trevor II

proof: for $x \in pA$, defn $f(x) = y$ iff $(x, y) \in A$ and

(**) $(\varphi_0(x,y), y(0), \varphi_1(x,y), y(1), \ldots)$ is lex-least.

such a y exists: for $n < \omega$, choose y_n s.t. $(x, y_n) \in A$ and

$(\varphi_0(x,y_n), y_n(0), \ldots, \varphi_{n-1}(x,y_n), y_n(n-1))$

is lex-least.

then $\exists y$ $(x, y_n) \longrightarrow (x,y)$ mod $\vec{\varphi}$

so $(x, y) \in A$ by semi scale property.

+ y makes (**) lex-least.

<u>thm.</u> (martin-steel, steel) if $J_\alpha(\mathbb{R}) \models AD$, then scale$(\Sigma_1^{J_\alpha(\mathbb{R})})$

(Σ_1 means $\Sigma_1(\mathbb{R})$, i.e. quantification over reals counts as bounded.)

complexity of scales on $\Pi_1^{J_\alpha(\mathbb{R})}$ sets is described using an "envelope" obtained by the following operation:

<u>definition.</u> (martin) for $\mathcal{O} \subset \mathcal{P}(\omega)$,

define $\overline{\mathcal{O}} = \{ A \in \mathcal{P}(\omega) :$

f.a. cntbl. $\sigma \subset \omega$ there is

$A' \in \mathcal{O}$ $\quad A' \cap \sigma = A \cap \sigma \}$.

note: $\mathcal{O} \subset \overline{\mathcal{O}} = \overline{\overline{\mathcal{O}}}$. so it's a closure operation.

define OD^α = set of all ~~pointclasses~~ pointsets 1st order definable on $(J_\alpha(\mathbb{R}), \in)$ from a finite seq. of ordinals.

$$OD^{<\kappa} = \bigcup_{\alpha<\kappa} OD^\alpha.$$

<u>lem</u>. let $\kappa \in OR$.

① $\overline{OD^{<\kappa}}$ is closed under rec. substitution, $\exists^\omega$, $\forall^\omega$, and all boolean operations.

② if $cf(\kappa) > \omega$, $\Sigma_1^{J_\kappa(\mathbb{R})} \subset \overline{OD^{<\kappa}}$

proof: ① e.g. for closure under rec. substitution. let $A \in \overline{OD^{<\kappa}}$, $f: \mathcal{N} \longrightarrow \mathcal{N}$ recursive. we claim $f^{-1}{}''A \in \overline{OD^{<\kappa}}$.

let $\sigma \subset \mathcal{N}$ be ctble.

take $A' \in OD^{<\kappa}$ with $A' \cap f''\sigma = A \cap f''\sigma$.

then $f^{-1}{}''A'$ $\in OD^{<\kappa}$ and

$f^{-1}{}''A' \cap \sigma = f^{-1}{}''A \cap \sigma$.

so $f^{-1}{}''A \in \overline{OD^{<\kappa}}$.

② say $A \subset \mathcal{N}$ is def. th. over $J_\kappa(\mathbb{R})$ by a Σ_1 fmla φ.

take $\sigma \subset W$ be ctbl.

because $cf(\kappa) > \omega$, can take $\alpha < \kappa$

s.t. $A \cap \sigma = A_\alpha \cap \sigma$, where

$A_\alpha = \{x : J_\alpha(\mathbb{R}) \models \varphi(x)\} \in OD^{<\kappa}$.

so $A \in \overline{OD^{<\kappa}}$.

<u>Cor.</u> $\bigcup_{x \in \mathbb{R}} \overline{OD_x^{<\kappa}}$ is closed under

cont. preimages

[N.B. $\bigcup_{x \in \mathbb{R}} \overline{OD_x^{<\kappa}} \neq \overline{\bigcup_{x \in \mathbb{R}} OD_x^{<\kappa}} = \mathcal{P}(\mathbb{R})$.]

<u>remark</u>. $\overline{OD^{<\kappa}}$ is called the

envelope of $\Sigma_1^{J_\kappa(\mathbb{R})}$, $Env(\Sigma_1^{J_\kappa(\mathbb{R})})$.

$\bigcup_{x \in \mathbb{R}} \overline{OD_x^{<\kappa}}$ is called $Env(\underset{\sim}{\Sigma}_1^{J_\kappa(\mathbb{R})})$,

or $\underset{\sim}{Env}(\Sigma_1^{J_\kappa(\mathbb{R})})$.

check out trevor's "the envelope of pointclasses under a local determinacy hypothesis."

<u>lem.</u> (anti-uniformization)

let $\kappa \in OR$ and $A = \{(x,y) : y \notin OD_x^{<\kappa}\}$.

(A is $\Pi_1^{J_\kappa(\mathbb{R})}$.)

assume $\forall x \exists y\ (x,y) \in A$.

[e.g., if $J_\kappa(\mathbb{R}) \models KP + AD$.]

let $f : \mathcal{N} \longrightarrow \mathcal{N}$ uniformize A.

then $R = \{(x,m,n) \in \mathcal{N} \times \omega \times \omega :$

$f(x)(n) = m\}$

is not $\overline{OD_x^{<\kappa}}$ for any $x \in \mathbb{R} = \mathcal{N}$.

proof: let $x \in {}^\omega\omega$.

WTS: $R \notin \overline{OD_x^{<\kappa}}$. Consider

$\sigma = \{x\} \times \omega \times \omega$.

$\ln$ $R' \in OD_x^{<\kappa}$, s.t. $R' \cap \sigma = R \cap \sigma$.

$R' \cap \sigma \in OD_x^{<\kappa}$.

but $R \cap \sigma = \{(x, m, n) : f(x)(m) = n\}$

$\notin OD_x^{<\kappa}$.

So $R' \cap \sigma \neq R \cap \sigma$. $\lightning$

remark. this is optimal: if there is a scale on A, whose norms are $\bigcup_{x \in \mathbb{R}} \overline{OD_x^{<\kappa}}$-norms, we can get, assuming $J_\kappa(\mathbb{R}) \models KP + AD$, a uniformization of A s.t. for all $n<\omega$,

$$R_n = \{(x, m) \in \mathcal{N} \times \omega : \cancel{f(x)} \; f(x)(n) = m\}$$

$$\in \bigcup_{x \in \mathbb{R}} \overline{OD_x^{<\kappa}}.$$

<u>theorem</u>. $(ZF + DC_{\mathbb{R}})$ if $J_\kappa(\mathbb{R}) \models KP + AD$

then

Σ_1 collection + Δ_1 comprehension + things which hold in all $J_\alpha(\mathbb{R})$ anyway.

① $\mathrm{Det}(\overline{OD^{<\kappa}})$

② $\overline{OD^{<\kappa}}$ is closed under $\exists^{\mathbb{R}}$, $\forall^{\mathbb{R}}$.

(relativizes: also holds for $\overline{OD_x^{<\kappa}}$, $x \in \mathcal{N}$)

<u>remark</u>. proved similarly to the Kechris-Woodin transfer thm. for "the equivalence of partition properties + determinacy."

also similar to martin, "the largest c.t.h. this, that, and the other."

combining this with scale constructions of moschovakis ("scales on coinductive sets") + steel ("scales in $L(\mathbb{R})$")

<u>Cor.</u> ($ZF + DC_{\mathbb{R}}$) if $J_\kappa(\mathbb{R}) \models KP + AD$, and κ begins a Σ_1-gap in $L(\mathbb{R})$ that ends in $L(\mathbb{R})$ (i.e., new gaps later).

then every $\Pi_1^{J_\kappa(\mathbb{R})}$ set $A \subset \mathcal{N}$ has a scale $\vec{\varphi}$ with

$$R_n = \{(x,y) \in A^2 : \varphi_n(x) \leq \varphi_n(y)\} \in \bigcup_{z \in \mathbb{R}} \overline{OD_z^{<\kappa}},$$

for all $n < \omega$.

[question: are they in ~~$OD^{<\kappa}$~~ $\overline{OD^{<\kappa}}$?]