Tutorial I: The derived model theorem

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Outline

1. Background
   - Remark
   - Universally Baire
   - Homogeneously Suslin
   - Some facts
2. Derived model theorem
   - Derived model theorem and lemmas
   - Proof of reflection

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This talk is expository. None of the theorems or definitions are due to the speaker. I have not included credits for various definitions.

Outline

Definition

G is $<\lambda$*-generic* iff G is $(V,\mathbb{P})$-generic for a poset $\mathbb{P}$ of size $<\lambda$.

Definition

Let T,U be trees on $\omega\times X$. Say (T,U) is λ*-absolutely complementing* iff whenever G is $<\lambda$-generic, then

$$V[G]\models \text{``}p[T]=\mathbb{R}\backslash p[U]\text{''}.$$

Given $A\subseteq\mathbb{R}$, we say that A is λ*-universally Baire* iff there is a λ-absolutely complementing pair (T,U) such that $A=p[T]$.

Lemma

Let (T, U) and (R, S) be two λ-absolutely complenting pairs. Suppose $p[T] = p[R]$. Let G be $< \lambda$-generic. Then

$$V[G] \models \text{“}p[T] = p[R]\text{”}.$$

Definition

Let $A = p[T]$ where (T, U) is λ-absolutely complementing.
Let G be $< \lambda$-generic.
We write $A^{V[G]}$ for $p[T]^{V[G]}$.
By the lemma, this notation is unambiguous.

Outline

Definition

By *measure* we mean *countably complete ultrafilter*.
Let $m \leq n < \omega$, μ_0 be a measure on X^m, and μ_1 a measure on X^n.
We say μ_1 *projects to* μ_0 iff for every $A \in \mu_0$, we have

$$A^\frown X^{n-m} \in \mu_1$$

where

$$A^\frown X^{n-m} = \{u^\frown v \mid u \in A \;\&\; v \in X^{n-m}\}.$$

Equivalently, for every $B \in \mu_1$, we have

$$\{u \upharpoonright m \mid u \in B\} \in \mu_0.$$

Definition

Let X, μ_0, μ_1 be as above and μ_1 project to μ_0.
We have the natural elementary embedding

$$k = k_{\mu_0,\mu_1} : \mathrm{Ult}(V, \mu_0) \to \mathrm{Ult}(V, \mu_1)$$

given by

$$k([f]^V_{\mu_0}) = [f^{m,n}_X]^V_{\mu_1}$$

where $f^{m,n} : X^n \to V$ is

$$f^{m,n}_X(u) = f(u \restriction m).$$

Moreover, $k \circ i_{\mu_0} = i_{\mu_1}$.

Definition

A *tower of measures* (on X) is a sequence $\vec{\mu} = \langle \mu_n \rangle_{n<\omega}$ such that for all $m \leq n < \omega$, μ_n projects to μ_m.
Given a tower of measures $\vec{\mu}$, define

$$\mathrm{Ult}(V, \vec{\mu}) = \mathrm{dirlim}_{m \leq n < \omega}(\mathrm{Ult}(V, \mu_m), \mathrm{Ult}(V, \mu_n), k_{\mu_m, \mu_n}).$$

We say the tower is *wellfounded* iff $\mathrm{Ult}(V, \vec{\mu})$ is wellfounded.

Lemma

Let $\vec{\mu} = \langle \mu_n \rangle_{n<\omega}$ be a tower of measures on X.
Then $\vec{\mu}$ is wellfounded iff for all sequences $\langle A_n \rangle_{n<\omega}$ such that

$$A_n \in \mu_n \text{ for all } n < \omega,$$

there is $f : \omega \to X$ threading $\langle A_n \rangle_{n<\omega}$, i.e. such that

$$f \restriction n \in A_n \text{ for all } n.$$

Proof Sketch.

Suppose there is a bad sequence $\langle A_n \rangle_{n<\omega}$ (there's no f threading the sequence).
May assume that $A_{n+1} \subseteq A_n {}^\frown X$ for each n.
Let T be the tree of attempts to build a threading f.
So T is wellfounded.
Let $\psi : T \to \mathrm{OR}$ be the rank function, and $\psi_n = \psi \upharpoonright X^n$.
Then $\langle [\psi_n]_{\vec{\mu}} \rangle_{n<\omega}$ illfounds the ordinals of $\mathrm{Ult}(V, \vec{\mu})$. □

Definition

Let T be a tree on $\omega \times X$.
For $s \in {}^{<\omega}\omega$, $T_s = \{u|(s,u) \in T\}$.
A *homogeneity system* for T is a system $\langle \mu_s \rangle_{s \in {}^{<\omega}\omega}$ such that:

- for all s, μ_s is a measure on T_s,
- if $s \subseteq t$ then μ_t projects to μ_s,
- for all $x \in p[T]$, the tower $\langle \mu_{x \restriction n} \rangle_{n<\omega}$ is wellfounded.

The homogeneity system is κ*-complete* iff every μ_s is κ-complete.
T is κ*-homogeneous* iff there is a κ-complete homogeneity system for T.

Note: *hom* will abbreviate *homogeneous/ly*. From now on assume there is a measurable cardinal.

Definition

Let $A \subseteq {}^\omega\omega$ and $\kappa \in \mathrm{OR}$.

A is *κ-homogeneously Suslin* iff $A = p[T]$ for some κ-hom tree T. Hom_κ denotes the collection of κ-homogeneously Suslin sets.

A is *$< \lambda$-homogeneously Suslin* iff A is κ-hom Suslin for all $\kappa < \lambda$. $\mathrm{Hom}_{<\lambda}$ denotes the collection.

A is *homogeneously Suslin* iff A is ω_1-hom Suslin.

A is *κ-weakly homogeneously Suslin* iff* there is a κ-hom Suslin set $B \subseteq {}^\omega\omega \times {}^\omega\omega$ such that $A = p[B]$.

A is *$< \lambda$-weakly homogeneously Suslin* iff A is κ-weakly hom Suslin for all $\kappa < \lambda$.

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Lemma

Hom_κ closed under Wadge reduction (continuous preimages), and countable intersections.

Theorem (Martin)

Every homogeneously Suslin set is determined.

Theorem (Martin, Solovay, Steel, Woodin)

Let λ be a limit of Woodins and $A \subseteq \mathbb{R}$.
Then the following are equivalent:

1. *$A \in \mathrm{Hom}_{<\lambda}$,*
2. *A is $< \lambda$-weakly homogeneously Suslin,*
3. *A is λ-universally Baire.*

$\mathrm{Hom}_{<\lambda}$ is closed under complementation and real quantifiers.

Theorem (Martin, Solovay)

Every κ-weakly hom Suslin set of reals is κ-universally Baire.

Theorem (Woodin)

Let δ be Woodin.
Then every δ^+-universally Baire set of reals is $<\delta$-weakly hom Suslin.

Theorem (Martin, Steel)

Let δ be Woodin. Let $A \subseteq \mathbb{R}$ be δ^+-weakly hom Suslin.
Then $\mathbb{R} \backslash A$ is $<\delta$-hom Suslin.

Theorem (Steel, Woodin)

Let λ be a limit of Woodins. Then there is $\gamma < \lambda$ such that $\mathrm{Hom}_\gamma = \mathrm{Hom}_{<\lambda}$.

Proof Sketch.

We have

$$\alpha < \beta \implies \mathrm{Hom}_\beta \subseteq \mathrm{Hom}_\alpha.$$

If the theorem fails we can pick a sequence $\langle A_n \rangle_{n<\omega}$ of sets of reals such that $A_{n+1} <_{\mathrm{w}} A_n$ and $\mathbb{R} \backslash A_{n+1} <_{\mathrm{w}} A_n$. Using the determinacy of sets in Hom, we can run Martin's proof that the Wadge order is wellfounded, for a contradiction. □

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Derived model theorem (old version) (Woodin, 1980s)

Let λ be a limit of Woodins and $G \subseteq \mathrm{Col}(\omega, <\lambda)$ be V-generic. Let $\mathbb{R}^* = \bigcup_{\alpha<\lambda} \mathbb{R}^{V[G\restriction\alpha]}$.
For A, α such that $A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}$, let $A^*_\alpha = \bigcup_{\alpha\leq\beta<\lambda} A^{V[G\restriction\beta]}$. (Here $A^{V[G\restriction\beta]}$ is in the sense of $V[G\restriction\alpha]$). Let

$$\mathrm{Hom}^* = \{A^*_\alpha \mid \alpha < \lambda \ \& \ A \in \mathrm{Hom}_{<\lambda}^{V[G\restriction\alpha]}\}.$$

Then:

1. $\mathbb{R}^* = \mathbb{R}^{V[G]} \cap L(\mathbb{R}^*, \mathrm{Hom}^*)$,
2. $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$,
3. Hom^* *is the set of all Suslin co-Suslin sets of reals of* $L(\mathbb{R}^*, \mathrm{Hom}^*)$.

Definition

AD$^+$ is the theory AD + DC$_{\mathbb{R}}$ + Ordinal determinacy + "All sets of reals are ∞-Borel".
When λ is a limit of Woodins, we call $L(\mathbb{R}^*, \mathrm{Hom}^*)$ as above the *derived model at* λ.

Remark: Woodin has also shown that AD$^+$ implies that Σ^2_1 has the scale property.

A key reflection theorem:

Theorem (Woodin)

Let λ be a limit of Woodins. Let $A \in \mathrm{Hom}_{<\lambda}$. Let φ be a formula and suppose there is $B \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ such that $B \subseteq \mathbb{R}^*$ and

$$(\mathbb{R}^*, A^*, B) \models \varphi.$$

Then there is $B \in \mathrm{Hom}_{<\lambda}$ (so $B \in V$) such that

$$(\mathbb{R}, A, B) \models \varphi.$$

Theorem (Steel)

Let λ be a limit of Woodins. Then every set in $\mathrm{Hom}_{<\lambda}$ has a $\mathrm{Hom}_{<\lambda}$ scale.

Steel's proof of this theorem uses the stationary tower.

Lemma

Let λ be a limit of Woodins. Let $A \in \mathrm{Hom}_{<\lambda}$. Let P be the set of all projective-in-A formulas.
Then there is a sequence $\langle T^\varphi, U^\varphi \rangle_{\varphi \in P}$ of λ-absolutely complementing pairs (T^φ, U^φ), such that for all $< \lambda$-generics G and $x \in \mathbb{R}^{V[G]}$ and $\varphi \in P$,

$$(\mathbb{R}^{V[G]}, A^{V[G]}) \models \varphi(x) \iff x \in p[T^\varphi]^{V[G]}.$$

Lemma

Let λ, A, G be as above. Let H be $< \lambda$-generic over $V[G]$. Then

$$(\mathbb{R}^{V[G]}, A^{V[G]}) \preccurlyeq (\mathbb{R}^{V[G][H]}, A^{V[G][H]}).$$

Therefore $(\mathbb{R}^{V[G]}, A^{V[G]}) \preccurlyeq (\mathbb{R}^*, A^*)$.

Proof of old Derived model theorem (modulo above facts):

Claim: $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \mathsf{AD}^+$.

First consider AD. Let φ be the natural formula expressing "B is determined", when interpreted over the structure $(\mathbb{R}, B)$. Suppose $L(\mathbb{R}^*, \mathrm{Hom}^*) \models \neg\mathsf{AD}$. Then

$$L(\mathbb{R}^*, \mathrm{Hom}^*) \models \exists B \subseteq \mathbb{R}^*[(\mathbb{R}^*, B) \models \neg\varphi].$$

So by Woodin's reflection theorem, there is $B \in \mathrm{Hom}_{<\lambda}$ such that

$$(\mathbb{R}, B) \models \neg\varphi.$$

But every set in $\mathrm{Hom}_{<\lambda}$ is determined, contradiction.
$\mathsf{DC}_{\mathbb{R}}$ is easier.

Claim: Every set in Hom^* is Suslin-co-Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$.

Let $A' \in \mathrm{Hom}^*$. We may assume $A' = A^*$ some $A \in \mathrm{Hom}_{<\lambda}$.
By Steel's scales theorem, there's a scale B on A s.t. $B \in \mathrm{Hom}_{<\lambda}$.
The statement "B is a scale on A" is projective in (A, B).
So by elementarity, B^* is a scale on A^* in $L(\mathbb{R}^*, \mathrm{Hom}^*)$.
Therefore A^* is Suslin in $L(\mathbb{R}^*, \mathrm{Hom}^*)$.
Since Hom^* is closed under complements, we are done.

Claim: Every Suslin-co-Suslin set of $L(\mathbb{R}^*, \mathrm{Hom}^*)$ is in Hom^*.

Let $X, T, U \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ be such that

$$X = p[T] \text{ and } \mathbb{R}^* \backslash X = p[U].$$

Let α and $A \in \mathrm{Hom}_{<\lambda}^{V[G \restriction \alpha]}$ be such that

$$(T, U) \text{ is } \mathrm{OD}_A^{L(\mathbb{R}^*, \mathrm{Hom}^*)}.$$

Then $T, U \in V[G \restriction \alpha]$ (use homogeneity of the forcing and homogeneous names for $\mathbb{R}^*, \mathrm{Hom}^*$).
Note T, U project to complements in $V[G \restriction \beta]$, for $\alpha < \beta < \lambda$.
It follows that T, U are λ-absolutely complementing in $V[G \restriction \alpha]$.
So $X \in \mathrm{Hom}^*$, as required.

The remaining axioms are proved somewhat like AD was proved, using that every set in Hom^* is Suslin in the derived model, together with some other AD results. However, these results are beyond the scope of the talk, so we omit further discussion.

Outline

We will focus on the proof of Woodin's reflection theorem, mostly following Steel's "A stationary tower free proof of the derived model theorem" (the only difference between the argument here and that in Steel's paper is that we use another tool in place of the "tower flipping" function).

We first discuss where we get $\mathrm{Hom}_{<\lambda}$ sets from.

Definition

An iteration tree $\mathcal{T}$ is:

1. $2^{\aleph_0}$-*closed* iff $M^{\mathcal{T}}_\alpha \models$ "$\mathrm{Ult}(V, E^{\mathcal{T}}_\alpha)$ is $2^{\aleph_0}$-closed" for all α,
2. *nice* iff $M^{\mathcal{T}}_\alpha \models$ "$\mathrm{strength}(E^{\mathcal{T}}_\alpha) = \mathrm{lh}(E^{\mathcal{T}}_\alpha)$ is inaccessible" for all α.

1. Every nice tree is $2^{\aleph_0}$-closed.
2. If $\mathcal{T}$ on V is $2^{\aleph_0}$-closed and $\mathrm{lh}(\mathcal{T}) \leq \omega$ then $M^{\mathcal{T}}_n$ is $2^{\aleph_0}$-closed for every n.

The key to getting $\mathrm{Hom}_{<\lambda}$ sets is the following theorem:

Theorem (Windßus)

Let (V_θ, δ) be a coarse premouse.
Let $\pi : M \to (V_\theta, \delta)$ be elementary, with M countable.
Let $\kappa \in \mathrm{OR}^M$.
Let W be the set of all (reals coding some) $\mathcal{T}$ such that:

1. *$\mathcal{T}$ is an iteration tree on M of length $\omega + 1$,*
2. *$\mathcal{T}$ is above κ (critical points $\geq \kappa$),*
3. *$\mathcal{T}$ is $2^{\aleph_0}$-closed,*
4. *$\pi\mathcal{T}$ has wellfounded models (equiv, $M_\omega^{\pi\mathcal{T}}$ is wellfounded).*

Then W is $\pi(\kappa)$-homogeneously Suslin.

Proof of Windßus' Theorem:

Assume $\kappa = 0$. Let N, τ be such that:

1. $\tau : N \to (V_\theta, \delta)$ is elementary,
2. N has cardinality $2^{\aleph_0}$,
3. $\mathrm{rg}(\pi) \subseteq \mathrm{rg}(\tau)$; so let $\sigma : M \to N$ be the factor,
4. for every tree $\mathcal{T}$ on M of length $\omega + 1$,

$$M_\omega^{\pi\mathcal{T}} \text{ wellfounded} \iff M_\omega^{\sigma\mathcal{T}} \text{ wellfounded.}$$

Let $\tau : N \to V_\theta$ be the uncollapse.
We define a tree S of attempts to build:

1. (1st coordinate) a $2^{\aleph_0}$-closed tree $\mathcal{T}$ on M of length $\omega + 1$,
2. (2nd coordinate) an elementary $\varrho : M_\omega^{\sigma\mathcal{T}} \to V_\theta$.

A non-empty node s of S specifies:

1. a $2^{\aleph_0}$-closed tree $\bar{\mathcal{T}}$ on M, of length $n+1<\omega$,
2. a sequence $\langle \varrho_k \rangle_{k \leq_{\bar{\mathcal{T}}} n}$, such that

$$\varrho_k : M_k^{\sigma\bar{\mathcal{T}}} \to V_\theta,$$

$$\varrho_n \circ i_{kn}^{\sigma\bar{\mathcal{T}}} = \varrho_k,$$

with $\mathrm{lh}(s) = \mathrm{card}([0,n]_{\bar{\mathcal{T}}})$.
(Here s is declaring that $n \in b^{\mathcal{T}}$, for the tree $\mathcal{T}$ being built.)

Claim: $p[S] = W$.

Proof that $p[S] = W$:

If $\mathcal{T} \in p[S]$ then $M_\omega^{\sigma\mathcal{T}}$ is wellfounded, so $M_\omega^{\pi\mathcal{T}}$ is wellfounded, so $\mathcal{T} \in W$.

Conversely, suppose $\mathcal{T} \in W$, so $M_\omega^{\pi\mathcal{T}}$ is wellfounded.
For $n \in b^\mathcal{T}$ let $\pi_n : M_n^{\sigma\mathcal{T}} \to M_n^{\pi\mathcal{T}}$ be the copy map.
We have $\pi_n \in M_n^{\pi\mathcal{T}}$ by $2^{\aleph_0}$-closure, and

$$\varrho'_n =_{\text{def}} i_{n\omega}^{\pi\mathcal{T}} \circ \pi_n = i_{n\omega}^{\pi\mathcal{T}}(\pi_n) \in M_\omega^{\pi\mathcal{T}}.$$

It follows that $\langle \varrho'_n \rangle_{n <_\mathcal{T} \omega}$ is a branch through $S'_\mathcal{T} = i_{0\omega}^{\pi\mathcal{T}}(S)_\mathcal{T}$.
Since $M_\omega^{\pi\mathcal{T}}$ is wellfounded, $S'_\mathcal{T}$ has a branch in $M_\omega^{\pi\mathcal{T}}$.
Therefore $S_\mathcal{T}$ has a branch in V. So $\mathcal{T} \in p[S]$, as required.

Now we define a homogeneity system

$$\langle \mu_{\bar{\mathcal{T}}} \rangle_{\bar{\mathcal{T}} \in \mathscr{F}}$$

on S, where $\mathscr{F}$ is the set of finite trees on M, as follows.
Let $\bar{\mathcal{T}} \in \mathscr{F}$, of length $n+1$. We define a measure $\mu_{\bar{\mathcal{T}}}$ on $S_{\bar{\mathcal{T}}}$.
Recall $S_{\bar{\mathcal{T}}}$ consists of tuples $\langle \varrho_k \rangle_{k \leq_{\bar{\mathcal{T}}} n}$ where $\varrho_k : M_k^{\sigma \mathcal{T}} \to V_\theta$.
Let $\bar{\mathcal{U}} = \pi \bar{\mathcal{T}}$.
Let $\pi_k : M_k^{\sigma \bar{\mathcal{T}}} \to M_k^{\bar{\mathcal{U}}}$ be the copy map.
Let $\pi'_k = i_{kn}^{\bar{\mathcal{U}}}(\pi_k)$, for $k \leq_{\bar{\mathcal{U}}} n$.
For $A \subseteq S_{\bar{\mathcal{T}}}$, put $A \in \mu_{\bar{\mathcal{T}}}$ iff

$$\langle \pi'_k \rangle_{k \leq_{\bar{\mathcal{U}}} n} \in i_{0,n}^{\bar{\mathcal{U}}}(A).$$

An easy modification of the Claim proof shows that this gives a homogeneity system for S, completing the proof.

Neeman's genericity iterations give the trees $\mathcal{T}$ above:

Theorem (Neeman)

Let M be countable transitive model of ZFC. Suppose $M \models$ "δ is Woodin". Then for every real x there is a length ω (nice) iteration tree $\mathcal{T}$ on M such that for every wellfounded $\mathcal{T}$-cofinal branch b, there is an $M_b^{\mathcal{T}}$-generic $g \subseteq \mathrm{Col}(\omega, i_c^{\mathcal{T}}(\delta))$ such that $x \in M_c^{\mathcal{T}}[g]$. Moreover, given $\alpha < \delta$ we may take $\mathcal{T}$ above α.

(See Neeman's "Determinacy in $L(\mathbb{R})$".) We will combine this with iterability results of Martin and Steel (see "Iteration trees"):

Theorem (Martin, Steel)

(a) Every finite length, putative iteration tree on V has wellfounded models.
(b) Let $\mathcal{T}$ be a length ω, nice iteration tree on V. Then there is a $\mathcal{T}$-cofinal branch b such that $M_b^{\mathcal{T}}$ is wellfounded.

Corollary

Let (V_θ, δ) be a coarse premouse and $\pi : M \to (V_\theta, \delta)$ be elementary, with M countable.
Let $\mathcal{T}$ be a length ω, nice iteration tree on M, so $\pi\mathcal{T}$ is nice on V.
Let b be such that $M_b^{\pi\mathcal{T}}$ is wellfounded. Then b is π-realizable.
That is, there is an elementary $\sigma : M_b^{\mathcal{T}} \to V_\theta$ such that $\sigma \circ i_b^{\mathcal{T}} = \pi$.

If M as above has infinitely many Woodins, we can combine these results repeatedly to form $\mathbb{R}^V$-genericity iterations, working in $V[G]$ where G collapses $\mathbb{R}$...

$\mathbb{R}^V$*-genericity iterations*: We have M as above with infinitely many Woodins. Work in $V[G]$ where G collapses $\mathbb{R}$. Let $\langle x_n \rangle_{n<\omega}$ enumerate $\mathbb{R}^V$. Let $\langle \delta_n \rangle_{n<\omega}$ be an increasing sequence of Woodins of M. Let $\mathcal{T}_0$ be a Neeman genericity iteration on $M_0 = M$, making x_0 generic at the image of δ_0. Let b_0 be a $\pi_0 = \pi$-realizable branch, witnessed by

$$\pi_1 : M^{\mathcal{T}_0}_{b_0} \to V_\theta.$$

Let $M_1 = M^{\mathcal{T}_0}_{b_0}$. Repeat with $M_1, \pi_1, x_1, i^{\mathcal{T}_0}_{b_0}(\delta_1)$, iterating above $i^{\mathcal{T}_0}_{b_0}(\delta_0)$.
Let $\mathcal{T} = \mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1 \,\widehat{}\, \ldots$. So $\mathcal{T}$ has a unique cofinal branch b_∞ and we have $\sigma : M^{\mathcal{T}}_{b_\infty} \to V$ commuting with all the π_n's. We have each $\mathcal{T}_n \in V$ (but maybe $\mathcal{T} \notin V$).

Proof of Woodin's reflection theorem: We mostly follow Steel's stationary tower free proof. We have $A \in \mathrm{Hom}_{<\lambda}$, and there is $B \in L(\mathbb{R}^*, \mathrm{Hom}^*)$ such that $B^* \subseteq \mathbb{R}^*$ and $(\mathbb{R}^*, A^*, B) \models \varphi$.

Claim 1: There is $B \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mathrm{Hom}_{<\lambda})$ s.t. $(\mathbb{R}, A, B) \models \varphi$.

Proof of Claim 1: Let $\pi : M \to V_\theta$ be elementary, θ high cofinality, $\lambda \in V_\theta \preccurlyeq_n V$, everything relevant in $\mathrm{rg}(\pi)$, M countable.

Notation: Write $\pi(\lambda^M) = \lambda$, $\pi(A^M) = A$, etc. Given iteration tree $\mathcal{T}$ on M and $P = M^{\mathcal{T}}_\alpha$, and a P-generic $H \subseteq \mathrm{Col}(\omega, <\lambda^P)$, write:

1. $i^{\mathcal{T}}_{MP} = i^{\mathcal{T}}_{0\alpha}$ and $\lambda^P = i^{\mathcal{T}}_{MP}(\lambda^M)$ and $A^P = i^{\mathcal{T}}_{MP}(A^M)$, etc,
2. $D^P_{\lambda^P}$ for the name for the derived model of P at λ^P,
3. $(\mathrm{Hom}^*)^P, ((A^P)^*)^P$, etc, for the names for associated objects,
4. $D^{P[H]}_{\lambda^P}$, $((A^P)^*)^{P[H]}$ for the H-interpretations of such names,

etc.

We have $M \models \psi(\lambda^M, A^M)$, where $\psi(\lambda', A')$ is the formula:

$$\mathrm{Col}(\omega, < \lambda') \Vdash \text{"}\exists B \in D_{\lambda^M} \text{ such that } (\mathbb{R}^*, (A')^*, B) \models \varphi\text{"}.$$

Let $\langle \delta_n \rangle_{n<\omega}$ be a strictly increasing sequence of Woodins of M cofinal in λ^M.
Working in $V[G]$ where G collapses $\mathbb{R}$, let $\mathcal{T} = \mathcal{T}_0 \frown \mathcal{T}_1 \frown \ldots$ be a π-realizable Neeman $\mathbb{R}^V$-genericity iteration on M, with $\mathcal{T}_n$ based on the interval $i^{\mathcal{T}_{n-1}}((\delta_{n-1}, \delta_n))$. We have $\mathcal{T}_n \in V$. We have $\langle \sigma_n \rangle_{n<\omega} \subseteq V$ such that $\sigma_0 = \pi$ and

$$\sigma_{n+1} : M^{\mathcal{T}_n}_\omega \to V_\theta \text{ and } \sigma_{n+1} \circ i^{\mathcal{T}_n} = \sigma_n.$$

Choose $\mathcal{T}$ with $i^{\mathcal{T}}$ continuous at λ^M.
Let $N = M^{\mathcal{T}}_\infty$ and $\sigma_\omega : N \to V_\theta$ be the limit. So $N \models \psi(\lambda^N, A^N)$.

We can realize $\mathbb{R}^V$ as the reals $(\mathbb{R}^*)^{N[H]}$ of some derived model D of N, via a generic $H \subseteq \mathrm{Col}(\omega, < \lambda^N)$. So for some α,

$$D = D^{N[H]}_{\lambda^N} = L_\alpha(\mathbb{R}^V, (\mathrm{Hom}^*)^{N[H]}).$$

Subclaim 1.1: $((A^N)^*)^{N[H]} = A$.
Proof: Let $T, U \in \mathrm{rg}(\pi)$ be λ-absolutely complementing trees such that $p[T] = A$. Then T^N, U^N are λ^N-absolutely complementing in N, and

$$((A^N)^*)^{N[H]} = p[T^N] \cap \mathbb{R}^V.$$

But $\sigma(T^N, U^N) = (T, U)$. So given $x \in \mathbb{R}^V$, either:

1. $x \in p[T^N] \backslash p[U^N]$, which implies $x \in p[\sigma(T^N)] = A$, or
2. $x \in p[U^N] \backslash p[T^N]$, which implies $x \in p[\sigma(U^N)] = \mathbb{R}^V \backslash A$.

This gives the subclaim.

Subclaim 1.2: $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge initial segment of $\mathrm{Hom}_{<\lambda}$.

With Subclaim 1.2 we can prove Claim 1. For $N \models \psi(\lambda^N, A^N)$; so in N it is forced by $\mathrm{Col}(\omega, <\lambda^N)$ that

$$\exists B \in D^N_{\lambda^N} \text{ such that } (\mathbb{R}^*, (A^N)^*, B) \models \varphi.$$

But by Subclaim 1.2,

$$(D_{\lambda^N})^{N[H]} = L_\alpha(\mathbb{R}^V, (\mathrm{Hom}^*)^{N[H]}) \in L(\mathbb{R}^V, \mathrm{Hom}_{<\lambda}),$$

and by Subclaim 1.1, $((A^N)^*)^{N[H]} = A$. Therefore

$$\exists B \in L(\mathbb{R}, \mathrm{Hom}_{<\lambda}) \text{ such that } (\mathbb{R}^V, A, B) \models \varphi,$$

giving Claim 1.

Subclaim 1.2: $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge initial segment of $\mathrm{Hom}_{<\lambda}$.

Proof of Subclaim 1.2:
It suffices to see that $(\mathrm{Hom}^*)^{N[H]} \subseteq \mathrm{Hom}_{<\lambda}$, because $(\mathrm{Hom}^*)^{N[H]}$ is closed under Wadge reducibility.

So let $X \in (\mathrm{Hom}^*)^{N[H]}$. We want $X \in \mathrm{Hom}_{<\lambda}$. Recall

$$\mathrm{Hom}_{<\lambda} = \mathrm{Hom}_{\gamma_0} \text{ for some } \gamma_0 < \lambda.$$

Let ξ_0 be the least Woodin $> \gamma_0$. It suffices to see that X is ξ_0^+-weakly hom Suslin. (For then $\mathbb{R}^V \backslash X$ is in $\mathrm{Hom}_{<\xi_0}$, hence in Hom_{γ_0}, hence in $\mathrm{Hom}_{<\lambda}$. But $(\mathrm{Hom}^*)^{N[H]}$ is self-dual, so we are done.) Note $\gamma_0, \xi_0 \in \mathrm{rg}(\pi)$.

Fix $n < \omega$ and H_n, Y, Z such that:

1. $H_n = H \restriction \delta_n^N$
2. $Y, Z \in N[H_n] \models$ "Y, Z are λ^N-absolutely complementing trees",
3. $X = p[Y] \cap \mathbb{R}^V$,
4. $\xi_0^N < \delta_n^N = i^{\mathcal{T}}(\delta_n)$.

Let $P = M_\omega^{\mathcal{T}_n}$, so $P|\delta_n^P = N|\delta_n^N$ and $\xi_0^P = \xi_0^N$. Let

$$i_{PN}^{\mathcal{T}+} : P[H_n] \to N[H_n]$$

be the canonical extension of $i_{PN}^{\mathcal{T}}$ ($\mathcal{T}_k$ is above $\delta_n^P + 1$ for $k > n$). We may assume we have $Y^P, Z^P \in P[H_n]$ such that

1. $i_{PN}^{\mathcal{T}+}(Y^P, Z^P) = (Y, Z)$.

We have $\sigma = \sigma_{n+1} : P \to V_\theta$.

Let W be the set of $2^{\aleph_0}$-closed above-$(\delta_n^P + 1)$ iteration trees $\mathcal{U}$ on P, of length $\omega + 1$, such that $\sigma\mathcal{U}$ is wellfounded. By Windßus' theorem, W is $\pi(\delta_n)$-hom Suslin, hence ξ_0^+-hom Suslin.

Let X' be the set of all reals x such that for some $\mathcal{U} \in W$, letting

$$i^{\mathcal{U}+} : P[H_n] \to M_\omega^{\mathcal{U}}[H_n]$$

be the elementary extension, then $x \in p[i^{\mathcal{U}+}(Y^P)]$. Then X' is ξ_0^+-weakly hom Suslin. We want that X is ξ_0^+-weakly hom Suslin, so it suffices to prove:

Subsubclaim 1.2.1: $X' = X$.

Subsubclaim 1.2.1: $X' = X$.
Proof:
Suppose $x \in X'$, as witnessed by $\mathcal{U}$. We need to see that $x \in X = p[Y]$. Let $m \in (n, \omega)$ be such that $x \in N[H_m]$ and let $Q = M_\omega^{\mathcal{T}_m}$. Extend $i^{\mathcal{T}}_{P,Q}$ and $i^{\mathcal{T}}_{QN}$ elementarily to

$$i^{\mathcal{T}+}_{PQ} : P[H_n] \to Q[H_n],$$

$$i^{\mathcal{T}+}_{QN} : Q[H_m] \to N[H_m].$$

Let $(Y^Q, Z^Q) = i^{\mathcal{T}+}_{PQ}(Y^P, Z^P)$. It suffices to see $x \in p[Y^Q]$, as $i^{\mathcal{T}+}_{QN}(Y^Q) = Y$. We have $x \in Q[H_m]$, so

$$x \in p[Y^Q] \cup p[Z^Q],$$

so suppose $x \in p[Z^Q]$.

In V we have the map

$$\varrho = \sigma_{n+1} : Q \to V_\theta$$

with commutativity $\varrho \circ i^{\mathcal{T}}_{PQ} = \sigma$.

Now let $\mathcal{V} = \tau\mathcal{U}$ where $\tau = i^{\mathcal{T}}_{PQ}$. By commutativity, $\varrho\mathcal{V} = \sigma\mathcal{U}$, and since $\mathcal{U} \in W$, therefore $M^{\varrho\mathcal{V}}_\omega = M^{\sigma\mathcal{U}}_\omega$ is wellfounded. Therefore $M^{\mathcal{V}}_\omega$ is wellfounded. We have

$$i^{\mathcal{U}+} : P[H_n] \to M^{\mathcal{U}}_\omega[H_n],$$

$$i^{\mathcal{V}+} : Q[H_n] \to M^{\mathcal{V}}_\omega[H_n],$$

$$\tau^+_\omega : M^{\mathcal{U}}_\omega[H_n] \to M^{\mathcal{V}}_\omega[H_n]$$

extending the iteration maps and the final copy map τ_ω respectively.

The maps commute, so

$$(\widetilde{Y}, \widetilde{Z}) =_{\text{def}} \tau_\omega^+(i^{\mathcal{U}+}(Y^P, Z^P)) = i^{\mathcal{V}+}(Y^Q, Z^Q).$$

We chose $\mathcal{U}$ such that $x \in p[i^{\mathcal{U}+}(Y^P)]$ (in V). As usual it follows that

$$x \in p[\tau_\omega^+(i^{\mathcal{U}+}(Y^P))] = p[\widetilde{Y}].$$

But because $x \in p[Z^Q]$, we also have

$$x \in p[i^{\mathcal{V}+}(Z^Q)] = p[\widetilde{Z}].$$

So $p[\widetilde{Y}] \cap p[\widetilde{Z}] \neq \emptyset$. But $M_\omega^{\mathcal{V}}[H_n]$ is wellfounded and

$$M_\omega^{\mathcal{V}}[H_n] \models \text{ZFC}^{-\varepsilon} + \text{“}p[\widetilde{Y}] \cap p[\widetilde{Z}] = \emptyset\text{”},$$

giving a contradiction. This completes the proof that $X' \subseteq X$.

Now suppose $x \in X$. We find $\mathcal{U}$ witnessing that $x \in X'$. Let δ be the least Woodin of P such that $\delta > \delta_n^P$. Let $\widetilde{\mathcal{U}}$ be a nice Neeman genericity iteration of P at δ, above δ_n^P, making a real y generic, where y computes (H_n, x). So $\widetilde{\mathcal{U}}$ is nice of length ω, and for every $\widetilde{\mathcal{U}}$-cofinal wellfounded b, y is $< \lambda^{M_b^{\widetilde{\mathcal{U}}}}$-generic over $M_b^{\widetilde{\mathcal{U}}}$.

We have $\sigma\widetilde{\mathcal{U}}$ on V. By Martin-Steel theorem, we may fix b such that $M_b^{\sigma\widetilde{\mathcal{U}}}$ is wellfounded. Let $\mathcal{U} = \widetilde{\mathcal{U}} \frown b$. We claim that $\mathcal{U}$ witnesses that $x \in X'$. For $\mathcal{U} \in W$. Extend $i^{\mathcal{U}}$ to

$$i^{\mathcal{U}+} : P[H_n] \to M_\omega^{\mathcal{U}}[H_n].$$

Now x is $< \lambda^{M_\omega^{\mathcal{U}}}$-generic over $M_\omega^{\mathcal{U}}[H_n]$, as y computes (H_n, x). So

$$x \in p[i^{\mathcal{U}+}(Y^P)] \cup p[i^{\mathcal{U}+}(Z^P)].$$

If $x \in p[i^{\mathcal{U}+}(Z)]$, argue as before for a contradiction. So $x \in p[i^{\mathcal{U}+}(Y)]$, so $\mathcal{U}$ witnesses that $x \in X'$.
This completes the proof of Claim 1.

In the final part of the proof, we will show that we get some such $B \in \mathrm{Hom}_{<\lambda}$. Thanks to Trevor Wilson for pointing out one issue that arises in this part, which I had initially overlooked. We will actually give two different arguments, which each deal with this issue somewhat differently. Argument 1 is based on Steel's argument. In order to deal with one issue that arises, we will prove an extra fact, the proof of which takes some extra work, postponed until after Argument 2. Argument 2 deals with this issue differently, and in a more elementary manner, avoiding the need for the extra fact. It instead uses a modification to Argument 1 due to Wilson (cf. "A model of the Axiom of Determinacy in which every set of reals is universally Baire", by Larson, Sargsyan and Wilson). (Actually we set things up slightly different to Wilson's original argument, but use the same idea.)

Argument 1:

By Claim 1, we can fix the Wadge least initial segment Γ_0 of $\mathrm{Hom}_{<\lambda}$ such that $A \in \Gamma_0$ and

$$L(\mathbb{R}, \Gamma_0) \models \exists B[(\mathbb{R}, A, B) \models \varphi]. \tag{1}$$

Fix the least α_0 such that there is a witness $B \in \mathcal{J}_{\alpha_0+1}(\mathbb{R}, \Gamma_0)$. Fix $C_0 \in \Gamma_0$ and a formula φ_0 such that

$$\exists! B_0 \in \mathcal{J}_{\alpha_0+1}(\mathbb{R}, \Gamma_0) \text{ such that } \mathcal{J}_{\alpha_0+1}(\mathbb{R}, \Gamma_0) \models \varphi_0(C_0, B_0),$$

and moreover, the unique B_0 witnesses (1). The theorem will now follow from:

Claim 2: $B_0 \in \mathrm{Hom}_{<\lambda}$.

Claim 2: $B_0 \in \mathrm{Hom}_{<\lambda}$.

Proof: Let $\gamma_0, \xi_0 < \lambda$ be as before. If λ is regular let $\nu = \xi_0$; otherwise let $\nu = \max(\mathrm{cof}(\lambda)^+, \xi_0)$. Let ξ_1 be the the least Woodin $> \nu$. Let $\pi : M \to V_\theta$ be as usual. Notation is as before, so $\pi(A^M) = A$, etc. Abbreviate $A^{R[g]} = (A^R)^{R[g]}$, etc.

Let W be the set of all above-ν^M, $2^{\aleph_0}$-closed trees $\mathcal{U}$ on M of length $\omega + 1$ such that $M_\omega^{\pi\mathcal{U}}$ is wellfounded. So W is ξ_0^+-hom Suslin.

Let X' be the set of all $x \in \mathbb{R}$ such that for some tree $\mathcal{U} \in W$, $\mathcal{U}$ is based below ξ_1^M, and letting $R = M_\omega^{\mathcal{U}}$, there is an R-generic $g \subseteq \mathrm{Col}(\omega, i^{\mathcal{U}}(\xi_1))$ such that $x \in R[g]$, and

$$R[g] \models \psi_0(\lambda^R, A^{R[g]}, C_0^{R[g]}, x),$$

where $\psi_0(\dot{\lambda}, \dot{A}, \dot{C}, \dot{x})$ says that $\mathrm{Col}(\omega, < \dot{\lambda})$ forces that $D_{\dot{\lambda}}$ satisfies:

1. There is B s.t. $(\mathbb{R}^*, (\dot{A})^*, B) \models \varphi$.
2. Let Γ' be the Wadge least initial segment of Hom^* such that $(\dot{A})^* \in \Gamma'$ and there is some such $B \in L(\mathbb{R}^*, \Gamma')$.
3. $(\dot{C})^* \in \Gamma'$.
4. Let α' be least such that some such B is in $L_{\alpha'+1}(\mathbb{R}^*, \Gamma')$.
5. There is a unique $\widetilde{B} \in L_{\alpha'+1}(\mathbb{R}^*, \Gamma')$ such that

$$L_{\alpha'+1}(\mathbb{R}^*, \Gamma') \models \varphi_0(\dot{C}^*, \widetilde{B}).$$

6. $\dot{x} \in \widetilde{B}$.

So X' is ξ_0^+-weakly homogeneously Suslin, so the following completes the proof of the theorem:

Subclaim 2.1: $X' = B_0$.

Proof: Let $x \in X'$, as witnessed by $\mathcal{U}, g$. Let $R = M_\omega^{\mathcal{U}}$. As before, in some $V[G]$, form a nice Neeman $\mathbb{R}^V$-genericity iteration $\mathcal{T}$ on R, iterating above ξ_1^R. Let $N = M_\infty^{\mathcal{T}}$ and $\sigma : N \to V_\theta$ be as before. Realize the derived model

$$D_{\lambda^N}^{N[H]} = L_\alpha(\mathbb{R}^V, (\mathrm{Hom}^*)^{N[H]})$$

as before. By the proof of Subclaim 1.2, $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge initial segment of $\mathrm{Hom}_{<\lambda}$.

We have $i^{\mathcal{U}+} : R[g] \to N[g]$ extending $i^{\mathcal{U}}$. So

$$N[g] \models \psi_0(\lambda^N, A^{N[g]}, C_0^{N[g]}, x)$$

(but $A^{N[g]} = A^{R[g]}$, etc). Let $\Gamma', \alpha', \widetilde{B}$ as mentioned in ψ_0 be their interpretations relative to $D_{\lambda^N}^{N[H]}$.

We have

$$(A^{N[g]})^* = A \text{ and } (C_0^{N[g]})^* = C_0$$

because there are λ-absolutely complementing trees $T, U \in \mathrm{rg}(\pi)$ such that $p[T] = A$, and likewise for C_0, and $\sigma \circ i^{\mathcal{T}} \circ i^{\mathcal{U}} = \pi$.

Now because $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge initial segment of $\mathrm{Hom}_{<\lambda}$, it clearly follows that *if* $\mathrm{OR}^N > \alpha_0$ then $\Gamma' = \Gamma_0$ and $\alpha' = \alpha_0$ and $\widetilde{B} = B_0$, and therefore $x \in B_0$, as desired.

So it suffices to verify the following:

Subsubclaim 2.1.1: $\mathrm{OR}^N > \alpha_0$.

Proof: Postponed until after Argument 2, which deals with this issue in a more elementary manner.

So finally suppose $x \in B_0$; we want $x \in X'$.

Let $\mathcal{U}$ be a nice Neeman genericity iteration on M, based on the interval (ν^M, ξ_1^M), making x generic. Let b be such that $\mathcal{U} = \widetilde{\mathcal{U}}\ \widehat{}\ b \in W$. Using an $\mathbb{R}^V$-genericity iteration on $M_\omega^{\mathcal{U}}$, as before (repeating Subsubclaim 2.1.1), and because $x \in B_0$, it easily follows that $\mathcal{U}$ witnesses that $x \in X'$, as required.

This completes the proof (via Argument 1), mod the Subsubclaim.

Before proving Subclaim 2.1.1, we give a modification of Argument 1, which gets around the issue of the subclaim in a different manner. This modification is due to Wilson (though we have reorganized his original argument somewhat):
Argument 2: We define a finite sequence of objects

$$(\Gamma_0, \alpha_0, C_0, B_0), \ldots, (\Gamma_z, \alpha_z, C_z, B_z).$$

By Claim 1, we can fix the Wadge least initial segment Γ_0 of $\mathrm{Hom}_{<\lambda}$ such that $A \in \Gamma_0$ and

$$L(\mathbb{R}, \Gamma_0) \models \exists B[(\mathbb{R}, A, B) \models \varphi]. \quad (2)$$

Fix the least α_0 such that there is a witness $B \in \mathcal{J}_{\alpha_0+1}(\mathbb{R}, \Gamma_0)$.
Fix $C_0 \in \Gamma_0$ and a formula φ_0 such that

$$\exists! B_0 \in \mathcal{J}_{\alpha_0+1}(\mathbb{R}, \Gamma_0) \text{ such that } \mathcal{J}_{\alpha_0+1}(\mathbb{R}, \Gamma_0) \models \varphi_0(C_0, B_0),$$

and moreover, the unique B_0 witnesses (2).

Now suppose we have defined $(\Gamma_n, \alpha_n, C_n, B_n)$.

Case 1: There is a Wadge segment Γ of $\mathrm{Hom}_{<\lambda}$ such that

$$L_{\alpha_n}(\mathbb{R}, \Gamma) \models \exists B[(\mathbb{R}, A, B) \models \varphi].$$

Then we set $n < z$. Let Γ_{n+1} be the least such Γ. Note that $\Gamma_n \subsetneq \Gamma_{n+1}$. Let α_{n+1} be the least α such that there is some such $B \in L_{\alpha+1}(\mathbb{R}, \Gamma_{n+1})$. Let $C_{n+1} \in \Gamma_{n+1} \backslash \Gamma_n$ be such that there is some such $B \in \mathrm{OD}_{C_{n+1}}^{L_{\alpha+1}(\mathbb{R}, \Gamma_{n+1})}$. Now pick φ_{n+1}, B_{n+1} relative to these things much as we chose φ_0, B_0.

Case 2: Otherwise.
Then $z = n$, so we are done.

Let $\psi(\dot{i}, \dot{\lambda}, \dot{A}, \dot{C}, \dot{C}^{+})$ be the formula asserting that $\mathrm{Col}(\omega, <\dot{\lambda})$ forces that $D_{\dot{\lambda}}$ satisfies:

1. There is B s.t. $(\mathbb{R}^*, (\dot{A})^*, B) \models \varphi$.
2. Let Γ' be the Wadge least initial segment of Hom^* such that $(\dot{A})^* \in \Gamma'$ and there is some such $B \in L(\mathbb{R}^*, \Gamma')$.
3. $(\dot{C})^* \in \Gamma'$ and if $\dot{i} < z$ then $(\dot{C}^{+})^* \notin \Gamma'$.

Let $C_{z+1} = \emptyset$.

Claim 3: There is $i \leq z$ such that $\psi(i, \lambda, A, C_i, C_{i+1})$.

Proof: By homogeneity, it suffices to see that some condition in $\mathrm{Col}(\omega, <\lambda)$ forces the statement. We establish this by showing that it is true of some countable elementary substructure of V_θ.

Let $\pi : M \to V_\theta$ be as usual. Notation is as before, so $\pi(C_i^M) = C_i$, etc. Abbreviate $A^{R[g]} = (A^R)^{R[g]}$, etc.

As before, in some $V[G]$, form a nice Neeman $\mathbb{R}^V$-genericity iteration $\mathcal{T}$ on M, above $\mathrm{cof}(\lambda)^+$ if λ is regular. Let $N = M_\infty^{\mathcal{T}}$ and $\sigma : N \to V_\theta$ be as before. Realize the derived model

$$D = D_{\lambda^N}^{N[H]} = L_\alpha(\mathbb{R}^V, (\mathrm{Hom}^*)^{N[H]})$$

as before. By the proof of Subclaim 1.2, $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge initial segment of $\mathrm{Hom}_{<\lambda}$.
Now $D \models \exists B[(\mathbb{R}^V, (A^*)^{N[H]}, B) \models \varphi]$. Let Γ' be the Wadge least segment of $(\mathrm{Hom}^*)^{N[H]}$ such that $D \models$"There is some such $B \in L(\mathbb{R}^V, \Gamma')$". Let α' be the least α such that there is some such $B \in L_{\alpha'+1}(\mathbb{R}^V, \Gamma')$.

Now Γ' is a Wadge segment of $\mathrm{Hom}_{<\lambda}$, and there is some φ-witness $B \in L_{\mathrm{OR}^N}(\mathbb{R}^V, \Gamma')$, so $\Gamma_0 \subseteq \Gamma'$. If $\mathrm{OR}^N > \alpha_0$ then it easily follows that $\Gamma' = \Gamma_0$ and $\alpha' = \alpha_0$. But (this was the issue with the argument noticed by Wilson) we might have $\mathrm{OR}^N \leq \alpha_0$. Suppose this is the case. Then note that $\Gamma_1 \subseteq \Gamma'$. If $\mathrm{OR}^N > \alpha_1$ then we get $\Gamma' = \Gamma_1$ and $\alpha' = \alpha_1$. Etc. So we get some $i \leq z$ such that $\Gamma' = \Gamma_i$ and $\alpha' = \alpha_i$.

But $((C_j^M)^*)^{N[H]} = C_j$ for each $j \leq z + 1$ (see proof of Claim 1), so

$$((C_i^M)^*)^{N[H]} = C_i \in \Gamma_i = \Gamma',$$

$$((C_{i+1}^M)^*)^{N[H]} = C_{i+1} \notin \Gamma_i = \Gamma',$$

which clearly gives Claim 3.

Fix the i from Claim 3. The reflection theorem follows from:

Claim 4: $B_i \in \mathrm{Hom}_{<\lambda}$.

Proof: Let $\gamma_0, \xi_0 < \lambda$ be as before. If λ is regular let $\nu = \xi_0$; otherwise let $\nu = \max(\mathrm{cof}(\lambda)^+, \xi_0)$. Let ξ_1 be the the least Woodin $> \nu$.

Let W be the set of all above-ν^M, $2^{\aleph_0}$-closed trees $\mathcal{U}$ on M of length $\omega + 1$ such that $M_\omega^{\pi\mathcal{U}}$ is wellfounded. So W is ξ_0^+-hom Suslin.

Let X' be the set of all $x \in \mathbb{R}$ such that for some tree $\mathcal{U} \in W$, $\mathcal{U}$ is based below ξ_1^M, and letting $R = M_\omega^{\mathcal{U}}$, there is an R-generic $g \subseteq \mathrm{Col}(\omega, i^{\mathcal{U}}(\xi_1))$ such that $x \in R[g]$, and

$$R[g] \models \psi_0(\lambda^R, A^{R[g]}, C_i^{R[g]}, x),$$

where $\psi_0(\dot{\lambda}, \dot{A}, \dot{C}, \dot{x})$ says that $\mathrm{Col}(\omega, < \dot{\lambda})$ forces that $D_{\dot{\lambda}}$ satisfies:

1. There is B s.t. $(\mathbb{R}^*, (\dot{A})^*, B) \models \varphi$.
2. Let Γ' be the Wadge least initial segment of Hom^* such that $(\dot{A})^* \in \Gamma'$ and there is some such $B \in L(\mathbb{R}^*, \Gamma')$.
3. $(\dot{C})^* \in \Gamma'$.
4. Let α' be least such that some such B is in $L_{\alpha'+1}(\mathbb{R}^*, \Gamma')$.
5. There is a unique $\widetilde{B} \in L_{\alpha'+1}(\mathbb{R}^*, \Gamma')$ such that
$$L_{\alpha'+1}(\mathbb{R}^*, \Gamma') \models \varphi_i(\dot{C}^*, \widetilde{B}).$$
6. $\dot{x} \in \widetilde{B}$.

So X' is ξ_0^+-weakly homogeneously Suslin, so the following completes the proof of the theorem:

Subclaim 4.1: $X' = B_i$.

Proof: Let $x \in X'$, as witnessed by $\mathcal{U}, g$. Let $R = M_\omega^{\mathcal{U}}$. As before, in some $V[G]$, form a nice Neeman $\mathbb{R}^V$-genericity iteration $\mathcal{T}$ on R, iterating above ξ_1^R. Let $N = M_\infty^{\mathcal{T}}$ and $\sigma : N \to V_\theta$ and the derived model

$$D_{\lambda^N}^{N[H]} = L_\alpha(\mathbb{R}^V, (\mathrm{Hom}^*)^{N[H]})$$

be as before. We have that $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge initial segment of $\mathrm{Hom}_{<\lambda}$.

We have $i^{\mathcal{U}+} : R[g] \to N[g]$ extending $i^{\mathcal{U}}$. So

$$N[g] \models \psi_0(\lambda^N, A^{N[g]}, C_i^{N[g]}, x)$$

(but $A^{N[g]} = A^{R[g]}$, etc). Let $\Gamma', \alpha', \widetilde{B}$ as mentioned in ψ_0 be their interpretations relative to $D_{\lambda^N}^{N[H]}$.

We have $(A^{N[g]})^* = A$ and $(C_j^{N[g]})^* = C_j$ for $j \leq z+1$. By the proof of Claim 3, there is j such that $\Gamma' = \Gamma_j$ and $\alpha' = \alpha_j$. So by Claim 3, $C_i \in \Gamma_j$ and if $i < z$ then $C_{i+1} \notin \Gamma_j$. Therefore $i = j$. Note then that $\widetilde{B} = B_i$. Therefore $x \in B_i$, as desired.

Finally suppose $x \in B_i$; we want $x \in X'$.

Let $\mathcal{U}$ be a nice Neeman genericity iteration on M, based on the interval (ν^M, ξ_1^M), making x generic. Let b be such that $\mathcal{U} = \widetilde{\mathcal{U}} \frown b \in W$. Using an $\mathbb{R}^V$-genericity iteration on $M_\omega^{\mathcal{U}}$, as before, and because $x \in B_i$, it easily follows that $\mathcal{U}$ witnesses that $x \in X'$, as required.

This completes the proof of Subclaim 4.1, and hence, Claim 4 and the reflection theorem (via Argument 2).

Finally, we complete:

Proof of Subsubclaim 2.1.1: We use the following theorem due to Steel and Van Wesep, and independently to Kechris and Woodin:

Theorem

Assume $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$*. Let* $A, B \subseteq \mathbb{R}$ *and suppose that* $A \notin L(\mathbb{R}, B)$*. Then* $B^{\#}$ *exists.*

See "Two consequences of determinacy consistent with choice", (Steel, Van Wesep), Theorem 1.3.3. Here we can consider $B^{\#}$ as an $\mathbb{R}$-sound, iterable premouse over $(\mathbb{R}, B)$ (with exactly one extender on its sequence, which is active). For a general discussion on these see "Scales in $K(\mathbb{R})$" (Steel); there are also some more details in "Scales in hybrid mice over $\mathbb{R}$" (Schlutzenberg, Trang). (All elements of $\mathbb{R} \cup \{B\}$ are added to all fine structural hulls, and the language will have symbols for $\mathbb{R}$ and B.)

One can alternatively consider $B^\#$ as the theory of $(\mathbb{R}, B)$-indiscernibles, but the properties must be set up carefully.

Now we also need the following simple variant of the theorem above, proved in almost the same way:

Theorem

Assume $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$. Let Γ be an initial segment of the Wadge hierarchy, let $A \subseteq \mathbb{R}$ and suppose that $A \notin L(\mathbb{R}, \Gamma)$. Then there is an elementary embedding $j : L(\mathbb{R}, \Gamma) \to L(\mathbb{R}, \Gamma)$, and $\Gamma^\#$ exists.

Proof: We will give most of the proof of this theorem here, though it is just a slight variant of the theorem mentioned above. Here $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, and $\Gamma^\#$ is defined as a premouse much like $B^\#$, but now the premouse is over $\mathbb{R} \cup \Gamma$, and we only require $\mathbb{R} \cup \Gamma$-soundness. (All elements of $\mathbb{R} \cup \Gamma$ are put into all fine structural hulls, and the language has symbols for $\mathbb{R}$ and Γ.)

Here are some further details. Our $\mathbb{R}\cup\Gamma$ premice will have the form

$$M = (\mathcal{J}_\alpha(\mathbb{R}\cup\Gamma), E)$$

for some $\alpha \in \mathrm{OR}$ and E, and for some $\kappa < \alpha$, $\alpha = (\kappa^{++})^M$, and E is an extender over M, with only one generator κ, and E is $\mathcal{P}(\mathbb{R})^M \times \kappa$-complete, and $\mathcal{P}(\mathbb{R})^M$-weakly amenable to M. Here $\mathcal{P}(\mathbb{R})^M \times \kappa$-*completeness* means that whenever $\alpha < \kappa$ and $f : \alpha \times \mathcal{P}(\mathbb{R})^M \to M$ and $f \in M$ and $f(\beta, X) \in E_{\{\kappa\}}$ for all β, X, then

$$\bigcap_{\beta,X} f(\beta, X) \in E_{\{\kappa\}}.$$

And $\mathcal{P}(\mathbb{R})^M$-*weakly amenable* means that whenever

$$f : \kappa \times \mathcal{P}(\mathbb{R})^M \to \mathcal{P}(\kappa^{<\omega})$$

and $f \in M$ and $\beta < \mathrm{OR}^M$, then

$$E\restriction(\mathrm{rg}(f) \times \beta^{<\omega}) \in M.$$

Let $\Theta_{\mathcal{P}}$ denote the least α not the surjective image of $\mathcal{P}(\mathbb{R})$.

Lemma

Assume $\mathsf{ZF} + \mathsf{AD}$. *Let* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$. *Let* $\mu > \Theta^{L(\Gamma,\mathbb{R})}$ *be regular. Then* $\mu \geq \Theta_{\mathcal{P}}^{L(\Gamma,\mathbb{R})}$.

Proof.

Suppose not. Let $\theta = \Theta_{\mathcal{P}}^{L(\Gamma,\mathbb{R})}$ and $M = L(\Gamma, \mathbb{R})$ and $P = \mathcal{P}(\mathbb{R}) \cap M$ and $f : P \to \mu$ be surjective, with $f \in M$. Let $g : \mu \to \theta$ be $g(\alpha)$ is the least β such that for some $X \in P$ of Wadge rank β in M, we have $f(X) = \alpha$. As μ is regular, there is $A \subseteq \mu$ unbounded in μ with $f \restriction A$ constant. Let $g``A = \{\beta\}$. Let $B \subseteq P$ be the set of all sets of Wadge rank $\beta$ in $M$. Then $A \subseteq f``B$. But B is the surjective image of $\mathbb{R}$ in M. But μ is regular, so μ is then surjective image of $\mathbb{R}$ in M, contradiction. □

We now prove the theorem on the existence of $\Gamma^{\#}$; we follow the Steel-Van Wesep proof mentioned above:

Proof: Write $M = L(\Gamma, \mathbb{R})$. We first produce an elementary $j : M \to M$, following almost identically the proof of Steel-Van Wesep. Because $A \notin M$, and by AD, every set of reals in M is Wadge below A, so there is a surjection from $\mathbb{R}$ onto $\mathcal{P}(\mathbb{R}) \cap M$. So $\Theta > \Theta^M$. Moschovakis showed that $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$ proves "Θ is a limit of cardinals which are weakly inaccessible and measurable" (see Steel-Van Wesep, Theorem 1.1.5). We are only assuming $\mathsf{DC}_{\mathbb{R}}$, but note we may assume $V = L(A, \mathbb{R})$, where $\mathsf{DC}_{\mathbb{R}}$ implies DC. So let $\Theta^M < \mu < \kappa < \Theta$ be such that μ is regular and κ is weakly inaccessible and measurable. By the lemma, $\Theta^M_{\mathcal{P}} \leq \mu < \kappa$. Let U be a κ-complete normal ultrafilter over κ.

Now we can form $\mathrm{Ult}(M, U)$, using functions $f : \kappa \to M$ with $f \in M$, and we get the ultrapower embedding

$$j : M \to \mathrm{Ult}(M, U).$$

We want to see that we have Los' theorem, and so j is elementary. For this, suppose we have some function $f \in M$ and a formula φ and

$$\{\alpha < \kappa \mid \exists x \varphi(x, f(\alpha))\} \in U.$$

We want to find some $g \in M$ such that

$$\{\alpha < \kappa \mid \varphi(g(\alpha), f(\alpha))\} \in U.$$

Now $M \models$"For every x there is some $Y \subseteq \mathbb{R}$ such that $x \in \mathrm{OD}_Y$". So working in M, let $h : \kappa \to \Theta^M$ be $h(\alpha) =$ the least β such that for some set Y of Wadge rank β, there is $x \in \mathrm{OD}_Y$ such that $\varphi(x, f(\alpha))$. Then there is $D_0 \in U$ and β such that $h``D_0 = \{\beta\}$.

In M, fix some surjection $b : \mathbb{R} \to W_\beta$, where W_β is the set of all sets of Wadge rank β. For $\alpha < \kappa$ and $y \in \mathbb{R}$, let $y \in A_\alpha$ iff there is $x \in \mathrm{OD}_{b(y)}$ such that $\varphi(x, f(\alpha))$. So for $\alpha \in D_0$, $A_\alpha \neq \emptyset$. Now we claim that there is $D_1 \in U$ such that $D_1 \subseteq D_0$ and for all $\alpha, \alpha' \in D_1$, we have $A_\alpha = A_{\alpha'}$. For otherwise note that there is $D_1 \in U$ such that $D_1 \subseteq D_0$ and for all $\alpha \in D_1$, we have $A_\alpha \neq A_{\alpha'}$ for all $\alpha' < \alpha$. Let $w : X_1 \to \Theta^M$ be $w(\alpha) =$ the Wadge rank of A_α. Then w is injective, but X_1 is unbounded in $\kappa > \Theta^M$, contradiction. So we have D_1 as claimed. Now let $y \in A_\alpha$, where $\alpha \in X_1$, and let $Y = b(y)$. Then for all $\alpha \in X_1$, there is some $x \in \mathrm{OD}_Y$ such that $\varphi(x, f(\alpha))$. So let $g : X_1 \to M$ be $g(\alpha) =$ the OD_Y-least $x \in \mathrm{OD}_Y$ witnessing this. Then g works.

So we have an elementary $j : M \to M$. We now want $\Gamma^\#$.

Assume $V = L(\mathcal{P}(\mathbb{R}))$. For $\kappa \in \mathrm{OR}$ we say κ is a *$\mathcal{P}$-cardinal* iff there is no $\alpha < \kappa$ and surjection $f : \mathcal{P} \times \alpha \to \kappa$; κ is an *ordinal-cardinal* iff there is no $\alpha < \kappa$ and surjection $f : \alpha \to \kappa$.

Lemma

Assume $V = L(\mathcal{P}(\mathbb{R}))$. Let $\kappa \geq \Theta_{\mathcal{P}}$. Then κ is a $\mathcal{P}$-cardinal iff κ is an ordinal-cardinal.

Proof.

Let $f : \mathcal{P} \times \alpha \to \kappa$ be a surjection with $\alpha < \kappa$. If $\kappa = \Theta_{\mathcal{P}}$ then because $\alpha < \kappa = \Theta_{\mathcal{P}}$, we get a surjection $\mathcal{P} \times \mathcal{P} \to \Theta_{\mathcal{P}}$, contradiction. So $\Theta_{\mathcal{P}} < \kappa$. For $\beta < \alpha$ let $f_\beta : \mathcal{P} \to \kappa$ be $f_\beta(X) = f(X, \beta)$. Note then that $\mathrm{rg}(f_\beta)$ has ordertype $\xi_\beta < \Theta_{\mathcal{P}}$. Let $g_\beta : \xi_\beta \to \mathrm{rg}(f_\beta)$ be the order-preserving bijection. Let $g : \Theta_{\mathcal{P}} \times \alpha \to \kappa$ be $g(\gamma, \beta) = g_\beta(\gamma)$ if $\gamma < \xi_\beta$, and $g_\beta(\gamma) = 0$ otherwise. Then $\kappa = \mathrm{rg}(f) \subseteq \mathrm{rg}(g)$, so κ is not an ordinal-cardinal. □

Therefore, if $V = L(\mathcal{P}(\mathbb{R}))$ and $\kappa \geq \Theta_{\mathcal{P}}$, we say that κ is a *cardinal* iff κ is an ordinal-cardinal iff κ is a $\mathcal{P}$-cardinal. Using the previous lemma, we easily get:

Lemma

Assume $V = L(\mathcal{P}(\mathbb{R}))$. Then forcing with $\mathrm{Col}(\omega, \mathcal{P}(\mathbb{R}))$ preserves all cardinals $\geq \Theta_{\mathcal{P}}$ (and collapses those $< \Theta_{\mathcal{P}}$ to ω).

Now we have $j : M \to M$ and want $\Gamma^{\#}$. Let $\mathbb{P} = \mathrm{Col}(\omega, \mathcal{P}(\mathbb{R}))^M$. Let x be V-generic for $\mathbb{P}$. We have $j \restriction \mathbb{P} \cup \{\mathbb{P}\} = \mathrm{id}$. So we can extend j elementarily to $j^+ : M[x] \to M[x]$. There is some real y such that $M[x] = L[y]$. So by Kunen, $y^{\#}$ exists. Let $\alpha = \mathrm{OR}^{y^{\#}}$ and E be the measure of $y^{\#}$ (we mean $y^{\#}$ in mouse form). Let $\lfloor N \rfloor = L_\alpha(\mathcal{P}(\mathbb{R})^M)$ and

$$N = (\lfloor N \rfloor, E \restriction \lfloor N \rfloor).$$

The next claim finishes the proof:

Claim: $N = \Gamma^{\#}$.

Proof Sketch: It is straightforward to see that the measure of N is $(\mathcal{P}(\mathbb{R})^M, \mu)$-complete and $\mathcal{P}(\mathbb{R})^M$-weakly amenable, where $\mu = \mathrm{cr}(E)$. (Note maybe $\mu < \kappa$.) By the lemma above, $L[x]$ and M have the same cardinals $\geq \Theta_{\mathcal{P}}$, so $\alpha = (\mu^{++})^{L[x]} = (\mu^{++})^M$. (Remark: We get Los' theorem from $(\mathcal{P}(\mathbb{R})^M, \mu)$-completeness: For example suppose there are measure one many $\alpha < \mu$ such that $N \models \exists x \varphi(x, f(\alpha))$, some Σ_0 formula φ, and for simplicity assume that φ does not mention the active extender. Let $f \in N|\beta$, $\beta < \mathrm{OR}^N$. We claim that there is $X \in \mathcal{P}(\mathbb{R})^M$ such that for measure one many $\alpha < \mu$, there is $x \in \mathrm{OD}_X^{N|\beta}$ such that $N \models \varphi(x, f(\alpha))$. For otherwise we get an $\mathbb{R}$-sequence of measure one sets whose intersection has measure 0.)

Now the usual small forcing calculation shows that for every $f : \mu \to \mathrm{OR}$ with $f \in L[x]$, there is $\tilde{f} : \mu \to \mathrm{OR}$ with $\tilde{f} \in M$, such that $f(\beta) = \tilde{f}(\beta)$ for E-measure one many β. With this it is easy to see that N is iterable.
Finally, we get $N = \mathrm{Hull}^N_{\Sigma_1}(\mathcal{P}(\mathbb{R})^M)$ using the soundness of $y^\#$, the forcing relation, and the fact that the active extender of $y^\#$ is just the small forcing extension of the active extender of N. So N is sound.
This completes the proof of the theorem.

We can now prove Subsubclaim 2.1.1. So suppose $\mathrm{OR}^N \leq \alpha_0$. (Recall that N is the $\mathbb{R}^V$-genericity iterate of M, where M was some countable hull of V.) Recall that $D = D^{N[H]}_{\lambda^N}$ has a φ-witness B, and recall the minimality of Γ_0, α_0 in V, and that $(\mathrm{Hom}^*)^{N[H]}$ is a Wadge segment of $\mathrm{Hom}_{<\lambda}$.

Because $\mathrm{OR}^N \leq \alpha_0$ and by the minimality of α_0, it therefore follows that there is $X \in (\mathrm{Hom}^*)^{N[H]}$ such that $X \notin L(\Gamma_0, \mathbb{R}^V)$. Since the derived model of V at λ satisfies AD^+, it satisfies the conclusions of the theorems above on sharp existence. Therefore D satisfies these same conclusions. Therefore $D \models$ "$\Gamma_0^\#$ exists". Because $\mathbb{R}^D = \mathbb{R}^V$, $(\Gamma_0^\#)^D$ is truly iterable, so $(\Gamma_0^\#)^D = \Gamma_0^\#$. Since $N \models$ "$L(\Gamma_0, \mathbb{R}^V)$ has no φ-witness" and $\Gamma_0^\# \in N$, it follows that $L(\Gamma_0, \mathbb{R}^V)$ really has no φ-witness, contradicting the choice of Γ_0.

This completes the proof of Subsubclaim 2.1.1, hence Argument 1.