grigor 1

a core model induction

thm. (trang + sargsyan)

PFA $\Rightarrow$ there is a model of LSA.

recall: $\Gamma_{max} =$

short extenders

$\{ A \subset \mathbb{R} : \exists$ a hod pair (P, Σ)
$A \leq_w \text{code}(\Sigma) \}$.

thm. assume PFA, and let $g \subset Col(\omega, \omega_1)$.
then in $V[g]$,
$L(\Gamma_{max}, \mathbb{R}) \models AD_{\mathbb{R}} + \theta$ reg.

thm. assume PFA. let $g \subset Col(\omega, \omega_1)$.
then in $V[g]$,
$\exists A \in \Gamma_{max}$ $L(A, \mathbb{R}) \models LSA$.

LSA : largest suslin axiom :

(a) AD^+

(b) there is a largest suslin cardinal, κ.

(c) $\forall A \subset \mathbb{R}$ $w(A) < \kappa$

$\neg \exists f \in OD_A$ $f : \mathbb{R} \xrightarrow[\text{onto}]{} \kappa$

ex. Solovay sequence :

$\theta_0 = \sup \{ \alpha : f : {}^\omega\omega \to \alpha,\ f$ OD f onto $\}$

$\theta_\alpha < \theta$: $\theta_{\alpha+1} = \sup \{ \beta : f : \theta_\alpha^\omega \to \beta,$ f OD, f onto $\}$

$\theta_\lambda = \sup \theta_\alpha$.

show gives the same sequence.

Some core model induction projects

① assume PFA. Let $g \subset Col(\omega, \omega_1)$

then in $V[g]$, $HOD^{L(\mathbb{R}_*, \Gamma_{max})} \models$ "there is a supercompact cardinal."

(2) change PFA to any reasonable theory.

e.g., $\neg \Box_\kappa$, κ sing. strong limit

$\Box_\kappa$ fails everywhere.

<u>thm.</u> (...) $\neg \Box_{\omega_2} + \neg \Box(\omega_2)$ in a

$\mathbb{P}_{max}$ extension over a determinacy hypo.

(weaker than LSA).

CMI. Supp. (P, Σ) is a hod pair.

<u>initial step:</u> $Lp^{\Sigma}(\mathbb{R}) \models AD^+$

$Lp^{\Sigma}(\mathbb{R}) = \bigcup \{ M : M$ is a sound

Σ-mouse over $\mathbb{R}$ proj. to

$\mathbb{R}$ whose ctbl. substructures

are $\omega_1 + 1$ it. ble $\}$

scale analysis goes thru.

$Lp^{\Sigma}(\mathbb{R})$.

<u>external step</u>:

we have a model of $AD_{\mathbb{R}}$,
and we want to get past it.
ie have $\Gamma \subset \mathcal{P}(\mathbb{R})$ s.t. Γ-det.
holds; think of Γ as Γ_{max}. <u>goal</u>:
show that there is a next set.

<u>idea</u>: $M = L(\Gamma, \mathbb{R})$

(even have to prove putting L on top
doesn't raise an issue.)

assume: $\mathcal{P}(\mathbb{R}) \cap M = \Gamma$

$M \models AD^{+}$.

set $V_\Theta^{HOD} = H^- =$ a hod mouse.

look for a hod pair (P, Σ) s.t.

$$M_\infty(P, \Sigma) \trianglerighteq H^- .$$

here, $M_\infty(P, \Sigma) =$ dir lim of all iterates of P via Σ.

why is this a contradiction?

we have $\Sigma \in \Gamma_{max}$.

ex. $A = \{ (Q, \alpha) : Q$ is a Σ-it. of P s.t. $\pi_{P,Q}$ exists, & $\alpha \in Q \cap OR \}$

Construct

$$f : \text{code}(A) \xrightarrow{\text{onto}} \Theta$$

case 1. $cf^V(\theta) = \omega$.

then fix $(P_i, \Sigma_i) \in \Gamma_{max}$,

s.t. $\bigcup M_\infty(P_i, \Sigma_i) = H^-$

let $A = \bigoplus_{i<\omega} \Sigma_i$.

hard case:

$\check{cf}(\theta) > \omega$.

we have tools for showing that

$M \models \theta$ regular.

<u>first</u>. we have a strategy for H^-

why?: fix $\mathcal{N} \trianglelefteq H^-$, a cutpoint

initial segment of H^-.

we have $\mathcal{N} = \mathcal{M}_\infty(Q, \Lambda)$, for (Q, Λ)

~~s.t.~~

take $\Sigma_{H^-} = \bigoplus_{\mathcal{N} \trianglelefteq H^-} \Sigma_{\mathcal{N}}$.

Set $H = Lp^{\Sigma_{H^-}}(H^-)$.

~~ifwe~~ assume we work in V,

κ is measurable, Γ_{max} is

defined in $V^{Col(\omega, <\kappa)}$

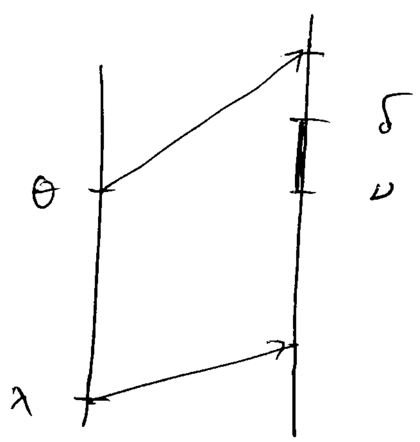
let $j : V \longrightarrow M$ be the ult embedding.

<u>idea.</u> we want to use $j \restriction H$ to get a strategy for H.

<u>ex.</u>: show that if
$M \models cf(\theta) < \theta$, then
$H \models cf(\theta) < \theta$.

let $\lambda = cf^H(\theta)$. let μ ~~be a~~ a measure on the H seq, with cnt λ.

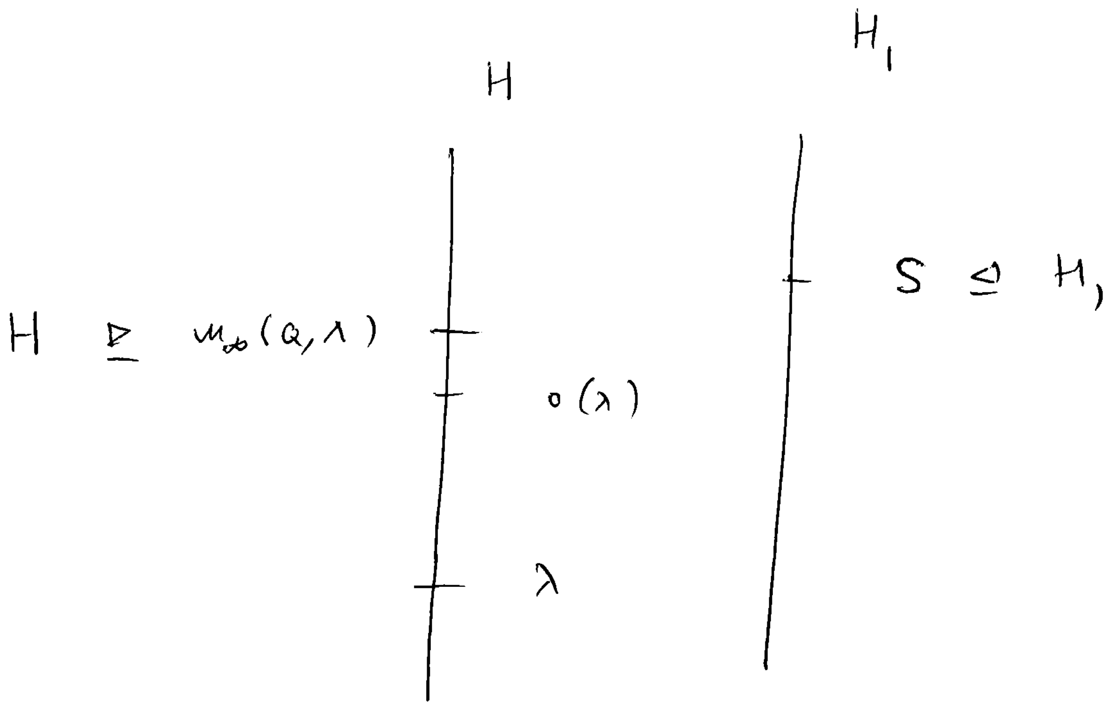
Consider $ult(H, \mu) = H_1$.
let δ be woodin, $\delta > \theta$.
~~at~~ cutpoint



want to define a strategy for $H_1 | \delta$ that (let's say, for simplicity) acts on the window $(\nu, \delta]$.

fix (Q, Λ) s.t. $M_\infty(Q, \Lambda)$ has all ext. with crit λ.

let $S = \mathrm{ult}(M_\infty(Q, \Lambda), \mu)$.



$$
\begin{tikzcd}
 & & j(\mathcal{M}_\infty(Q,\Lambda)) \\
\mathcal{M} = \mathcal{M}_\infty(Q,\Lambda) \arrow[urr, "j"] \arrow[rr, "\pi_\mu"'] & & S \arrow[u, "\pi_{S,\infty}^{\Lambda_S}"']
\end{tikzcd}
$$

ex. $j \upharpoonright \mathcal{M}_\infty(Q,\Lambda) =$ the iteration embedding via Λ.

ex. lift the diagram along H.

$$
\begin{tikzcd}
 & j(H) & \\
H \arrow[ur, "j"] \arrow[r, "\pi_\mu"] \arrow[d, phantom, "\unrhd"] & H_1 & \\
\mathcal{M} \arrow[r, "\pi_\mu"'] & S \arrow[uu, bend right=60, "\pi_{S,\infty}^{\Lambda_S}"'] &
\end{tikzcd}
$$

lift $\pi_{S,\infty}^{\Lambda_S}$ to $\sigma : H_1 \longrightarrow j(H)$:

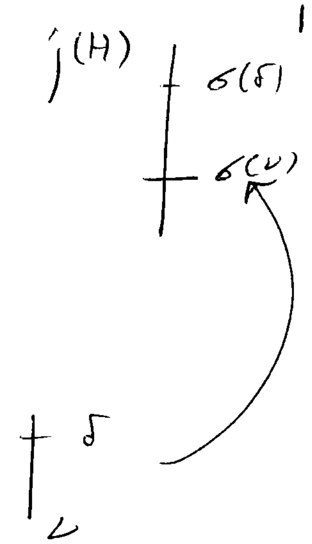
~~let E = extender from $\pi_{S,\infty}^{\Lambda_S}$~~

$$x \in H_1 \implies x = \pi_\mu(f)(a), \quad a \in [\nu_\mu]^{<\omega}, \quad f \in H$$

$$\sigma(x) = j(f)\left(\pi^{\Lambda_s}_{s,\infty}(a)\right)$$

we have $j = \sigma \circ \pi_\mu$.

we now set $\Sigma_{H_1 | \sigma} = \sigma$-pullback of $j(\Sigma_H)$.



⌜justification for ex. 6.

$$\begin{aligned}(a, A) \in E &\iff \pi^{\Lambda_s}_{s,\infty}(a) \in j(A) \\ &\iff \pi^{\Lambda_s}_{s,\infty}(a) \in \pi^{\Lambda_s}_{s,\infty}(\pi_\mu(A)) \\ &\iff a \in \pi_\mu(A)\end{aligned}$$

⌟

<u>idea.</u> $j : V \longrightarrow \mathrm{ult}(V; \mu)$

we want to show $\mathrm{ult}(V;\mu) \models$ "there is a hod pair (P, Σ) s.t. $\mathcal{M}_\infty(P, \Sigma) = j(H)$"

try to show that

$$N \models \text{"}\mathcal{M}_\infty(H, \Sigma) = j(H).\text{"}$$

why is $\Sigma \in N$.

PFA, ... etc. imply $|H| = \kappa$.

$\Rightarrow$ $j \restriction H \in \mathrm{ult}(V;\mu)$,

and Σ is defined from $j \restriction H$.

for this, in fact just need $|H| < \kappa^+$.

2008-12: it was believed that you need PFA, — to show that $|H| < \kappa^+$.

however:

<u>thm.</u> Con (LSA) $\Longrightarrow$

Con ($\exists\,\kappa$ measurable s.t.

H of the max. model at

κ has size $< \kappa^+$) .

So just having $\bar{\bar{H}} < \kappa^+$ doesn't give

more strength than LSA.

~~these cost~~ we need more structure on the

top of H.

<u>conjecture</u>: Supp. κ is a measurable

limit of woodins, and strongs.

and $|H| < \kappa^+$.

$g \subset \mathrm{Col}(\omega, <\kappa)$-gen.

then in $V[g]$, $L(H^\omega, \mathrm{Hom}^*) \models AD^+$.

ω-sequences of

elts. of H.

under those hypo's.

<u>Covering with derived models</u> (no model with a superstrong)

Then is $M_0 \lhd M_1$, hod premice, s.t.

- $L(M_0^\omega, \mathrm{Hom}^*) \models AD^+$
- $\eta = o(M_0)$, then η is the largest card. of M_1.
- $\Box_\eta^{M_1}$ holds + is not threadable.

($\Rightarrow$ PFA gives a superstrong).