Least branch hod pairs

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Problem: Analyze HOD in models of determinacy.

Conjecture 1. Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R}))$; then $\mathrm{HOD} \models \mathsf{GCH}$.

Conjecture 2. There is $M \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$ such that $\mathrm{HOD}^M \models$ "there is a subcompact cardinal".

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Conjecture 2. There is $M \models \mathsf{AD}^+ + V = L(P(\mathbb{R}))$ such that $\mathrm{HOD}^M \models$ "there is a subcompact cardinal".

Definition

"No long extenders" (NLE) is the assertion: there is no countable, iterable pure extender mouse with a long extender on its sequence.

Theorem

Suppose that κ is supercompact, and there are arbitrarily large Woodin cardinals. Suppose that V is uniquely iterable above κ; then

1. *for any $\Gamma \subseteq \mathrm{Hom}_\infty$ such that $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$, $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models \mathsf{GCH}$, and*
2. *there is a $\Gamma \subseteq \mathrm{Hom}_\infty$ such that $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ "there is a subcompact cardinal".*

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Suppose that κ is supercompact, and there are arbitrarily large Woodin cardinals. Suppose that V is uniquely iterable above κ; then

(1) for any $\Gamma \subseteq \mathrm{Hom}_\infty$ such that $L(\Gamma, \mathbb{R}) \models$ NLE, $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ GCH, and

(2) there is a $\Gamma \subseteq \mathrm{Hom}_\infty$ such that $\mathrm{HOD}^{L(\Gamma,\mathbb{R})} \models$ "there is a subcompact cardinal".

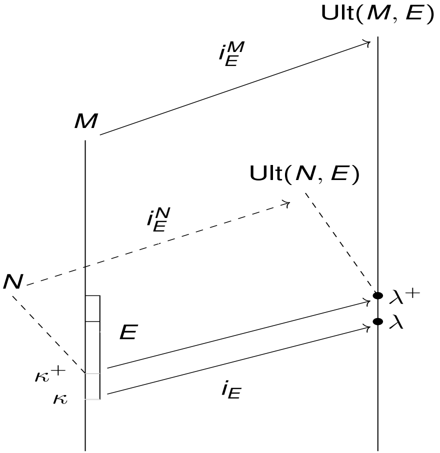
Moral: Below long extenders, there is a simple general notion of *hod pair*, and a general comparison theorem for them. They have a fine structure. *Modulo the existence of iteration strategies*, they can be used to analyze HOD, and they can have subcompact cardinals.

A Glossary

(a) An *extender E over M* is a system of measures on M coding an elementary $i_E \colon M \to \mathrm{Ult}(M, E)$. E is *short* iff all its component measures concentrate on $\mathrm{crit}(i_E)$.

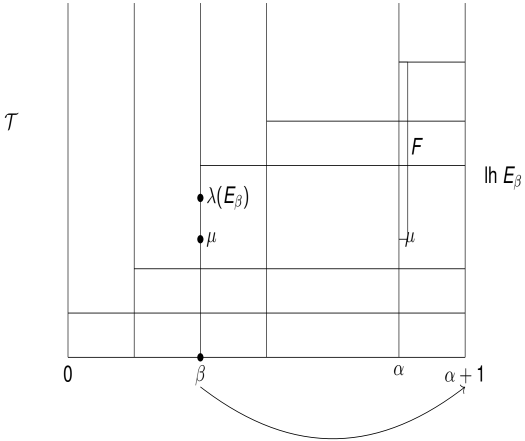
$$\mathrm{Ult}(M, E) = \{[a, f]^M_E \mid f \in M \text{ and } a \in [\lambda]^{<\omega}\},$$

where $\lambda = \lambda(E) = i_E(\mathrm{crit}(E))$.

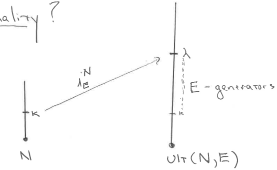


M agrees with Ult(M, E) and Ult(N, E) to $(\lambda^+)^{\mathrm{Ult}(M,E)}$.

(b) A *normal iteration tree on M* is an iteration tree $\mathcal{T}$ on M in which the extenders used have increasing strengths, and are applied to the longest possible initial segment of the earliest possible model. (So along branches of $\mathcal{T}$, generators are not moved.)

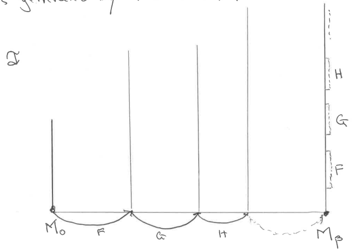

$\mathcal{T}$
F
lh E_β
$\lambda(E_\beta)$
μ
μ
0
β
α
$\alpha+1$

Why normality?



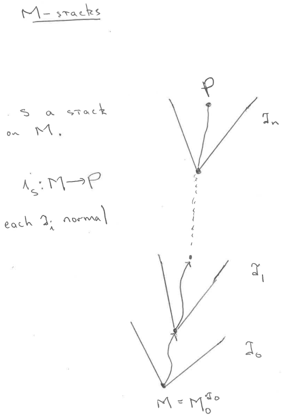
$\text{Ult}(N,E) = \{ i(f)(a) \mid f \in N \text{ and } a \in [\lambda]^{<\omega} \}$.

It's generated by $\lambda \cup \text{ran}(i)$.



The individual extenders used going from M_0 to M_β can be recovered from $i^{\mathcal{T}}_{0\beta}$.

(c) An *M-stack* is a sequence $s = \langle \mathcal{T}_0, ..., \mathcal{T}_n \rangle$ of normal trees such that $\mathcal{T}_0$ is on M, and $\mathcal{T}_{i+1}$ is on the last model of $\mathcal{T}_i$.



(d) An *iteration strategy* Σ for M is a function that is defined on M-stacks s that are by Σ whose last tree has limit length, and picks a cofinal wellfounded branch of that tree.

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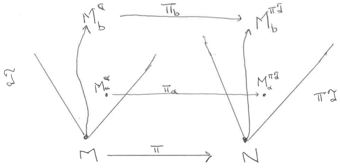
(f) It $\pi \colon M \to N$ is elementary, and Σ is an iteration strategy for N, then Σ^π is the *pullback strategy* given by: $\Sigma^\pi(s) = \Sigma(\pi s)$.

Pullback strategies

Given Σ for N, and $\pi : M \to N$



if $b = \Sigma(\pi\mathcal{T})$

then $\Sigma^\pi(\mathcal{T}) = b$

Least branch hod pairs

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A *least branch premouse* (lpm) is a structure $\mathcal{M}$ constructed from a coherent sequence $\dot{E}^{\mathcal{M}}$ of extenders, and a predicate $\dot{\Sigma}^{\mathcal{M}}$ for an iteration strategy for $\mathcal{M}$.

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Remarks

(a) $\mathcal{M}$ has a hierarchy, and a fine structure. By convention, there is a $k = k(\mathcal{M})$ such that $\mathcal{M}$ is *k-sound*. (I.e., $\mathcal{M} = \text{Hull}_k(\rho_k^{\mathcal{M}} \cup p_k^{\mathcal{M}})$.)

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(b) We use Jensen indexing for the extenders in $\dot{E}^{\mathcal{M}}$.

(c) At strategy-active stages α, we consider the $\mathcal{M}|\alpha$-least $\langle \nu, k, \mathcal{T} \rangle$ such that $\mathcal{T}$ is a normal tree of limit length on $\mathcal{M}|\langle \nu, k \rangle$ that is by $\dot{\Sigma}^{\mathcal{M}|\alpha}$, and $\dot{\Sigma}^{\mathcal{M}|\alpha}(\mathcal{T})$ is undefined. Then $\dot{\Sigma}^{\mathcal{M}|(\alpha+1)} = \dot{\Sigma}^{\mathcal{M}|\alpha} \cup \{\langle \nu, k, \mathcal{T}, b \rangle\}$, where b is some cofinal branch of $\mathcal{T}$.

Definition

A *least branch hod pair* (lbr hod pair) with with scope Z is a pair (P, Σ) such that

(1) P is an lpm,

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(4) Σ is self-consistent, normalizes well, and has strong hull condensation.

Σ is *self-consistent* iff the part of Σ that is a strategy for $\mathcal{M}|\langle \nu, k \rangle$ is consistent with the part of Σ that is a strategy for $\mathcal{M}|\langle \mu, j \rangle$.

Normalizing well

For $\langle \mathcal{T}, \mathcal{U} \rangle$ a stack on P, and $\mathcal{W} = W(\mathcal{T}, \mathcal{U})$ its embedding normalization, we have

$$\begin{tikzcd} P \arrow[r, "i^{\mathcal{T}}"] \arrow[drr, "i^{\mathcal{W}}"'] & Q \arrow[r, "i^{\mathcal{U}}"] & R \arrow[d, "\pi"] \\ & & S \end{tikzcd}$$

Then Σ *2-normalizes well* iff

$$\langle \mathcal{T}, \mathcal{U} \rangle \text{ is by } \Sigma \text{ iff } W(\mathcal{T}, \mathcal{U}) \text{ is by } \Sigma,$$

and

$$\Sigma^{\pi}_{\langle \mathcal{W} \rangle} = \Sigma_{\langle \mathcal{T}, \mathcal{U} \rangle}.$$

for all such stacks $\langle \mathcal{T}, \mathcal{U} \rangle$. Σ *normalizes well* iff all its tails 2-normalize well.

$W(E, F)$

Let $\mathcal{T} = \langle E \rangle$ and $\mathcal{U} = \langle F \rangle$, with $\mathrm{crit}(F) < \mathrm{crit}(E)$.

$$\begin{tikzcd}
N \arrow[r, "F"] & Q \arrow[r, "\tau"] & i^M_F(N) = \mathrm{Ult}_0(P, i^M_F(E)) \\
M \arrow[u, "E"] \arrow[r, "F"] & P \arrow[ur, "i^M_F(E)"'] &
\end{tikzcd}$$

τ is the natural embedding from $\mathrm{Ult}(N, F)$ to $i^M_F(N)$. That is,

$$\tau([a, g]^N_F) = [a, g]^M_F.$$

The extenders used in $W(E,F)$ are E, then F, then $i_F^M(E)$.

$$\begin{tikzcd} N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & i_F^M(N) = \mathrm{Ult}_0(P, i_F^M(E)) \\ M \arrow[u, "E"] \arrow[r, "i_F^M"'] & P \arrow[u, "i_F^N(E)"] \arrow[ur, "i_F^M(E)"'] & \end{tikzcd}$$

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M \arrow[u, "E"] \arrow[r, "i^M_F"'] & P \arrow[u, "i^N_F(E)"] \arrow[ur, "i^M_F(E)"'] &
\end{tikzcd}
$$

The *full normalization* $X(E, F)$ uses E, then F, then $i^N_F(E)$.Ult(M, F) has $i^N_F(E)$ on its sequence by Condensation.

The situation when $\mathrm{crit}(E) < \mathrm{crit}(F) < \lambda(E)$ is summarized by:

$$
\begin{tikzcd}
N \arrow[r, "i_F^N"] & Q \arrow[r, "\tau"] & R \\
M \arrow[u, "E"] \arrow[ur, "i_F^N(E)"] \arrow[urr, "i_F^M(E)"'] \arrow[r, "i_F^M"'] & P &
\end{tikzcd}
$$

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Remark So there are two ways F can appear in the branch to the final model: as itself, or buried inside $i_F(E)$.

$W(\mathcal{T}, F)$

Let $\mathcal{T}$ have last model $M_\theta^{\mathcal{T}}$, with F on its sequence. Let

$$\alpha = \text{least } \xi \text{ such that } F \text{ is on the } M_\xi^{\mathcal{T}}\text{-sequence},$$

and

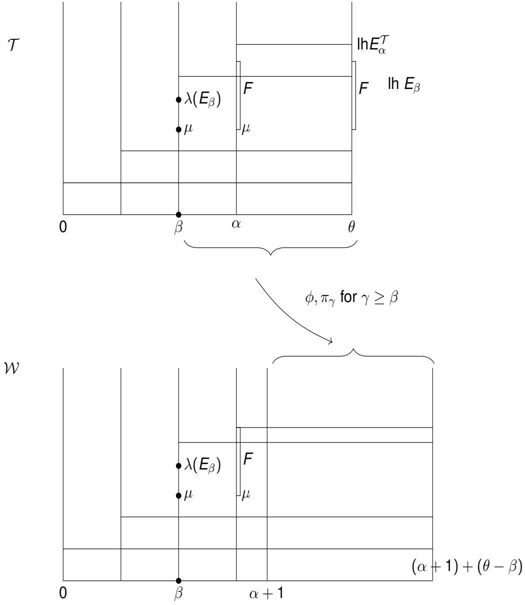
$$\beta = \text{least } \xi \text{ such that } \mathrm{crit}(F) < \lambda(E_\xi^{\mathcal{T}}).$$

Then

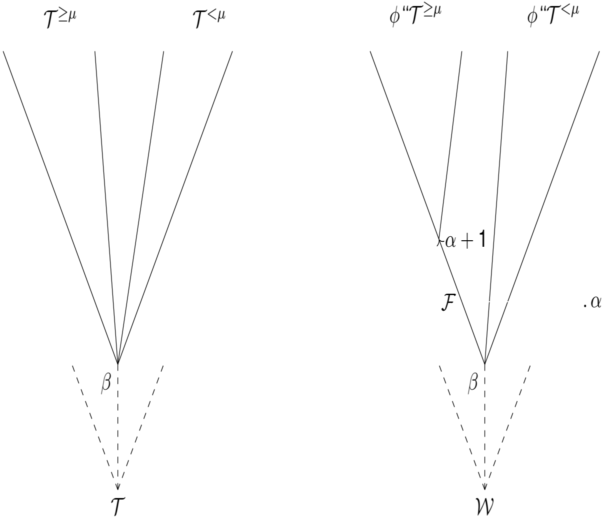
$$W(\mathcal{T}, F) = \mathcal{T} \restriction (\alpha + 1)^\frown \langle F \rangle^\frown \Phi``\mathcal{T} \restriction (\theta + 1 - \beta).$$

Here Φ comes from a copying construction.

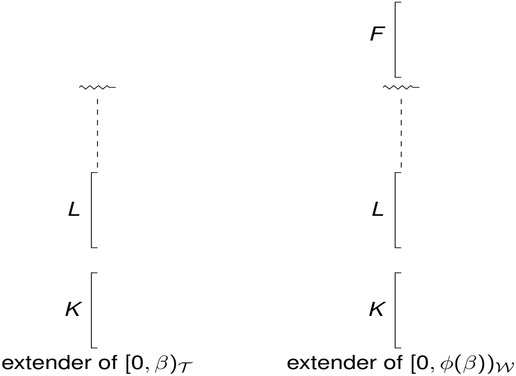
$W(\mathcal{T}, F)$


$\mathcal{T}$
$\mathrm{lh} E_\alpha^{\mathcal{T}}$
F
F
$\mathrm{lh}\ E_\beta$
$\lambda(E_\beta)$
μ
μ
0
β
α
θ
ϕ, π_γ for $\gamma \geq \beta$
$\mathcal{W}$
$\lambda(E_\beta)$
F
μ
μ
$(\alpha + 1) + (\theta - \beta)$
0
β
$\alpha + 1$

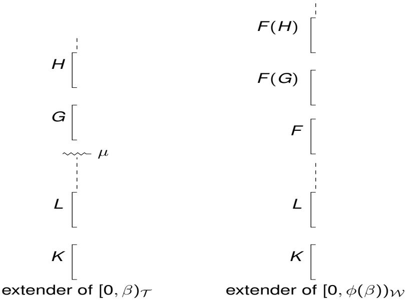
Next we have the tree order picture,



We show how F gets inserted into the extender of the branch $\mathcal{T}$ ending at $M^{\mathcal{T}}_{\xi}$.
For $\xi = \beta$:

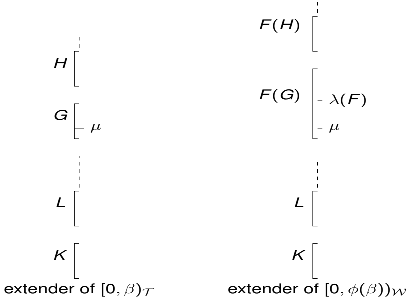


For $\xi > \beta$, let G be the first extender used in $[0, \xi)_T$ such that $\nu(G) \geq \nu(E_\beta^{\mathcal{T}})$. The picture depends on whether $\mu \leq \mathrm{crit}(G)$. If $\mu \leq \mathrm{crit}(G)$, it is



In this case, F is used on $[0, \phi(\xi))_W$, and the remaining extender used are the images of old ones under copy maps.

If $\mathrm{crit}(G) < \mu < \nu(G)$, the picture is



In this case, the two branches use the same extenders until G is used on $[0, \xi)_T$. At that point and after, $[0, \phi(\xi))_W$ uses the images of extenders under the copy maps.

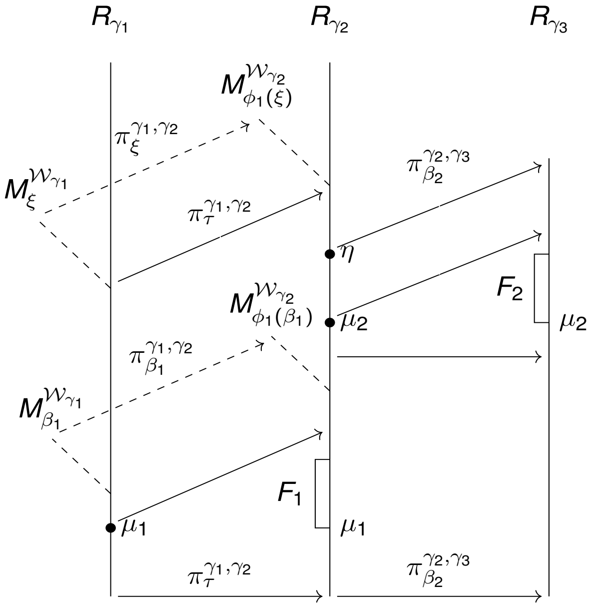
$W(\mathcal{T},\mathcal{U})$

We define $W_\gamma = W(\mathcal{T},\mathcal{U} \restriction \gamma + 1)$ by induction on γ. It has last model R_γ, and we have σ_γ from $M^{\mathcal{U}}_\gamma$ to R_γ.

$W(\mathcal{T}, \mathcal{U})$

We define $W_\gamma = W(\mathcal{T}, \mathcal{U} \restriction \gamma + 1)$ by induction on γ. It has last model R_γ, and we have σ_γ from $M_\gamma^{\mathcal{U}}$ to R_γ.

The W_γ's constitute a tree of iteration trees, under the order $<_U$ on the γ's. If $\gamma_1 <_U \gamma_2 <_U \gamma_3$, the picture is:


R_{γ_1}
R_{γ_2}
R_{γ_3}
$M^{\mathcal{W}_{\gamma_2}}_{\phi_1(\xi)}$
$\pi^{\gamma_1,\gamma_2}_{\xi}$
$M^{\mathcal{W}_{\gamma_1}}_{\xi}$
$\pi^{\gamma_1,\gamma_2}_{\tau}$
$\pi^{\gamma_2,\gamma_3}_{\beta_2}$
η
F_2
$M^{\mathcal{W}_{\gamma_2}}_{\phi_1(\beta_1)}$
μ_2
μ_2
$\pi^{\gamma_1,\gamma_2}_{\beta_1}$
$M^{\mathcal{W}_{\gamma_1}}_{\beta_1}$
F_1
μ_1
μ_1
$\pi^{\gamma_1,\gamma_2}_{\tau}$
$\pi^{\gamma_2,\gamma_3}_{\beta_2}$

Another view of $W(\mathcal{T},\mathcal{U})$:

$$
\begin{tikzcd}
R_0 = M_0^{\mathcal{U}} \arrow[rr] \arrow[drr] & & M_\eta^{\mathcal{U}} \arrow[r, "i^{\mathcal{U}}_{\eta,\gamma}"] \arrow[d, "\sigma_\eta"] & M_\gamma^{\mathcal{U}} \arrow[dd, "\sigma_\gamma"] \\
& & R_\eta \arrow[dr, "\pi^{\eta,\gamma}_{z(\eta)}"] & \\
& M_\sigma^{\mathcal{W}_\eta} \arrow[dr, "\pi^{\eta,\gamma}_\sigma"] & & R_\gamma \\
& & M^{\mathcal{W}_\gamma}_{\phi_{\eta,\gamma}(\sigma)} & \\
M \arrow[uuuu, "i^{\mathcal{W}_0}"] \arrow[uur, "i^{\mathcal{W}_\eta}_{0\sigma}"] \arrow[uuurr, "i^{\mathcal{W}_\eta}"] \arrow[urr] \arrow[uurrr, bend right, "i^{\mathcal{W}_\gamma}"'] & & &
\end{tikzcd}
$$

Strong hull condensation

Roughly, Σ has *strong hull condensation* iff $\mathcal{T}$ and $\mathcal{U}$ are normal trees on P, and $\mathcal{U}$ is by Σ, and $\pi\colon \mathcal{T} \to \mathcal{U}$ is appropriately elementary, then $\mathcal{T}$ is by Σ.

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One must be careful about the elementarity required of π, and in particular, the extent to which π is required to preserve exit extenders. There are several possible condensation properties here: hull condensation (Sargsyan), strong hull condensation, and still stronger ones.

Strong hull condensation means condensing under *psuedo-hull embeddings*. The natural embedding of $\mathcal{T}$ into $W(\mathcal{T},\mathcal{U})$ is an example of a psuedo-hull embedding.

Definition

A *pseudo-hull embedding* of $\mathcal{T}$ into $\mathcal{U}$ is a system

$$\langle u, \langle t^0_\beta \mid \beta < \operatorname{lh}\mathcal{T}\rangle, \langle t^1_\beta \mid \beta + 1 < \operatorname{lh}\mathcal{T}\rangle, p\rangle$$

with various properties, including:

$$\begin{aligned} p(E^{\mathcal{T}}_\alpha) &= t^1_\alpha(E^{\mathcal{T}}_\alpha) \\ &= E^{\mathcal{U}}_{u(\alpha)}. \end{aligned}$$

The diagram related to successor steps in $\mathcal{T}$ is:

$$
\begin{tikzcd}
M^{\mathcal{T}}_{\alpha+1} \arrow[rr, "t^0_{\alpha+1}"] & & M^{\mathcal{U}}_{v(\alpha+1)} \\
& M^{\mathcal{U}}_{u(\beta)} & \\
& & M^{\mathcal{U}}_{\beta^*} \arrow[ul] \arrow[uu, "E^{\mathcal{U}}_{u(\alpha)}"'] \\
M^{\mathcal{T}}_{\beta} \arrow[uuu, "E^{\mathcal{T}}_{\alpha}"] \arrow[uur, "t^1_{\beta}"] \arrow[urr, "\rho"] \arrow[rr, "t^0_{\beta}"] & & M^{\mathcal{U}}_{v(\beta)} \arrow[u] \\
M^{\mathcal{T}}_{\alpha} \arrow[rr, "t^1_{\alpha}"] & & M^{\mathcal{U}}_{u(\alpha)}
\end{tikzcd}
$$

Elementary properties of lbr hod pairs

Lemma

Let (P, Σ) be an lbr hod pair with scope Z, and suppose $\pi\colon Q \to P$ is elementary; then (Q, Σ^π) is an lbr hod pair with scope Z.

Elementary properties of lbr hod pairs

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Lemma

(Pullback consistency) Let (P, Σ) be an lbr hod pair with scope Z, and let s be a P-stack by Σ giving rise to the iteration map $\pi \colon P \to Q$; then $(\Sigma_s)^\pi = \Sigma$.

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Lemma

(Dodd-Jensen) The Σ-iteration map from (P, Σ) to (Q, Ψ) is pointwise a pointwise minimal embedding of (P, Σ) into (Q, Ψ).

Comparison

Theorem (Comparison)

Assume AD^+, and let (P, Σ) and (Q, Ψ) be lbr hod pairs with scope HC; then there are normal trees $\mathcal{T}$ on P by Σ and $\mathcal{U}$ on Q by Ψ with last models R and S respectively, such that either

(1) $R \trianglelefteq S$, and $\Sigma_{\mathcal{T}} \subseteq \Psi_{\mathcal{U}}$, or

(2) $S \trianglelefteq R$, and $\Psi_{\mathcal{U}} \subseteq \Sigma_{\mathcal{T}}$.

Corollary

Assume AD^+; then the mouse order $\leq^*$ on lbr hod pairs with scope HC is a prewellorder.

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Proof of theorem. Let N^* be a countable, Γ-correct model with a Woodin cardinal, where (P, Σ) and (Q, Ψ) are in Γ.Let (N, Ω) be a level of the lbr hod pair construction done inside N^*. We compare of P with N by iterating away least extender disagreements, and show:

(a) no extenders on the N side are used,and

(b) no strategy disagreements show up.

That Σ normalizes well and has strong hull condensation are crucial here.

Proof of theorem. Let N^* be a countable, Γ-correct model with a Woodin cardinal, where (P, Σ) and (Q, Ψ) are in Γ.Let (N, Ω) be a level of the lbr hod pair construction done inside N^*. We compare of P with N by iterating away least extender disagreements, and show:

(a) no extenders on the N side are used,and

(b) no strategy disagreements show up.

That Σ normalizes well and has strong hull condensation are crucial here.
Since N^* has a Woodin cardinal, (P, Σ) cannot iterate past all such (N, Ω), and hence, some such (N, Ω) is an iterate of (P, Σ). Similarly for (Q, Ψ), and we are done.

Why are there no strategy disagreements?

Suppose we have produced an iteration tree $\mathcal{T}$ on P with last model R, and that $R|\alpha = N|\alpha$, and that $\mathcal{U}$ is a tree on $R|\alpha = N|\alpha$ played by both $\Sigma_{\mathcal{T},R|\alpha}$ (the tail of Σ) and Ω, the N^*-induced strategy for N. Let $\mathcal{U}$ have limit length, and let $b = \Omega(\mathcal{U})$. We must see $b = \Sigma(\langle \mathcal{T}, \mathcal{U} \rangle)$.

For this, we look at the embedding normalization $W(\mathcal{T}, \mathcal{U})$ of $\langle \mathcal{T}, \mathcal{U} \rangle$, which also has limit length. Then for $b = \Omega(\mathcal{T})$:

(i) b generates (modulo $\mathcal{T}$) a unique cofinal branch a of $W(\mathcal{T},\mathcal{U})$.

(ii) Letting $i^*_b\colon N^* \to N^*_b$ come from lifting $i^{\mathcal{U}}_b$ to N^* via the iteration-strategy construction, $W(\mathcal{T},\mathcal{U})^\frown\langle a\rangle$ is a pseudo-hull of $i^*_b(\mathcal{T})$.

(iii) But $i^*_b(\Sigma) \subseteq \Sigma$ because Σ was Suslin-co-Suslin captured by N^*, so $i^*_b(\mathcal{T})$ is by Σ.

(iv) Thus $W(\mathcal{T},\mathcal{U})^\frown\langle a\rangle$ is by Σ, because Σ has strong hull condensation.

(v) But a determines b, so since Σ normalizes well, $\Sigma(\langle\mathcal{T},\mathcal{U}\rangle) = b$, as desired.

Phalanx comparisons work too. From this we get

Theorem

Assume AD^+*, and let* (P, Σ) *be an lbr hod pair with scope HC; then the standard parameter of* P *is solid and universal, and hence* (P, Σ) *has a core.*

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Phalanx comparisons also give

Theorem
Assume AD^+*, and let* (P, Σ) *be an lbr hod pair with scope HC; then*

(1) Σ *is positional,*
(2) Σ *has very strong hull condensation, and*
(3) Σ *fully normalizes well.*

Hod pair capturing

Hod pairs can be used to compute HOD, provided that there are enough of them.

Definition

(AD^+) HOD *pair capturing* (HPC) is the statement: for every Suslin, co-Suslin set of reals A, there is an lbr hod pair (P, Σ) with scope HC such that A is Wadge reducible to Code(Σ).

Remark. Under AD^+, if (P, Σ) is an lbr pair with scope HC, then Code(Σ) is Suslin and co-Suslin.

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Remark. Under AD^+, if (P, Σ) is an lbr pair with scope HC, then $\mathrm{Code}(\Sigma)$ is Suslin and co-Suslin.

Theorem

Assume there is a supercompact cardinal, and arbitrarily large Woodin cardinals. Suppose V is uniquely iterable. Let $\Gamma \subseteq \mathrm{Hom}_\infty$ be such that $L(\Gamma, \mathbb{R}) \models \mathsf{NLE}$; then $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$.

Theorem

Assume $\mathsf{AD}^+ + V = L(P(\mathbb{R})) + \mathsf{HPC}$*; then* $\mathrm{HOD}\,|\theta$ *is an lpm. Thus* $\mathrm{HOD} \models \mathsf{GCH}$*.*

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Remark. Woodin showed θ_0 and the $\theta_{\alpha+1}$ are Woodin in HOD. He proved an approximation to their being cutpoints.

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HPC localizes:

Theorem

Assume $\mathsf{AD}^+ + \mathsf{HPC}$*, and let* $\Gamma \subseteq P(\mathbb{R})$*; then* $L(\Gamma, \mathbb{R}) \models \mathsf{HPC}$*.*

The key to localization of HPC is to compute optimal Suslin representations for the iteration strategies in lbr hod pairs.

Hod pairs vs. Suslin cardinals

Definition

(AD^+) For (P, Σ) an lbr hod pair with scope HC, $M_\infty(P, \Sigma)$ is the direct limit of all nondropping Σ-iterates of P, under the maps given by comparisons.

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A tree $\mathcal{T}$ by Σ is M_∞*-relevant* iff there is a normal $\mathcal{U}$ by Σ extending $\mathcal{T}$ with last model Q such that the branch P-to-Q does not drop. Σ^{rel} is the restriction of Σ to M_∞-relevant trees.

Recall that A is κ-*Suslin* iff $A = p[T]$ for some tree T on $\omega \times \kappa$.

Theorem

(AD^+) Let (P, Σ) be an lbr hod pair with scope HC; then Code(Σ^{rel}) is κ-Suslin, for $\kappa = |M_\infty(P, \Sigma)|$.

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Remark. Code(Σ^{rel}) is not α-Suslin, for any $\alpha < |M_\infty(P, \Sigma)|$, by Kunen-Martin. So $|M_\infty(P, \Sigma)|$ is a Suslin cardinal.

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Proof sketch. $M_\infty(P,\Sigma)$ is the direct limit along a generic stack s of trees by Σ.But s can be fully normalized, so there is a single normal tree $\mathcal{W}$ on P with last model $M_\infty(P,\Sigma)$ such that every countable "weak hull" of $\mathcal{W}$ is by Σ.But then for $\mathcal{T}$ countable and M_∞-relevant,

$$\mathcal{T} \text{ is by } \Sigma \Leftrightarrow \mathcal{T} \text{ is a weak hull of } \mathcal{W}.$$

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The right-to-left direction follows from very strong hull condensation for Σ.
For left-to-right direction, we may assume $\mathcal{T}$ has last model Q, and P-to-Q does not drop. We then have a normal $\mathcal{U}$ on Q with last model $M_\infty(P,\Sigma)$ such that all countable weak hulls of $\mathcal{U}$ are by Σ.

We have

$$\begin{tikzcd}
P \arrow[rr, "\mathcal{W}"] \arrow[dr, "\mathcal{T}"'] & & M_\infty(P,\Sigma) \\
& Q \arrow[ur, "\mathcal{U}"'] &
\end{tikzcd}$$

Then

$$\mathcal{W} = X(\mathcal{T},\mathcal{U})$$

is the full normalization of $\langle \mathcal{T},\mathcal{U}\rangle$. The construction of $X(\mathcal{T},\mathcal{U})$ produces a weak hull embedding from $\mathcal{T}$ into $X(\mathcal{T},\mathcal{U})$, which is what we want.

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Thus our Suslin representation verifies that $\mathcal{T}$ is in the M_∞-relevant part of Σ by producing a weak hull embedding of $\mathcal{T}$ into $\mathcal{W}$.

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Definition

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(c) Let $\mathcal{T}$ be a normal tree on P with last model Q. We say $\mathcal{T}$ is *short* iff P-to-Q drops, or for $\pi\colon P \to Q$ the iteration map, $\eta^Q < \pi(\eta^P)$.

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Theorem

(AD^+) Let (P, Σ) be an lbr hod pair with scope HC, and suppose P has a top block. Let Ψ be the restriction of Σ^{rel} to short trees, and $\pi\colon P \to M_\infty(P, \Sigma)$ be the iteration map; then Code(Ψ) is $\pi(\beta^P)$-Suslin, but not α-Suslin for any $\alpha < |\pi(\beta^P)|$.

We believe that under $\mathsf{AD}^+ + \mathsf{HPC}$, all Suslin cardinals κ arise in one of these two ways. That is, the set that is Suslin for the first time at κ is either a complete iteration strategy for an lpm, or a short tree strategy for an lpm.

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This suggests proving HPC, assuming $\mathrm{AD}^+ + \mathrm{NLE}$, via an induction on Suslin cardinals, or equivalently, pointclasses with the Scale Property. Crossing gaps in scales is not actually a problem:

Theorem

Assume AD^+*, and let* Γ *be an inductive-like pointclass with the scale property. Suppose that the iteration strategies of lbr hod pairs are Wadge cofinal in* $\Gamma \cap \check{\Gamma}$*; then*

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(a) there is a short-tree-strategy pair (P, Ψ) *such that Code(*Ψ*) is in* $\Gamma \setminus \check{\Gamma}$*, and*

(b) if all sets in $\check{\Gamma}$ *are Suslin, then there is an lbr hod pair* (P, Σ) *such that Code(*Σ*) is not in* Γ*.*

Determinacy models from hod pairs

Theorem

(Sargsyan,S.) Assume AD^+*, and that there is an lbr hod pair* (P, Σ) *such that* $P \models \mathsf{ZFC}$ *+ "*δ *is a Woodin limit of Woodin cardinals + "there are infinitely many Woodin cardinals above* δ*". Then there is a pointclass* Γ *such that*

(1) $L(\Gamma, \mathbb{R}) \models$ *"the largest Suslin cardinal exists, and belongs to the Solovay sequence" (*LSA*), and*

(2) $L(\Gamma, \mathbb{R}) \models$ *"if A is a set of reals that is OD(s) for some* $s \colon \omega \to \theta$*, then A is Suslin and co-Suslin".*

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Part (1) is due to Sargsyan, and requires weaker hypotheses on P. The insight that Woodin limits of Woodins are what you need for (2) is due to Sargsyan. Part (2) is a step toward a model of $\mathsf{AD}_{\mathbb{R}}$ that satisfies "ω_1 is X-supercompact, for all sets X".