trevor.

thm. (woodin) assume $ZF + DC_{\mathbb{R}}$, and let M_1, M_2 are two models of AD^+ with $\mathbb{R} \cup OR \subset M_1, M_2$ that are divergent, i.e. $\mathcal{P}(\mathbb{R}) \cap M_1 \not\subset M_2$ and $\mathcal{P}(\mathbb{R}) \cap M_2 \not\subset M_1$.

let $M = L(\mathcal{P}(\mathbb{R})^{M_1} \cap \mathcal{P}(\mathbb{R})^{M_2})$.

then $M \models AD^+$ + every set of reals is Suslin.

note: woodin's pf. used pointed Sacks forcing.

recall. for a pt. class Θ,

$$\overline{\Theta} = \{A \subset \mathbb{R} : \text{for all ctbl } \sigma \subset \mathbb{R} \text{ there is } A' \in \Theta \text{ s.t. } A \cap \sigma = A' \cap \sigma\}.$$

or, $\mathbb{R} \times \mathbb{R}$, etc.

lem. $(ZF + DC_{\mathbb{R}})$ let $M \models ZF + AD^+$ be transitive w/ $\mathbb{R} \subset M$.

let $z \in M$. then every set of reals in $\overline{OD_z^M}$ is determined.

proof: similar to $\overline{OD^{<\kappa}}$ (+ to Kechris–Woodin, Kechris–Solovay). $\dashv$

Corollary. $(ZF + DC_{\mathbb{R}}$; $M + z$ as above$)$ for any $A, B \in \overline{OD_z^M}$, either $A \leq_W B$ or $B \leq_W \mathbb{R} \setminus A$.

pf.: Wadge's lem. the payoff set $\{ a \oplus b : a \in A \leftrightarrow b \in B \}$ is in $\overline{OD_z^M}$. $\dashv$

pf. of thm. $\mathcal{P}(\mathbb{R})^M = \mathcal{P}(\mathbb{R})^{M_1} \cap \mathcal{P}(\mathbb{R})^{M_2}$. AD^+ can be written in a Π^2_1 way, so $M \models AD^+$.

if $M \not\models$ every set of reals is suslin, then M has a largest suslin cardinal, κ, and not every suslin set is co-suslin.

it suff. to show: $M \models$ "every suslin set of reals is co-suslin".

let T be a tree on $\omega \times \kappa$ in M.

want to see: $\mathbb{R} \setminus p[T]$ is suslin in M.

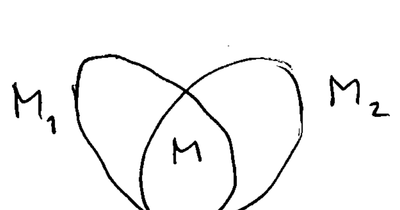
let $\boxed{\theta = OD^M_T .}$

in V, $\bar{\theta}$ is determined, and for all $A, B \in \bar{\theta}$, $A \leq_W B$ or $B \leq_W \neg A$.

therefore $\bar{\theta}^{M_1} \subset \bar{\theta}^{M_2}$ or $\bar{\theta}^{M_1} \subset \bar{\theta}^{M_2}$.

(no divergent envelopes)

we may assume WLOG $\bar{\theta}^{M_1} \subset \bar{\theta}^{M_2}$.



then $\overline{\mathcal{O}}^{M_1}$ is not wadge cofinal in M_1, as o.w. $\mathcal{P}(\mathbb{R})^{M_1} \subset M_2$.

<u>claim</u>. in M_1, there is a semiscale $\vec{\varphi}$ on $\mathbb{R} \setminus p[T]$ whose norm relations $\leq_{\varphi_n}$, $n<\omega$, are in $\overline{\mathcal{O}}$ (meaning $\overline{\mathcal{O}}^{M_1}$).

proof: work in M_1.

for $s \in {}^{<\omega}k$ define $\leq_s$ on $\mathbb{R} \setminus p[T]$ by $x \leq_s y$ iff $\mathrm{rank}_{T_x}(s) \leq \mathrm{rank}_{T_y}(s)$ (rank $= 0$ iff $s \notin$ the tree)

each $\leq_s$ is in $\mathcal{O}$.

therefore, $\overline{\{\leq_s \,:\, s \in {}^{<\omega}k\}} \subset \overline{\mathcal{O}}$, which is not wadge-cofinal in M_1.

so there is a fine c.c. measure on

$\mathcal{P}_{\omega_1}(\{\leq_s : s \in {}^{<\omega}\kappa\})$.

(push forward martin's measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$.)

by the lea from last time, the is a semiscale $\vec{\varphi}$ on $\mathbb{R} \setminus p[T]$ whose norm relatis are in $\overline{\{\leq_s : s \in {}^{<\omega}\kappa\}}$

$\subset \bar{\vartheta}$.

now for all $n<\omega$, $\leq_{\varphi_n} \in \bar{\vartheta}^{M_1}$ and $\bar{\vartheta}^{M_1} \subset \bar{\vartheta}^{M_2}$, so $\leq_{\varphi_n} \in M_2$.

$\{\leq_{\varphi_n} : n<\omega\}$ is not wadge cofinal in M_2, as o.w. we would have

$\mathcal{P}(\mathbb{R})^{M_2} \subset M_1$.

so by $\underbrace{\text{countable choice}}_{CC_{\mathbb{R}}}$ in V,

$(\leq_{\varphi_n} : n<\omega) \in M_2$.

also, $(\leq_{\varphi_n} : n < \omega) \in M_1$ by

construction.

so $(\leq_{\varphi_n} : n<\omega) \in M$, so

$\mathbb{R} \setminus p[T]$ is suslin in M. $\dashv$

lu's also prove. [we may assume DC by working in $L(A, \mathbb{R})$ for some A.]

<u>thm</u>. (woodin) assume $ZF + DC_{\mathbb{R}} + AD$. if T is a tree on $\omega \times \kappa$ (some κ) + $\theta_T < \theta$, then $p[T]$ is co-suslin

$\theta_T = \sup\{\alpha : \exists f : \alpha \xrightarrow{\text{onto}} OR,\ f\ OD_T\}$

<u>proof</u>: as before, it suff. to show $\{\leq_s : s \in {}^{<\omega}\kappa\}$ is not wadge cofinal.

$\theta_T < \theta$, so it suff. to show every elem of $\{\leq_s : s \in {}^{<\omega}\kappa\}$ is OD_T.

<u>define (from T)</u> a well-ordering of

<u>$\{ \leq_s : s \in {}^{<\omega}\kappa \}$</u>. for $R_1, R_2 \in$

$\{ \leq_s : s \in {}^{<\omega}\kappa \}$, define $R_1 < R_2 \iff$

$\forall^* d \in \mathcal{D}_T$, defining $\sigma = \{ (x,y) : x, y \leq_T d \}$,

turing degrees

the $<_{lex}$-least $s \in {}^{<\omega}\kappa$ $R_1 \cap \sigma = \leq_s \cap \sigma$

$<_{lex}$

the $<_{lex}$-least $t \in {}^{<\omega}\kappa$ $R_2 \cap \sigma = \leq_t \cap \sigma$.

(when $<_{lex}$ is supp. to denote a w.o.)

by martin's cone theorem, we have

$<$, $=$, or $>$ on a cone of d.

if $R_1 \neq R_2$, we can't have $=$ on a cone.

$<$ is well-fdd. by countable completeness.

so $<$ is a well-ordering.

$\dashv$

rmk. Woodin used a slightly different argument, counting measures on $<\omega_\kappa$.

<u>Corollary</u>. (Woodin)

assume $ZF + AD^+$. if $\theta_0 < \theta$, then every Π^2_1 set is Suslin.

(using definable trees for Σ^2_1 sets.)