# M. Zeman – Tutorial III, part 3

2017/07/21

<u>Recall</u>. We would like to let $\bar\alpha \in B_\alpha$ iff

i) $n_{\bar\alpha} = n_\alpha$

ii) there is a map $\sigma : N_{\bar\alpha} \to N_\alpha$ s.t.

a) $\operatorname{crit}(\sigma) = \bar\alpha$, $\sigma(\bar\alpha) = \alpha$

b) $\sigma(p_{\bar\alpha}) = p_\alpha$

c) $\sigma$ is $\Sigma_0^{(n_\alpha)}$-preserving

d) for each $\beta \in p_\alpha$ there is some generalized solidity witness $(Q, r)$ for $\beta$ wrt. $p_\alpha$ in $N_\alpha$ s.t. $(Q, r) \in \operatorname{rng}(\sigma)$

Let $\bar\alpha \in B_\alpha$

. if $\alpha^* \in B_\alpha \cap \bar\alpha$ then we have $\sigma_{\alpha^*\alpha} : N_{\alpha^*} \to N_\alpha$

and $\sigma_{\bar\alpha\alpha^*} : N_{\bar\alpha} \to N_{\alpha^*}$

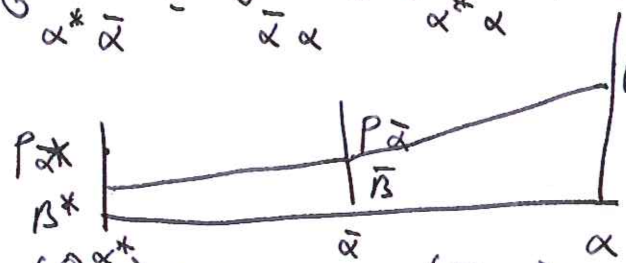
By Fact 3: $\operatorname{rng}(\sigma_{\alpha^*\alpha}) \subseteq \operatorname{rng}(\sigma_{\bar\alpha\alpha})$ so

we can define

$$\sigma_{\alpha^*\bar\alpha} = \sigma_{\bar\alpha\alpha}^{-1} \circ \sigma_{\alpha^*\alpha} : N_{\alpha^*} \longrightarrow N_{\bar\alpha}$$

$\beta \in p_\alpha$

$(Q, r) \in \operatorname{rng}(\sigma_{\alpha^*\bar\alpha})$




We have $(Q, r) \in \operatorname{rng} \subseteq \operatorname{rng}(\sigma_{\bar\alpha\alpha})$.

So $\sigma^{-1}_{\bar{\alpha}\alpha}(Q,r)$ is defined and is a generalized solidity witness for $\bar{\beta}$ w.r.t $p_{\bar{\alpha}}$ in $N_{\bar{\alpha}}$.

Also $\sigma^{-1}_{\bar{\alpha}\alpha}(Q,r) \in \operatorname{rng}(\sigma_{\alpha^*\bar{\alpha}})$.

This shows: $\sigma_{\alpha^*\bar{\alpha}}$ satisfies clause d).

Hence $\alpha^* \in B_{\bar{\alpha}}$. We have $B_\alpha \cap \bar{\alpha} \subseteq B_{\bar{\alpha}}$.

Now assume $\alpha^* \in B_{\bar{\alpha}}$. If $\alpha^* \geq \min(B_\alpha)$

We still have some $(Q,r)$ for $\beta$ s.t.

$(Q,r) \in \operatorname{rng}(\sigma_{\min(B_\alpha),\alpha})$.

So we can form $\sigma^{-1}_{\alpha^*\alpha}(Q,r)$ and thereby show that $\sigma_{\alpha^*\alpha}$ satisfies a), so $\alpha^* \in B_\alpha$.

All we know is that

$$B_\alpha \cap \bar{\alpha} = B_{\bar{\alpha}} \setminus \min(B_\alpha).$$

To get coherence define $\cancel{B^{\alpha^*}_{\alpha}}$ $B'_\alpha$ as follows

$\alpha_0 := \min(B_\alpha)$

$\alpha_{i+1} := \min(B_{\alpha_i})$ if smaller than $\alpha_i$.

Let $l$ = the last $i$. Then let

$$B'_\alpha := B_{\alpha_0} \cup B_{\alpha_1} \cup \dots \cup B_{\alpha_\ell} \cup B_\alpha .$$

<u>Easy</u>: The sets $B'_\alpha$ are coherent.

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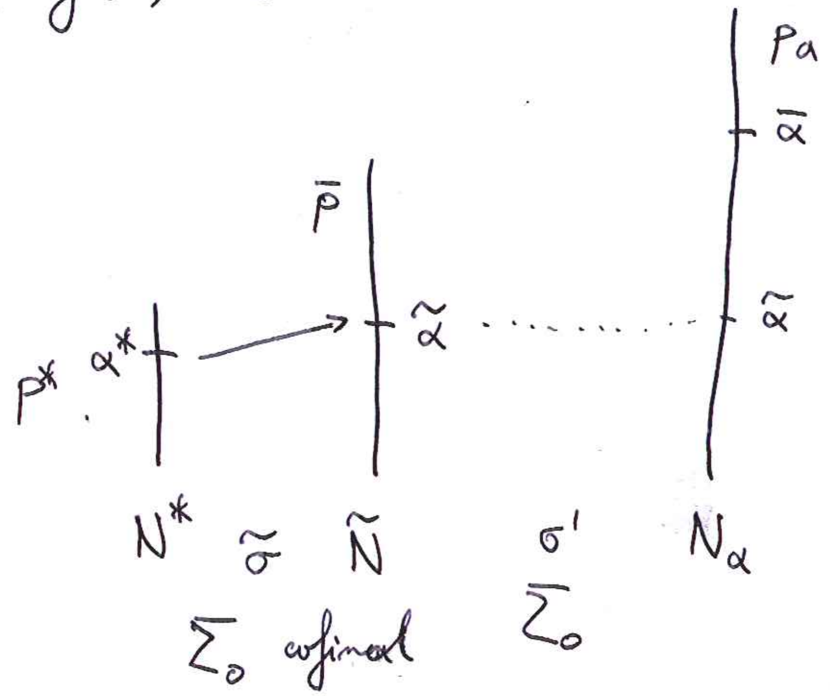
Unboundedness of $B_\alpha$: Interpolation argument.
Assume $\mathrm{cf}(\alpha) > \omega$.




- picture + critical points + preservation degree as in L.
- $h_{\tilde N}(\kappa \cup \{\bar p\}) = \tilde N$ as in L.
- if $\beta \in P_\alpha$ then there is some gen. solidity witness $(Q,r)$ for $\beta$ w.r.t. $P_\alpha$ in $N_\alpha$ s.t. $(Q,r) \in \mathrm{rng}(\sigma')$.

<u>Important point:</u> If we can prove that $\tilde{N}$ is a premouse then we can apply condensation and prove $\tilde{N} = N_{\tilde{\alpha}}$ – provided $\tilde{\alpha}$ does not index an extender – see later.

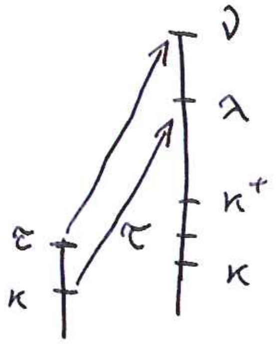
. If $n_\alpha > 0$ then $\tilde{\sigma}$ is $\Sigma_2$-preserving. In this case OK.

Revise def of coherent structure.

$$M = (J_\nu^E, F)$$

F an extender with critical point $\kappa$, $\lambda_F = \lambda$ s.t. there is some $\tau \leq \kappa^{+M}$ s.t.

$$J_\nu^E = Ult(J_\tau^E, F)$$

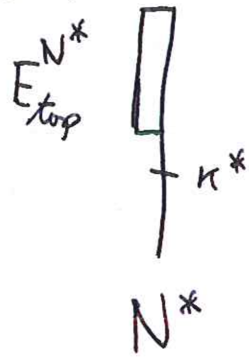



<u>Note:</u> F is weakly amenable iff $\tau = \kappa^{+M}$.

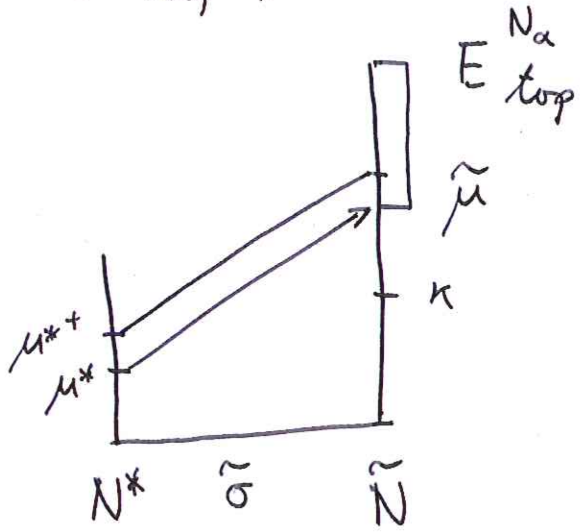
<u>Note:</u> Being a coherent structure is a Q-property.

Back to interpolation and $n_\alpha = 0$:

• if $\mathrm{crit}(E^{N_\alpha}_{top}) \geq \kappa$




$\mathrm{crit}(E^{N_\alpha}_{top}) \geq \kappa$ <u>OK</u>



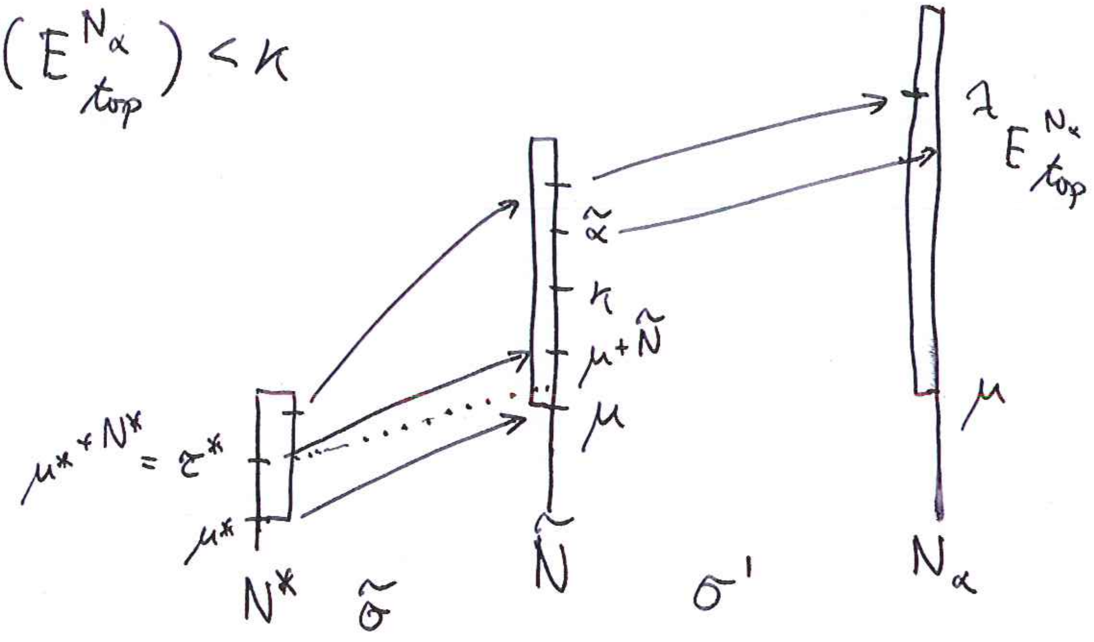

coarse

ultrapower

$\tilde{\sigma}[\mu^{*+N^*}] \subseteq \tilde{\mu}^{+\tilde{N}}$ cofinal

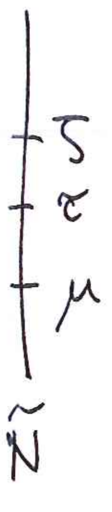
$$E^{\tilde{N}}_{top} = \bigcup_{z \in N^*} \tilde{\sigma}(E^{N^*}_{top} \cap z)$$

$\mathrm{crit}(E^{N_\alpha}_{top}) < \kappa$




$$\tau = \sup \tilde{\sigma}[\tau^*].$$

<u>Summary</u>: If $\mathrm{crit}(E^{N_\alpha}_{top}) < \kappa$ then the top extender of $\tilde{N}$ is not weakly amenable.




$\mathrm{dom}(E^{\tilde{N}}_{top}) \subseteq J^E_\tau$ and $\tau$ is the least such.
Also: $\mu$ is the largest cardinal in $J^E_\tau$.
Let $\tilde{N}\|\zeta$ be the collapsing level of $\tilde{N}$ for $\tau$.

<u>Important fact</u>: $\mathrm{Ult}^*(\tilde{N}\|\zeta, E^{\tilde{N}}_{top}) \lhd N_\alpha$.

In the long extender model this is unfortunately not known in general.

Let $M$ be a coherent structure and $\tau(M)$ be the least $\tau$ s.t. $\mathrm{dom}(E^M_{top}) \subseteq J^{E^M}_\tau$.

We let $N^*(M)$ be the collapsing level of $M$ for $\tau$ – if defined.

<u>Def</u>. Let $M$ be a coherent structure s.t. $\tau(M) < \mu^{+M}$, $\mu := \mathrm{crit}(E^M_{top})$.

~~Assume $\mu < \kappa$ and Ult* (M~~

~~$\mathrm{Ult}^*(N^*(M), E^M_{top}) \lhd L[E]$.~~

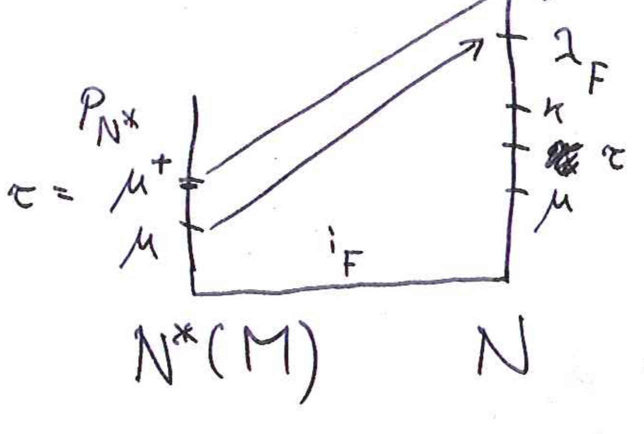
In this case we say that $M$ is a protomouse.

Let $M = (J^E_\nu, F)$ be a protomouse s.t.

$$N = \mathrm{Ult}^*(N^*(M), F)$$

is a sound + solid premouse.

(Think of $N$ as an initial segment of $L[E]$.)




• $i_F(p_{N^*}) =: r$ is a top segment of $p_N$.

The map $i_F$ has the following properties:

- is $\Sigma_1^{(m)}$-preserving where

$$\varsigma_{N^*(M)}^{m+1} \leq \mu < \varsigma_{N^*(M)}^{m}$$

- maps $\varsigma_{N^*(M)}^{m}$ cofinally into $\varsigma_N^m$.

~~a) Def. Let $N \triangleleft L[E]$ be a collapsing level for $\alpha$~~

Def. Let $N \triangleleft L[E]$ be such that $\varsigma_N^{m+1} = \kappa < \varsigma_N^m$ and $q, r$ be s.t.

a) $q \cup r = p_N$ and $\max(q) < \min(r)$ $\quad \begin{array}{l} | \, r \\ | \, q \end{array}$

b) $\tilde{h}_N^{m+1}(\mu \cup r) \cap \varsigma_N^m$ is cofinal in $\varsigma_N^m$,

c) $\tilde{h}_N^{m+1}(\mu \cup r) \cap \max(q) \subseteq \mu$

The pair $(q, \mu)$ is called a <u>divisor</u> for $N$.