The difficulties of $\square_\Lambda$ in long extender models

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Expectation

Let W be an iterable plus-one premouse. Then $\square_\Lambda$ holds at all and only those cardinals Λ which are neither subcompact nor the successor of a 1-subcompact cardinal.

The argument should follow the global structure of Schimmerling-Zeman.

Theorem (Voellmer)

Let W be an iterable plus-one premouse. Assume that the extenders of W have finitely many long generators. Then $\square_{\Lambda,2}$ holds at all and only those cardinals Λ which are neither subcompact nor the successor of a 1-subcompact cardinal.

Any uncredited lemmas or theorems to follow are either due to Neeman & Steel, Schimmerling & Zeman, or Voellmer (correct me if I am wrong).

Outline

1. Plus-one premice
2. Long Protomice
3. Unstable Levels and $\square_{\Lambda,2}$

Definition (Neeman-Steel)

A *plus-one potential premouse* is a J-structure N constructed from a sequence $\vec{E}$ of extenders such that whenever (M, G) is an active level of N, either

1. G is a short extender over M and (M, G) satisfies the Jensen conditions, or
2. G is a long extender with space $(\kappa_G^+)^M$ and
   - (a) $M = \mathrm{Ult}(M, G)|(\lambda_G^+)^{\mathrm{Ult}(M,G)}$
   - (b) $G \restriction \lambda_G \in M$
   - (c) G has a largest generator ν_G

Definition

A plus-one potential premouse (M, G) is *type* Z_1 iff

1. G is long
2. (M, G) satisfies the *weak initial segment condition*: $\forall \xi < \nu_G (G \upharpoonright \xi \in M)$
3. there is a short extender F indexed at ν_G such that
   (a) $\lambda_G = \lambda_F$
   (b) $\kappa_G < \kappa_F$
   (c) $(\kappa_F^+)^M$ is *not* the space of an extender on the M-sequence
   (d) for cofinally many $\gamma < (\kappa_F^+)^M$, we have that $i_F(E_\gamma^M) \subseteq G$
   (e) $(\nu_G^+)^{\mathrm{Ult}(M,F)} = (\nu_G^+)^{\mathrm{Ult}(M,G \upharpoonright \nu_G)} := \eta$ and $\mathrm{Ult}(M, F)|\eta = \mathrm{Ult}(M, G \upharpoonright \nu_G)|\eta$.

Definition

An extender $\bar{G}$ is *pseudo-indexed at* α in $L[E]$ if there is a type Z_1 level (M, G) with stretching extender F such that $\bar{G}$ is the long extender defined by

$$\text{if } \gamma < (\kappa_F^+)^M, \text{then } E_\gamma^M \subseteq \bar{G} \text{ iff } i_F^M(E_\gamma^M) \subseteq G$$

and $\alpha = (\kappa_F)^{++}$.

Definition

Let κ be a regular uncountable cardinal. Then κ is *1-subcompact* iff for any $A \subseteq H_{\kappa^{++}}$, there exists a cardinal μ, a $\bar{A} \subseteq H_{\mu^{++}}$, and an elementary embedding $j : \langle H_{\mu^{++}}, \bar{A} \rangle \to \langle H_{\kappa^{++}}, A \rangle$ with $\textsc{crit}(j) = \mu$.

Lemma

If κ is 1-subcompact, then $\neg\square_{\kappa^+}$.

Lemma (Voellmer)

Suppose κ is a cardinal in $L[E]$ such that

$$\{\alpha : \kappa^+ < \alpha < \kappa^{++} \wedge \exists \beta > \alpha \, (L[E]|\beta \text{ is type } Z_1 \wedge \alpha = (\kappa_F^{++})^{L[E]|\beta})\}$$

is stationary in κ^{++}. Then κ is 1-subcompact.

These are equivalences.

Definition

Suppose (M, G) is a plus-one potential premouse, $\eta < \lambda_G$, and

- $H = G \upharpoonright \eta$, if G is short
- $H = G \upharpoonright (\eta \cup \{\nu_G\})$, if G is long.

Then H is *whole* iff $i_H(\kappa_H) = \eta$. η is called a *cutpoint* of G.

(M, G) has the *Jensen Initial Segment Condition (JISC)* iff $H \in M$ whenever H is a whole initial segment of G.

Definition

A plus-one ppm (M, G) with G a long extender is *Dodd-solid* iff $G \upharpoonright \nu_G \in M$.

Lemma

If M is a type Z_1 premouse, then M is not Dodd-solid.

Definition

Let M be a plus-one potential premouse. Then M has *projectum-free spaces (PFS)* iff whenever G is a long extender on the M-sequence which is total on M,

1. if there is a k such that $\varrho_k(M) \leq (\kappa_G^+)^M$, then $\varrho_k(M) \leq \kappa_G$ for the least such k
2. if M is active with short extender H such that $\kappa_H = \kappa_G$, then $\varrho_1(M) > (\kappa_G^+)^M$.

Definition

A plus-one potential premouse M is a *plus-one premouse* iff

1. every proper initial segment of M is fully sound and satisfies PFS
2. M satisfies PFS
3. every initial segment of M satisfies the JISC
4. if (N, G) is an initial segment of M and G is long, then (N, G) is either Dodd-solid or type Z_1.

Theorem (Neeman-Steel)

Let M and N be iterable plus-one premice. Then there are iterates P and Q of M and N resp. such that $P \trianglelefteq Q$ or $Q \trianglelefteq P$.

Neeman and Steel also proved a version of solidity which suffices for preservation of the standard parameter for k-sound $(\omega_1 + 1)$-iterable plus-one premice.

Theorem (Condensation Lemma)

Suppose H and M are both of the same type and $n+1$ sound. Suppose there is a $\Sigma_0^{(n)}$-elementary and cardinal preserving map $\sigma : H \to M$ with $\mathrm{CRIT}(\sigma) = \alpha$. Suppose $\alpha \geq \varrho_{n+1}^H$ and that Anomalous Case 4 does not hold. Then one of the following hold:

1. $H = M$
2. $H \lhd M$
3. $H \unlhd Ult_n(M, E_\alpha^M)$.
4. $H \unlhd Ult_n(M, F)$, *where F is pseudo-indexed in the M-sequence at α.*

Definition (Anomalous Case 4)

α is not a cardinal of M, and setting $\langle \eta, k \rangle$ to be the lex-least such that $\varrho_{k+1}(M|\eta) < \alpha$, $F = \dot{F}^{M|\eta}$ is short, $\alpha = (\kappa_F^{++})^{M|\eta}$, $k = 0$, and there are total long extenders on the M-sequence with $\mathrm{CRIT} = \kappa_F$.

Work in W. Let $\mathcal{S} \subseteq \Lambda^+$ consist of all τ such that

1. Λ is the largest cardinal in J^E_τ
2. J^E_τ is fully elementary in $J^E_{\Lambda^+}$
3. $E_\tau = \varnothing$
4. τ is not a *pseudoindex*

We aim to produce a sequence $\mathfrak{C} = \langle \mathfrak{C}_\tau : \tau \in \mathcal{S} \rangle$, where each $\mathfrak{C}_\tau$ contains one or two sets C_τ, such that:

1. $C_\tau \subseteq \mathcal{S} \cap \tau$ is closed
2. C_τ is unbounded in τ whenever τ is a limit point of $\mathcal{S}$ and $\mathrm{cof}(\tau) > \omega$
3. $C_\tau \cap \bar{\tau} \in \mathfrak{C}_{\bar{\tau}}$ for $\bar{\tau} \in C_\tau$
4. $\mathrm{otp}(C_\tau) \leq \Lambda$

Definition

Suppose $M = (M, G)$ is a sound coherent structure. Then a J-structure $\tilde{M} = (\tilde{M}, \tilde{G})$ is an *interpolant* of M if

1. there is a map $\sigma : \tilde{M} \to M$ which is $\Sigma_0^{k(M)}$-preserving with respect to the language of coherent structures
2. $\sigma(p(\tilde{M})) = p(M)$
3. for every $\alpha \in p(M)$, there is a generalized solidity witness $Q_M(\alpha)$ in $\mathrm{ran}(\sigma)$
4. $\varrho(\tilde{M}) = \varrho(M)$.

In the situations of interest, $\mathrm{CRIT}(\sigma) = \bar{\tau}$ for some $\bar{\tau} \in \mathcal{S}$.

Definition (Pluripotent)

Let N_τ be a level of W. Then N_τ is *short pluripotent* iff N_τ is active with a short top extender G, $\kappa_G < \Lambda$, and $\varrho_1(N_\tau) = \Lambda$.

Let N_τ be a level of W. Then N_τ is *long pluripotent* iff N_τ is active with a long top extender G, $\kappa_G^+ < \Lambda$, and $\varrho_1(N_\tau) = \Lambda$.

Such levels give rise to protomice via interpolation.

Lemma

Suppose H is an interpolant of N_τ with interpolation embedding σ. Let $\text{CRIT}(\sigma) = \bar{\tau}$ and suppose $\bar{\tau}$ is not an index or pseudoindex in W. Suppose that N_τ is not pluripotent.

Then H is a level of W.

Definition

A *short protomouse* is a J-structure $M = (M, \tilde{G})$ (considered in the language of coherent structures) such that

1. $|M|$ is a passive premouse with $k(|M|) = 0$
2. $\tilde{G}$ is a short extender such that there is an ordinal $\theta < \kappa^+_{\tilde{G}}$ such that $\tilde{G}$ measures exactly the $x \in P(\kappa_{\tilde{G}}) \cap M|\theta$, and $\theta = (\kappa^+_{\tilde{G}})^{M|\theta}$
3. $|M| = \mathrm{Ult}_n(M||\theta, \tilde{G})$
4. Let $\langle N^*, n \rangle$ be the collapsing level for θ in M. Then $\varrho_1(M)$ is not the space of a long extender on the sequence of $\mathrm{Ult}_n(N^*, \tilde{G})$.

Definition

Let N_τ be the collapsing level for τ in W, $\varrho_{n+1}(N_\tau) = \Lambda < \varrho_n(N_\tau)$, and $\tau = (\Lambda^+)^N$. Then (κ, q) is a *(strong) short divisor* of N_τ if

1. κ is a cardinal $< \Lambda$
2. there is an ordinal $\lambda(\kappa, q)$ such that $\Lambda < \lambda(\kappa, q) < \varrho_n(N_\tau)$
3. setting $r = p(N_\tau) - q$,
   - (a) $q = p(N_\tau) \cap \lambda(\kappa, q)$
   - (b) $\mathcal{H}^{N_\tau}_{n+1}(\kappa \cup r) \cap \varrho_n(N_\tau)$ is cofinal in $\varrho_n^{N_\tau}$
   - (c) $\lambda(\kappa, q)$ is the least ordinal in $\mathcal{H}^{N_\tau}_{n+1}(\kappa \cup r) - \kappa$
   - (d) $P(\kappa) \cap \mathcal{H}^{N_\tau}_{n+1}(\kappa \cup r) = P(\kappa) \cap \mathcal{H}^{N_\tau}_{n+1}(\kappa \cup p(N_\tau))$

$\mathcal{H}^{N_\tau}_{n+1}(\kappa \cup r)$ is the *divisor hull* associated with (κ, q).

Definition

A *long protomouse* $M = (M, \tilde{G})$ is a J-structure (considered in the language of coherent structures) such that

1. $|M|$ is a passive premouse with degree of soundness $k(|M|) = 0$
2. $\tilde{G}$ is a long extender over $|M|$ such that there is an ordinal θ such that $\kappa^+_{\tilde{G}} < \theta < \kappa^{++}_{\tilde{G}}$ and $\tilde{G}$ measures exactly the subsets of $\kappa^+_{\tilde{G}}$ in $M|\theta$, and $\theta = (\kappa^{++}_{\tilde{G}})^{M|\theta}$. θ is denoted $\mathrm{dom}(\tilde{G})$.
3. $|M| = (\mathrm{Ult}_n(M||\theta, \tilde{G}))|o(M)$
4. $\kappa^+_{\tilde{G}} < \varrho_1(M)$
5. $\varrho_1(M)$ is not the space of a long extender on the $\mathrm{Ult}_n(N^*, \tilde{G})$-sequence, where $\langle N^*, n\rangle$ is the collapsing level for θ in M
6. $\tilde{G}$ has largest generator $\nu = \nu^M$
7. $\tilde{G} \restriction \lambda_{\tilde{G}}$ is on the M-sequence.

Definition

Let $M = (M, \tilde{G})$ be a long protomouse and $\theta^M = \text{dom}(\tilde{G})$. Let $(N^*)^M$ be the collapsing level for θ in M, i.e., $(N^*)^M = \langle N^*, n \rangle$ where n is such that $\varrho_{n+1}(N^*) = \kappa_{\tilde{G}}^+ < \varrho_n(N^*)$.

Definition

A long protomouse (M, G) is *type 2* if $(N^*)^M = \langle N^*, n \rangle$ is active with short top extender F, $n = 0$, and $\kappa_F = \kappa_G$.

Definition

Let M be a type 1 long protomouse with top extender $\tilde{G}$ and $(N^*)^M = \langle N^*, n \rangle$. The *associated ppm* of M is $\mathrm{Ult}_n(N^*, \tilde{G})$.

Definition

Let M be a type 2 long protomouse. Then the *associated quasi-protomouse* of M is $(P, F) = \mathrm{Ult}_0(N^*, \tilde{G})$.

Let μ be the least long generator of $\tilde{G}$. Then $\mathrm{Ult}_0((P|\mu), F)$ is the *associated ppm* of M.

Definition

Let N_τ be the collapsing level for τ in W and $\varrho_{n+1}(N_\tau) = \Lambda < \varrho_n(N)$ and $\tau = (\Lambda^+)^N$. Then an ordinal $\nu \in p(N_\tau)$ is a *long divisor* of N_τ if

1. There is an extender E_ν on the N-sequence such that $\kappa_{E_\nu} < \Lambda < \lambda_{E_\nu}$ and $\lambda^+_{E_\nu} = (\lambda^+_{E_\nu})^{N_\tau} < \varrho_n(N_\tau)$
2. $\mathrm{Hull}^{N_\tau}_{n+1}(Z \cup r) \cap \varrho_n(N_\tau)$ is cofinal in $\varrho_n(N_\tau)$
3. $\mathrm{Hull}^{N_\tau}_{n+1}(Z \cup r) \cap \lambda^+_{E_\nu} = Z$
4. $\lambda^+_{E_\nu}$ is not the space of an extender on the N sequence

where

- $r = p(N_\tau) - (\nu + 1)$,
- $E_\mu = E_\nu \restriction \lambda$ is the short part of E_ν, and
- $Z = i_{E_\nu}``(\kappa^+) = i_{E_\mu}``(\kappa^+)$.

Suppose we are N. We want to know how we can be recovered as the associated ppm of a long protomouse $(M, \tilde{G})$.

If N does arise this way, then letting N^* be the collapsing structure for $\mathrm{dom}(\tilde{G})$, $N = \mathrm{Ult}(N^*, \tilde{G})$. We would like to recover N^* and $\tilde{G}$ as a hull in N.

$\varrho_1(N^*) = \kappa^+_{\tilde{G}}$ (because M is long), so to specify the map from N^* to N, we need to know how $\kappa^+_{\tilde{G}}$ is moved. From the perspective of N, this map and N^* can be recovered as the hull of a set of ordinals and a piece of the parameter, namely $\mathrm{Hull}^N_{n+1}(i_{\tilde{G}}``(\kappa^+_{\tilde{G}}) \cup i_{\tilde{G}}(p(N^*)))$. But it's hard to guess a set of ordinals.

Instead, the notion of long divisor above assumes that the largest generator of $\tilde{G}$ is a successor generator. *This is a smallness assumption.*

If we know that the largest generator of $\tilde{G}$, $\nu_{\tilde{G}}$, is a successor generator, then from $\nu_{\tilde{G}}$ alone we can recover $i_{\tilde{G}}``(\kappa^+_{\tilde{G}})$: the extender indexed in N at $\nu_{\tilde{G}}$ is $\tilde{G} \restriction \nu_{\tilde{G}}$.

This is enough to determine $i_{\tilde{G}}``(\kappa^+_{\tilde{G}})$.

When $\nu_{\tilde{G}}$ is a limit generator, no such extender is indexed at $\nu_{\tilde{G}}$.

By remarks of Martin yesterday, the standard parameter of N and the Dodd parameter of M are correlated. In particular, $\nu_{\tilde{G}}$ is the greatest element of the Dodd parameter of M, which implies that it is in the standard parameter of N. Thus we "guess" $\nu_{\tilde{G}}$ by choosing ν from the standard parameter of N.

Type 2 Long Divisors

There is a more complicated notion of type 2 long divisor. We need to be able to recover the short top extender F of $\mathrm{Ult}_0(N^*, \tilde{G})$ as a predicate, and that complicates things. We can do so, and N^* is recoverable as a Σ_1-hull in the language with this predicate, so we can get back to $(M, \tilde{G}) = (N || i_{\tilde{G}}``\kappa^+, \tilde{G})$.

Fortunately, the long divisors and the type 2 long divisors form disjoint subsets of the standard parameter, so there will not be conflict between them.

Lemma

Let N be a level of W, and let ν be a long divisor of N with N^* the transitive collapse of the divisor hull associated with ν. Let π be the inverse of the collapse map.

Then $N^* \lhd N$ and $p(N^*) = \pi^{-1}(r)$.

Long divisors are "strong" by Voellmer's definition.

Premouse to protomouse

Definition

Let N be a level of W and ν a long divisor of N with $\pi : N^* \to N$ the uncollapse map associated with the divisor hull. Let $\eta = (\lambda^+_{E_\nu})^N$. Then $N(\nu) = (J^E_\eta, G)$ is the *long protomouse associated with* ν, where G is the long extender of length $\lambda^+_{E_\nu}$ derived from π.

Lemma

Let N be a level of W and ν a long divisor of N with associated protomouse $N(\nu)$. Then $N(\nu)$ is a long protomouse of type 1.

Lemma

Let N be a level of W and $N(\nu)$ the long protomouse associated with a divisor ν of N. Then N is the associated ppm of $N(\nu)$.

Lemma (Long Protomouse Condensation)

Let M_τ be either a long pluripotent level of W or the protomouse associated with a (type 2) long divisor ν of collapsing level N_τ. Let M be an interpolant of M_τ such that the critical point of the interpolation embedding is $\bar{\tau} \in \mathcal{S}$. Let N be the associated ppm of M.

Then N is a level of W.

Definition

Let N be a level of W with a canonical short divisor (κ, q) and at least one long divisor. Let ν be the least long divisor. Then N is *unstable* iff $\nu = max(q)$ and, if ν is a long divisor, $\kappa_{E_\nu} < \kappa$.

I.e., unstable levels have two canonical associated protomice.

Lemma (Instability)

Let (κ, q) be a short divisor of N, and let $r = p(N) - q$. Then any long divisor $\nu \in r$ is such that $\kappa_{E_\nu} < \kappa$.

If $\nu \in q$ is a long divisor, then either $\nu = max(q)$ or $\kappa_{E_\nu} \geq \kappa$.

Definition (Canonical divisor)

Let N be a stable level of W with at least one divisor. Let (κ, q) be the canonical short divisor, if there is one. Let $\nu \in p(N)$ be the least long divisor, if there is one. Then

(a) If ν is undefined or $max(q) < \nu$, then (κ, q) is the canonical divisor of N.

(b) If (κ, q) is undefined or $\nu \leq max(q)$, then ν is the canonical divisor of N.

The canonical protomouse associated with N is the protomouse associated with the canonical divisor of N.

Think of canonical divisors as giving the largest divisor hulls.

Definition

Let N_τ be the collapsing level for τ in W, and suppose N_τ is stable. Then

1. If there is a canonical protomouse associated with N_τ, set this protomouse to be M_τ.
2. If N_τ does not have an associated canonical protomouse but N_τ is pluripotent, set $M_\tau = N_\tau$.
3. If N_τ does not have an associated canonical protomouse and N_τ is not pluripotent, M_τ is undefined.

Suppose N_τ is unstable. Then

4. $M_\tau^{\text{short}} = N(\kappa, q)$.
5. $M_\tau^{\text{long}} = N(\nu)$.

This leads to sets B_τ^{short} and B_τ^{long} for unstable levels, and hence to a $\square_{\Lambda,2}$ sequence.

The Difficulties

Find a way to get from $\square_{\Lambda,2}$ to full $\square_{\Lambda}$.

Accommodate long generators which are limit generators.