in A \} \text{ is } \Pi^1_3 .$$

<u>dfn.</u> $O^{T_2} = \{ (\ulcorner \varphi \urcorner, \alpha) : \varphi \text{ is } \Sigma_1, L_{u_3}(T_2) \models \varphi(\alpha) \}$.

this is a level-2 analogue of kleene's O, a Π^1_3 subset of $\omega \times u_\omega$, universal.

<u>def.</u> $(0^{2\#})_n = \{ (\ulcorner\varphi\urcorner, \ulcorner\psi\urcorner) :$

φ, ψ are Σ_1, $\exists \alpha < u_n$

$(\ulcorner\varphi\urcorner, \alpha) \in 0^{T_2} \wedge$

$\forall \eta < \alpha \; (\ulcorner\psi\urcorner, \eta) \notin 0^{T_2} \}$

$(0^{2\#})_n$ is obtained by the difference operation on the $2^{\underline{nd}}$ coordinate of $0^{T_2} \cap (\omega \times u_n)$.

<u>def.</u> $0^{2\#} = \bigoplus_{n<\omega} (0^{2\#})_n$.

<u>th.</u> $0^{2\#} \equiv_m M_1^{\#}$

many-one-deg. the

$x^{2\#} \equiv_m M_1^{\#}(x)$, uniformly in x.

<u>th</u>: (martin – neeman)

$$0^{\#} \equiv_m \bigoplus_{\substack{n \\ \text{eff}}} \left(\partial (\omega n - \Pi^1_1) \text{ univ. subset of } \omega \right)$$

$$M_1^{\#} \equiv_m \bigoplus_{\substack{n \\ \text{eff}}} \left(\partial^2 (\omega n - \Pi^1_1) \text{ univ. subset of } \omega \right)$$

by BK – KM,

$$0^{2\#} = \bigoplus_{\substack{n \\ \text{eff}}} \left((u_n - \Pi^1_3) \text{ univ. subset of } \omega \right)$$

<u>def</u>. $\beta < \omega_1^{CK}$. $A \subset \mathbb{R}$ is $\beta - \Pi^1_1$ iff

$\exists \, \Pi^1_1 \; B \subset \beta \times \mathbb{R}$ s.t.

$$x \in A \iff \exists \alpha \, (\alpha \text{ odd} \wedge (\alpha, x) \in B \wedge \forall \eta < \alpha \; (\eta, x) \notin B)$$

$\beta < u_\omega$. $A \subset \mathbb{R}$ is $\beta - \Pi^1_3$ iff there is

a Π^1_3 $B \subset \beta \times \mathbb{R}$ s.t.

$$x \in A \iff \exists \alpha \, (\alpha \text{ odd} \wedge (\alpha, x) \in B \wedge \forall \eta < \alpha \; (\eta, x) \notin B).$$

<u>thm.</u> $\mathcal{O}^2(<\omega^2 - \Pi^1_1) = <u_\omega - \Pi^1_3$.

<u>th.</u> $0^{3\#} \equiv_m \bigoplus_{n \text{ eff}} (\mathcal{O}(u_n - \Pi^1_3)$ subsets of $\omega)$

<u>cor.</u> $0^{3\#} \equiv_m M_2^\#$,

$x^{3\#} \equiv_m M_2^\#(x)$, uniformly in x.

higher up:

level-3 uniform ind. $(u_{\xi;3} : \xi \le \omega^{\omega^\omega})$.

every ordinal in $u_{\omega^{\omega^\omega}}$ is represented by a level-3 sharp code $(\ulcorner \tau \urcorner, R, x^{3\#})$.

T_4 = level-4 martin-solovay tree.

<u>th.</u> $A \subset \mathbb{R} \times u_{\omega^{\omega^\omega}}$ is Π^1_5 iff

$\exists \varphi$ Σ_1 s.t.

$(x,\alpha) \in A \iff L_{\kappa_5^x}(T_4, x) \models \varphi(T_4, x, \alpha)$.

Thm. (Jackson) AD + DC. then

$\underset{\sim}{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1}$. let A be the minimum set coding $\{\omega, u_n : n<\omega\}$, closed under $+, \cdot$ (ordinal arithmetic),

$otp(A) = \omega^{\omega^\omega}$.

for $\xi < \omega^{\omega^\omega}$, let $\hat{\xi} = \xi^{th}$ member of A.

$\hat{0} = \omega$, $\hat{\omega} = \omega_1$, $\widehat{\omega^\omega} = u_2$.

$cf(\aleph_{\omega+1+\xi+1}) = \aleph_{\omega+1+\bar{\xi}+1}$, where

$cf(\hat{\xi}) = \hat{\bar{\xi}}$.

eff. w.o. cardinals!

$\alpha \leq \underset{\sim}{\delta}^1_{2n+1}$ is a $\underset{\sim}{\Delta}^1_{2n+1}$ cardinal iff

there is no $\beta < \alpha$ and $\underset{\sim}{\Delta}^1_{2n+1}$ norms

$\varphi : \mathbb{R} \to \beta$, $\psi : \mathbb{R} \to \alpha$, and $\pi : \beta \twoheadrightarrow \alpha$

s.t. $\mathrm{Code}_{\varphi,\psi}(\pi)$ is $\underset{\sim}{\Delta}^1_{2n+1}$.

assume $\underset{\sim}{\Delta}^1_{2n}$-det. then the $\underset{\sim}{\Delta}^1_{2n+1}$-card.

have otp $E(2n+1) + 1$

$E(0) = 0$.

$E(i+1) = \omega^{E(i)}$

they are: finite, ω, ω_1, u_2, $u_3, \ldots u_\omega$

$\underset{\sim}{\delta}^1_3, \ldots, \underset{\sim}{\delta}^1_{2n-1}$, $u_{\xi;2n+1}$ for

$\xi \leq E(2n+1)$, $\underset{\sim}{\delta}^1_{2n+1}$.

<u>thm.</u> (Silver, Hjorth, Schlicht, Zhu)

~~E is a Π^1_k thin equivalence relation~~

~~on $\mathbb{R}$~~. $M \models ZFC$ a proper class.

(1) $k = 2n+1$ odd. TFAE.

(a) $\forall x\ \exists y \in M\ \ x E y$ for all E, thin Π^1_k e. relation

(b) $M \prec_{\Sigma^1_{2n+1}} V$, and

$\left(\underset{\sim}{\Delta}^1_{2n+1}\text{-card} < \delta^1_{2n+1}\right)^V =$

$\left(\quad\text{———}\quad \right)^M$.

(2) $k = 2n+2$ even. TFAE.

(a) $\forall x \ \exists y \in M \quad x E y$ for all E, Π^1_k e. rel.

(b) $M \prec_{\Sigma^1_{2n+3}} V$ and

$$\left((\underset{\sim}{\Delta}^1_{2n+1}\text{-card}) \le \delta^1_{2n+1} \right)^V =$$

$$\left(\text{——————} \right)^M .$$