## trevor IV

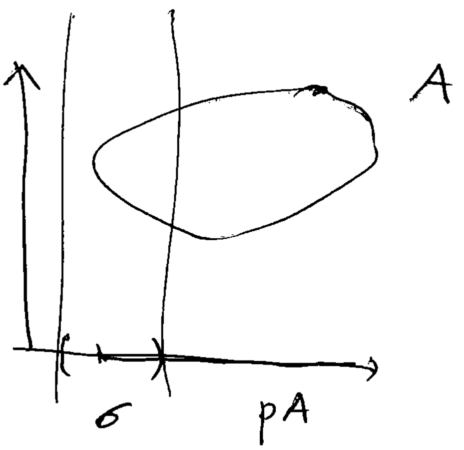
proof of (2) (closure of $\overline{OD^{<\kappa}}$ under $\exists^{\mathbb{R}}$)

let $A \subset \mathbb{R} \times \mathbb{R}$, $A \in \overline{OD^{<\kappa}}$

$$pA = \{ x : \exists y \in \mathbb{R} \ (x, y) \in A \} .$$

we show $pA$ satisfies $(**)$ from the lemma, i.e., f.a. ctbl. $\sigma \subset \mathbb{R}$ there is a cone of turing degrees $d$ s.t.

$\exists A' \in OD_d^{<\kappa}$ $pA \cap \sigma = A' \cap \sigma$.




let $\sigma \subset \mathbb{R}$ be ctbl. using $CC_{\mathbb{R}}$, for each $x \in pA \cap \sigma$, choose $y_x \in \mathbb{R}$ with $(x, y_x) \in A$.

take $d_0 \in \mathbb{R}$ coding $(y_x : x \in pA \cap \sigma)$

Then $d \geq_T d_0$.

take $A' \in OD^{<\kappa}$ s.t. $A' \cap (\sigma \times \{y : y \leq_T d\})$

$= A \cap (\sigma \times \{y : y \leq_T d\})$.

take $A^* = A' \cap (\mathbb{R} \times \{y : y \leq_T d\}) \in OD_d^{<\kappa}$.

$pA^* \in OD_d^{<\kappa}$, and $pA^* \cap \sigma = pA \cap \sigma$.

so $pA$ satisfies $(**)$, so $pA \in \overline{OD^{<\kappa}}$.

**Theorem.** (Woodin) ZF + DC + AD.

if there is a large Suslin cardinal $\kappa$, and $T_h$ is a tree on $\omega \times \kappa$ projecting to a complete $\kappa$-Suslin set, then f.a. $A \subset \mathbb{R}$ there is a cone of $x$ s.t. $A \cap L[T, x] \in L[T, x]$.

This will follow from:

**Le.** (ZF + DC) if $T$ is a tree on $\omega \times \kappa$ + we define $R = \{ \leq_s : s \in {}^{<\omega}\kappa \}$, where $\leq_s$ is a p.w.o. on $\mathbb{R} \setminus p[T]$

$x \leq_s y$ iff $\mathrm{rank}_{T_x}(s) \leq \mathrm{rank}_{T_y}(s)$,

and there is a fine ctbly. complete measure on $\mathcal{P}_{\omega_1}(\overline{R})$, then there is a semiscale on $\mathbb{R} \setminus p[T]$ whose norm rel. are in $\overline{R}$.

the thm. follows fr the lea :

$S(x)$ is not self-dual.

so $\mathbb{R} \setminus p[T]$ is not Sushin.

supp. towards a contradicti that

$$\underset{\sim}{\Lambda} = \{ A \subset \mathbb{R} : \forall^* x \; A \cap L[T,x] \in L[T,x] \}$$

$$\neq \mathcal{P}(\mathbb{R}).$$

closed udr wadge reducti.

hence get a surjecti $f : \mathbb{R} \longrightarrow \underset{\sim}{\Lambda}$ (wadge)

push forward martin's measure to get a fine, c.c. measure on $\mathcal{P}_{\omega_1}(\underset{\sim}{\Lambda})$.

note: $R \subset \underset{\sim}{\Lambda}$.

also, $L[T,x] \cap \mathbb{R}$ is ctble.,

so $\overline{R} \subset \underset{\sim}{\Lambda}$.

then there is a fine c.c. measure on $\mathcal{P}_{\omega_1}(\overline{R})$. by the lea, $\mathbb{R} \setminus p[T]$ is

Suslin, contradiction.

note: Woodin proved something equivalent to the lemma, assuming also $\exists$ fine c.c. measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

<u>proof</u> of the lemma. assume towards contradiction that $\forall \sigma \in \mathcal{P}_{\omega_1}(\bar{\mathbb{R}})$, $\sigma$ does not give a semiscale on $\mathbb{R} \setminus p[T]$.

then $\forall \sigma \in \mathcal{P}_{\omega_1}(\bar{\mathbb{R}})$, player I has a w.s. in the game

$$
G_\sigma \quad
\begin{array}{c|ccc}
\mathrm{I} & \underset{\in \omega}{x(0)}, \underset{\in \mathrm{OR}}{f(0)} & & g(0), x(1), f(1) \\
\hline
\mathrm{II} & & \underset{\in \sigma}{\leq_0} & & \leq_1
\end{array}
$$

where: $(x, f) \in [T]$ (so $f$ witnesses $x \in p[T]$)

$(x, g) \in [T_{\vec{\varphi}}]$ $\quad \vec{\varphi}$ = the norms given by II's moves

$T_{\vec{\varphi}}$ = tree for "scale"

there is then a canonical w.s. $\sigma^{\mathcal{C}}$ for I in $G_e$ (assuming no semi-scale).

we define a seq. $(\leq_n : n<\omega) \subset \overline{R}$ + show that this is a winning play for II against $\sigma^{\mathcal{C}}$ for $\mu$-a.e. $\mathcal{C}$.

for $\mathcal{C} \in P_{\omega_1}(\overline{R})$, define $x^{\mathcal{C}}(0)$, $f^{\mathcal{C}}(0)$ to be the initial play by $\sigma^{\mathcal{C}}$.

define $\leq_0$ on $\mathbb{R} \setminus p[T]$:

$$x \leq_0 y \text{ iff } \forall^*_\mu \mathcal{C} \ \text{rank}_{T_x}(f^{\mathcal{C}}(0)) \leq \text{rank}_{T_y}(f^{\mathcal{C}}(0)).$$

$\leq_0 \in \overline{R}$,

now for $\mathcal{C} \in P_{\omega_1}(\overline{R})$, define $g^{\mathcal{C}}(0)$, $x^{\mathcal{C}}(1)$, $f^{\mathcal{C}}(1)$ to be $\sigma^{\mathcal{C}}$'s response to $\underline{\text{II}}$ plays $\leq_0$ .

define $\leq_1$ on $\mathbb{R} \setminus p[T]$ by

$$x \leq_1 y \text{ iff } \forall^*_\mu \mathcal{C} \ \operatorname{rank}_{T_x}(f^{\mathcal{C}}(0), f^{\mathcal{C}}(1))$$
$$\leq \operatorname{rank}_{T_y}(f^{\mathcal{C}}(0), f^{\mathcal{C}}(1))$$

$\leq_1 \ \in \ \overline{R}$ .

etc.

continue, getting $(\leq_0, \leq_1, \ldots)$ .

let $\vec{\varphi}$ be the putative semiscale on $\mathbb{R} \setminus p[T]$ ass. to this seq. of p.w.o.

note: $\forall^*_\mu \mathcal{C} \ (\leq_0, \leq_1, \ldots) \subset \mathcal{C}$, so is a legal play for $\underline{\text{II}}$ in $G^{\mathcal{C}}$.

using Coutable compactness twice,

$\forall_\mu^* \mathcal{C} \quad x^{\mathcal{C}} = x$, some fixed $x$.

$x \in p[T]$, witnessed by $f^{\mathcal{C}}$,
for $\forall_\mu^* \mathcal{C}$.

$x \in p[T_{\vec{\varphi}}^x]$, as being witnessed by $g^{\mathcal{C}}$,
$\forall_\mu^* \mathcal{C}$.

take a seq. of reals $(x_k : k < \omega) \longrightarrow x$
mod $\vec{\varphi}$, $x_k \notin p[T]$, (but $x \in p[T]$).
(as $\vec{\varphi}$ not a Scale)

So for all $n$, the rank of $x_k$ in
$\leq_n$ is eventually constant.

So $\forall_\mu^* \mathcal{C}$, $\mathrm{rank}_{T_{x_k}}(f^{\mathcal{C}}(0), -, f^{\mathcal{C}}(n))$
is eventually constant, as $k \longrightarrow \infty$.

fix such a typical $\mathcal{C}$ + define

$$h(n) = \lim_{k \to \infty} \operatorname{rank}_{T_{x_k}} \left( f^{\mathcal{C}}(0), \ldots, f^{\mathcal{C}}(n) \right).$$

$h(n+1) < h(n)$ for all $n < \omega$.

$\dashv$