New Partition and Non-partition Results

S. Jackson

Department of Mathematics
University of North Texas

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This is joint work with A. Apter, A. Blass and B. Löwe.

We discuss some new partition results around the level of an inaccessible Suslin cardinal.

We assume $\mathrm{AD} + \mathrm{DC}$ throughout.

The picture at an inaccessible Suslin cardinal is somewhat similar to that at a projective ordinal.

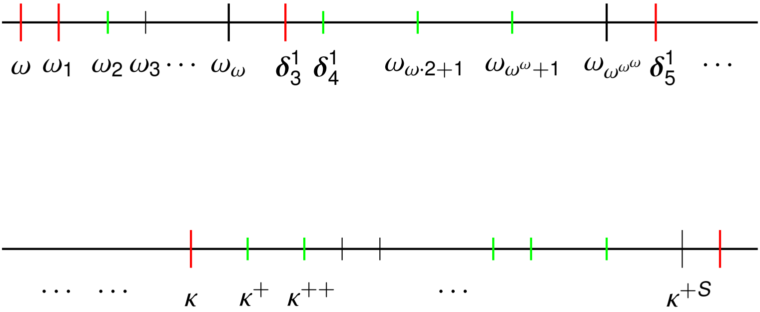

Figure: Overview of the regular cardinals. Regular Suslin cardinals in red, other regular cardinals in green.

Let κ an inaccessible Suslin cardinal (i.e., a regular, limit Suslin cardinal).

So, κ^+ and κ^{++} are regular, and κ^{+n} for $n \geq 3$ are singular of cofinality κ^{++}.

A basic fact (Kechris, Kleinberg, Moschovakis, Woodin) is that we have the strong partition relation at any inaccessible Suslin κ.

Fact (KKMW)

Let $\kappa = o(\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}$ is closed under quantifiers, $\vee$, $\wedge$, and cof$(\kappa) > \omega$. Assume the Steel pointclass $\boldsymbol{\Gamma}$ at κ is closed under $\vee$. Then $\kappa \rightarrow (\kappa)^\kappa$.

With Apter and Löwe we showed:

Theorem (AJL)

$\kappa^+ \to (\kappa^+)^\kappa$ and $\kappa^{++} \to (\kappa^{++})^\kappa$. In fact, we have the polarized partition property

$$(\kappa, \kappa^+, \kappa^{++}) \to (\kappa, \kappa^+, \kappa^{++})^\kappa.$$

Question

What is the exact partition strength of κ^+ and κ^{++}? What is the partition strength of other cardinals in the gap?

Main Results

Theorem

$\kappa^+ \rightarrow (\kappa^+)^{<\kappa^+}$ but $\kappa^+ \not\rightarrow (\kappa^+)^{\kappa^+}$.

Theorem

$\kappa^{++} \rightarrow (\kappa^{++})^{<\kappa^+}$ but $\kappa^{++} \not\rightarrow (\kappa^{++})^{\kappa^+}$.

We don't have the partition strength at κ^{++} to guarantee that the κ^+-cofinal c.u.b. filter is a normal measure. Nevertheless we get this fact directly.

Theorem

The κ^+-cofinal c.u.b. filter on κ^{++} is a normal measure.

The situation is somewhat similar to that in the projective hierarchy:

Theorem

Let ρ be a regular cardinal with $\delta^1_{2n+1} < \rho < \delta^1_{2n+3}$. Then

$$\rho \to (\rho)^{\delta^1_{2n+1}} \quad \textit{but} \quad \rho \nrightarrow (\rho)^{\delta^1_{2n+2}}.$$

In summary: we know the exact partition strength of the regular cardinals in the projective hierarchy, and for κ, κ^+, κ^{++} where κ is an inaccessible Suslin cardinal.

Let Γ be the Steel class at κ, P a Γ-complete set, and $\{\varphi_n\}$ a Γ-scale on P. Let $|x| = \varphi(x)$.

Let $P_\alpha = \{x : |x| < \alpha\}$.

Let G be the ω-c.u.b. set of Suslin cardinals $< \kappa$ of cofinality ω which are sufficiently closed:

- For all $\alpha' < \alpha$, $\sup\{\varphi_n(x) : |x| = \alpha'\} < \alpha$.
- $|P_\alpha|_W = \alpha$.

Let $\mathbf{\Sigma}_0^\alpha = \bigcup_\omega(\cup_{\alpha'<\alpha}\mathbf{\Delta}_{\alpha'})$, etc.

- We uniformly in $\alpha \in G$ have $\mathbf{\Sigma}_0^\alpha$-complete sets P'_α and $\mathbf{\Sigma}_0^\alpha$ scales $\{\varphi_n^\alpha\}$. Note that if $\alpha \leq \beta$ then $P'_\alpha \subseteq P'_\beta$.
- We uniformly in $\alpha \in G$ have $\mathbf{\Pi}_1^\alpha$ sets Q_α and $\mathbf{\Pi}_1^\alpha$-scales $\{\psi_n^\alpha\}$ on Q_α. We can take then so that if $\alpha \leq \beta$ then $Q_\alpha \supseteq Q_\beta$.

For example, $Q_\alpha = \{x \colon (P'_\alpha)_x \text{ is wellfounded }\}$.

We will consider the following “types” of functions.

Definition

A type-1 block function is a function $f: \kappa \to \kappa$ such that for all $\alpha \in G$ we have $f(\alpha) \in (\alpha, \alpha^+)$.

A type-2 block function is a function f with domain the (α, β) with $\alpha \in G$ and $\beta < \alpha^+$. We have $f(\alpha, \beta) \in (\alpha, \alpha^+)$. We sometimes just write $f(\beta)$.

There are trees T^+ and T^{++} on $\omega \times \kappa$ with the following properties.

1. For any type-1 block function $f : \kappa \to \kappa$, there is an $x \in \omega^\omega$ such that T_x^+ is wellfounded and $|T_x^+ \restriction \alpha| > f(\alpha)$ for all good α with $\mathrm{cof}(\alpha) = \omega$.
2. If $f : \kappa \to \kappa$ is a type-2 block function, then there is an x such that T_x^{++} is wellfounded and for all $\alpha \in G$ there is an ω c.u.b. set of $\beta < \alpha^+$ such that $|T_x^{++} \restriction \beta| > f(\beta)$.

Remark

Let μ be the ω-cofinal normal measure on κ. For $\alpha \in G$, let μ_α be the ω-cofinal normal measure on α^+.

It follows from the existence of T^+ and T^{++} that

$$[\alpha \mapsto \alpha^+]_\mu = \kappa^+.$$

$$[\alpha \mapsto [\beta \mapsto \alpha^+]_{\mu_\alpha}]_\mu = \kappa^{++}.$$

(equivalently $[\alpha \mapsto \alpha^{++}]_\mu = \kappa^{++}$)

Remark

From the proof that $\kappa^+ \to (\kappa^+)^\kappa$ and $\kappa^{++} \to (\kappa^{++})^\kappa$ we get a version of the block partition property:

If $\mathcal{P}$ is a partition of the type-1 (or type-2) block functions which only depends on $[f]$, then there are c.u.b. sets $C_\alpha \subseteq \alpha^+$ which are homogeneous for $\mathcal{P}$.

Question

Do we get the full partition property for block functions of type-1 or type-2?

We first show:

Theorem
$\kappa^+ \not\to (\kappa^+)^{\kappa^+}$.

Proof: Consider $\mathcal{P}$: we partition $F\colon \kappa^+ \to \kappa^+$ of the correct type according to whether $F \in \mathrm{Ult}_\mu(V)$.

Suppose $C \subseteq \kappa^+$ is homogeneous for the contrary side.

Lemma
There are c.u.b. $C_\alpha \subseteq \alpha^+$ with $[\alpha \mapsto C_\alpha]_\mu \subseteq C$.

Proof: Consider the partition $\mathcal{P}_1$ of pairs (f, g) of type-1 block functions according to whether $C \cap ([f]_\mu, [g]_\mu) \neq \emptyset$.

This is an invariant partiton.

We cannot have $C_\alpha \subseteq \alpha^+$ homogeneous for the contrary side. So, let C_α be homogeneous for the stated side.

Let $C'_\alpha \subseteq C_\alpha$ be the closure points of C_α. Then any type-1 f into the C'_α is such that $[f]_\mu$ is a limit of C and so in C. □

Let $h: \kappa \to \kappa$ be a type-2 function of the correct type with $f(\beta) \in C_\alpha$ for all $\beta \in (\alpha, \alpha^+)$. Then $H = [h]_\mu$ is a function from κ^+ to κ^+ which is of the correct type and takes values in $[\alpha \to C_\alpha]_\mu \subseteq C$. However, $H \in \mathrm{Ult}_\mu(V)$, a contradiction.

So, fix a c.u.b. $C \subseteq \kappa^+$ which is homogeneous for the stated side of $\mathcal{P}$.

Fix $F: \kappa^+ \to C$ of the correct type, and let $A = \mathrm{ran}(F)$. By the homogeneity of C, for any $B \subseteq A$ of size κ^+ we have that $B \in \mathrm{Ult}_\mu(V)$.

Claim

For any $B \subseteq \kappa^+$ of size κ^+, $B \in \mathrm{Ult}_\mu(V)$.

Proof.

$\mathrm{Ult}_\mu(V)$ can compute the transitive collapse map $\pi \colon A \to \kappa^+$. If $B \subseteq \kappa^+$, then $B \in \mathrm{Ult}_\mu(V)$ iff $\pi^{-1}(B) \in \mathrm{Ult}_\mu(V)$. However, $\pi^{-1}(B) \subseteq A$, and so $\pi^{-1}(B) \in \mathrm{Ult}_\mu(V)$. □

To get a contradiction we prove the following.

Lemma

Let $A_\omega \subseteq (\kappa, \kappa^+)$ be the points of cofinality ω. Then $A_\omega \notin \mathrm{Ult}_\mu(V)$.

Proof: Suppose $A_\omega = [g]_\mu$ where wlog:

$$\forall\alpha \in G\ g(\alpha) \subsetneq \mathrm{Cof}_\omega \cap (\alpha, \alpha^+).$$

Claim

$\forall^*_\mu \alpha\ \forall^*_{\mu_\alpha} \beta\ (\beta \in g(\alpha))$.

Proof.

If not, we can get $C_\alpha \subseteq \alpha^+$ to the contrary (use a coding lemma argument). Then take a type-1 function f with range in the C_α and of uniform cofinality ω. □

By the coding lemma argument again, let C_α be such that for μ almost all α, $C_\alpha \cap \mathrm{Cof}_\omega \subseteq g(\alpha)$.

Fix a type-1 block function f with $f(\alpha) \in C_\alpha$ and $f(\alpha)$ of uniform cofinality α.

- Since for μ almost all α we have $\mathrm{cof}(\alpha) = \omega$, we have $f(\alpha) \in C_\alpha \cap \mathrm{Cof}_\omega \subseteq g(\alpha)$, and so $\mathrm{cof}([f]_\mu) = \omega$.
- Since $f(\alpha)$ has uniform cofinality α, $\mathrm{cof}([f]_\mu) = \kappa$.

This contradiction completes the proof that $\kappa^+ \not\rightarrow (\kappa^+)^{\kappa^+}$. □

The proof that $\kappa^{++} \not\rightarrow (\kappa^{++})^{\kappa^+}$ is similar.

$\mathcal{P}$: partition $f\colon \kappa^+ \to \kappa^{++}$ of the correct type according to whether $f \in \mathrm{Ult}_\mu(V)$.

Suppose $C \subseteq \kappa^{++}$ were homogeneous for the contrary side.

For f of type-2, let $\delta_f = [\alpha \mapsto [\beta \mapsto f(\alpha,\beta)]_{\mu_\alpha}]_\mu < \kappa^{++}$.

Claim

There are $C_\alpha \subseteq \alpha^+$ such that for any type-2 f with range in the C_α of uniform cofinality ω, we have $\delta_f \in C$.

Proof.

Similar to before, now partitioning pairs (f, g) of type-2 functions according to $(\delta_f, \delta_g) \cap C \neq \emptyset$. □

Fix then f a type-2 function with range in the C_α and with $f(\alpha, \beta)$ of uniform cofinality β. Let $f'(\alpha, \beta, \gamma)$ induce f (so $\gamma < \beta$).

Then f' gives a function $F' : \kappa^+ \to C$ of the correct type which is in $\mathrm{Ult}_\mu(V)$, a contradiction.

So, let $C \subseteq \kappa^{++}$ be homogeneous for the stated side of $\mathcal{P}$. Fix $F : \kappa^+ \to C$ of the correct type.

Any $A \subseteq \kappa^+$ can be coded into F, and so $A \in \mathrm{Ult}_\mu(V)$. This contradicts $A_\omega \notin \mathrm{Ult}_\mu(V)$.

We sketch the proof that the κ^+-cofinal c.u.b. filter is a normal measure.

This follows from the *n*-fold version of the following partition result:

Theorem

Let $\mathcal{P}$ be a partition on the type-2 block functions with $f(\beta)$ of uniform cofinality β. Suppose that $\mathcal{P}$ depends only on $[f] = [\alpha \mapsto [\beta \mapsto f(\beta)]_{\mu_\alpha}]_\mu$. Then there are c.u.b. $C_\alpha \subseteq \alpha^+$ which are homogeneous for $\mathcal{P}$.

We code functions f with domain the triples (α, β, γ) where $\alpha \in G$, $\beta < \alpha^+$, and $\gamma < \beta$.

Definition

For $\alpha \in G$, $\beta < \alpha^+$, and $\gamma < \beta$, we say (x, y, z) is (α, β, γ)-good if:

1. $z \in P$.
2. $|z| \in \mathrm{wf}(T_y^+ \restriction \alpha)$.
3. $\gamma \subseteq \mathrm{wf}(T_x^{++} \restriction \beta(|T_y^+ \restriction \alpha(|z|))$.

Then we define:

$$r_{x,y,z}(\alpha, \beta, \gamma) = \sup\{|T_x^{++} \restriction \beta(\gamma')| \colon \gamma' < \gamma \wedge \gamma' <_{T_x^{++} \restriction \beta} |T_y^+ \restriction \alpha(|z|)|\}.$$

Claim

If f is a type-2 block function with $f(\beta)$ of uniform cofinality β, then there is an (x, y, z) with $z \in P$, T_y^+ and T_x^{++} wellfounded, and such that

$$\forall_\mu \alpha \ \forall^*_{\mu_\alpha} \beta < \alpha^+ \ f(\beta) = \sup_{\gamma < \beta} r_{x,y,z}(\alpha, \beta, \gamma).$$

Proof.

Use properties of T^+ and T^{++} to get x, y, z with $f(\beta) = |T_x^{++} \restriction \beta \,(|T_y^+ \restriction \alpha(|z|)|)|$ almost everywhere.

Then use the fact that if $f(\beta)$ has uniform cofinality β then f cannot have any smaller uniform cofinality to show x, y, z works. □

We play the game G where I plays out (x_1, y_1, z_1), II plays out (x_2, y_2, z_2). The payoff set for G is defined as follows.

First suppose there is a lexicographically least (α, β, γ) such that either (x_1, y_1, z_1) or (x_2, y_2, z_2) is not (α, β, γ)-good. Then II wins if (x_1, y_1, z_1) is not (α, β, γ)-good.

Suppose that both (x_1, y_1, z_1) and (x_2, y_2, z_2) are (α, β, γ)-good for all (α, β, γ). Let $r_1 = r_{x_1, y_1, z_2}$ and likewise for r_2. Let

$$r(\alpha, \beta) = \sup_{\gamma < \beta} \max\{r_1(\alpha, \beta, \gamma), r_2(\alpha, \beta, \gamma)\}.$$

Then II wins the run iff $\mathcal{P}(r) = 1$.

For $\alpha \in G$, $\beta < \alpha^+$, $\gamma < \beta$, and $\delta < \alpha^+$ we define:

$(x, y, z) \in A_{\alpha,\beta,\gamma,\delta}$ iff

1. $(z \in P \wedge |z| < \alpha)$
2. $\forall \alpha' < \alpha\ (|T_y^+ \restriction \alpha'| < \alpha \wedge |T_x^{++} \restriction \alpha'| < \alpha)$
3. $\forall \beta' < \beta\ (|T_x^{++} \restriction \beta'| < \beta)$
4. $(|T_y^+ \restriction \alpha(|z|)| < \beta)$
5. $\forall \gamma' \leq \gamma\ (\gamma' \in T_x^{++} \restriction \beta\,(T_y^+ \restriction \alpha(|z|)) \rightarrow |T_x^{++} \restriction \beta\,(\gamma')| \leq \delta)$

From the closure of $\mathbf{\Delta}^{\alpha}_1$ under $< \alpha^+$ length unions and intersections (Martin's theorem) we get that $A_{\alpha,\beta,\gamma,\delta} \in \mathbf{\Delta}^{\alpha}_1$.

A boundedness argument then gives:

Claim

For $\alpha \in G$, $\beta < \alpha^+$, and $\gamma < \beta$,

$$h_\tau(\alpha,\beta,\gamma) = \sup\{r_{x_2,y_2,z_2}(\alpha,\beta,\gamma) \colon (x_2, y_2, z_2) \in \tau[A_{\alpha,\beta,\gamma,\delta}]\} < \alpha^+.$$

This defines the c.u.b. sets homogeneous for $\mathcal{P}$.