Omer II

defn $R(\kappa)$. conditions $p \in R(\kappa)$ are

fin. sequences $p = d_0 {}^\frown d_1 {}^\frown \cdots {}^\frown d_\ell$, $\ell < \omega$.

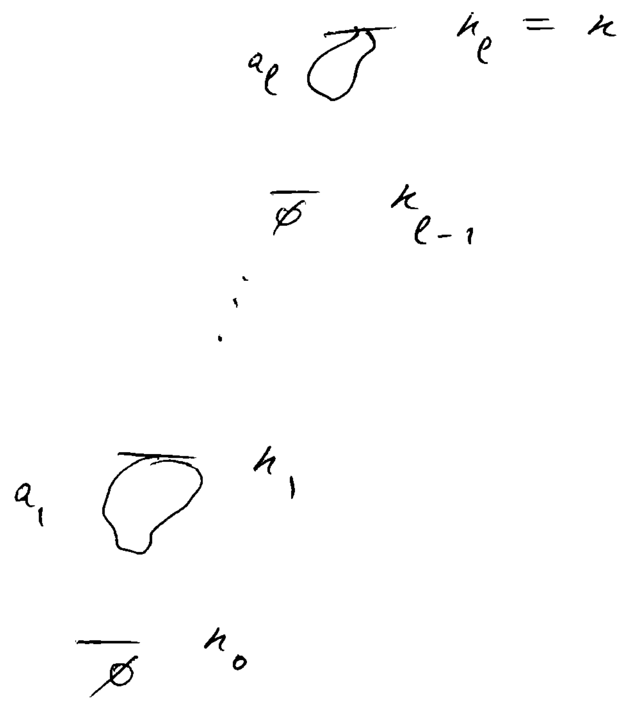
s.t. each $d_i = (\kappa_i, a_i)$ where

$\kappa_i \leq \kappa$, $a_i \in \mathcal{F}_{\kappa_i}$.

$(\kappa_i : i \leq \ell)$ is increasing.

$\kappa_\ell = \kappa$.

for any $i > 0$, $a_i \subset \kappa_i \setminus \kappa_{i-1}$.



notation : for p as above, with

$\ell = \ell^p \qquad \forall i \leq \ell \qquad \kappa_i = \kappa_i^p$

$a_i = a_i^p$ etc.

Supp. $p, p^* \in R(\mathcal{U})$.

p^* is a direct extens of p,
denoted $p^* \leq^* p$.
iff $\ell^p = \ell^{p^*}$, $\kappa_i^p = \kappa_i^{p^*}$, all $i \leq \ell^p$
$a_i^{p^*} \subset a_i^p$.

a condn $p' \in R(\mathcal{U})$ is a 1-point
extens of p if
$\exists j \leq \ell^p \ \exists \nu \in a_j^p$ s.t. or $\langle \nu, \emptyset \rangle$ if $o(\nu) = 0$

$$p' = d_0^p \frown d_1^p \frown \ldots d_{j-1}^p \frown \langle \nu, a_j^p \cap \nu \rangle \frown$$
$$\langle \kappa_j, a_j^p \setminus \nu \rangle \frown d_{j+1}^p \frown \ldots \frown d_\ell^p$$

def. ~~let~~ let $p, q \in R(\mathcal{U})$, q extends
p iff it is obtained by fin. many

one point extension and direct extension.

equiv, $q \leq^* p^\frown \vec{\nu}$, some $\vec{\nu} = \langle \nu_0, \ldots, \nu_n \rangle$

↑ put them in somewhere, not nec. at the end.

def. let $G \subset R(\mathcal{U})$ be a generic filter. the radin generic club is

$$C = \{ \alpha < \kappa : \exists p \in G \ \exists i < \ell^p \ \ \alpha = \kappa_i^p \}.$$

then, $V[G] = V[C]$.

$\boxed{\text{examples}}$

(1) suppose $o^{\mathcal{U}}(\kappa) = 1$. ($\mathcal{U}_{\kappa,0}$ is the only measure on κ in $\mathcal{U}$).

$\mathcal{F}_\kappa = \mathcal{U}_{\kappa,0}$

let $p = (\kappa, A_0)$ $A_0 = \{\alpha < \kappa : o^{\mathcal{U}}(\alpha) = 0\} \in \mathcal{F}_\kappa$.

force w/ $R(U)$ below $p = \langle \kappa, A_0 \rangle$.

then $C = (\kappa_n : n < \omega)$

is cof. in κ.

C is a Prikry seq. w.r.t. $U_{\kappa,0}$

$cf^{V[C]}(\kappa) = \omega$.

(2) $o^U(\kappa) = 2$.

$\mathcal{F}_\kappa = U_{\kappa,0} \cap U_{\kappa,1}$.

let $A_1 = \{ \alpha < \kappa : o^U(\alpha) \le 1 \}$,

$A_1 \in \mathcal{F}_\kappa$.

force below (κ, A).

will add a C of otp ω^2.

$cf^{V[C]}(\kappa) = \omega$.

③ $o^{\mathcal{U}}(\kappa) = \tau < \kappa$, τ reg. cardinal, $\tau > \omega$.

$otp(C) = \omega^{\tau} = \tau$.

$cf^{V[C]}(\kappa) = \tau$.

④ $o^{\mathcal{U}}(\kappa) = \kappa$.

then $otp(C) = \kappa$.

& $cf^{V[C]}(\kappa) = \omega$.

<u>why?</u> for each $\nu_0 < \kappa$ define a seq.

$(\nu_n : n < \omega)$ by $\nu_{n+1} = \min(\nu \in C \setminus (\nu_n + 1) :$
$o(\nu) \geq \nu_n)$.

$\nu_\omega = \sup \nu_n$.

$\nu_\omega = \kappa$ or $\nu_\omega \in C$.

since $o(\kappa) = \kappa$, then $A = \{\alpha < \kappa :$
$o(\alpha) < \alpha\} \in \mathcal{F}_\kappa$.

by density, there is $\nu_0 < \kappa$ s.t.

$C \setminus \nu_0 \subset A$.

$$o(\nu_\omega) \geq \bigcup_n o(\nu_{n+1})$$

$$\geq \bigcup_n \nu_n = \nu_\omega .$$

if $\nu_\omega \in C$, then $o(\nu_\omega) \geq \nu_\omega$. $\cancel{\S}$.

<u>theorem</u> 1 (mitchell) if $cf(o^U(\kappa)) \geq \kappa^+$, then κ remains regular in $V[C]$.

<u>theorem</u> 2 if $cf(o^U(\kappa)) \geq \kappa^+$ and $S \in \mathcal{P}(\kappa) \cap V$.
then S is stationary in $V[C]$ iff $S \in U_{\kappa,\tau}$ for unboundedly many $\tau < o(\kappa)$.

<u>Corollary</u>.

① if $o^U(\kappa) = \kappa^+ \cdot \kappa^+$, then κ is mahlo in $V[C]$.

why? let $S = \{\alpha < \kappa : o(\alpha) = \alpha^+ \cdot (\tau + 1)$, some $\tau < \alpha^+\}$.

S is stationary in $V[C]$ by th. 2.

also $\mathrm{cf}(o(\alpha)) = \alpha^+$ for any $\alpha \in S \cap C$

$\Rightarrow$ α is regular in $V[C]$ by the proof of th. 1.

<u>basic properties of $R(\mathcal{U})$.</u>

① $(R(\mathcal{U}), \leq, \leq^*)$ is a Prikry type forcing.

namely, for any $p \in R(\mathcal{U})$ and σ, a statement in the forcing language, then ex. $p^* \leq^* p$ s.t. p^* decides σ.

② f.a. $p, q \in R(\mathcal{U})$, if $\ell^p = \ell^q$ and $\kappa_i^p = \kappa_i^q$ for all $i \leq \ell^p$, then p, q are compatible.

$\Rightarrow$ $R(\mathcal{U})$ has the κ^+-c.c.

③ for every $p = d_0{}^\frown \dots {}^\frown d_\ell$ and $m < \ell$

$$R(\mathcal{U})/p \cong R(\mathcal{U} \restriction \kappa_m + 1)/d_0{}^\frown \dots {}^\frown d_m$$

$$\times \; R(\mathcal{U} \setminus \kappa_m + 1)/d_{m+1}{}^\frown \dots {}^\frown d_\ell .$$

also, $\le^*$ of $R(\mathcal{U} \setminus \kappa_m + 1)$ is $(2^{\kappa_m})^+$ – closed

exercise ~~example~~. ① + ② + ③ $\Longrightarrow$ $R(\mathcal{U})$ preserves cardinals.