Grigor II

last time.

κ is a measurable cardinal that is a limit of Woodins + strongs.

$g \subset \mathrm{Col}(\omega, <\kappa)$ – generic.

$\Gamma = \mathrm{Hom}^*$ in $V(\mathbb{R}^*)$

assume $\neg \square_\kappa$.

$H^- =$ dir. limit representation of $V_\Theta^{\mathrm{HOD}^{L(\Gamma, \mathbb{R}^*)}}$

$H = Lp^\Sigma(H^-)$, Σ is the canonical strategy of H^-.

also last time

Covering with derived models (~~no~~ NLE)

there are two hod pm $\mathcal{M}_0 \triangleleft \mathcal{M}_1$

s.t. ① $L(\mathcal{M}_\infty^\omega, \Gamma) \models AD^+$

ω-seq. of elts. of $\mathcal{M}_\infty$

② if $\eta = o(\mathcal{M}_\infty)$, then η is the largest cardinal of $\mathcal{M}_1$

(3) $\square_\eta^{\mathcal{M}_1}$ and is not threadable.

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conj: $L(H^\omega, \mathrm{Hom}^*) \models AD^+$.

proof. (1) say Lp cond holds at a singular strg lim κ
if $\exists A \subset \kappa \quad Lp(A) \cap OR = \kappa$.

(a) show that Lp cond fails in the min. mouse with a strg after coll. the strg to ω.

$[\, \forall \kappa$ sing., $\forall A \subset \kappa\ \exists B\ A \in L_1(A)$
$Lp(B) \cap OR < \kappa^+ \,]$

E can be recovered from

$\pi_E \restriction \kappa^+$ and $ult(M; E) \mid \pi_E(\kappa^+)$

and $\pi_E \restriction \kappa^+$ is coded by the thy of indiscernibles.

i.e., E is given by some "nice" operator.

(a) say u.b. cong holds at $\kappa \geq \omega_2$ if there is $A \subset \kappa$ and some operator F s.t. $\mathrm{cf}(Lp^F(A)) \geq \kappa$.

(b) show that u.b. cong fails in $\mathcal{M}[G]$, where

a) $\mathcal{M}$ is a hod premouse with a sing + a proper class of woodins

b) G $\mathrm{Col}(\omega, \kappa)$, κ the sing κ.

reason CMI is hard is because we have been using $\neg$u.b. cong to get strength which is weak.

<u>last time.</u>

<u>Cam !</u>. $\mathrm{cf}(\Theta^{L(\Gamma, \mathbb{R}^*)}) = \omega$.

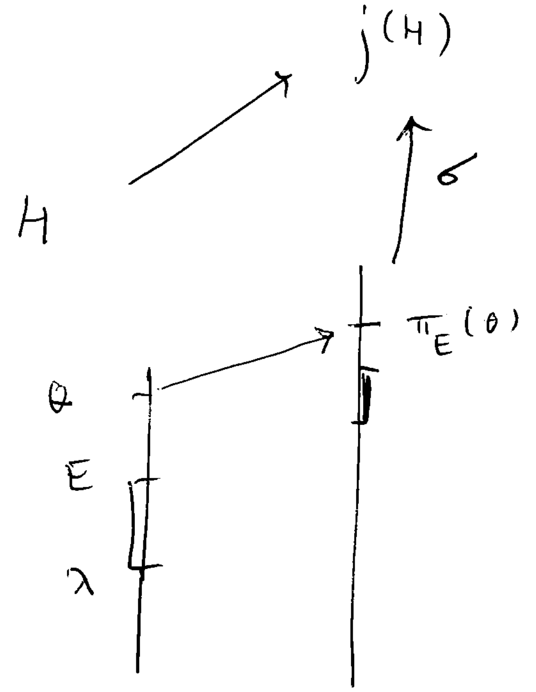
case 2. $L(\Gamma, \mathbb{R}^*) \models$ "$\varphi(\theta) < \theta$,

$\varphi(\theta) > \omega$."

get $H \models \varphi(\theta) < \theta \wedge \varphi(\theta)$ is measurable

recall: $j : V \longrightarrow M$

we want to show that $\mathcal{M}_\infty(\mathcal{H}, \Sigma) = j(\mathcal{H})$



we know : $\mathcal{M}_\infty(H, \Sigma) \trianglelefteq_{hod} j(H)$.

$\uparrow$ i.e., cutpoint

want to see :

$$\left.\begin{array}{l} \exists k : \mathcal{M}_\infty(H,\Sigma) \longrightarrow j(H) \\ \text{s.t. } \operatorname{crit}(k) = \beta^{\mathcal{M}_\infty(H,\Sigma)} \\ (\text{rather } k \restriction \beta = id) \end{array}\right\} \quad (*).$$

notation : $\beta^{\mathcal{N}}$ = the largest cardinal of $\mathcal{N}$.

then $(*) \Rightarrow \mathcal{M}_\infty(H,\Sigma) \models \beta^{\mathcal{M}_\infty(H,\Sigma)}$ is regular,

or $\mathcal{M}_\infty(H,\Sigma) = j(H)$.

as we assume $\operatorname{cf}(\theta) < \theta$, have $\mathcal{M}_\infty(H,\Sigma) = j(H)$

again contradiction, because we must have $\mathcal{M}_\infty(H,\Sigma) \triangleleft j(H)$, as in M there is a hod pair beyond $\mathcal{M}_\infty(H,\Sigma)$.

$$\begin{tikzcd}
 & j(H) & & \\
H \arrow[ur, "j"] \arrow[r, "\pi^{\Sigma}_{H,Q}"'] & Q \arrow[u, "\sigma"] \arrow[r, "\pi^{\Sigma}_{Q,Q_1}"'] & Q_1 \arrow[ul, "\sigma_1"'] \arrow[r, dashed] & M_\infty(H,\Sigma) \arrow[ull, bend right, "\sigma_\infty"']
\end{tikzcd}$$

pf. of $(*)$: why is $\sigma_\infty = id$ on

$\beta^{M_\infty(H,\Sigma)}$.

Let $\alpha < \beta^{M_\infty(H,\Sigma)}$.

~~$\sigma_\infty(\alpha) =$~~

$$\sigma_\infty(\alpha) = \sigma_n\Big(\underbrace{\pi^{-1}_{Q_n, M_\infty(H,\Sigma)}(\alpha)}_{\bar\alpha}\Big)$$

want to see $\sigma_n(\bar\alpha) = \alpha$.

follows from $\sigma_n = \pi_{Q_n, M_\infty(H,\Sigma)}$.

idea: make sure that the realizatn maps agree with the iteration maps.

this motivates the following defin,

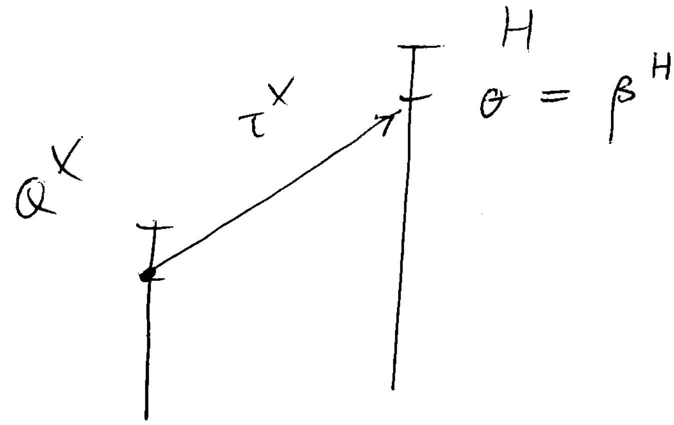
work in $V[g]$. let $X \in P_{\omega_1}(H)$.

$$Q^X = \mathrm{Hull}^H(X).$$

$$\tau^X : Q^X \longrightarrow H.$$

def: X is full iff $Q^X = Lp^{\Sigma^X}(Q^X | \beta^{Q^X})$,

where Σ^X = pullback of Σ by τ^X.



let $Y \in P_{\omega_1}(H|\theta)$.

Say Y is an extension of X iff

$$X \cap (H|\theta) \subset Y$$

let $Q^X_Y = \mathrm{Hull}(X \cup Y)$.

$$\pi^*_Y : Q^X \longrightarrow Q^X_Y \xrightarrow{\tau^X_Y} H$$

$$\pi^*_Y = (\tau^X_Y)^{-1} \circ \tau^X$$

$$\Sigma^X_Y = \tau^X_Y\text{-pullback of } \Sigma.$$

fix $X \in P_{\omega_1}(H)$, $Y \in P_{\omega_1}(H|\theta)$, Y an extension of X, let

$$\sigma^*_Y : Q^X_Y \,|\, \rho^{Q^X_Y} \longrightarrow H$$

$$\| $$

$$\pi^{\Sigma^X_Y}_{Q^X_Y, \mathcal{M}(H,\Sigma)}$$

let $\sigma_Y : Q^X_Y \longrightarrow H$ be given by

$$\sigma^X_Y(a) = \tau^X_Y(f)(\sigma^*_Y(y)), \quad \text{where}$$

$$a = \pi^X_Y(f)(y), \quad f \in Q^X, \quad y \in [\beta^{Q^X_Y}]^{<\omega}.$$

defn. X is a weakly condensing set if

1. if Y is an extension of X, then $X \cup Y$ is full

2. $\tau^X = \sigma^X_Y \circ \pi^X_Y$, where again Y is an ext. of X.

$$\begin{tikzcd} & H \\ Q^X \arrow[ur, "\tau^X"] \arrow[r, "\pi^X_Y"'] & Q^X_Y \arrow[u, "\sigma^X_Y"'] \end{tikzcd}$$

X is condensing iff $X \cup Y$ is weakly condensing ~~iff~~ for any Y extending X.

grigor III.

thm. there is a condensing set.

idea: fix $a \in H$ + $X \in \wp_{\omega_1}(H)$.

let Y be an extension of X.

let $T_Y^{X,a} = \{(\varphi, s) :$

$s \in [\beta^{Q_Y^X}]^{<\omega} \wedge H \models \varphi(a, \sigma_Y^X(s))\}$

if $X \cup Y$ is full, then $T_Y^{X,a} \in Q_Y^X$,

by mouse set capturing.

X is *a-condensing* if $\forall Y_0 \forall Y_1$ extensions of X,

$Y_0 \subset Y_1$, then

$$\begin{tikzcd}
 & H & \\
Q^X \arrow[ur, "\tau^X"] \arrow[r] & Q^X_{Y_0} \arrow[u, "\tau^X_{Y_0}"'] \arrow[r] & Q^X_{Y_1} \arrow[ul, "\tau^X_{Y_1}"']
\end{tikzcd}$$

$$\pi^{X}_{Y_0 Y_1}\left(T^{X,a}_{Y_0}\right) = T^{X,a}_{Y_1}.$$

call this a-condensing.

we now look at $j : V \longrightarrow M$

show if $\alpha \in j''H$, then $j''H$ is α-condensing.

exercise $\Rightarrow$ $j''H$ is condensing.

Supp. not. ~~let $m: M \to N$~~.

let $Y_0, Y_1 \in \mathcal{P}_{\omega_1}(j(H))$ extending $j''H$.

let $m : M \longrightarrow N$ be an iterate of j.

<u>notice</u>: the fact that $\pi^{X}_{Y_0 Y_1}\left(T^{X,\alpha}_{Y_0}\right) \neq T^{X,\alpha}_{Y_1}$

can be formulated in $L(\Gamma^N, \mathbb{R}^*)$

and $\alpha^* \in m(j(H))$ where $\alpha^* = \pi^{j(\Sigma)}_{j(H),\infty}(\alpha)$

$X = j''H$.

$$\begin{tikzcd} & & m \circ j(H) \arrow[d, phantom, sloped, "\trianglerighteq"] \\ & & \mathcal{M} \ni \alpha^* \\ & j(H) \arrow[uur] \arrow[ur, "j(\theta)"] & \\ H \arrow[ur, "j"] \arrow[r] & Q^X_{Y_0} \arrow[u] \arrow[r] & Q^X_{Y_1} \arrow[ul] \end{tikzcd}$$

$\mathcal{M} = \mathrm{ult}(j(H), E)$

E is the $(\beta^{j(H)}, \sup m'' \beta^{j(H)})$ extender from m.

We can now construct (H^+, Λ) s.t.

1. $H \trianglelefteq H^+$, H^+ is a Σ-hod pm over H.

2. ~~der~~ $D(H^+, \Lambda)$ reaches $\mathcal{M}$.

derived model

($\mathcal{M} \trianglelefteq$ HOD of $D(H^+, \Lambda)$).

key point. we can make sure that $T^{m''X, \alpha^*}$ is def. th ~ H^+.

$(\varphi, s) \in T^{m''X, \alpha^*} \iff$ iff

$H^+ \models$ if S_∞ is the image of s in the hod lim of my derived model, then $M \models \varphi(s_\infty, a^*)$.

$$\begin{tikzcd}
 & & m(j(H)) & & \\
H^+ \triangleright Q^X \arrow[urr] \arrow[rr] & & H^+_{Y_0} \triangleright Q^X_{Y_0} \arrow[u] \arrow[rr] & & H^+_{Y_1} \triangleright Q^X_{Y_1} \arrow[ull]
\end{tikzcd}$$

$$T^{X,\alpha^*} \longrightarrow T = \pi^X_{Y_0}(T^{X,\alpha^*})$$

$$T^{X,\alpha^*}_{Y_0}$$

how to get to LSA :

LSA : largest suslin axiom.

a) AD^+ + there is a largest suslin cardinal

b) if κ is the largest suslin, then $\forall A$, $w(A) < \kappa \Rightarrow$ no OD_A surjection of $\mathbb{R}$ onto κ.

we can analyze the min. model of LSA.

the hod pair (P, Λ) that corresponds to the min. model of LSA has the form

- β^P is woodin
- κ = the least $<\beta^P$-strong is a lim of woodins.

$$P = Lp^{\Lambda}(P \mid \beta^P)$$

for Λ is a short-tree-strategy for $P^+ = (P \mid \beta^P)^\#$.

Supp. $\mathcal{M}$ = proper class Λ-mouse over P with ω Woodin's above β^P.

the new derived model $\models$ LSA.

the problem with Λ-mice :

<u>problem</u> : max trees come down to short trees.

we certify shortness by certifying Q-structures.

0-sts mice : add all obvious branches.

1-sts mice : add all branches that are given by 0-sts mice.

(choose the one given by "my desired model has a strategy")

a) $j : V \longrightarrow M$.

b) $H \models \beta^H$ is regular.

two sorts of extenders

- exts with crit β^H
- exts with crit $> \beta^H$.

we now do a mixed K^c-constr. over H.

(*) supp. we are at some $(\mathcal{M}_\xi, \mathcal{N}_\xi)$.

we show we have the short tree strategy for $\mathcal{N}_\xi$.

picture: $\mathcal{N}_\xi$
$\trianglerighteq$
H

Supp. j gives an ext. E on $\mathcal{N}_\xi$ with ~~entry~~ crit β^H.

$$\begin{tikzcd} W_\xi & & j(H) \\ \unlhd & & \\ H \arrow[uurr] & & \end{tikzcd}$$

$(a, A) \in E$ iff

$\pi^{\wedge}_{W_\xi, \infty}(a) \in j(A)$.

if (W_ξ, E) makes sense, then add it.

Supp. we have E which is ctbly. certified + (W_ξ, E) makes sense. then add E.

<u>ex.</u> let $A \in \mu$ iff $\sup j''\beta^H \in j(A)$.

show that

① μ is a normal, β^H-complete ultrafilter on H,

② μ is amenable.