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More structural consequences of AD

Richard Ketchersid

Abstract. Woodin and Steel showed that under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$ the Suslin cardinals are closed below their supremum; Woodin devised an argument based on the notion of *strong ∞-Borel code* which is presented here. A consequence of the closure of the Suslin cardinals below their supremum is that the Suslin cardinals and the reliable cardinals coincide, the proof of this fact is also included.

Woodin's argument yields that AD^+ implies that the Suslin cardinals are closed below Θ. It turns out that this characterizes AD^+. We include a sketch of this argument as well.

Contents

1. Introduction

This is an expository paper based on notes from the set theory seminar at UC Berkeley in Fall 1994 and Spring 1995, personal communications with Woodin, and handwritten notes from a seminar on AD^+ at UCLA given by John Steel. The results here are due to many people including, but not limited to Howard Becker, Steve Jackson, Alexander Kechris, Tony Martin, Yiannis Moschovakis, John Steel, Robert Solovay, and Hugh Woodin. I will make attributions when known, however,

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I do not have a full account of the history and hence cannot attribute every fact, non-attributed facts are, as far as I know, folklore, come from private communication, or are unattributed in various hand written notes. The goal is to present some of the theory of AD^+ which, heretofore, has not been published, although see [**Ike10**]. References given are typically the ones most readily available to the author, and most likely the reader as well, and almost certainly are not the original sources.

I have placed some of the more technical descriptive set theoretic material to an appendix so as not to deter the reader less familiar with descriptive set theory. The primary tools used are forcing, ultraproducts, and absoluteness arguments. This paper, with the exception of §7, is intended to be self contained for those *familiar* with set theory in general and the axiom of determinacy in particular. I will begin with a few preliminaries in §2.

§3 concerns degree measures and associated ultrapowers and ultraproducts. Here we will also discuss generic ultrapowers formed by forcing with the positive sets associated to a filter and see that this forcing is essentially a version of Sacks forcing.

§4 introduces the hierarchy of degree notions.

§5 discusses many structural consequences of the degree structure developed in §4. In particular, the cases of having or not having a maximal degree are considered. Various implications concerning AD, AD^+, and $\mathrm{AD}_{\mathbb{R}}$ are considered. This section also contains a proof due to Woodin that $\mathrm{AD} +$ uniformization implies that all sets are ∞-Borel.

§6 develops the critical notion, due to Woodin, of strong infinity Borel code. The main results concerning the closure of the Suslin cardinals below their supremum, under $\mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$, and closure below Θ, under AD^+, are discussed in this section. Subsection §6.1 contains the short proof that the reliable cardinals and the Suslin cardinals are the same. The key to this result is the closure from §6.

§7 contains a sketch of the equivalence between AD^+ and the closure of the Suslin cardinals below Θ leaving the existence of a maximal model of AD^+ which contains all the Suslin sets, part of Woodin's derived model theorem, as a black box.

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2. Preliminaries

Throughout we will be assuming $\mathrm{ZF}+\mathrm{AD}+\mathrm{DC}_{\mathbb{R}}$. As is typical, in this area, the term *real* is used to mean either an element of Baire space, ω^ω, or of Cantor space, 2^ω; in general the spaces X^ω admit simple pairing functions that the Euclidean reals do not posses. I will use $\boldsymbol{x}$ to indicate a $\leq\omega$-sequence from a set X, i.e., $\boldsymbol{x} = \langle x_i : i < |\boldsymbol{x}|\rangle$. For a sequence, $\langle \boldsymbol{x}_i : i < n\rangle \in (X^\omega)^n$, I will often identify the sequence with its corresponding element in X^ω via the canonical isomorphism of X^ω with $(X^\omega)^n$ and I will write $(\boldsymbol{x})_i^n$ for the element in X^ω so that $\langle (\boldsymbol{x})_i^n : i < n\rangle = \boldsymbol{x}$ via this identification. Finally, when n is understood from context, I will write $(\boldsymbol{x})_i$ rather than $(\boldsymbol{x})_i^n$.

Given a set $A \subseteq X^\omega$, the game $G^X(A)$ consists of ω-rounds, where in the i^{th} round player I plays $x_{2i} \in X$ and then player II plays $x_{2i+1} \in X$. When the game is over, a play $\boldsymbol{x} \in X^\omega$ has been constructed. I wins if $\boldsymbol{x} \in A$, otherwise II wins. I will refer to $(\boldsymbol{x})_0 \in X^\omega$ as I's play and $(\boldsymbol{x})_1 \in X^\omega$ as II's play.

A strategy σ for player I (II) is a function $\sigma : X^{<\omega} \to X$ telling player I (II) how to move. For $f \in X^\omega$ and σ a strategy for I (II) write $\sigma(f)$ for the play resulting from II (I) playing f and I (II) using σ. The strategy σ is winning for I iff $\sigma(f) \in A$ for every valid play f by II; similarly define σ is winning for II. If $X \in \{\omega, 2\}$, then a strategy is easily coded by a real.

The game $G^X(A)$ is determined provided one of the players has a winning strategy. AD is the assertion that for all $A \subseteq \mathbb{R}$, $G(A)$ is determined.

Following on the heals of joint work with Martin, Moschovakis, and Kechris [**KKMW81**, §2], Woodin showed that if $M \supseteq \mathbb{R}$ is a transitive model of ZF and every set of reals in M is Suslin in some larger transitive model of ZF + AD, then the following hold in M:

- (InfBorel) All sets are ∞-Borel.
- (OrdDet) $<\Theta$-ordinal determinacy.
- $\mathrm{DC}_{\mathbb{R}}$.

Woodin eventually isolated the theory AD^+, which is now taken to be

$$\mathrm{ZF} + \mathrm{OrdDet} + \mathrm{InfBorel} + \mathrm{DC}_{\mathbb{R}}.$$

AD^+ is intended to axiomatize those sentences φ that hold in M where M is a transitive model of ZF containing $\mathbb{R}$ and such that every set of reals of M is Suslin in some possibly larger transitive model of ZF + AD with the same reals. This *downward absoluteness* is discussed below (see page 33 and Lemma 2.5). One immediate consequence is that AD+ all sets are Suslin $\implies \mathrm{AD}^+$. It is open whether $\mathrm{AD} \implies \mathrm{AD}^+$. The arguments contained here should illustrate how $\mathrm{DC}_{\mathbb{R}}$, ∞-Borel representations for sets of reals, and ordinal determinacy are used to investigate the structure of models of AD^+.

For facts about Descriptive Set Theory and models of AD refer to [**Mos09**] and to [**Jac10**]. For facts and references concerning AD^+ see [**Woo99, CK09, Ste94**]. I will define and discuss the Suslin sets and cardinals and ∞-Borel sets and codes in the following two subsections. I have placed many of the more technical facts involving descriptive set theory in models of AD in an appendix to be referred to when needed. The key results in this paper use forcing, ultrapowers/products, and absoluteness, the reader should not be deterred by the descriptive set theory that enters in now and again.

2.1. Suslin sets and cardinals. A *tree* on X is a subset of $X^{<\omega}$ closed under restriction. For T a tree on X, $[T]$ is the set of all infinite branches through T. Topologically, if X is a discrete space and X^ω is given the usual product topology, then the closed subsets of X^ω are of the form $[T]$ for some tree T on X. In particular the closed subsets of ω^ω or 2^ω are of this form. For $s \in X^{<\omega}$, $T_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

For T a tree on $X \times Y$, and $s \in X^{<\omega}$, $T_s = \{(t, u) \in T : t \subseteq s \vee s \subseteq t\}$ and for $f \in X^\omega$ set $T_f = \{u : \exists n\, (f|n, u) \in T\}$. The set

$$p[T] = \{f \in X^\omega : \exists g \in Y^\omega\, (f, g) \in [T]\} = \{f \in X^\omega : T_f \text{ is illfounded}\}$$

is the *projection* of T. Here I am identifying sequences $\langle (x_0, y_0), \ldots, (x_n, y_n)\rangle \in (X \times Y)^{\leq\omega}$ with sequences $s = (\langle x_0, \ldots, x_n\rangle, \langle y_0, \ldots, y_n\rangle) \in X^{\leq\omega} \times Y^{\leq\omega}$ such that $\mathrm{lh}(s_0) = \mathrm{lh}(s_1)$.

A set $A \subseteq X^\omega$ is *Y-Suslin* iff $A = p[T]$ for some tree on $X \times Y$.

We will only be interested in the case where Y is wellorderable. If $<_Y$ is a wellordering of Y and T is a tree on Y, with $[Y] \neq \emptyset$, then b^T is the *leftmost branch of T (w.r.t. $<_Y$)* and is defined by

$$b^T|i = <_Y^{\text{lex}}\text{-least } s \text{ such that } T_s \text{ is illfounded}$$

From now on just take Y to be an ordinal.

If T is a tree on $X \times \kappa$, then define $\varphi_i^T(x) = b^{T_x}(i)$ for $x \in p[T]$. So $\varphi_i^T(x) : p[T] \to \kappa$. The sequence $\{\varphi_i^T\}_i$ is the *semiscale associated to T*. In general we have

Definition 2.1. A sequence of functions, $\{\theta_i : A \to \mathrm{OR}\}_i$, is a *semiscale on A* iff whenever

(1) $x_i \in A$ for all $i \in \omega$,
(2) $\lim_i x_i = x$, i.e., $\langle x_i(j) : i \in \omega\rangle$ is eventually constant, and
(3) $\forall j \exists \lambda_j \forall^\infty i\, \theta_j(x_i) = \lambda_j$, i.e., $\langle \theta_j(x_i) : i \in \omega\rangle$ is eventually constant,

then $x \in A$. ⊣

In other-words, $\{\theta_i\}_i$ a semiscale on A iff $A = p[T^{\{\theta_i\}_i}]$, where

$$T^{\{\theta_i\}_i} \stackrel{\text{df}}{=} \{(x|n, \langle \theta_0(x), \dots, \theta_{n-1}(x)\rangle) : n \in \omega \wedge x \in A\}.$$

A semiscale is *regular* if for each i, $\theta_i : A \xrightarrow{\text{onto}} \kappa_i$. Any semiscale $\{\theta_i\}_i$ generates a regular semiscale $\{\theta_i\}_i$ by collapsing the range. If $\{\theta_i\}_i$ with $\theta_i : A \xrightarrow{\text{onto}} \kappa_i$ is a regular semiscale and $A \subseteq \mathbb{R}$, then $\kappa_i < \Theta$, being the rank of a prewellordering on A. Similarly, $\sup_i \kappa_i \leq \sum_i \kappa_i < \Theta$, since defining $(i, x) \leq (j, y) \stackrel{\text{df}}{\iff} i < j \vee [i = j \wedge \theta_i(x) \leq \theta_i(y)]$ is a prewellordering of $\omega \times A$. As a consequence, we see that if $A = p[T]$ for T a tree on $\omega \times \kappa$, then $A = p[T']$ for T' a tree on $\omega \times \kappa'$ with $\kappa' < \Theta$. In the future κ-Suslin will entail $\kappa < \Theta$.

Define $\boldsymbol{S}_\kappa$ to be the collection of all κ-Suslin sets. A set is *co-κ-Suslin* if its complement is κ-Suslin. $\boldsymbol{S}_\kappa$ is closed under continuous preimages, real existential quantification (projection), in fact closed under $\exists f \in \kappa^\omega$. Under $\mathrm{AD} + \mathrm{DC}_\mathbb{R}$, $\boldsymbol{S}_\kappa$ is also closed under countable union and intersection, where the coding lemma 8.4 is used in conjunction with $\mathrm{DC}_\mathbb{R}$ to pick a countable sequence of Suslin representations given a countable sequence of Suslin sets.

It might seem as though $\boldsymbol{S}_\kappa$ should be closed under κ-length wellordered unions; however this would require the ability to pick a κ-sequence of κ-Suslin representations and as we shall see, this amount of choice fails under AD.

A cardinal κ is a *Suslin cardinal* if $\boldsymbol{S}_\kappa \setminus \boldsymbol{S}_{<\kappa} \neq \emptyset$, where $\boldsymbol{S}_{<\kappa} = \bigcup_{\lambda<\kappa} \boldsymbol{S}_\lambda$. Notice κ a Suslin cardinal implies $\kappa < \Theta$. DC implies that the Suslin cardinals form an omega-club, since one needs only to pick a suitable sequence of trees. Define $\boldsymbol{\kappa}_\infty = \sup\{\kappa < \Theta : \kappa \text{ is a Suslin cardinal}\} \leq \Theta$ and $\boldsymbol{S}_\infty$ to be the set of all Suslin sets. Thus $\boldsymbol{\kappa}_\infty$ is a Suslin cardinal iff $\boldsymbol{S}_\infty \setminus \boldsymbol{S}_{<\boldsymbol{\kappa}_\infty} \neq \emptyset$. If $\boldsymbol{\kappa}_\infty < \Theta$, then $\mathrm{DC}_\mathbb{R}$ suffices to show that the Suslin cardinals form an ω-club. Again, the coding lemma is used to reduce DC to $\mathrm{DC}_\mathbb{R}$. Under $\mathrm{DC}_\mathbb{R}$ it is possible for $\mathrm{cf}(\boldsymbol{\kappa}_\infty) = \omega$, but only if $\boldsymbol{\kappa}_\infty = \Theta$.

Suslin subsets of $\mathbb{R}^2$ are uniformizable as follows. Suppose $A \subseteq \mathbb{R}^2$ with $A = p[T]$ for T a tree on $\omega^2 \times \kappa$, then define $A^*(x, y) \iff (b^{T_x})_0 = y$, that is, the left hand side of the leftmost branch through T_x is y. A^* is a uniformization of A. This shows that being Suslin entails a bit of choice. Since choice conflicts with determinacy, AD together with "there are a lot of Suslin sets", should be stronger than just AD and this is indeed the case.

Borel sets are precisely the ω-Suslin/co-ω-Suslin sets. Consider $\mathrm{WO}_\alpha = \{x : x$ codes a wellorder of rank $\alpha\}$. Each WO_α is Borel; hence has a ω-Suslin representation. However, assuming AD, there is no sequence of trees T_α (on any ordinal) witnessing WO_α is Suslin. If such a sequence existed, then setting $x_\alpha = b_0^{T_\alpha}$ would give an uncountable sequence of distinct reals. This example shows that the passage from Borel representation to Suslin representations is non-trivial. One of the main goals of this paper is to understand the passage from ∞-Borel code of a set to a Suslin representation, if such exists. Conversely, passing from Suslin to ∞-Borel representations is relatively straightforward; an old result of Sierpiński is that any co-κ-Suslin set is the κ^+-union of $<\kappa^+$-Borel sets and the passage from Suslin representation to Borel representation is simply definable.

We will need the following fact regarding the rank of λ-Suslin wellfounded relations.

Theorem 2.2 (Kunen-Martin [**Mos09, Jac10**]). *Let $\prec$ be a λ-Suslin wellfounded relation, then $|| \prec || < \lambda^+$.* $\dashv$

2.2. ∞-Borel sets. In section 2.2.3 of [**CK09**], ∞-Borel codes are defined as well as several equivalent notions. We need two of these here. The official definition takes an ∞-Borel code to be a wellfounded tree, T, which describes how to build a set of reals, A_T, via well ordered unions, complements, etc., beginning with basic open sets. Let BC_κ be the collection of ∞-Borel codes where the tree is on κ. It should be fairly clear that for $\alpha \geq \omega$, BC_α is closed under continuous substitution since given $\pi : \omega^\omega \to \omega^\omega$ continuous we just need for each $i, j \in \omega$ a code for $\pi^{-1}[\{x : x(i) = j\}]$.

Our official definition of ∞-Borel code is equivalent to considering the infinitary propositional calculus, $\mathcal{L}_\infty(\dot{x})$, with basic propositions $\dot{x}_{i,j}$ with intended interpretation being $\{x \in \omega^\omega : x(i) = j\}$, so instead of writing $\dot{x}_{i,j}$ I write $\dot{x}(i) = j$. Negation and wellordered conjunction/disjunction is allowed. The standard definition of $z \models S(\dot{x})$ is used and clearly $z \models S(\dot{x}) \iff z \in A_S$. I will utilize this notation and write "$S(z)$" in place of "$z \models S(\dot{x})$" or "$z \in A_S$" when useful. Similarly $\mathcal{L}_\infty(\dot{x}_0, \ldots, \dot{x}_{n-1})$ is used to describe subsets of $\mathbb{R}^n$ with $\mathcal{L}_\infty^{<\omega}(\{\dot{x}_i : i \in \omega\})$ being the union of the $\mathcal{L}_\infty^n$. This allows easy manipulation of variables to derive new codes from old and I will utilize this when useful.

There is a fixed Σ_1-formula, Φ, so that for $T \in \mathrm{BC}_\kappa$,

$$x \in A_T \iff L_{\alpha_{T,x}}[T, x] \models \Phi(T, x)$$

where $\alpha_{T,x}$ is the least α so that $L_\alpha(T, x)$ is a model of Kripke-Platek (KP) set theory [**Bar75**].

Consequently, one variant of the coding is to take a code to be a triple (α, φ, S), for $S \subseteq \mathrm{OR}$ and φ a formula of set theory, and sets

$$x \in A_{(\alpha,S,\varphi)} \iff L_\alpha[S, x] \models \varphi(S, x).$$

Call (α, φ, S) an ∞-Borel* code and let BC^*_κ be those codes (α, φ, S) with $S \subseteq \alpha \subseteq \kappa$. The relative sizes of these two notions of code and in which inner models they exists will be used.

Lemma 2.3. *For any κ, $\mathrm{BC}_\kappa \subseteq \mathrm{BC}^*_{\kappa^+}$ while for any κ such that $\omega\kappa = \kappa$, $\mathrm{BC}^*_\kappa \subseteq \mathrm{BC}_\kappa$.*

Proof. For $S \in \mathrm{BC}_\kappa$ let $\alpha_S = \sup_{x\in\mathbb{R}} \alpha_{S,x}$, then

$$x \in A_S \iff L_{\alpha_S}[S,x] \models \Phi(S,x).$$

Here we use the fact that Φ is Σ_1. Letting $\alpha_\kappa = \sup_{S\in\mathcal{P}(S)} \alpha_S$ and $\alpha_{S,x} < \kappa^+$ for $S \subseteq \kappa$ so $\alpha_S \leq \kappa^+$ and we have

$$\mathrm{BC}_\kappa \subseteq \mathrm{BC}^*_{\alpha_\kappa} \subseteq \mathrm{BC}^*_{\kappa^+}.$$

The other direction requires quite a significant detour and will be omitted here. The point is essentially that $J_\alpha[S]$ can "see" $J_\alpha[S,\dot{x}]$ (as a collection of "uninterpreted names/descriptions for sets"), where the J-hierarchy is Jensen's version of L (see [**Jen72**].) ❑

The following two lemmas say something about how far one must look to interpret a given ∞-Borel code and conversely how far one must look to find a code for a given ∞-Borel set. It would be good to consult Lemma 8.3 for the definitions and facts used concerning $\delta(A)$, $\Pi(A)$, and $\Delta(A)$.

Lemma 2.4. For any set A, $\mathcal{B}_{<\delta(A)} \subseteq \Delta(A)$.

Proof. If $S \in \mathrm{BC}_\kappa$ and $\leq$ is are prewellorder of rank κ, then $\delta(A)$ is measurable, hence regular, and $\Delta(A)$ is closed under $<\delta(A)$-wellordered unions, so $\mathcal{B}_{<\delta(A)} \subseteq \Delta(A)$. ❑

Lemma 2.5. Any ∞-Borel set, A, has a code in $\Delta(A)$, with respect to any $\Pi(A)$-norm on a complete $\Pi(A)$ set.

Proof. Let S be a code for A and consider the relation $\sim$ on BC_∞ given by $T \sim T' \iff A_T = A_{T'}$. Let $M = L[\sim, S]$ and look at $\mathrm{BC}^M_\infty/\sim$. If $\mathrm{BC}^M_{<\delta(A)} = \mathrm{BC}^M_\infty$, then there is a code S' of size $<\delta(A)$ with $S' \sim S$ and such a code is in $\Delta(A)$.

If $\mathrm{BC}^M_{<\delta(A)} \neq \mathrm{BC}^M_\infty$, then there is a $\delta(A)$-antichain, $\langle S_\alpha : \alpha < \delta(A)\rangle$. This gives a prewellorder $\leq^*$ of length $\delta(A)$ with a code $S^* = \bigvee_{\alpha\leq\beta} S_\alpha \times S_\beta$ in $\mathrm{BC}_{\delta(A)}$ and thus has a $\Delta(A)$ code. Clearly, $\leq^* \not\leq_w A$, since otherwise $\|\leq^*\| < \delta(A)$. Since $\leq^* \not\leq A$, either $A \leq_w \leq^*$ or $A \leq_w \not\leq^*$. Since BC_α is closed under continuous substitution for $\alpha \geq \omega$. ❑

This gives the downward absoluteness of being ∞-Borel.

It is shown in [**CK09**], under $\mathrm{AD} + \mathrm{DC}_\mathbb{R}$, or more precisely, assuming there is a countably complete fine ultrafilter on $\mathcal{P}_{\omega_1}(\mathbb{R})$ and $\mathrm{DC}_\mathbb{R}$, that A is ∞-Borel iff A appears in a model of the form $L(S,\mathbb{R})$ for some $S \subseteq \mathrm{OR}$. In fact Woodin has shown more:

Theorem 2.6 (Woodin)**.** Work in ZF. Suppose μ is a fine measure on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$ and $\prod \mathrm{OR}/\mu$ is wellfounded. Let $S \subseteq \mathrm{OR}$ and $A \subseteq \mathbb{R}$ with $A \in \mathrm{OD}^{L(S,\mathcal{P}(\gamma))}_{S,T}$ for $T \subseteq \gamma$, then A is ∞-Borel with code in $\mathrm{OD}^V_{S,T,\mu}$. ⊣

Assuming AD for each $\gamma < \Theta$ there is an OD, fine, σ-complete measure, μ_γ, on $\mathcal{P}_{\omega_1}(\mathcal{P}(\gamma))$. So μ can be dropped in the definability estimate for the code in this case.

This theorem shows that under $\mathrm{AD} + \mathrm{DC}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$, $L(\mathcal{B}_\infty, \mathbb{R}) \cap \mathcal{P}(\mathbb{R}) = \mathcal{B}_\infty$ so there is a largest model of $\mathrm{AD} + \mathrm{DC}_\mathbb{R} + \mathrm{InfBorel}$. The point is that if $\mathcal{B}_\infty \neq \mathcal{P}(\mathbb{R})$, then $\mathcal{B}_\infty = \mathcal{B}_\lambda$ for a $\lambda < \Theta$ and thus $\mathcal{B}_\infty \subseteq L(\mathcal{P}(\gamma))$.

3. Cone ultrafilters and ultraproducts.

Let

$$x \leq_S y \iff L[S,x] \subseteq L[S,y]$$

be the partial order of S-constructibility degrees. Let

$$x \equiv_S y \iff x \leq_S y \,\&\, y \leq_S x$$

be the corresponding equivalence relation with classes $[x]_S = \{y : x \equiv_S y\}$ and let $\mathcal{D}_S$ be the set of S-degrees. We will use $x \leq y$ to mean x is Turing reducible to y. Notice, in particular, that $x \leq_\emptyset y$ is different from $x \leq y$.

A set of reals, A, is *S-invariant* iff

$$x \in A \,\&\, x \equiv_S y \Longrightarrow y \in A$$

An S-invariant set can be viewed as a subset of $\mathcal{D}_S$. A function $f : \mathbb{R} \to V$ is S-invariant if

$$x \equiv_S y \Longrightarrow f(x) = f(y)$$

so f can be viewed as $f : \mathcal{D}_S \to V$. More generally, for a formula φ, define φ is S-invariant iff

$$\boldsymbol{x} \equiv_S \boldsymbol{y} \Longrightarrow \big(\varphi(\boldsymbol{x}) \iff \varphi(\boldsymbol{y})\big).$$

A set, A, of reals *contains an S-cone* iff $\exists x_0\, \forall x \geq_S x_0\, (x \in A)$. I will write this as $\forall^*_S x\, (x \in A)$. More generally, write $\forall^*_S x\, \varphi(x)$ for a formula φ and say φ *holds on an S-cone*, if $\exists x_0 \forall x \geq_S x_0\, \varphi(x)$.

The collection of sets of reals containing S-cones forms a σ-complete filter under $\mathrm{DC}_{\mathbb{R}}$, denoted μ_S and called the S-cone filter or Martin measure on S-degrees. μ will denote the Martin measure on Turing degrees, so μ and $\mu_\emptyset$ are distinct.

Theorem 3.1 (Martin). μ_S when restricted to S-invariant sets is an ultrafilter.

Proof. Let A be S-invariant. Play the game where I plays x and II plays y (bit by bit). Player II wins if $y \geq_S x$ and $y \in A$. If I wins with strategy σ, then when II plays $y \geq_S \sigma$, $x = \sigma(y) \notin A$ since $x \leq_S \sigma \oplus y \equiv_S y$ and A is S-invariant. This shows that the S-cone above σ is contained in $\neg A$.

If σ is a II winning strategy, then for $x \geq_S \sigma$, $\sigma(x) \in A$, but $\sigma(x) \equiv_S x$ since $\sigma(x) \geq_S x$, so $x \in A$ by the S-invariance of A. ❑

Clearly for S-invariant φ

$$\forall^* x\, \varphi(x) \iff \forall^*_S x\, \varphi(x)$$

and we will use this without mention throughout. In particular we do not have to be careful about using the expression "S-cone" and can just use "cone" in most cases.

We give a partial order to $\mathcal{P}(\mathrm{OR})$ as follows:

$$S \preceq T \iff \forall^* x\, (L[S,x] \cap \mathbb{R} \subseteq L[T,x])$$
$$S \approx T \iff \forall^* x\, (L[S,x] \cap \mathbb{R} = L[T,x] \cap \mathbb{R})$$

I will refer to $S/\approx$ as the *degree notion corresponding to S*.

When looking at reduced products of S-invariant functions, whether μ or μ_S is used is irrelevant; what does matter is the class of functions used. For S-invariant $f : \mathbb{R} \to \mathrm{OR}$ define $[f]_S$ recursively by

$$[f]_S = \prod_S f/\mu = \{[g]_S : g \text{ is } S\text{-invariant} \,\&\, \forall^* x\, g(x) < f(x)\}$$

One gets the same notion if μ is replaced by μ_S and $\forall^*$ is replaced by $\forall^*_S$.

If $S \preceq T$, then $\pi_{S,T}(\mu_S) = \mu_T$, where $\pi_{S,T} : \mathcal{D}_S \to \mathcal{D}_T$ is given by $\pi_{S,T}([x]_S) = [x]_T$, this is extended to the measures by defining $X \in \pi_{S,T}(\mu_S) \overset{\text{df}}{\iff} \pi^{-1}_{S,T}[X] \in \mu_S$. If f is S-invariant and $S \prec T$, then $[f]_T \leq [f]_S$, but equality will not hold in general, for example, LEMMA 3.3 shows $\prod_T \omega_1^{L[T,x]} = \omega_1^V$ while $\prod_S \omega_1^{L[T,x]} > \prod_S \omega_1^{L(S,x)}$, since $\omega_1^{L(S,x)} < \omega_1^{L(T,x)}$ on a S-cone of x by THEOREM 4.4.

I will assume DC^-, the statement that $\prod_\mu \mathrm{OR}/\mu$ is wellfounded, for the remainder of the paper. It is a non-trivial fact due to Woodin that DC^- follows from AD^+. It is clear that DC^- sits somewhere between $\mathrm{DC}_\mathbb{R}$ and DC. Since all of the other measures we use reduce to μ, DC^- gives $\prod_S \mathrm{OR}/\mu_S$ is wellfounded. Assuming DC^- could be avoided throughout a large portion of the paper. Assuming $\mathrm{DC}_\mathbb{R}$ we have that $L(A, \mathbb{R})$ is a model of DC and where we can get by working locally in a model of the form $L(A, \mathbb{R})$, we could get by also with just $\mathrm{DC}_\mathbb{R}$.

The notation $[f]_S$ can be extended to S-invariant functions $f : \mathbb{R} \to \mathcal{P}(\mathrm{OR})$ and more generally to the situation where we have S-invariant map $x \mapsto (M^x, \lhd^x)$ where M^x is a transitive (set- or proper class-sized) structure which carries a natural well ordering $\lhd^{M^x}$. Form $\prod_S M^x/\mu$ using S-invariant functions f such that $f(x) \in M^x$ on a cone of x. This will yield

$$M^\infty_S = \prod_S M^x/\mu$$

a transitive structure with well ordering $\lhd^{M^\infty}$ and Łos's Lemma will hold:

$$M^\infty_S \models \varphi([f]_S) \iff \forall^* x\, M^x \models \varphi(f(x)),$$

for all formulas φ. This uses DC or $\mathrm{DC}_\mathbb{R}$ in some appropriate model as described above.

Let $[T]_S$ be the object corresponding to the constant function with value T, and let $j_S : \mathcal{P}(\mathrm{OR}) \to \mathcal{P}(\mathrm{OR})$ be the function $T \mapsto [T]_S$. The critical point of this embedding is ω_1^V.

Extending slightly the notation from [**CK09**], for $S \preceq T$, set

$$H^x(T) = \mathrm{HOD}_T^{L[T,x]}$$

and set

$$H^\infty_S(T) = \prod_S H^x(T)/\mu.$$

More generally, I will use S^∞ for $j_S(S)$ and T^∞_S ambiguously for $j_S(T)$ or $[x \mapsto T^x]_S$ where $x \mapsto T^x$ is S-invariant.

We will see below that $H^\infty_S(T)$ can be viewed as $H^{x^\infty_S}(T^\infty_S) = \mathrm{HOD}_{T^\infty_S}^{L[T^\infty_S, x^\infty_S]}$ for a generic real x^∞_S which can be viewed as a kind of Sacks generic over V.

Set $\delta^x_S = \omega_2^{L[S,x]}$ and $\delta^\infty_S = \prod_S \delta^x_S/\mu$. Recall from [**CK09**] that on a cone of x, GCH holds below ω_1^V in $L[S,x]$, ω_1^V is inaccessible in $L[S,x]$, and δ^x_S is inaccessible (in fact Woodin [**KW10**]) in $H^x(S)$. Let GCH^* denote "GCH holds below the least inaccessible" (ω_2, "least measurable", etc. would work just as well as "least inaccessible" here.) We will primarily use

$$\mathrm{GCH}^* \implies 2^\omega = \aleph_1 \text{ and } 2^{\omega_1} = \aleph_2.$$

We will show below that δ^∞_S depends on S, see THEOREM 5.16 and COROLLARY 6.4. In contrast, we have $\prod_S \omega_1^{L[S,x]} = \omega_1^V$. First we need the following:

Lemma 3.2. Suppose $f : \mathbb{R} \to \mathrm{OR}$ is S-invariant, then either f is monotonically increasing on a cone, i.e., for a cone of x, $y > x \to f(y) > f(x)$, or else, f is constant on a cone.

Proof. If monotonically increasing fails on a cone, then on a cone of x there is $y \geq x$, with $f(y) \leq f(x)$. Since there is not an infinite descending sequence of ordinals we get a cone on which f is constant. ❑

Lemma 3.3. For all S, $\prod_S \omega_1^{L[S,x]}/\mu = \omega_1^V$.

Proof. Let $f \in \prod_S \omega_1^{L[S,x]}$. If f is constantly α on a cone, then $[f]_S = \alpha$, so suppose f is monotonic increasing (see Lemma 3.2). Consider the game where I plays x and II plays y, z. II wins iff $y \geq_S x$ and $z \in \mathrm{WO}^{L[S,y]}$ codes $f(y)$, where WO is just the collection of reals that code ordinals below ω_1.

Suppose I follows a strategy σ and II plays y, z with y above σ and such that y is in the cone on which f is monotonic, and $z \in \mathrm{WO}^{L[S,y]}$ coding $f(y)$. This is clearly a win for II, so no strategy is winning for I.

Let σ be a II winning strategy and set $Y = \{(\sigma(x))_1 : x \in \mathbb{R}\}$. This is a $\utilde{\Sigma}^1_1$ subset of WO and hence $Y \subseteq \mathrm{WO}_\alpha$ for some $\alpha < \omega_1$. Let $x \geq_S \sigma$, then $(\sigma(x))_0 \equiv_S x$ and $f(x) = f((\sigma(x))_0) = ||(\sigma(x))_1||$. This shows that on a cone of x, $f(x) < \alpha$, but then f must be constant on a cone. ❑

Viewing μ_S as a filter on $\mathcal{P}(\mathbb{R})$ rather than an ultrafilter on S-invariant sets we have $A \subseteq \mathbb{R}$ is *S-positive* provided $\forall^* x\, [x]_S \cap A \neq \emptyset$. Equivalently, A is S-positive if $[A]_S$ contains a cone. Let $\mathbb{P}_S$ be the notion of forcing with conditions being S-positive sets and with $A \leq_{\mathbb{P}_S} B$ iff $A \subseteq B$. $\mathbb{P}_S$-generics are V-ultrafilters on $\mathcal{P}(\mathbb{R})^V$. The next lemma shows that the map $j_G : V \to \mathrm{ult}(V, G)$ agrees with j_S on $\mathcal{P}(\mathrm{OR})$. Recall $\mathrm{ult}(V, G)$ is formed in $V[G]$ using functions $f : \mathbb{R} \to V$ in V. Without choice in V, j_G need not be elementary and we will be more interested in ultraproducts of canonically well-ordered structures like $L[T, x]$ where $T \subseteq \mathrm{OR}$.

Lemma 3.4. For $G \subseteq \mathbb{P}_S$ generic, the map $k([f]_S) = [f]_G$ is an isomorphism of $\prod_S \mathrm{OR}/\mu$ with $\prod_\mathbb{R} \mathrm{OR}/G$.

Proof. It is clear that $k([f]_S) = [f]_G$ is an embedding. We want to see that k is onto. Let $f : A \to \mathrm{OR}$ for A S-positive. Let $B \leq_{\mathbb{P}_S} A$ and define $\hat{f} : [B]_S \to \mathrm{OR}$ by $\hat{f}([x]_S) = \inf f[[x]_S \cap B]$.

Define $C = \{x \in B : f(x) = \inf f[[x]_S \cap B]\}$, then C is an S-positive subset of B and $C \Vdash_{\mathbb{P}_S} k([\hat{f}]_S) = [f]_G$. Since $B \leq_S A$ is arbitrary, $A \Vdash [f]_G \in \mathrm{rng}(k)$. ❑

If G is $\mathbb{P}_S$-generic, then define

$$x_S^\infty(G) = \bigcup \{s : [s] \in G\}$$

where $[s] = \{x \in \mathbb{R} : x \supset s\}$. When G is understood, I will simply write x_S^∞. We have, by Łos's Lemma, that

$$\prod_\mathbb{R} \langle L[T, x], H^x(T), T, x\rangle/G = \langle L[T_S^\infty, x_S^\infty], H_S^\infty(T), T_S^\infty, x_S^\infty\rangle$$

and

$$L[T_S^\infty, x_S^\infty] \models H_S^\infty(T) = \mathrm{HOD}_{T_S^\infty}^{L[T_S^\infty, x_S^\infty]}$$

In particular, $H_S^\infty(T) = H^{x_S^\infty}(T_S^\infty)$.

We will use the fact that $\mathbb{P}_S$ can be recast as a version of Sacks forcing. Call a tree a on ω *S-pointed perfect* iff a is perfect and, identifying a with a real in a natural way, $a \leq_S x$ for every branch $x \in [a]$. A proof of the following appears in **[Kec88]**.

Theorem 3.5 (Martin)**.** For $A \subseteq \mathbb{R}$, A is S-positive iff there is an S-pointed perfect tree a such that $[a] \subseteq A$.

Proof. Players I and II play x and y respectively. Player II wins if $x \leq_S y$ and $y \in A$. If σ is a winning strategy for II, then $\sigma[\mathbb{R}]$ contains a perfect subset and a tree, a, witnessing this can be found in $L[S,\sigma]$. (In fact a is very simply definable from σ, but one still needs to go to $L[S,y]$ to compute a from $y \in [a]$.) One way to see this as follows. In $L[S,\sigma]$ define $s \mapsto t_s, n_s$ where $t_s = \big(x_s \oplus \sigma\big)|n_s$ for some x_s subject to the constraints that

(1) $\mathrm{lh}(s) = \mathrm{lh}(s') \to n_s = n_{s'}$
(2) $s \perp s' \to \sigma(t_s) \perp \sigma(t'_s)$

Let a be the tree $\{\sigma(t_s) : s \in 2^{<\omega}\}$. For $y \in [a]$ there is $b \in 2^\omega$ and $x_{b|i}$ so that $\lim_i x_{b|i} \oplus \sigma = x_b \oplus \sigma$ and $y = \sigma(x_b \oplus \sigma)$. Since σ is winning for II, $x_b \oplus \sigma \leq_S y$ so in particular $\sigma \leq_S y$ for $y \in [a]$ and so $a \leq_S y$ since $a \leq_S \sigma$.

If player I wins with σ, then if $y \geq_S \sigma$ we have $y \geq_S \sigma(y)$ so $y \notin A$. This shows $\neg A$ contains a cone. ❑

So forcing with $\mathbb{P}_S$ is equivalent to the version of Sacks forcing where S-pointed perfect trees are used, call this $\mathbb{S}_S$. If G is a generic ultrafilter of S-positive sets, then $x_S^\infty(G)$ as defined previously is the corresponding Sacks generic.

The next theorem will be used later and is our main application of the fact that $\mathbb{P}_S$ is essentially S-pointed perfect Sacks forcing, the point being that literally S is not necessarily in $L(S^\infty, \mathbb{R})$. By passing to the S-pointed Sacks forcing, however, we can see that if x_S^∞ is $\mathbb{S}_S$ generic, then x_S^∞ is $\mathbb{S}_{S^\infty}$ generic over $L(S^\infty, \mathbb{R})$.

Theorem 3.6. Let x_S^∞ be $\mathbb{P}_S$-generic, then $\delta_S^\infty \leq \Theta$ and x_S^∞ is $\mathbb{S}_{S^\infty}$-generic over $L(S^\infty, \mathbb{R})$.

Proof. First notice that $S^\infty \approx S$ since for any $y, x \in \mathbb{R}$:

$$y \in L[S^\infty, x] \Longleftrightarrow \forall^* z\big(y \in L[S,x]\big) \leftrightarrow y \in L[S,x]$$

So, while $L(S^\infty, \mathbb{R})$ might not see S, it does see the corresponding S^∞-pointed perfect forcing and this is equivalent to S-pointed perfect forcing. So x_S^∞ is $\mathbb{S}_{S^\infty}$-generic over $L(S^\infty, \mathbb{R})$.

In $L(S^\infty, \mathbb{R})[x_S^\infty]$ there is no map from $\mathbb{R}^V$ onto Θ^V since if $\dot\tau$ were a name for such a map, look at $B_\alpha = \{(a,x) : a \Vdash \dot\tau(x) = \alpha\}$. The sequence B_α determines a prewellordering, $\leq \overset{\mathrm{df}}{=} \bigcup_{\alpha \leq \beta < \Theta^V} B_\alpha \times B_\beta$, in V, of length Θ^V, which cannot exist.

If $\Theta < \delta_S^\infty = \omega_2^{L[S^\infty, x_S^\infty]}$, then $|\Theta|^{L[S^\infty, x_S^\infty]} = \omega_1^{L[S^\infty, x_S^\infty]} = \omega_1^V$ by Lemma 3.3. But, then $L(S^\infty, \mathbb{R})[x_S^\infty] \models |\Theta| = \omega_1^V$ contradicting the previous paragraph, since $L(S^\infty, \mathbb{R})$ has a map from $\mathbb{R}^V$ onto ω_1^V. ❑

4. Compatibility of constructibility degrees.

In this section we reproduce Woodin's proof that for any two sets of ordinals S and T either $L[S,x]$ sees a large initial segment of $L[T,x]$ or else $L[T,x]$ sees a large initial segment of $L[S,x]$ in $H(\omega_1)^{L[T,x]}$. For this Woodin utilized a generalized notion of Prikry forcing.

Work in ZF and let U be an ultrafilter on a set X. A condition in $\mathbb{P}_U$ is a pair $p = (p_0, p_1)$ where $p_0 \in X^{<\omega}$ and $p_1 : X^{<\omega} \to U$ with order defined by $p \leq_{\mathbb{P}_U} q$ iff

(1) $p_0 \supseteq q_0$,
(2) for all $i \in \mathrm{dom}(p_0) \setminus \mathrm{dom}(q_0)$, $p_0(i) \in q_1(p_0 \restriction i)$, and
(3) $p_1(t) \subseteq q_1(t)$ for all $t \in X^{<\omega}$.

We may, and do, work with conditions with the property that for all subsequences s of t, $p_1(t) \subseteq p_1(s)$. (Here I mean subsequences, not initial segments.) Only finite additivity is required. This is almost a tree forcing with p_1 determining U-large splitting above a trunk p_0, however, for the Mathias Condition below we require p_1 to be defined on $X^{<\omega}$ and not just a subtree. Using finite additivity of the measure it is clear that any two conditions p, p' with $p_0 = p_0'$ are compatible.

A generic, G, determines an element, g, of X^ω with $g = \bigcup\{p_0 : p \in G\}$. Conversely, $G = \{p : p_0 \subseteq g \text{ and } \forall i > |p_0|\, g(i) \in p_1(g \restriction i)\}$. We shall refer to g as the generic sequence from X.

Two key properties of Prikry forcing extend to this setting, these are the *Prikry Property* and the *Mathias Property*. We follow Woodin and use a rank function in the proofs. In the case of the Prikry Property, the rank function turns out to only take the value 0 or ∞.

Theorem 4.1 (Prikry Property)**.** Given any condition p and sentence φ of the forcing language, there is a condition $p' \leq_{\mathbb{P}_U} p$ such that $p_0 = p_0'$ and p' decides φ.

Proof. It suffices to show that there is some f so that $(\emptyset, f)$ decides φ, for then we can take $p_1'(t) = f(t) \cap p_1(t)$.

Define a rank function $\rho_\varphi : X^{<\omega} \to \mathrm{OR} \cup \{\infty\}$

$$\begin{aligned}
\rho_\varphi(t) = 0 &\iff \text{there is } p \text{ such that } p_0 = t \text{ and } p \Vdash \varphi \\
\rho_\varphi(t) = \alpha &\iff \{x : \rho_\varphi(tx) < \alpha\} \in U \text{ and } \rho_\varphi(t) \not< \alpha \\
\rho_\varphi(t) = \infty &\iff \rho_\varphi(t) \neq \alpha \text{ for any } \alpha
\end{aligned}$$

Define

$$f_\varphi(s) = \begin{cases} \{x : \rho_\varphi(sx) = 0\} & \text{if } \rho_\varphi(s) = 0 \\ \{x : \rho_\varphi(sx) < \rho_\varphi(s)\} & \text{if } 0 < \rho_\varphi(s) < \infty \\ \{x : \rho_\varphi(sx) = \infty\} & \text{if } \rho_\varphi(s) = \infty \end{cases}$$

So $(\emptyset, f_\varphi) \in \mathbb{P}_U$ and we will see $(\emptyset, f_\varphi)$ decides φ.

If $\rho_\varphi(\emptyset) < \infty$, then take $q \leq_{\mathbb{P}_U} (\emptyset, f_\varphi)$ with q deciding φ. For $i \in \mathrm{dom}(q_0)$, $q_0(i) \in f_\varphi(q_0 \restriction i)$, so $\rho_\varphi(q_0) \neq \infty$, so $\rho_\varphi(q_0|i) \geq \rho_\varphi(q_0|i+1)$ and equality only occurs if both values are 0. We can extend q_0 to q_0' so that $\rho_\varphi(q_0') = 0$ and set $q' = (q_0', q_1)$ so $q' \leq_{\mathbb{P}_U} q$ and $q' \Vdash \varphi$. This means that $q \Vdash \varphi$.

If $\rho_\varphi(\emptyset) = \infty$, then take $q \leq_{\mathbb{P}_U} (\emptyset, f_\varphi)$ which decides φ. In this case it must be that $q \Vdash \neg\varphi$ since $\rho_\varphi(q_0) = \infty$. In this case $(\emptyset, f_\varphi) \Vdash \neg\varphi$. ❑

Clearly the proof shows that for all t

$$\begin{aligned}
\rho_\varphi(t) = 0 &\iff \{x : \rho_\varphi(tx) = 0\} \in U \\
\rho_\varphi(t) = \infty &\iff \{x : \rho_\varphi(tx) = \infty\} \in U
\end{aligned}$$

So for all t, either $\rho_\varphi(t) = 0$ or $\rho_\varphi(t) = \infty$.

Notice we have shown that to each φ there is a canonical f_φ so that letting $p_{\varphi,t} = (t, f_\varphi)$, $\rho_\varphi(t) < \infty \to p_{\varphi,t} \Vdash \varphi$ and $\rho_\varphi(t) = \infty \to p_{\varphi,t} \Vdash \neg\varphi$. The map $t \mapsto f_\varphi$ requires no choice.

Say that $g \in X^\omega$ has the Mathias Condition if for any $(\emptyset, f) \in \mathbb{P}_U$, there is i such that $\forall j \geq i, g(j) \in f(g \restriction i)$.

Theorem 4.2 (The Mathias Property)**.** Assuming some choice, any $g \in X^\omega$ with the Mathias condition is generic.

Proof. The proof is as in the Prikry property. Define a rank function $\rho_D(t)$ for D an open dense set. The base case is

$$\begin{aligned}
\rho_D(t) = 0 &\iff \exists p \in D\,\bigl(p_0 = t\bigr)\\
\rho_D(t) = \alpha &\iff \{x : \rho_D(tx) < \rho_D(t)\} \in U \text{ and } \rho_D(t) \not< \alpha\\
\rho_D(t) = \infty &\iff \rho_D(t) \text{ is undefined}
\end{aligned}$$

Define f_D analogous with f_φ above, except when $s \in D^*$, where

$$s \in D^* \overset{\text{df}}{\iff} \rho_D(s) = 0 \wedge \forall i \in \operatorname{dom}(s)\, \rho_D(s|i) > 0$$

Here we need enough choice to choose one member of $\mathbb{P}_u$ for each member of D^*.

$$f_D(s) = \begin{cases} f_t(s) & \text{if } t \subseteq s \wedge t \in D^* \\ \{x : \rho_D(sx) < \rho_\varphi(s)\} & \text{if } 0 < \rho_D(s) < \infty \\ \{x : \rho_D(sx) = \infty\} & \text{if } \rho_D(s) = \infty \end{cases}$$

First we show that $\rho_D(t) \neq \infty$ for all t. If $\rho_D(t) = \infty$ let $p_{D,t} = (t, f_D)$. Let $q \leq_{\mathbb{P}_U} p_{D,t}$ with $q \in D$. Then $\rho_D(q_0) = 0$; however, the fact that $q \leq_{\mathbb{P}_U} p_{D,t}$ implies $\rho_D(q_0) = \infty$. This contradiction shows $\rho_D(t) \neq \infty$ to begin with.

The amount of choice here is $\mathrm{AC}_{\mathbb{P}_U}^{D^*}$ and $|\mathbb{P}_U| = \left|\left(2^X\right)^{X^{<\omega}}\right| = 2^{X^{<\omega}}$ so $\mathrm{AC}_{2^{X^{<\omega}}}^{X^{<\omega}}$ suffices. For $X = \omega$, which is the case we use, $\mathrm{AC}_{\mathbb{R}}^{\omega}$ is what is requires. This is weaker than $\mathrm{DC}_{\mathbb{R}}$.

Fix i so that for all $k \geq i$, $g(k) \in p_{D,1}(g \restriction k)$. By definition of f_D we have that $\rho_D(g \restriction k+1) < \rho_D(g \restriction k)$ if $\rho_D(g \restriction k) \neq 0$. Thus $\rho_D(g \restriction k) = 0$ for some $k \geq i$. We have for all $l \geq k$, $g(l) \in f_{g\restriction k}(g \restriction l)$ and thus $(g \restriction k, f_{g\restriction k}) \in D$ so $G_g \cap D \neq \emptyset$. ❑

From the Mathias property it follows that if g is $\mathbb{P}_U$-generic and g' is any infinite subsequence of g, then g' is also $\mathbb{P}_U$-generic. This is why we restricted $\mathbb{P}_U$ to conditions satisfying $p_1(s) \subseteq p_1(t)$ for t a subsequence of s.

These facts are all that we shall require concerning generalized Prikry forcing.

For the next theorem we need the following lemma due to Hausdorff.

Lemma 4.3. There is a recursive Lipschitz continuous $\pi : 2^\omega \to [\omega]^\omega$ so that if $x_1, \ldots, x_n$ are distinct elements of 2^ω, then $\pi(x_1), \ldots, \pi(x_n)$ are mutually independent.

Proof. $a_0, \ldots, a_{n-1} \in [\omega]^\omega$ are *mutually independent* if $\bigcap_{i<n} a_i^{s(i)}$ is infinite for all $s \in 2^n$ where $a_i^1 = a_i$ and $a_i^0 = \omega \setminus a_i$. One way to accomplish this is to define $\pi_n : 2^n \to [m_n]^{<\omega}$ so that these functions cohere reasonably, i.e., $\pi_{n+1}(t) \cap m_n = \pi_n(t|n)$ and

$$\left|\bigcap_{s \in 2^n} \pi_n(s)^{t(s)}\right| \geq n$$

where $t \in 2^{2^n}$, $\pi_n(s)^1 = \pi_n(s)$, and $\pi_n(s)^0 = m_n \setminus \pi_n(s)$. Suppose π_n is defined, then we take $m_{n+1} = m_n + 2^{2^{n+2}}$ and enumerate all $\sigma : 2^{n+1} \to 2$ by $i \in [m_n, m_{n+1})$. For σ_i we put $i \in \pi_{n+1}(t)$ iff $\sigma_i(t) = 1$. ❑

Define the filter $\mathcal{F}_\pi$ as follows:

$$a \in \mathcal{F}_\pi \iff \{y : a \supseteq^* \pi(y) \vee a \supseteq^* \omega \setminus \pi(y)\} \text{ is infinite}$$

where $a \supseteq^* b$ iff $|b \setminus a| < \omega$. Let $a_1, \dots, a_n \in \mathcal{F}_\pi$ and let $y_1, \dots, y_n$ be so that $a_i \supset^* \pi^{n_i}(y_i)$ where $n_i \in 2$ and $\pi^1(y_i) = \pi(y_i)$ while $\pi^0(y_i) = \omega \setminus \pi(y_i)$. Now $\bigcap a_i \supseteq^* \bigcap \pi^{n_i}(y_i)$ and $\bigcap \pi^{n_i}(y_i)$ is infinite. So $\mathcal{F}_\pi$ generates a filter on $\mathcal{P}(\omega)$ all of whose elements are infinite. The following characterizes being $\mathcal{F}_\pi$-positive:

$$a \text{ is } \mathcal{F}_\pi\text{-positive} \iff \{y : a \subseteq^* \pi(y) \vee a \subseteq^* \omega \setminus \pi(y)\} \text{ is finite.}$$

So being $\mathcal{F}_\pi$-positive is Π^1_1 in a fixed parameter $y_1, \dots, y_n$ and thus is absolute.

The following theorem is the main theorem of this section. The theorem states that either (on a cone) $L[S,x]$ and $L[T,x]$ agree for quite a while, or else, one of the models is "much" larger than the other (on a cone). Let κ^S_x be the least inaccessible of $L[S,x]$ (or $\omega_2^{L[S,x]}$, or least Mahlo of $L[S,x]$, or any other uniformly, in x, definable "large" cardinal.)

Theorem 4.4. Let S and T be sets of ordinals. Then one of the following hold on a Turing cone:

(1) $\kappa = \kappa^S_x = \kappa^T_x$ and $H(\kappa^+)^{L[S,x]} = H(\kappa^+)^{L[T,x]}$ or
(2) $L[S,x] \cap \mathcal{P}(\kappa^S_x) \in H(\omega_1)^{L[T,x]}$, or else,
(3) $L[T,x] \cap \mathcal{P}(\kappa^T_x) \in H(\omega_1)^{L[S,x]}$, or

Proof. First suppose $\{x : L[S,x] \cap \mathbb{R} \subseteq L[T,x]\}$ contains a cone. Let x_0 be the base of such a cone and $x \geq x_0$. Let g be $L[T,x]$ generic for collapsing κ^S_x to ω. Since every $a \subseteq \kappa^S_x$ can be coded by a real, a_g, using g, we essentially have, $L[S,x] \cap \mathcal{P}(\kappa^S_x) \subseteq L[S,x,g] \cap \mathbb{R} \subseteq L[T,x,g]$. Since g is any $L[T,x]$-generic (in V) we have $L[S,x] \cap \mathcal{P}(\kappa^S_x) \subseteq L[T,x]$. Notice that this also shows $\kappa^S_x \leq \kappa^T_x$.

If it is the case that $\{x : L[S,x] \cap \mathbb{R} = L[T,x] \cap \mathbb{R}\}$ contains a cone, then the same argument shows that on a cone, $\kappa = \kappa^S_x = \kappa^T_x$ and $H(\kappa^+)^{L[T,x]} = H(\kappa^+)^{L[S,x]}$.

If it is not the case that $\{x : L[S,x] \cap \mathbb{R} = L[T,x] \cap \mathbb{R}\}$ contains a cone, then either on a cone $\{x : L[S,x] \cap \mathbb{R} \not\subseteq L[T,x]\}$ or on a cone $\{x : L[T,x] \cap \mathbb{R} \not\subseteq L[S,x]\}$. Suppose the former.

Fix x_0 so that for $x \geq x_0$, $L[S,x] \cap \mathbb{R} \not\subseteq L[T,x]$. Fix π as in the previous lemma. Let U be an ultrafilter in $L[T,x]$ extending the filter $\mathcal{F}_\pi^{L[T,x]}$. For $a \in U$ are positive and since this is absolute a is positive in V. So if $z \notin L[S,x]$, $\pi(z) \cap a$ and $a \cap \pi(z)$ are both infinite for all $a \in U$.

Build g, $\mathbb{P}_U^{L[T,x]}$ generic over $L[T,x]$ such that $g \cap \pi(z)$ and $g \cap \omega \setminus \pi(z)$ are both infinite. For any $b \in 2^\omega$ we can shrink g to g_b so that $b(i) = 1 \iff g_b(i) \in \pi(z)$. In this way we get $b \in L[S,x,g_b]$. Of course b can be chosen so as to code $L[T,x_0] \cap \mathcal{P}(\kappa^T_x)$ and thus $L[T,x] \cap \mathcal{P}(\kappa^T_x) \in L[S,x,g_b]$ and countable there.

Consider $L[T,x,g_b]$. In this model g_b is $\mathbb{P}_U$-generic so $\kappa = \kappa^T_x = \kappa^T_{x,g_b}$ and $H(\kappa^+)^{L[T,x]}[g_b] = H(\kappa^+)^{L[T,x,g_b]}$ so $H(\kappa^+)^{L[T,x,g_b]} \in H(\omega_1)^{L[S,x,g_b]}$. We have shown that either (2) or (3) hold for some $y \geq x$, for all $x \geq x_0$ ($y = x \oplus g_b$ above). This implies that either (2) or (3) holds on a cone. ❑

Corollary 4.5. (ZF + AD) For all $S, T \subseteq \mathrm{OR}$

$$S \preceq T \to \text{ on a cone of } x, L[S,x] \cap \mathcal{P}(\kappa_x^S) \subseteq L[T,x]$$
$$S \prec T \to \text{ on a cone of } x, L[S,x] \cap \mathcal{P}(\kappa_x^S) \in H(\omega_1)^{L[T,x]}$$

From here on $\kappa_S^x = \delta_S^x = \omega_2^{L[S,x]}$, so I will dispense with κ_S^x in favor of δ_S^x.

5. Constructibility degrees and the structure of AD models

5.1. Uniformization and the non-existence of a maximal degree notion. *Uniformization* (UNIF) is the assertion that for all $R \subseteq \mathbb{R} \times \mathbb{R}$, there is R^* so that

$$\exists y\, R(x,y) \iff \exists! y\, R^*(x,y).$$

$\mathrm{UNIF}(\Gamma, \Gamma')$ is the same assertion, except R is taken from the pointclass Γ and R^* can be found in the pointclass Γ'.

The relation R is *uniformizable on a cone* means that there is R^* so that for a cone of x,

$$\exists y\, R(x,y) \iff \exists! y\, R^*(x,y).$$

Write $\mathrm{UNIF}^*(\Gamma, \Gamma')$ if for all relations R in Γ, there is a relation in Γ' that uniformizes R on a cone. The following shows that there is essentially no difference between uniformization and uniformization on a cone.

Lemma 5.1. For Γ and Γ' pointclasses

$$\mathrm{UNIF}(\Gamma, \Gamma') \iff \mathrm{UNIF}^*(\Gamma, \Gamma')$$

Proof. Let $R \in \Gamma$ and let $R^* \in \Gamma'$ uniformize R on a cone. Set

$$S(u \oplus x, y) \iff R(x,y)$$

So $S \in \Gamma$. Let $S^* \in \Gamma'$ uniformize S on the cone above a. Then $R^*(x,y) \iff S^*(a \oplus x, y)$ uniformizes R. ❑

Because of this fact I will not distinguish between uniformization and uniformization on a cone.

The method of proof in Theorem 4.4 can be extended to yield uniformization on ∞-Borel relations whose codes are not maximal. The general idea is as follows, suppose $S \prec T$ and S is an ∞-Borel code of a relation. We will show that on a cone of x

$$\exists y\, A_S(x,y) \iff \exists y \in L[T,x](A_S(x,y))$$

This gives a canonical uniformization of A_S using the natural well ordering of $L[T,x]$. For this we need to review the construction of the ∞-Borel code $\exists^{\mathbb{R}} S$ for $\exists^{\mathbb{R}} A_S$ described in [**CK09**] and reviewed below. It turns out that for the argument $S \prec T$ does not suffice, we must replace S with S_*^∞ described below.

For each z consider the "Vopenka-like" algebra

$$\mathbb{Q}^z(S) \stackrel{\mathrm{df}}{=} \mathrm{BC}_\infty^{H^z(S)} / \sim_S^z$$

where for $T, T' \in \mathrm{BC}_\infty^{H^z(S)}$,

$$T \sim_S^z T' \iff (A_T)^{L[S,z]} = (A_{T'})^{L[S,z]}.$$

It is shown in [**CK09**] that, for a cone of z,

$$\mathbb{Q}^z(S) = \mathrm{BC}_{\delta_S^z}^{H^z(S)} / \sim_S^z .$$

Because of this last fact, on a cone of z, $\mathbb{Q}^z(S)$ can be viewed as a complete Boolean algebra on δ^z_S in $H^z(S)$.

Every $\mathbb{Q}^z(S)$-generic, G, generates a real x_G with $x_G(i) = j \iff c_{i,j}/\sim^z_S \in G$, where $c_{i,j}$ is the code for the basic open set $A_{c_{i,j}} = \{x : x(i) = j\}$. It can be shown that $G = \{T/\sim^z_S : x_G \in A_T\}$, so $H^z(S)[x_G] = H^z(S)[G]$. Moreover as with the usual Vopenka algebra, every real, x, in $L[S,z]$, generates a $\mathbb{Q}^z(S)$-generic $G_x = \{T/\sim^z_S \in \mathbb{Q}^z(S) : x \in A_T^{L[S,z]}\}$. That G_x is generic follows from the fact that

$$\mathcal{A} \subseteq \mathbb{Q}^z(S) \text{ is predense } \iff \bigcup_{p\in\mathcal{A}} A_p^{L[S,z]} = \mathbb{R}^{L[S,z]}.$$

To put this in context, recall Vopenka's algebra (see, e.g., [**Jec03**]) $\mathbb{Q}^z(S)^*$ takes $\mathrm{OD}^{L[S,z]}$ subsets of $\mathbb{R}$ as conditions (actually a $H^z(S)$ copy is used.) Every $\mathbb{Q}^z(S)^*$-generic G^* gives rise to a real x_{G^*} just as above and every real, x, in $L[S,z]$ gives rise to a generic G^*_x and $H^z(S)[G^*_x] = H^z(S,x) = \mathrm{HOD}^{L[S,z]}_{S,x}$.

The two algebras are related as follows: $\mathbb{Q}^z(S)$ is the complete subalgebra of $\mathbb{Q}^z(S)^*$ generated by the basic open sets $[s]$ for $s \in \omega^{<\omega}$. For $x \in L[S,x]$,

$$H^z(S)[G_x] = H^z(S)[x] = \mathrm{HOD}^{L[S,z]}_S[x] \subseteq \mathrm{HOD}^{L[S,z]}_{S,x} = H^z(S,x) = H^z(S)[G^*_x]$$

Below I use $M|\kappa$ to mean V^M_κ, so $L[S,x]|\kappa$ does not mean $L_\kappa[S,x]$. The following facts are discussed thoroughly in [**CK09**]:

- For all x, x is $\mathbb{Q}^z(S)$-generic over $H^z(S)$ for all $z \geq_S x$.
- On a cone of z:
  - $\delta^z_S = \omega_2^{L[S,z]}$ is inaccessible in $H^z(S)$.
  - $\mathbb{Q}^z(S) \subseteq H^z(S)|\delta^z_S$ and hence, essentially, $\mathbb{Q}^z(S) \subseteq \delta^z_S$.
  - $\mathbb{Q}^z(S)$ is δ^z_S-cc in $H^z(S)$.

So if $\mathcal{D}^z(S)$ is the collection of maximal antichains of $\mathbb{Q}^z(S)$ in $H^z(S)$, then

$$\mathcal{D}^z(S) \subseteq H^z(S)|\delta^z_S$$

and so the structure, $\langle \mathbb{Q}^z(S), \mathcal{D}^z(S)\rangle$, is definable in $\langle H^z(S)|\delta^z_S, \sim^z_S\rangle$. Let S_z be the least ∞-Borel code in $H^z(S)$ so that $S \sim^z_S S_z$. Set

$$N^z(S) = \langle H^z(S)|\delta^z_S, \sim^z_S, S_z\rangle$$

and

$$S^z_* = \bigwedge_{\mathcal{A}\in\mathcal{D}^z(S)} \bigwedge_{T,T'\in\mathcal{A}} \neg(T \wedge T') \wedge S_z \wedge \bigwedge_{\mathcal{A}\in\mathcal{D}^z(S)} \bigvee \mathcal{A}$$

The code S^z_* is a member of $\mathrm{BC}^{H^z(S)}_{\delta^z_S}$, here we are using the fact that $H^z(S)$ has a canonical function $e : \delta^z_S \xrightarrow{\text{onto}} \mathcal{D}^z(S) \cup \mathrm{BC}^{H^z(S)}_{\delta^z_S}$ which depends on $L[S,z]$ and S, but not on z. So $z \mapsto S^z_*$ is S-invariant. Take S^∞_* to be the corresponding object in the ultra power. A similar comment applies to $N^z(S)$ and we define $N^\infty(S)$. Notice that for x (in any wellfounded model) there is a Σ_1-formula, Φ, so that

$$L_{\alpha(N^z(S),x)}[N^z(S), x] \models \Phi(x, N^z(S)) \iff x \in A_{S^z_*}$$

and thus for any x (in V), x is generic over $L[N^\infty(S)]$ and

$$\begin{aligned} A_S(x) &\iff L_{\alpha(N^\infty(S),x)} \models \Phi(x, N^\infty(S)) \\ &\iff L_{\delta^{\infty}_S{}^+} \models \Phi(x, N^\infty(S)). \end{aligned}$$

So $(\delta_S^{\infty+}, N^\infty(S), \Phi)$ is an ∞-Borel* code which is essentially the same as S_*^∞; as is made explicit in the next paragraph.

Clearly, $S_*^z \in H^z(N^z(S))$ and so $S_*^\infty \in H^\infty(N^\infty(S))$. The converse is also true, but this requires a bit of work. For now it suffices to notice that for any x, on a cone of z, $x \in L[S_*^z, z] \iff x \in L[N^z(S), z]$. This amounts to showing

$$x \in L[S_*^z, z] \iff x \in \mathrm{HOD}_S^{L[S,z]}$$

The left-to-right direction is clear. Conversely, if $x \in \mathrm{HOD}_S^{L[S,z]}$, then x "is" a code for itself in $\mathbb{Q}^z(S)$ and thus all we need is that $\mathbb{Q}^z(S) \subseteq L[S_*^z, z]$ on a cone. I leave this as an exercise. So we have

- $N^\infty(S) \approx S_*^\infty$, in fact, $L[N^\infty(S)] = L[S_*^\infty]$.
- In V, $A_S = A_{S_*^\infty} = A_{(\delta_S^{\infty+}, N^\infty(S), \Phi)}$.

It would be nice to see that $T \approx S \implies T_*^\infty \approx S_*^\infty$, however, I do not see how to show this. It is possible to prove something slightly weaker.

Lemma 5.2. If $S \in H^z(T)$ and $T \in H^z(S)$ on a cone of z, so that $H^z(S) = H^z(T)$ on a cone of z, then $S_*^\infty \approx T_*^\infty$.

Proof. Notice that the hypothesis readily implies $L[T, x] = L[S, x]$ on a cone so this is a strengthening of $S \approx T$. To prove $S_*^\infty \approx T_*^\infty$ it suffices to show that for a cone of z

$$x \in L[S_*^z, z] \iff x \in L[T_*^z, z]$$

We will actually see that for a cone of z, $L[S_*^z, z] = L[T_*^z, z]$. As explained above we may show that on a cone of z, $L[N^z(S), z] = L[N^z(T), z]$. Since $H^z(S) = H^z(T)$, this is trivial. ❑

The following lemma appears in **[CK09]**

Lemma 5.3. The ∞-Borel code S_*^z satisfies

$$x \in A_{S_*^z} \iff x \text{ is } \mathbb{Q}^z(S)\text{-generic over } H^z(S) \text{ and } H^z(S)[x] \models x \in A_S.$$

Correspondingly,

$$x \in A_{S_*^\infty} \iff x \text{ is } \mathbb{Q}^\infty(S)\text{-generic over } H^\infty(S) \text{ and } H^\infty(S)[x] \models x \in A_{S^\infty}.$$

In both x can come from any transitive model containing S_*^∞ and $H^\infty(S)$ or S_*^z and $H^z(S)$ respectively. ⊣

Lemma 5.4. For all reals, x, in V:

- x is $\mathbb{Q}^\infty(S)$-generic over $L[S_*^\infty]$. This by Łos' Lemma together with the fact that $N^z(S)$ is a rank initial segment of $L[N^z(S)]$.
- $A_S = A_{S_*^\infty}$
- $S_*^\infty \subseteq \delta_S^\infty$
- $S \preceq S_*^\infty$.

Proof. The first three facts are immediate. For the last fact show $S^\infty \preceq S_*^\infty$ and use $S \approx S^\infty$ (see the first line in the proof of Theorem 3.6.) We want to see that on a cone of z, $L[S^\infty, z] \cap \mathbb{R} \subseteq L[S_*^\infty, z]$. In fact, for all z, for a cone of $x \geq z$, $L[S, z] \cap \mathbb{R} \subseteq L[S_*^x, z]$, since $L[S_*^x, z] = L[H^x(S)|\delta_S^x][z]$ and $L[H^x(S)|\delta_S^x][z]|\delta_S^x = H^x(S)[z]|\delta_S^x$. Of course $H^x(S)[z] \supseteq L[S, z]$ and $H^x(S)[z]|\delta_S^x \supseteq \mathbb{R} \cap H^x(S)[z]$. ❑

Arguments reminiscent of those here appear in [**KMS83**, §13]. In particular, it is shown that if T is a tree on a complete Π^1_{2n+1} set, so that T as a degree notion is essentially C_{2n+2}, then T^∞_* is essentially Q_{2n+3} as a degree notion, so in these cases $T \prec T^\infty_*$.

By collapsing δ^z_S to be countable we can produce $H^z(S)$-generic subsets of $\mathbb{Q}^z(S)$. It is shown in [**CK09**, Lemma 4.6] that:

$$\begin{aligned}\exists y\, A_S(t,y) &\iff \forall^* z\, L[S^z_*][t]^{\mathrm{Col}(\omega,\delta^z_S)} \models \exists y\, A_{S^z_*}(t,y)\\ &\iff L[S^\infty_*][t]^{\mathrm{Col}(\omega,\delta^\infty_S)} \models \exists y\, A_{S^\infty_*}(x,y)\end{aligned}$$

If g is $\mathrm{Col}(\omega,\delta^\infty_S)$-generic over $L[S^\infty_*,t]$, then "$\exists y\, A_{S^\infty_*}(x,y)$" is $\Sigma^1_1(x,$ code for $S^\infty_*)$ and so absolute to $L[S^\infty,t][g]|\omega_1^{L[S^\infty,t][g]}$. Since

$$\omega_1^{L[S^\infty,t][g]} = \left(\delta^{\infty}_S{}^+\right)^{L[S^\infty,t]} < \left(\delta^{\infty}_S{}^+\right)^V,$$

the supremum of these ordinals is $\le \delta^{\infty}_S{}^+$ and so

$$\exists y\, A_S(t,y) \iff L_{\delta^\infty_S{}^+}[S^\infty_*,t]^{\mathrm{Col}(\omega,\delta^\infty_S)} \models \exists y\, A_{S^\infty_*}(x,y).$$

From Lemma 2.3 this gives an ∞-Borel code $\exists^{\mathbb{R}} S \in \mathrm{BC}_{\delta^\infty_S{}^+} \in L[S^\infty_*]$ such that

$$\exists y\, A_S(x,y) \iff A_{\exists^{\mathbb{R}} S}(x).$$

This shows that $\exists^{\mathbb{R}} S \preceq S^\infty_*$ and that if κ is a limit cardinal such that for all $S \subseteq \lambda < \kappa$, $\delta^\infty_S < \kappa$, then $\mathcal{B}_{<\kappa}$ is closed under real quantification.

Finally we can prove the promised uniformization result which is in essence just a variant of the construction of the code $\exists^{\mathbb{R}} S$.

Theorem 5.5. Assume $S \prec T$, then

$$\begin{aligned}A_S(x,y) &\iff \exists y \in H^\infty(S,T)[x] A_S(x,y)\\ &\iff \exists y \in L[N^\infty(S,T)][x] A_S(x,y)\end{aligned}$$

This produces an ∞-Borel uniformization of A_S, with code in $\mathrm{BC}_{\delta^\infty_{S,T}{}^+}$.

Proof. Clearly

$$\forall^* z\, \exists y \in L[S,z] A_S(x,y)$$

Recall every real in $L[S,z]$ is $\mathbb{Q}^z(S)$-generic over $H^z(S)$. Let $g \subseteq \mathrm{Col}(\omega,\delta^z_S)$ be $H^z(S)[x]$-generic. I claim that in $H^z(S)[x][g]$ there is y so that $A_S(x,y)$. If not, then this is forced by some $p \in \mathrm{Col}(\omega,\delta^z_S)$. We can in V build a $H^z(S)[x]$ generic through p which would allow us to build a $H^z(S)$-generic, y, for $\mathbb{Q}^z(S)$ such that $A_S(x,y)$. So we have

$$\forall^* z H^z(S)[x]^{\mathrm{Col}(\omega,\delta^z_S)} \models \exists y\, A_S(x,y).$$

Since $H^z(T,S)[x]$ can find $H^z(S)[x]$ generics for $\mathrm{Col}(\omega,\delta^z_S)$ on a cone, we have that if $\exists y\, A_S(x,y)$, then

$$\forall^* z H^z(S,T)[x] \models \exists y\, A_S(x,y),$$

hence,

$$H^\infty(S,T)[x] \models \exists y\, A_{S^\infty}(x,y).$$

This produces an S-invariant uniformization of A_S using the canonical wellorder of $H^\infty(S,T)$. The argument where $H^z(S,T)[x]$ is replaced by $L[N^z(S,T),x]$ is the same. ❑

If the relation A_S starts out with countable slices, then we get a slightly better uniformization result.

Lemma 5.6. Suppose S is an ∞-Borel code for a many-countable relation, then for all x, for a cone of z, $(A_S)_x \subseteq H^z(S)[x]$. Consequently, for all x, $(A_S)_x \subseteq H^\infty(S)[x]$.

Proof. For any x, fix z_0 so that for any $z \geq z_0$, $(A_S)_x \in H(\omega_1)^{L[S,z]}$. We can find a $g \subseteq \mathrm{Col}(\omega, \delta^z_S)$ generic g over $H^z(S)[x]$ and name $\dot{u}$ so that $(A_S)_x = \{(\dot{u}[g])_i : i \in \omega\}$. Assume there is i so that $\Vdash^{H^z(S)[x]}_{\mathrm{Col}(\omega,\delta^z_S)} A_S(x, (\dot{u})_i) \wedge (\dot{u})_i \notin V$. Then we could use $(\dot{u})_i$ to build a perfect set of y so that $A_S(x,y)$. So for all $z \geq z_0$, $(A_S)_x \subseteq H^z(S)[x]$.

We can replace $H^z(S)$ and $H^\infty(S)$ by $N^z(S)$ and $N^\infty(S)$ respectively and thus get $(A_S)_x \subseteq L[N^\infty(S), x] = L[S^\infty_*, x]$. ❑

It essentially follows from this that $S \prec T$ iff $H^\infty(S,T)$ (or $L[N^\infty(S,T)]$) can "uniformize" $y \notin L[S^\infty_*, x]$ in a sense made precise below. Set

$$D_S(x,y) \overset{\text{df}}{\iff} y \notin L[S,x].$$

Theorem 5.7. If T is an ∞-Borel code for a uniformization of D_S, then $S \prec T^\infty_*$. If $S \prec T$, then, $(S,T)^\infty_*$ gives a code for a uniformization of D_S.

Proof. For the first claim, just apply the preceding lemma to get that on a cone of z

$$(D_S)_x \subseteq L[T^\infty_*, x]$$

So $L[T^\infty_*, x] \cap \mathbb{R} \not\subset L[S^\infty_*, x]$ on a cone of x, i.e., $T^\infty_* \not\preceq S^\infty_*$, and hence $S^\infty_* \prec T^\infty_*$.

For the converse, note that there is code S' for D_S so that $S' \in L[S]$ so $S' \preceq S \prec T$ and Theorem 5.5 gives $N^\infty(S', T) \approx (S', T)^\infty_*$ yields a code for a uniformization of S'. ❑

While it need not be the case that $S \approx S^\infty_*$ (recall $S \approx S^\infty$), it is true that if S is a non-maximal degree, then S^∞_* is also non-maximal. This gives the following corollary:

Corollary 5.8. The following are equivalent

(1) S is a non-maximal degree.
(2) D_S is uniformized by an ∞-Borel set.
(3) $\delta^\infty_S < \Theta^{L(\mathcal{B}_\infty, \mathbb{R})}$. (See Theorem 5.13 and Theorem 5.16) $\dashv$

5.2. The extent of Suslin sets under the non-existence of a maximal degree. Following Becker [**Bec85**], a strongly closed pointclass, Λ (recall $\delta_\Lambda = w_\Lambda$), of countable Wadge cofinality has the *Kunen-Martin property* iff

$$(\delta_\Lambda)^+ = \delta^1_1(\Lambda)$$

or equivalently if

$$w(A)^+ = \delta^1_1(A)$$

for $A = \bigoplus_{i \in \omega} A_i$ where $\langle A_i : i \in \omega \rangle$ is Wadge cofinal in Λ.

If δ_Λ is a Suslin cardinal, then $\Pi^1_1(\Lambda)$ is scaled and $\boldsymbol{S}_{\delta_\Lambda} = \Sigma^1_1(\Lambda)$. The Kunen-Martin theorem gives that $\sigma^1_1(\Lambda) = (\delta_\Lambda)^+$, recall $\delta^1_1(\Lambda) = \sigma^1_1(\Lambda)$ in this case, so the Kunen-Martin property holds. Becker showed in [**Bec85**], assuming some additional closure for Λ, that

$$\Lambda\text{-Uniformization} \ + \ \text{the Kunen-Martin property holds} \iff \delta_\Lambda \text{ is Suslin}$$

The following question is apparently still open and relevant for part of our analysis.

QUESTION. Does the Kunen-Martin property hold for all type I hierarchies?

Woodin answered this question assuming $\Lambda \subseteq \mathcal{B}_\infty$.

THEOREM 5.9 (Woodin)**.** For Λ strongly closed with countable cofinality, if $\Lambda \subseteq \mathcal{B}_\infty$, then the Kunen-Martin property holds, i.e., $\delta^1_1(\Lambda) = (\delta_\Lambda)^+$. $\dashv$

Woodin also showed that AD+uniformization $\Longrightarrow$ all sets are ∞-Borel, so that the Kunen-Martin property holds for all strongly closed pointclasses of countable cofinality. The proof, as far as I know, does not appear in print and its techniques are similar to those being discussed here, so I am including it. His argument works in a more general setting: Rather than AD + DC we shall assume the existence of a fine σ-complete measure, μ, on $\mathcal{P}_{\omega_1}(\mathbb{R})$, uniformization, and that $\prod_{\mathcal{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu$ is wellfounded.

The existence of a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ is guaranteed by Turing-determinacy, that is, determinacy for all Turing invariant sets. Turing determinacy is equivalent to the cone filter on Turing degrees being an ultrafilter and this ultrafilter readily induces a fine σ-complete measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$, under $\mathrm{DC}_\mathbb{R}$. The assumption that "$\prod_{\mathcal{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu$ is wellfounded" is implied by DC and implies $\mathrm{DC}_\mathbb{R}$, but not DC. This is discussed in [**CK09**]. The existence of a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$ implies that ω_1 is measurable, hence choice fails, and that ω_1^V is Mahlo in any inner model of choice. This is also discussed in [**CK09**], where these same hypotheses are used to derive several results.

For $\mathbb{P}$ a poset let the collection of canonical names for reals, $\mathcal{R}^\mathbb{P}$, be the collection of $\mathbb{P}$-names satisfying:

(1) For all n, for densely many p there is $m \in \omega$ with $(p,(n,m)) \in \tau$.
(2) For $(p,(n,m)),(q,(n,m')) \in \tau$ with $p \parallel_\mathbb{P} q \Longrightarrow m = m'$.

If τ is any $\mathbb{P}$-name for a real, then $\tau^* = \{(p,(n,m)){:}p \Vdash \tau(n) = m\} \in \mathcal{R}^\mathbb{P}$, conversely, whenever $g \subseteq \mathbb{P}$ is a "sufficiently generic" filter and $\tau \in \mathcal{R}^\mathbb{P}$, then $\tau_g \in \mathbb{R}$, here g need only meet each $D^\tau_n = \{p : \exists m\,(p,(n,m)) \in \tau\}$.

For $A \subseteq \mathbb{R}$, the *term relation* for A is defined as

$$\begin{aligned}\dot{A}(\mathbb{P},p,\tau) \overset{\text{df}}{\Longleftrightarrow} &(1)\, \mathbb{P} \text{ a poset}\\ &(2)\, \tau \in \mathcal{R}_\mathbb{P}\\ &(3)\, \forall^* g \subseteq \mathbb{P}\,\big(p \in g \to \tau_g \in A\big),\end{aligned}$$

where $\forall^* g \subseteq \mathbb{P}\,\varphi(g)$ means that there is a countable collection of $\mathbb{P}$-dense sets, $\mathcal{D}$, in V, such that if g is a $\mathcal{D}$-generic filter, then $\varphi(g)$ holds, in V. For M a transitive inner model, the M-term relation for A is defined by $\dot{A}^M \overset{\text{df}}{=} \dot{A} \cap M$. In general, there is no reason for $\dot{A}^M$ to be in M (or amenable to M) and when this occurs M is called *weakly A-closed*. Similarly, M is *weakly $A,\mathbb{P}$-closed* iff $\dot{A}_\mathbb{P} \cap M \in M$.

If M is weakly $A,\mathbb{P}$-closed and weakly $\neg A,\mathbb{P}$-closed, and

(†) for all $\tau \in \mathcal{R}_\mathbb{P} \cap M$, $D^A_{\mathbb{P},\tau} = \{p : p \Vdash^*_\mathbb{P} \tau \in A \vee p \Vdash^*_\mathbb{P} \tau \notin A\}$ is dense in $\mathbb{P}$,

then

$$\forall^* g \subseteq \mathbb{P}\; A^M_\mathbb{P}[g] = A \cap M[g].$$

Above "$p \Vdash^*_\mathbb{P} \tau \in A$" means there is a countable collection, $\mathcal{D}$, of $\mathbb{P}$-dense sets so that for any $\mathcal{D}$-generic filter, $\tau_g \in A$, similarly for "$\notin$".

CLAIM. The condition (†) is guaranteed by all sets having the Baire property.

Proof. This is essentially a standard forcing fact. There is a dense embedding $\pi : \omega^{<\omega} \xrightarrow{\text{onto}} \mathbb{P}_0 \subseteq \mathbb{P}$ with $\mathbb{P}_0$ a dense sub-poset of $\mathbb{P}$, and a dense G_δ set $E \subseteq \omega^\omega$ so that $\pi : E \xrightarrow{\text{onto}} X_\mathbb{P}$ where $X_\mathbb{P}$ is the set of $\mathbb{P}$-filters. Now define $f_\tau : E \to \mathbb{R}$ by $x \mapsto \tau_{\pi(x)}$. For $p \in \mathbb{P}_0$, look at $E_{\tau,p} = \{x \in E : p \in \pi(x) \wedge f_\tau(x) \in A\}$. If this set is comeager in $[\pi^{-1}(p)]$, then let $\mathcal{E}$ be a countable collection of open dense sets in $\omega^{<\omega}$ so that if x is $\mathcal{E}$-generic, then $x \in E$ and $f_\tau(x) \in A$. Let $\mathcal{D} = \pi[\mathcal{E}]$, then if g is $\mathcal{D}$-generic, then $\tau_g \in A$, and hence $\mathcal{D}$ witnesses $p \Vdash^*_\mathbb{P} \tau \in A$. If $E_{\tau,p}$ is not comeager, then there is $q \leq_\mathbb{P} p$ so that $\neg E_{\tau,p}$ is comeager in $[\pi^{-1}[q]]$ and one argues as above that there is $\mathcal{D}$ so that if g is $\mathcal{D}$-generic, then $q \in g \to \tau_g \notin A$. ❑

If we strengthen weak $A, \mathbb{P}$-closure to

$$\dot{A}^M_\mathbb{P} \in M \text{ and for all } M\text{-generic } g \subseteq \mathbb{P} \text{ (in } V\text{)},\ A^M_\mathbb{P}[g] = A \cap M[g],$$

then call M *strongly $A, \mathbb{P}$-closed*. M is said to be weakly (strongly) A-closed iff M is weakly (strongly) $A, \mathbb{P}$-closed for all $\mathbb{P} \in M$.

It is a relatively simple matter to produce weakly A-closed structures, namely, $N^A_x = L_{\omega_1}[\dot{A}, x]$ and $M^A_x = \mathrm{HOD}^{N^A_x}_{\dot{A}}$ are such, since $\dot{A}^{N^A_x} = \dot{A} \cap N^A_x$.

To produce strongly A-closed structures is trickier since we must ensure that the model has "enough" dense sets, this is where uniformization is used. Notice, in the argument that follows, essentially it is proved that assuming

(1) ZF,
(2) all sets have the Baire property,
(3) the existence of a fine measure, μ, on $\mathcal{P}_{\omega_1}(\mathbb{R})$, and
(4) $\prod_{\mathcal{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu$ is wellfounded,

then the following are equivalent:

(1) A is ∞-Borel.
(2) There is a uniform sequence $\langle M_\sigma : \sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})\rangle$ of strongly $A, \neg A$-closed transitive substructures of $H(\omega_1)$, such that each $y \in \sigma$ is generic over M_σ for a poset $\mathbb{P}_\sigma \in M_\sigma$.

Theorem 5.10 (Woodin)**.** Work in ZF. Suppose all subsets of $\mathbb{R}$ have the property of Baire, there is a fine measure , μ, on $\mathcal{P}_{\omega_1}(\mathbb{R})$, and that $\prod_{\mathcal{P}_{\omega_1}(\mathbb{R})} \omega_1/\mu$ is wellfounded. Then UNIF implies that all sets are ∞-Borel.

Proof. Fix a set $A \subseteq \mathbb{R}$ with the aim being to show that A is ∞-Borel. Define the relation B_A on reals by $B_A(x, y)$ iff x codes $(\mathbb{P}, \tau)$, where $\mathbb{P}$ is a countable poset and τ is a canonical $\mathbb{P}$-name for a real, and y codes $\mathcal{D}$ with $D^A_{\mathbb{P},\tau} \in \mathcal{D}$ such that for g that is $\mathcal{D}$-generic and $p \in g \cap D^A_{\mathbb{P},\tau}$

$$\tau_g \in A \iff p \Vdash^*_\mathbb{P} \tau \in A \text{ (and so also } \tau_g \notin A \iff p \Vdash^*_\mathbb{P} \tau \notin A),$$

or, equivalently, for $g \subseteq \mathbb{P}$ that is $\mathcal{D}$-generic,

$$(\ddagger_\tau) \qquad \tau_g \in A \iff \exists p \in g\, \dot{A}_\mathbb{P}(p, \tau) \text{ and } \tau_g \notin A \iff \exists p \in g\, \neg\dot{A}_\mathbb{P}(p, \tau).$$

Uniformization of B_A can be used to select witnesses to weak closure. Let B^*_A uniformize B_A and set $A^*(x, i, j) \overset{\text{df}}{\iff} B^*_A(x)(i) = j$.

Set

$$N_\sigma = L_{\omega_1}[\dot{A}, \neg\dot{A}, \dot{A}^*, \sigma] \text{ and } M_\sigma = \mathrm{HOD}^{N_\sigma}_{\dot{A}, \neg\dot{A}, \dot{A}^*}.$$

These are model of ZFC since ω_1 is measurable in V.

Claim. M_σ is strongly A-closed.

Proof. Let $\mathbb{P} \in M_\sigma$ and let $\mathbb{Q} = \mathrm{Col}(\omega, \mathbb{P})$ and fix $\tau \in \mathcal{R}_\mathbb{P} \cap M_\sigma$. Let $\dot{z}$ be the induced generic coding of $(\mathbb{P}, \tau)$. Pick $\mathcal{E}_\tau$ a countable collection of dense subsets of $\mathbb{Q}$ so that for any $\mathcal{E}_\tau$-generic, G, for all $i \in \omega$, there is $j \in \omega$ such that

$$(\dot{A}^*)^{M_\sigma}_\mathbb{P}[G] \subseteq A^* \cap M_\sigma[G] \text{ and } (\dot{A}^*)^{M_\sigma}[G](z_G, i, j)$$

This is essentially just the weak $A^*, \mathbb{Q}$-closure of M_σ. For G that is both M_σ-generic and $\mathcal{E}_\tau$-generic, $B^*_A(z_G) \in M_\sigma[G]$.

Let $\mathcal{D}_G$ be the countable collection of $\mathbb{P}$-dense sets coded by $B^*_A(z_G)$, then $\mathcal{D}_G \in M_\sigma[G]$. If $g \subseteq \mathbb{P}$ is $M_\sigma[G]$-generic, then as $\mathcal{D}_G \subseteq M_\sigma[G]$, g is $\mathcal{D}_G$-generic and thus $(\ddagger_\tau)$ holds for every canonical $\mathbb{P}$-name, τ, in M_σ.

The order of the forcings can be inverted so that, in $M_\sigma[g][G]$, $(\ddagger_\tau)$ holds for all $\tau \in \mathcal{R}_\mathbb{P} \cap M_\sigma$. This is independent of G and thus $(\ddagger_\tau)$ holds in $M_\sigma[g]$ for all $\tau \in \mathcal{R}_\mathbb{P} \cap M_\sigma$. Thus for $g \subseteq \mathbb{P}$ which are M_σ-generic, for all $\tau \in \mathcal{R}_\mathbb{P} \cap M_\sigma$,

$$\tau_g \in A \iff \exists p \in g\, \dot{A}(\mathbb{P}, p, \tau) \text{ and } \tau_g \notin A \iff \exists p \in g\, \neg\dot{A}(\mathbb{P}, p, \tau)$$

so

$$\dot{A}^{M_\sigma}_\mathbb{P}[g] = A \cap M_\sigma[g]$$

and hence M_σ is strongly $A, \mathbb{P}$-closed. ❑

Now let $\mathbb{P}_\sigma = \mathcal{B}^{M_\sigma}_\infty / \sim_\sigma$ where for $T, T' \in \mathcal{B}^{M_\sigma}_\infty$, $T \sim_\sigma T' \overset{\mathrm{df}}{\iff} A^{N_\sigma}_T = A^{N_\sigma}_{T'}$. This is just the ∞-Borel version of the Vopenka algebra, and as with the Vopenka algebra, for any $y \in N_\sigma$, $G^y_\sigma = \{T/\sim_\sigma : y \in A^{M_\sigma}_T\}$ is M_σ-generic. For any y with $y \in N_\sigma$, we have from the claim that, $y \in A \iff y \in \dot{A}^{M_\sigma}_{\mathbb{P}_\sigma}[G_y]$. Letting $T^\sigma_A = \bigvee\{T \in P_\sigma : \dot{A}^{M_\sigma}_{\mathbb{P}_\sigma}(T, \tau)\}$, it follows that

$$y \in \dot{A}^{M_\sigma}_{\mathbb{P}_\sigma}[G_y] \iff y \in A_{T^\sigma_A}$$

and hence $A \cap N_\sigma = A_{T^\sigma_A} \cap N_\sigma$ so that , locally anyway, A is ∞-Borel.

Let $(M_\infty, \mathbb{P}_\infty, T^\infty_A) = \prod_\sigma (M_\sigma, \mathbb{P}_\sigma, T^\sigma_A)$, then for $y \in V$, y is M_∞-generic for $\mathbb{P}_\infty$ and

$$y \in A \iff M_\infty[y] \models y \in A_{T^\infty_A}$$

Thus T^∞_A is an ∞-Borel code for A. ❑

Working in $\mathrm{ZF} + \mathrm{AD} + \text{uniformization}$, let μ be the measure induced from the cone measure on Turing degrees, then the ultrapower taken in the proof could be taken in $L(\mathbb{R}, A, A^*)$ and since uniformization implies $\mathrm{DC}_\mathbb{R}$, and since $\mathrm{DC}_\mathbb{R}$ in V gives DC in $L(\mathbb{R}, A, A^*)$, it follows that

$$\mathrm{ZF} + \mathrm{AD} + \text{uniformization} \implies \text{all sets are } \infty\text{-Borel}$$

Corollary 5.11. In the case that there is no maximal degree notion, $L(\mathcal{B}_\infty, \mathbb{R})$ is the maximal model of $\mathrm{AD} + \text{uniformization}$. ⊣

Corollary 5.12. The following are equivalent under $\mathrm{ZF} + \mathrm{DC}$:

(1) $\mathrm{AD} + \text{uniformization}$.
(2) $\mathrm{AD} + \text{all sets are Suslin}$.
(3) $\mathrm{AD}^+ + \text{all sets are Suslin}$.
(4) $\mathrm{AD}_\mathbb{R}$.

Proof. (2) $\implies$ (3) follows from the discussion of AD^+ in the preliminary section, namely the discussion of downward absoluteness (see page 33 and Lemma 2.5). That (2) $\implies$ (4) (and hence (3) $\implies$ (4)) is due independently to Martin and Woodin (unpublished). (4) $\implies$ (1) is a simple exercise, in fact each player need only make a single real move.

Only (1) $\implies$ (2) needs discussion here. Given any set A, uniformization gives that we can find a minimal strongly closed pointclass Λ containing A such that Λ-uniformization holds. DC implies $w(\Lambda) < \Theta$ and by the results of Woodin, $\Lambda \subseteq \mathcal{B}_\infty$ and Λ has the Kunen-Martin property. So Becker's result gives that $w(\Lambda)$ is a Suslin cardinal. ❑

5.3. Bounds on $\boldsymbol{\delta}^\infty_S$. Let f be an S-invariant function with $f(x) = M^x \in H(\omega_1)^{L[T,x]}$ a transitive structure (on a cone). Say that y codes M^x iff $(\omega, \in_y) \simeq (M^x, \in)$ and set

$$R_f(x,y) \overset{\text{df}}{\iff} y \in \mathbb{R} \cap L[T,x] \text{ codes } f(x).$$

R_f can be uniformized as follows:

$$R^*_f(x,y) \overset{\text{df}}{\iff} y \text{ is the } L[T,x]\text{-least } y, \text{ such that } R_f(x,y).$$

Let $f^*(x) = y \iff R^*_f(x,y)$, f^* need not be S-invariant. If $R^*_f(x,y)$ let $\pi_x : (\omega, \in_{f^*(x)}) \simeq M^x$ be the induced isomorphism and set $n_x = \pi_x(n) \in M^x$. For h any S-invariant function such that on a cone, $h(x) \in M_x$, there $n \in \omega$, such that $A = \{x : n_{f(x)} = h(x)\}$ is S-positive. This n need not be unique. Together with f, the pair (n, A) "codes" h since for a cone of x,

$$h(x) = n_{f(z)} \text{ for any } z \in [x]_S \cap A.$$

A can be replaced by an S-pointed perfect tree, a, such that $[a] \subseteq A$, so that the codes are reals. Let C_f be the set of all such codes, that is,

$$\begin{aligned}(n,a) \in C_f \iff\ & a \text{ is an } S\text{-pointed perfect tree and} \\ & \text{for } x, y \in [a], \text{ if } x \equiv_S y, \text{ then } n_x = n_y\end{aligned}$$

For $(n,a) \in C_f$ set $h_{(n,a)}([x]_S) = n_z$, where $z \in [x]_S \cap [a]$. Set $(n,a) \sim (n', a')$ iff $h_{(n,a)} = h_{(n',a')}$ on a cone. Let $[n,a]_f$ be the equivalence class of (n,a) and set

$$[n,a]_f \in_f [n',a']_f \overset{\text{df}}{\iff} \forall^* x\, (h_{(n,a)}(x) \in h_{(n',a)'}(x))$$

Then $(C_f/\sim, \in_f)$ is isomorphic to $\prod_S M^x/\mu = M^\infty_S$. By looking at the complexity of this coding we get bounds on δ^∞_S.

Theorem 5.13. If $S \prec T$ with $S, T \subseteq \gamma < \lambda$ with λ a measurable cardinal and $\leq$ is a prewellordering of length λ, then $\delta^\infty_S \leq \delta(\leq)$.

Proof. Let $f(x) = H^x_S$, then $f(x) \in H(\omega_1)^{L[T,x]}$ and R^*_f is ∞-Borel since

$$R^*_f(x,y) \iff L[S,T,x] \models \text{“} y = <_L \text{-least } z \text{ such that } (\omega, \in_z) \simeq (H^x_S, \in)\text{”}$$

Let (φ, S, T) be the corresponding code in $\mathrm{BC}^*_\gamma \subseteq \mathrm{BC}_\kappa$. Since $\mathcal{B}_\kappa \subseteq \mathcal{B}_{<\delta(\leq)} \subseteq \Delta(\leq)$, the relation $R^*_f(x,y)$ is $\Delta(\leq)$. Similarly, $x \leq_S y$ is $\Delta(\leq)$, since $x \leq_S y \iff x \in L[S,y] \iff L[S,x,y] \models x \in L[S,y]$.

First compute the complexity of $(n,a) \in C_f$:

(1) a is S-pointed perfect.
(2) $x, y \in [a]$ and $x \equiv_S y \to n_x = n_y$. ($S$-invariance.)

For (1): a is S-pointed perfect iff

$$\forall s \in [a] \exists s', s'' \in [a]\, (s' \perp s)\ \&\ \forall x\, (x \in [a] \to a \leq_S x)$$

For (2):

$$\begin{aligned}\forall x, y \big[&\big(x, y \in [a] \text{ and } x \equiv_S y\big) \to \\ &\exists u, v\, (R_f^*(x, u)\ \&\ R_f^*(y, v) \text{ and } (\omega, \in_x) \simeq (\omega, \in_y))) \big]\end{aligned}$$

So both (1) and (2) are $\Pi(\leq)$ and thus $C_f \in \Pi(\leq)$.

Similarly, $\in_f$ is $\Pi(\leq)$ in the codes. Restricting to codes for ordinals, i.e., $h(x) \in \delta_S^x$, produces a $\Pi(\leq)$ norm on C_f and thus $\delta_S^\infty \leq \delta(\leq)$. ❑

All we actually use below is the following:

Corollary 5.14. If Λ is strongly closed and there is no maximal degree in Λ, then for all $S \in \Lambda$, $\delta_S^\infty < \delta_\Lambda$. If S is not a maximal degree, then $\delta_S^\infty < \Theta$. ⊣

Corollary 5.15. If κ is a limit of Suslin cardinals, then for all $S \subseteq \lambda < \kappa$, $\delta_S^\infty < \kappa$.

Proof. The point is that $\boldsymbol{S}_{<\kappa} = \Lambda$ is strongly closed and each $A \in \Lambda$ has a Suslin representation in Λ. If $S \in \Lambda$, then $D_S(x, y) \overset{\text{df}}{\iff} y \notin L[S, x]$ is in Λ. So D_S is Suslin by a tree in Λ and hence uniformized in Λ. Let D_S^* be the uniformization and let S' be a tree in Λ projecting to D_S^*. We see $S \prec S'$ since for any x, $\exists y\, (x, y) \in [S'] \iff \exists y \in L[S', x]\, (x, y) \in [S']$. (This is similar to 5.7.) ❑

Compare these corollaries to Theorem 5.16 which shows that if S is maximal, then $\delta_S^\infty = \Theta$ and Corollary 6.4 which shows that if $\kappa < \boldsymbol{\kappa}_\infty$ is Suslin, then $\delta_S^\infty \geq \lambda$ where λ is the next Suslin past κ.

5.4. Maximal degree. Call a degree notion, S, *strongly maximal* iff

$$\text{For all } A \subseteq \mathbb{R}, \text{ on a cone of } x\ A \cap L[S, x] \in L[S, x]$$

If S is maximal and every set of reals is ∞-Borel, then S is strongly maximal. Conversely, if S is strongly maximal, then $L(\mathcal{P}(\mathbb{R})) = L(S^\infty, \mathbb{R})$ and so all sets are ∞-Borel.

Theorem 5.16. If S is strongly maximal, then $L(\mathcal{P}(\mathbb{R})) = L(S^\infty, \mathbb{R})$ and $\delta_S^\infty = \Theta$.

Proof. We have that $\forall^* x\, (A \cap L[S, x] \in L[S, x])$. Set $A(x) = L[S, x] \cap A$, this is an S-invariant function. Unfortunately the well ordering of $A(x)$ is not S-invariant. Let $h^{L[S,x]|\delta_S^x} : \delta_S^x \xrightarrow{\text{onto}} L[S, x]|\delta_S^x$ be the canonical Σ_1-Skolem function. GCH* holds on a cone so $A(x) = h(\alpha(x))$ for some minimal $\alpha(x) < \delta_S^x$. The function $\alpha(x)$ is not S-invariant, since the well ordering of $L[S, x]$ depends on x. So let G be $\mathbb{P}_S$ generic and let $\alpha_S^\infty = [x \mapsto \alpha(x)]_G$. We want to see that

$$h^{L[S^\infty, x_S^\infty]|\delta_S^\infty}(\alpha_S^\infty) \cap \mathbb{R}^V = A$$

For $y \in \mathbb{R}^V$, we have

$$\begin{aligned} y \in h^{L[S^\infty, x_S^\infty]|\delta_S^I}(\alpha_S^\infty) &\iff \{x : y \in h^{L[S,x]|\delta_S^\infty}(\alpha(x))\} \in G \\ &\iff \{x : y \in A \cap L[S, x]\} \in G \end{aligned}$$

Since $\{x : y \in A \cap L[S,x]\}$ is S-invariant we have

$$\{x : y \in A \cap L[S,x]\} \in G \iff y \in A$$

This is what we wanted to see.

We have shown here that for all sets of reals A, for any S^∞-pointed perfect Sacks real x_S^∞, $A \in L(S^\infty, \mathbb{R})[x_S^\infty]$ and thus $A \in L(S^\infty, \mathbb{R})$ to begin with. So $L(\mathcal{P}(\mathbb{R})) = L(S^\infty, \mathbb{R})$.

From Theorem 3.6, $\delta_S^\infty = \omega_2^{L[S^\infty, x_S^\infty]} \leq \Theta$ and GCH^* holds in $L[S^\infty, x_S^\infty]$. For each $A \subseteq \mathbb{R}$ (in V) we have $A = A^\infty \cap \mathbb{R}^V$ for some $A^\infty \in L[S^\infty, x_S^\infty]$ and so $|\mathcal{P}(\mathbb{R})^V| \leq |\mathcal{P}(\mathbb{R})^{L[S^\infty, x_S^\infty]}| = \omega_2^{L[S^\infty, x_S^\infty]} = \delta_S^\infty$ and thus $\Theta \leq \delta_S^\infty$. ❑

The following theorem is due to Woodin and appears in [**Ste94**].

Theorem 5.17 (Woodin)**.** If $V = L(S, \mathbb{R})$ and $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$ holds, then there is $T \subseteq \Theta$ such that $\mathrm{HOD}_{S,x} = L[T, S, x]$ for all x. ⊣

As a consequence of this theorem T, S is a largest degree notion in $L(S, \mathbb{R})$. This $D_{S,T}(x,y) \iff y \notin \mathrm{HOD}_{S,x}^{L(S,\mathbb{R})}$ can not be uniformized, since any uniformization, F, must be $\mathrm{OD}_{S,x}^{L(S,\mathbb{R})}$ for some x and hence $F(x) \in \mathrm{HOD}_{S,x}^{L(S,\mathbb{R})}$. This yields

Corollary 5.18. $(\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD})$ If there is no maximal degree notion, then $V \neq L(S, \mathbb{R})$ for any $S \subseteq \mathrm{OR}$. ⊣

For S a maximal degree, Theorem 5.16 and variants give that δ_S^∞ is large:

(1) Assuming S is a "strongly maximal degree notion" in the sense that $A \cap L[S,x] \in L[S,x]$ for all $A \subseteq \mathbb{R}$ we have $\delta_S^\infty = \Theta$.
(2) If $V = L(T, \mathbb{R})$, then every set is ∞-Borel so $V = L(S^\infty, \mathbb{R})$ and $\delta_S^\infty = \Theta$ by (1).
(3) In general, if there is a maximal degree notion S, then $\mathcal{B}_\infty \subseteq L(S^\infty, \mathbb{R})$ and, conversely, $L(S^\infty, \mathbb{R})$ is a model of all sets are ∞-Borel. Hence $L(\mathcal{B}_\infty, \mathbb{R}) = L(S^\infty, \mathbb{R})$ and letting $\Theta^{\mathcal{B}_\infty} = \Theta^{L(\mathcal{B}_\infty, \mathbb{R})}$, we have $\delta_S^\infty = \Theta^{\mathcal{B}_\infty}$.

In general, if there is a largest degree notion S, then it need not be the case that S has size $\boldsymbol{\kappa}_\infty$, however, if $\boldsymbol{\kappa}_\infty$ is a Suslin cardinal, then any tree, S, on $\omega \times \boldsymbol{\kappa}_\infty$ witnessing this is strongly maximal. This follows from the following theorem.

Theorem 5.19 (Woodin)**.** Suppose $\boldsymbol{\kappa}_\infty$ is Suslin and S is a tree on $\omega \times \boldsymbol{\kappa}_\infty$ witnessing this, then S is strongly maximal.

This will require some results of Martin which appear in [**Jac10**]. For a non-selfdual pointclass Γ and $\bar{A} \in \Gamma^\kappa$ set

$$N(\bar{A}) = \{A : \forall \sigma \in \mathcal{P}_{\omega_1}(\mathbb{R}) \exists \alpha < \kappa\, (A \cap \sigma = A_\alpha \cap \sigma)\}$$

and

$$\mathrm{Env}(\Gamma, \kappa) = \{B : B \leq_w A \text{ such that } \exists \bar{A} \in \Gamma^\kappa\, A \in N(\bar{A})\}.$$

Call Γ *nice* if Γ has the prewellordering property and is closed under $\forall^{\mathbb{R}}$ and $\vee$. For nice Γ set $\mathrm{Env}(\Gamma) = \mathrm{Env}(\Gamma, \delta_\Gamma)$. It is shown in [**Jac10**] that for nice Γ

$$\mathrm{Env}(\Delta) = \mathrm{Env}(\Gamma) = \mathrm{Env}(\exists^{\mathbb{R}}\Gamma).$$

Let $\varphi : C \xrightarrow{\text{onto}} \delta_\Gamma$ be a Γ-norm where $C \in \Gamma \setminus \neg\Gamma$. Let U be universal $\exists^{\mathbb{R}}\Gamma$ and D be the set of codes for subsets of δ_Γ; so for $t \in D$

(1) $U(t, x) \to x \in C$ and

(2) $U(t,x)$ & $\varphi(x) = \varphi(x') \to U(t,x')$.

For $t \in D$ set $\pi(t) = \{\varphi(x) : U(t,x)\}$. For U a measure on δ_Γ set

$$U^* = \{t : \pi(t) \in U\}$$

It is shown in [**Jac10**] that $U^* \in \mathrm{Env}(\Gamma)$ for nice Γ.

For S a set of ordinals define

$$\mathrm{Env}(S) = \{A \subseteq \mathbb{R} : \forall^* x\, A \cap L[S,x] \in L[S,x]\}.$$

Since GCH* holds on a cone, we have

$$\mathrm{Env}(S) = \{A \subseteq \mathbb{R} : \forall^* x\, A \cap L[S,x]|\delta_S^x \in L[S,x]|\delta_S^x\}.$$

For nice Γ such that $\exists^{\mathbb{R}}\Gamma$ is scaled, $\exists^{\mathbb{R}}\Gamma = \boldsymbol{S}_\kappa$ for $\kappa = \delta_\Gamma$ and

$$\mathrm{Env}(S) = \mathrm{Env}(\Gamma)$$

where S is the tree of a scale on a complete Γ-set.

Letting meas(κ) be the set of measures on κ and let meas$^*(\kappa)$ be the set of codes, we have:

Lemma 5.20. Suppose κ is a Suslin cardinal, $\boldsymbol{S}_\kappa = \exists^{\mathbb{R}}\Gamma$ where Γ is nice. Then meas$^*(\kappa) \subseteq L(S^\infty, \mathbb{R})$, for S the tree of a scale on a complete Γ set. $\dashv$

Martin and Woodin [**MW08**] have shown that if meas$^*(\kappa)$ is bounded in the Wadge degrees, then any tree on κ is weakly homogeneous and thus κ is not the largest Suslin cardinal.

Proof of Theorem 5.19. Assume $\boldsymbol{\kappa}_\infty$ is Suslin and let $\Gamma = \boldsymbol{S}_{\boldsymbol{\kappa}_\infty}$, then Γ is nice and $\exists^{\mathbb{R}}\Gamma = \Gamma$. If S is a tree on $\omega \times \boldsymbol{\kappa}_\infty$ witnessing $\boldsymbol{\kappa}_\infty$ is Suslin, then $\delta_S^\infty = \Theta$. This means that $\Theta^{L(S^\infty,\mathbb{R})} = \Theta$ and thus $L(S^\infty,\mathbb{R}) = L(\mathcal{P}(\mathbb{R}))$. Thus S is strongly maximal. ❑

6. Strong ∞-Borel codes

Definition 6.1. An ∞-Borel code $S \subseteq \kappa$ is *strong* if player II wins the following game $G_{\mathrm{strong}}(S)$. Player I and II take turns playing ordinals below κ. In the end $f \in \kappa^\omega$ is played and we let S_f be the collapse of $(f[\omega], S \cap f[\omega])$. II wins if S_f is a Borel code, i.e., in $\mathrm{BC}_{<\omega_1}$ and $A_{S_f} \subseteq A_S$ as computed in V.

In a world with the axiom of choice this amounts to saying that a club of $\sigma \in [\kappa]^\omega$ satisfies $A_{S_\sigma} \subseteq A$. Without choice, the club must be witnessed by a strategy.

The game described can be cast appropriately so that ordinal determinacy will yield the determinacy of the game either assuming AD$^+$ or that κ is below the supremum of the Suslin cardinals. See [**CK09**, §2.2.4] for more on AD$^+$, ordinal determinacy and references. The point is that the map $f \mapsto \hat{S}_f$ is continuous where $\hat{S}_f$ is the canonical coding of S_f by a real given f as input, i.e., an enumeration of $f[\omega]$. The winning condition is that $\hat{S}_f$ be a Borel code and $A_{\hat{S}_f} \subseteq A_S$.

Clearly if T is a tree on $\omega \times \kappa$, then T is a strong ∞-Borel code for $p[T]$ since II need only ensure that $T \cap \sigma$ be sufficiently elementary in S, in this case, $T \cap \sigma$ must be a tree on $\omega \times \sigma$. Then $p[T_\sigma] = p[T \cap \sigma] \subseteq p[T]$ just by absoluteness. That the converse holds is the content of the next theorem.

Theorem 6.2. If $S \subseteq \kappa$ is a strong ∞-Borel code, then there is a tree T on κ with $p[T] = A_S$, in particular, A_S is κ-Suslin.

Remark. The map $S \mapsto T$ depends on a winning strategy in the game described above and there is no uniform way to produce those winning strategies.

Proof. Let $\sigma : \kappa^{<\omega} \to \kappa$ be II's winning strategy in $G_{\text{strong}}(S)$. Then let $(x, f) \in [T]$ iff $S \cap f_0[\omega]$ is σ-closed and f_1 witnesses $x \in A_{S_{f_0}}$. We need to show $x \in A_S \iff x \in p[T]$.

If $x \in p[T]$, then let $(x, f) \in [T]$ and let g be a play of the game with II using σ and $g[\omega] = f_0[\omega]$. Then $x \in A_{S_{f_0}} \subseteq A_S$. If $x \in A_S$, then take $\alpha > \kappa$ so that $L_\alpha[S, x, \sigma]$ is a model of some reasonable fragment of ZFC and $L_\alpha[S, x, \sigma] \models$ "S is an ∞-Borel code & $x \in A_S$". Let $\{S, x, \sigma\} \subseteq N \prec L_\alpha[S, x, \sigma]$. Choose f_0 so that $f_0[\omega] = N \cap \kappa$. Since $L_{\bar\alpha}[\bar S, x, \bar\sigma] \models$ "$\bar S$ is an ∞-Borel code & $x \in A_{\bar S}$", we have $\bar S = S_{f_0[\omega]}$, $S_{f_0[\omega]}$ is an ∞-Borel code, and $x \in S_{f_0[\omega]}$, so we can choose f_1 witnessing this. This f satisfies $(x, f) \in [T]$, as desired. ❑

While the passage from a strong ∞-Borel code to the corresponding Suslin representation is not uniform, it turns out that the passage from ∞-Borel code to corresponding strong code is uniform and this will yield the closure of the Suslin cardinals. Recall the codes S_*^x and S_* from Lemma 5.3.

Theorem 6.3. If δ_S^∞-determinacy holds, then S_* is strong.

Proof. By assumption $G_{\text{strong}}(S_*)$ is determined so we need only show that I does not win. Suppose σ is a I winning strategy. We aim to produce a play f consistent with σ so that $(S_*)_f = S_*^x$ for some x. Since $A_{S_*^x} \subseteq A_S = A_{S_*}$ this will yield a contradiction.

On a cone of x, consider the closed game $\mathcal{G}^x$ where in round i player I plays $\alpha_{2i} < \delta_S^x$ and II plays $\alpha_{2i+1} < \delta_S^x$, $\beta_i < \delta_S^\infty$, and $k_i \in 2$. In the end $f \in (\delta_S^x)^\omega$, $g \in (\delta_S^\infty)^\omega$, and $x \in 2^\omega$ are played. For II to win, $g[\omega]$ must be σ-closed, and x must determine a map $\pi_x : f[\omega] \to g[\omega]$ which must be an embedding of $S_*^x \cap f[\omega]$ into S_*. This game is closed for II.

Let $\mathcal{G}^\infty$ be the corresponding game played in $H_S^\infty(\sigma_S^\infty, S_*, S_*^\infty)$. Let G be generic over V for collapsing δ_S^∞ to ω and have II play so that $f[\omega] = \delta_S^\infty$, $g[\omega] = j_S[\delta_S^\infty]$, and x codes $j_S : \delta_S^\infty \xrightarrow[\text{1-1}]{\text{onto}} j_S[\delta_S^\infty]$. This play is winning against any play by I. By absoluteness for winning a closed game, II wins $\mathcal{G}^\infty$ in $H_S^\infty(\sigma_S^\infty, S_*, S_*^\infty)$. So by Łos' lemma, II wins $\mathcal{G}^x$ on a cone of x with canonical strategy τ^x.

Now fix x in the cone where II wins and have II play τ^x against an enumeration of δ_S^x and let $\pi : \delta_S^x \to \delta_S^\infty$ be II's isomorphism and $g \in (\delta_S^\infty)^\omega$ be II's enumeration of $\pi[\delta_S^x]$. Since $g[\omega]$ is σ-closed, $\sigma(g)[\omega] = g[\omega]$ and so $S_*^x \simeq_\pi S_*$. So g is a play by II that defeats σ. ❑

Corollary 6.4. If $\kappa < \boldsymbol{\kappa}_\infty$ is Suslin as witnessed by a tree S on $\omega \times \kappa$, then $\lambda \leq \delta_S^\infty$ where λ is the next Suslin cardinal after κ.

Proof. Since $\kappa < \boldsymbol{\kappa}_\infty$, $\delta_S^\infty < \boldsymbol{\kappa}_\infty$ by Theorem 5.13 and the corollaries following it. Set $x \in A \iff x \notin p[S]$, and note that A is not κ-Suslin. Fix φ so that $A = A_{(\varphi, S)}$ and let $\hat S$ be the corresponding ∞-Borel code. Then $\hat S_*$ is a strong ∞-Borel code of size δ_S^∞ so A is δ_S^∞-Suslin and so $\delta_S^\infty \geq \lambda$. ❑

Theorem 6.5. Suppose that κ is a limit of Suslin cardinals and that κ-ordinal determinacy holds, then κ is Suslin.

The hypotheses are satisfied if either $\kappa < \boldsymbol{\kappa}_\infty$ or $\kappa < \Theta$ and AD^+ holds. The Suslin cardinals do form an ω-club, so if $\mathrm{cf}(\kappa) = \omega$, there is nothing to do, hence assume $\mathrm{cf}(\kappa) > \omega$. The following lemma is reminiscent of 2.5.

Lemma 6.6. Let λ be a cardinal and suppose $\mathcal{B}_{<\lambda} \subsetneq \mathcal{B}_\infty$. Then $\mathcal{B}_{<\lambda}$ has a λ-length antichain.

Proof. First suppose λ is regular. Let $\sim$ be the relation on BC_∞ given by $T \sim T' \overset{\mathrm{df}}{\Longleftrightarrow} A_T = A_{T'}$. Let γ be least such that $\mathcal{B}_\gamma \setminus \mathcal{B}_{<\lambda} \neq \emptyset$ and let $S \in \mathrm{BC}_\gamma \setminus \mathrm{BC}_{<\lambda}$ code such a set. Work now in $L[S,\sim]$. For readability let $\mathbb{Q}_{<\lambda} = \mathrm{BC}_{<\lambda}^{L[S,\sim]}/\sim$ and let $\mathbb{Q} = \mathrm{BC}_{\lambda}^{L[S,\sim]}/\sim$. The point is that $\mathbb{Q}$ is a countably generated complete Boolean algebra in $L[S,\sim]$ and hence for any cardinal λ of $L[S,\sim]$, the "λ^{th} level" of $\mathbb{Q}$, essentially $\mathbb{Q}_{<\lambda}$, is either already complete or has a $\mathrm{cf}^{L[S,\sim]}(\lambda)$-sized antichain unbounded in $\mathbb{Q}_{<\lambda}$.

In $L[S,\sim]$, λ is regular and there is a least $\gamma' \leq \gamma$ so that $L[S,\sim] \models \mathbb{Q}_{\gamma'} \setminus \mathbb{Q}_{<\lambda} \neq \emptyset$. Choose $S' \in \mathrm{BC}_{\gamma'}^{L[S,\sim]}$ witnessing this. Then $S'/\sim = \bigvee_{\alpha<\gamma'} S_\alpha/\sim$ with $S_\alpha \in \mathrm{BC}_{<\lambda}^{L[S,\sim]}$. Now inside $L[S,\sim]$ take $S'_\alpha \in \mathrm{BC}_{<\lambda}^{L[S,\sim]}$ so that $S'_\alpha/\sim = \bigvee_{\alpha'<\alpha} S_\alpha$. We can thin out this sequence to a strictly increasing sequence in $\mathbb{Q}_{<\lambda}$, $\langle S'_{\alpha_\xi} : \xi < \rho\rangle$, where $\rho \geq \mathrm{cf}(\lambda)$. This gives an antichain of length $\mathrm{cf}^{L[S,\sim]}(\lambda)$ of codes in $\mathrm{BC}_{<\lambda}^{L[S,\sim]}$. Since λ is regular in $L[S,\sim]$ this does it.

The preceding paragraph actually showed that for any λ, if $\mathbb{Q}_{<\lambda} \neq \mathbb{Q}$, then a sequence $\langle S_\alpha \in \mathrm{BC}_{\lambda_\alpha} : \alpha < \rho\rangle$, where $\rho = \mathrm{cf}^{L[S,\sim]}(\lambda)$, can be found in $L[S,\sim]$ such that $S_\alpha/\sim$ and $S_{\alpha'}/\sim$ are incompatible for $\alpha < \alpha' < \rho$.

Suppose now that λ is singular in $L[S,\sim]$ with $\sup_{\alpha<\rho} \lambda_\alpha = \lambda$ $\lambda_\alpha < \lambda_{\alpha'}$ for $\alpha < \alpha' < \rho$ where $\rho = \mathrm{cf}^{L[S,\sim]}(\lambda)$. We may assume the S_α from the preceding paragraph are of the form $S_\alpha = \bigvee_{\gamma<\lambda_\alpha} S_{\alpha,\gamma}$ with $\langle S_{\alpha,\gamma} : \gamma < \lambda_\alpha\rangle$ an antichain in $\mathbb{Q}_{\lambda_\alpha}$. This gives us an antichain in $\mathbb{Q}_{<\lambda}$ of length λ. ❑

Corollary 6.7. If $\kappa < \boldsymbol{\kappa}_\infty$ is a limit of Suslin cardinals, then there is a κ-length antichain in $\mathcal{B}_{<\kappa}$.

Proof. The only point is that $\mathcal{B}_{<\kappa} \neq \mathcal{B}_\infty$. ❑

Lemma 6.8. If $S_\alpha \in \mathrm{BC}_{<\kappa}$ is a strong code for all $\alpha < \kappa$, then $S = \bigvee S_\alpha$ is strong.

Proof. All we need to do is see that player I cannot win $G_{\mathrm{strong}}(S)$. Suppose I did win with σ. Since $\mathrm{cf}(\kappa) > \omega$ take α closed under σ so that $S_\alpha \in \mathrm{BC}_\alpha$. Now have II play a winning strategy σ_α in $G_{\mathrm{strong}}(S_\alpha)$ against σ. Let $f \in \alpha^\omega$ be the resulting play. $A_{(S_\alpha)_f} \subseteq A_{S_\alpha} \subseteq A_S$. So this is a win for II. This contradiction shows that II must win $G_{\mathrm{strong}}(S)$. ❑

Now we can easily prove the theorem:

Proof of Theorem 6.5. Let κ be a limit of Suslin cardinals and assume κ-ordinal determinacy holds and $\mathrm{cf}(\kappa) > \omega$. Let $\langle S_\alpha : \alpha < \kappa\rangle$ be an antichain by Lemma 6.6. For each α let $S_{\alpha,*}$ be the associated strong code. $S_{\alpha,*} \in \mathrm{BC}_{<\kappa}$ by Corollary 5.15. By Lemma 6.8, $T = \bigvee_{\beta<\kappa}\big(\bigvee_{\alpha<\beta}(S_{\alpha,*} \times S_{\beta,*})\big)$ is strong and hence A_T is κ-Suslin. A_T is a prewellordering of length κ and hence is not λ-Suslin for any $\lambda < \kappa$, by the Kunen-Martin theorem. So A_T witnesses that κ is a Suslin cardinal. ❑

6.1. Equivalence of Suslin cardinals and reliable cardinals. Recall Definition 2.1 and the discussion around it. For a tree T on $\omega \times \kappa$ set $T' \overset{\text{df}}{=} T^{\{\varphi_i^T\}_i}$, that is, $T' \subseteq T$ is the tree induced by the semiscale associated to T, i.e.,

$$T' = \{(x|i, b_x^T|i) : x \in p[T] \text{ and } i \in \omega\}$$

A tree T, on $\omega \times \kappa$, is a *tree of a scale* if $T = T'$. A cardinal κ is called *reliable* if there is a tree on $\omega \times \kappa$ such that T is the tree of a semiscale and $|T| = \kappa$. Call a tree T, as in the definition of reliability, a *witness to the reliability of* κ. If κ is reliable, then it is possible to find a reliability witness such that $\forall \alpha < \kappa \exists x \in p[T]\, b_x^T(0) = \alpha$, i.e., $\varphi_0^T : p[T] \xrightarrow{\text{onto}} \kappa$, call such a reliability witness a *good witness to the reliability of* κ.

A Suslin cardinal is reliable; just take T on $\omega \times \kappa$ witnessing that κ is Suslin, then T' is a reliability witness. The closure of the Suslin cardinals below their supremum, Theorem 6.5 yields that every reliable cardinal is Suslin, thus providing a direct "structural" way of recognizing the Suslin cardinals.

Theorem 6.9. Every reliable cardinal is Suslin.

Proof. Suppose there is a reliable cardinal, $\kappa < \boldsymbol{\kappa}_\infty$, that is not a Suslin cardinal. By the closure of the Suslin cardinals below $\boldsymbol{\kappa}_\infty$, κ is not a limit of Suslin/reliable cardinals and hence there is a largest Suslin cardinal $\lambda < \kappa$ and $\boldsymbol{S}_\lambda = \boldsymbol{S}_\kappa$.

If γ is the next Suslin cardinal after λ, then $\lambda < \kappa < \gamma$, so $\gamma \neq \lambda^+$ and thus [**Jac10**, Lemma 3.7] gives that $\mathrm{cf}(\gamma) = \omega$ and [**Jac10**, Theorem 3.28] yields that λ is regular and $\mathrm{Scale}(\boldsymbol{S}_\lambda)$. In particular $\boldsymbol{S}_\lambda$ has the prewellordering property and hence is closed under arbitrary wellordered unions. This in turn means that there can be no λ^+-sequence of mutually disjoint sequence of sets in $\boldsymbol{S}_\lambda$, since if $\langle A_\alpha : \alpha < \lambda^+ \rangle$ were such a sequence, $x \prec y \overset{\text{df}}{\iff} (x, y) \in \bigcup_{\beta < \xi < \lambda^+} A_\beta \times A_\xi$ would be a $\boldsymbol{S}_\lambda$ wellfounded relation of rank λ^+ and this violates the Kunen-Martin theorem 2.2.

Let T be a good reliability witness for κ. Then $A_\alpha = \{x \in p[T] : \varphi_0^T(x) = \alpha\}$ is a sequence of disjoint non-empty sets of length κ in $\boldsymbol{S}_\kappa = \boldsymbol{S}_\lambda$. This contradicts the preceding paragraph. ❑

7. Equivalence of AD^+ with the closure of the Suslin cardinals below Θ

Recall, working in ZFC for the moment, that for δ a strong limit cardinal and $G \subseteq \mathrm{Col}(\omega, {<}\delta)$ generic, the set $\mathbb{R}_G^* = \bigcup_{\alpha < \delta} \mathbb{R}^{V[G|\alpha]}$ is called the set of symmetric reals for $\mathrm{Col}(\omega, {<}\delta)$. That a certain set of reals $\mathbb{R}^*$ is the symmetric reals for some generic $G \subseteq \mathrm{Col}(\omega, {<}\delta)$ can be axiomatized as follows:

(1) Every real $x \in \mathbb{R}^*$ is in $V[g]$ for generic $g \subseteq \mathbb{P}$ for some $\mathbb{P} \in V_\delta$.
(2) $\sup\{||x|| : x \in \mathrm{WO} \cap \mathbb{R}^*\} = \delta$.
(3) For $x, y \in \mathbb{R}^*$, $L[x, y] \cap \mathbb{R} \subseteq \mathbb{R}^*$.

If $\mathbb{R}^*$ is the set of symmetric reals for $\mathrm{Col}(\omega, {<}\delta)$, then $V(\mathbb{R}^*) \cap \mathbb{R} = \mathbb{R}^*$ is a model of ZF. The following is known as the Derived Model Theorem.

Theorem 7.1 (Woodin)**.** Assume ZFC and that δ is a limit of Woodin cardinals and $\mathbb{R}^*$ is the set of symmetric reals for $\mathrm{Col}(\omega, {<}\delta)$. Define

$$\Gamma^*_{\mathrm{AD}^+} = \{A \subseteq \mathbb{R}^* : A \in V(\mathbb{R}^*) \text{ and } L(A, \mathbb{R}^*) \models \mathrm{AD}^+\}$$

Then $L(\Gamma^*_{\mathrm{AD}^+}, \mathbb{R}^*) \models \mathrm{AD}^+$. ⊣

The model $L(\Gamma^*_{\mathrm{AD}^+}, \mathbb{R}^*)$ is called the derived model. For $\mathbb{R}^*$ the symmetric reals for $\mathrm{Col}(\omega, {<}\delta)$, in $V(\mathbb{R}^*)$ define

$$A \in \mathrm{Hom}^* \overset{\mathrm{df}}{\iff} A = p[T] = \mathbb{R} \setminus p[S] \text{ for } T, S \text{ trees on } \omega \times \delta$$

Then $\mathrm{Hom}^* = \boldsymbol{S}_{<\infty}^{L(\Gamma^*_{\mathrm{AD}^+}, \mathbb{R}^*)}$.

The derived model theorem gives a way of producing models of AD^+ from large cardinals. Starting with a model of $\mathrm{ZF} + \mathrm{AD} + \mathrm{DC}_{\mathbb{R}}$, define

$$\Gamma_{\mathrm{AD}^+} = \{A : L(A, \mathbb{R}) \models \mathrm{AD}^+\}.$$

Woodin produced an inner model N in a generic extension of V, so that

(1) $\delta = \omega_1^V$ is a limit of Woodin cardinals in N,
(2) $\mathbb{R}^V$ is a set of symmetric reals over N for $\mathrm{Col}(\omega, {<}\delta)$,
(3) $(\Gamma^*_{\mathrm{AD}^+})^{N(\mathbb{R})} = (\Gamma_{\mathrm{AD}^+})^V$, and
(4) $(\boldsymbol{S}_\infty)^V = (\mathrm{Hom}^*)^{N(\mathbb{R})}$
(5) For $S \in (\Gamma^*_{\mathrm{AD}^+})^{N(\mathbb{R})}$, $\prod_S \omega_2^{L(S,x)}/\mu$ is the same computed in V or in $L(\Gamma^*_{\mathrm{AD}^+}, \mathbb{R})^{N(\mathbb{R})}$.

The item (5) is a little technical, but we need it below. In particular, it implies that if S is a tree in $L(\Gamma^*_{\mathrm{AD}^+}, \mathbb{R})^{N(\mathbb{R})}$ witnessing that $\boldsymbol{\kappa}_\infty$ is Suslin, then $L(S^\infty, \mathbb{R})$ is the same computed in V or in $L(\Gamma^*_{\mathrm{AD}^+}, \mathbb{R})^{N(\mathbb{R})}$.

These results imply that every AD^+ model is a derived model and more generally, and more importantly for us here, every model of AD contains a maximal class inner model of AD^+ containing the reals, and this maximal model of AD^+ contains all of the Suslin sets. So we have the following:

$$L(\boldsymbol{S}_\infty, \mathbb{R}) \subseteq L(\Gamma_{\mathrm{AD}^+}, \mathbb{R}) \subseteq L(\mathcal{B}_\infty, \mathbb{R}) \subseteq L(\mathcal{P}(\mathbb{R})),$$

where in the case that there is no largest degree notion, $L(\mathcal{B}_\infty, \mathbb{R})$ is also the maximal model of AD+uniformization. Moreover, if $\boldsymbol{\kappa}_\infty$ is Suslin, then $L(S^\infty, \mathbb{R}) = L(\Gamma_{\mathrm{AD}^+}, \mathbb{R})$.

The desired characterization of AD^+ models follows almost immediately:

Theorem 7.2 (Woodin)**.** The following are equivalent under $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$

(1) AD^+
(2) $\mathrm{AD} +$ The Suslin cardinals are closed below Θ.

Proof. (1) $\implies$ (2) has already been discussed. So assume (2) holds. If $\boldsymbol{\kappa}_\infty = \Theta$, then we have $\mathrm{AD} +$ all sets are Suslin and this easily gives AD^+. So assume $\boldsymbol{\kappa}_\infty < \Theta$. Fix a tree, S, witnessing $\boldsymbol{\kappa}_\infty$ is Suslin. Theorem 5.19 shows that S is a strongly maximal degree and hence $L(S^\infty, \mathbb{R}) = L(\mathcal{P}(\mathbb{R}))$ for S a tree on $\omega \times \boldsymbol{\kappa}_\infty$ witnessing $\boldsymbol{\kappa}_\infty$ is Suslin. From the derived model theorem, $L(S^\infty, \mathbb{R})$ is the maximal model of AD^+. ❑

8. Appendix

Under AD, $\mathcal{P}(\mathbb{R})$ has a fair amount of structure. One facet of this is the Wadge hierarchy. For A and B sets of reals, A *is Wadge reducible to* B, denoted $A \leq_w B$, if there is a continuous reduction of A to B, that is, there is continuous $f : \mathbb{R} \to \mathbb{R}$ so that $A = f^{-1}[B]$. Wadge showed that, assuming AD,

$$\text{either } A \leq_w B \text{ or } \neg B \leq_w A$$

where $\neg A = \mathbb{R} \setminus A$. This gives a quasi-linear-order to $\mathcal{P}(\mathbb{R})$ with classes

$$[A]_w = \{B : B =_w A \text{ or } B =_w \neg A\}$$

A is *selfdual* if $A =_w \neg A$, otherwise A is *non-selfdual*.

Martin showed that $<_w$ is wellfounded. Set $w(A)$ to be the rank of A in $<_w$. Notice $w(A) < w(B) \iff A <_w B$, but $w(A) \leq w(B) \iff A \leq_w B$ or $A \leq_w \neg B$. Van Wesep showed that for A, $\mathrm{cf}(w(A)) = \omega$ iff A is selfdual [**VW78**].

The height of the Wadge hierarchy is denoted Θ; this ordinal can be alternatively defined by

$$\Theta = \sup\{\alpha : \exists f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}.$$

A *pointclass* (*Wadge class*), is any collection of sets of reals closed under continuous reduction, in other words, a Wadge initial segment of $\mathcal{P}(\mathbb{R})$. To any pointclass Γ let $\neg\Gamma = \{\neg A : A \in \Gamma\}$ and $\Delta_\Gamma = \Gamma \cap \neg\Gamma$. A pointclass, Γ, is *selfdual* if $\Gamma = \neg\Gamma (= \Delta_\Gamma)$.

A set $A \in \Gamma$ is called Γ-*complete* iff $\Gamma = \{B : B \leq_w A\}$. In the case that Γ is non-selfdual, then Γ not only has a complete set, but even has a universal set, that is, a set $U \subseteq \mathbb{R} \times \mathbb{R}$ such that $A \in \Gamma \iff A = U_x$ where $U_x = \{y : U(x, y)\}$. There are even nice collections of universal sets that have the "s-n-m" property (see [**Jac10**]) and thus "light face", or "effective", arguments from descriptive set theory lift to Γ, once a collection of nice universal sets is fixed. Selfdual classes can have complete sets, but a diagonal argument shows that they can never have a universal set.

For any pointclass Γ, let $w_\Gamma = \sup\{w(A) + 1 : A \in \Gamma\}$. If Λ is selfdual and $\mathrm{cf}(w_\Lambda) = \omega$, then for any sequence A_i Wadge cofinal in Λ, letting $A = \bigoplus_i A_i$ we have $w(A) = w_\Lambda$ and $\Lambda' = \{B : B \leq_w A\}$ is the first pointclass past Λ and is selfdual. If $\mathrm{cf}(w_\Lambda) > \omega$, then there is a non-selfdual Γ so that $\Delta_\Gamma = \Lambda$. This follows from the result of Van Wesep mentioned above.

I will use $\vee$, $\wedge$, etc., to operate on pointclasses. So

$$\Gamma \wedge \Gamma' = \{A \cap A' : A \in \Gamma \,\&\, B \in \Gamma'\}, \textstyle\bigwedge_\kappa \Gamma = \left\{\bigcap_{\alpha<\kappa} A_\alpha : \langle A_\alpha : \alpha < \kappa\rangle \in \Gamma^\kappa\right\},$$

etc. For example, Γ is closed under finite unions iff $\Gamma \vee \Gamma \subseteq \Gamma$, and Γ is closed under countable unions iff $\bigvee_\omega \Gamma \subseteq \Gamma$. Notice that as long as Γ has a complete set, then closure under $\bigvee_\omega$ is equivalent to closure under $\exists^\omega$ and I will use these two notions of closure interchangeably.

There are several ordinals other than w_Γ associated to pointclasses, two important ones are:

$$\delta_\Gamma = \sup\{|| \leq || : \leq \in \Delta_\Gamma \text{ is a prewellordering}\}$$
$$\sigma_\Gamma = \sup\{|| \prec || : \prec \in \Gamma \text{ is a wellfounded relation}\}$$

Here I use $|| \prec ||$ to mean the ordinal rank of $\prec$. I will write $||x||_\prec$ for the rank of x in $\prec$. For a pointclass Γ, if $\Delta \wedge \Delta \subseteq \Delta$, then $\delta_\Gamma \leq \sigma_\Gamma$, since for any prewellordering $\leq \in \Delta$ we can define $x \prec y \overset{\text{df}}{\iff} x \leq y \wedge y \not\leq x$.

8.1. The generalized projective hierarchy. A pointclass Λ is *strongly closed* if it is closed under real quantification and finite Boolean operations. Notice that if Λ is strongly closed, then Λ is selfdual. The smallest strongly closed pointclass is the pointclass of projective sets, $\bigcup_i \Sigma^1_i$. If Λ is strongly closed, then all three ordinals w_Λ, δ_Λ, and σ_Λ are the same [**KSS81**].

We will use the hierarchy of Levy classes more than the Wadge hierarchy. A pointclass is a *Levy class* if it is non-selfdual and closed under one or both of the real quantifiers. If Γ is a Levy class closed under $\exists^{\mathbb{R}}$ and not $\forall^{\mathbb{R}}$, then Γ is a Σ-Levy class and its complement is the corresponding Π-Levy class. In [**Ste81b**], Steel shows that for any non-selfdual pointclass Γ exactly one of Γ or $\neg\Gamma$ satisfies separation, we will use this to distinguish Σ-Levy classes and Π-Levy classes in case Γ is closed under both real quantifiers. If a Levy class, Γ, is closed under both real quantifiers, then take the Σ-Levy class to be the side on which separation fails; it turns out that the Σ-class actually has the prewellordering property. Facts about Levy classes and the associated ordinals are taken from [**KSS81, Ste81a, Bec85**]; a good source is [**Jac10**, "Wadge degrees and abstract pointclasses"].

By Wadge comparability we get a generalized projective hierarchy $(\Sigma^1_\alpha, \Pi^1_\alpha)$, for $\alpha < \Theta$, where Σ^1_α is the α^{th} Σ-Levy class and Π^1_α is the corresponding α^{th} Π-Levy class. When discussing the general theory of Levy classes we will assume that if Λ is strongly closed, then there is κ so that $\bigvee_\kappa \Lambda \not\subseteq \Lambda$. Steel showed that the least such κ is $\mathrm{cf}(w_\Lambda)$ [**Ste81b**]. If $\mathrm{cf}(w_\Lambda) = \omega$, then $\bigvee_\omega \Lambda \not\subseteq \Lambda$. Under the assumption, if $\mathrm{cf}(w_\Lambda) > \omega$, then $\Lambda = \Delta_\Gamma$ for some pointclass Γ by the result of Van Wesep mentioned above; Steel showed that Γ is a Levy class closed under $\forall^{\mathbb{R}}$ with the prewellordering property [**Ste81b, Ste81a, Jac10**]. Conversely, if Γ is non-selfdual and has the prewellordering property, then $\bigvee_\kappa \Delta_\Gamma \not\subseteq \Delta_\Gamma$ for κ least such that there is a Γ-norm of length κ on a set in $\Gamma \setminus \Delta_\Gamma$. If Λ is selfdual and properly contained in the ∞-Borel sets, then, Λ is not closed under arbitrary wellordered unions. From results below, if Λ is strongly closed and properly contained in the ∞-Borel sets, then a fair amount of the Levy hierarchy above Λ is also contained in the ∞-Borel sets.

We will quickly review a few of the relevant facts concerning the generalized projective hierarchy. Set

$$\delta^1_\alpha = \delta_{\Sigma^1_\alpha} \quad \sigma^1_\alpha = \sigma_{\Sigma^1_\alpha} \quad w^1_\alpha = w_{\Delta^1_\alpha}$$

For λ a limit I will also use $\delta^1_{<\lambda}$, $\sigma^1_{<\lambda}$, and $w^1_{<\lambda}$ in the obvious way corresponding to the class $\Delta^1_{<\lambda} = \bigcup_{\alpha<\lambda} \Delta^1_\alpha$. Recall $\delta^1_{<\lambda} = \sigma^1_{<\lambda} = w^1_{<\lambda}$ since $\Delta^1_{<\lambda}$ is strongly closed; I will use $\delta^1_{<\lambda}$ when there is no particular reason to use one of the other two. Notice $\delta^1_\alpha = \delta_{\Pi^1_\alpha} = \delta_{\Delta^1_\alpha}$, but, in general, $\sigma^1_\alpha \neq \sigma_{\Pi^1_\alpha}$. Clearly, $\delta^1_\alpha \leq \sigma^1_\alpha$ and $w^1_\alpha \leq \delta^1_{\alpha+1}$, since for $A \in \Delta^1_\alpha$, the relation $x \leq y$ iff $f_x^{-1}[A] \leq_w f_y^{-1}[A]$ can be seen to be $\Delta^1_{\alpha+1}$ where $f_x : \mathbb{R} \to \mathbb{R}$ is a (Lipschitz) continuous function coded by x. The following lemma summarizes several properties of the generalized projective sets.

Lemma 8.1. For $\alpha < \Theta$ the following hold:

(1) If Σ^1_α has the prewellordering property, then it is closed under arbitrary wellordered unions.
(2) If Σ^1_α is closed under finite intersections, then there is a δ^1_α complete measure on σ^1_α. (The argument for this essentially appears in [**Kec78**, §5].)
(3) If Π^1_α is closed under finite unions and has the prewellordering property, then
   (a) $\delta^1_\alpha = ||\leq||$, where $\leq$ is any Π^1_α norm on a complete Π^1_α set.
   (b) $\sigma^1_\alpha = \delta^1_\alpha$ and so δ^1_α is measurable.
   (c) Δ^1_α is closed under $<\delta^1_\alpha$-wellordered unions. $\dashv$

To determine whether or not, for example, Π^1_α is closed under finite unions, has the prewellordering property, etc., depends on the nature of the projective hierarchy to which Π^1_α belongs.

For λ a limit, $\Delta^1_{<\lambda}$ is the largest strongly closed pointclass contained in Δ^1_λ; $\Delta^1_{<\lambda}$ is called the *base* of the projective hierarchy $\mathrm{Proj}_\lambda = \langle(\Sigma^1_{\lambda+n}, \Pi^1_{\lambda+n}) : n \in \omega\rangle$. By First Periodicity, the behavior of the prewellordering property in the λ^{th} projective hierarchy, Proj_λ, is determined by what happens on $(\Sigma^1_\lambda, \Pi^1_\lambda)$ and alternates between the Σ-side and the Π-side.

The projective hierarchy Proj_λ is *type I* if $\mathrm{cf}(\lambda) = \mathrm{cf}(w^1_{<\lambda}) = \omega$. In this case, $\Delta^1_{<\lambda} \subsetneq \bigoplus_\omega \Delta^1_{<\lambda} \subsetneq \Delta^1_\lambda$. In type I hierarchies:

$$\Sigma^1_\lambda = \bigvee\nolimits_\omega \Delta^1_{<\lambda} = \Sigma^0_1(\Delta^1_{<\lambda}) = \Sigma^0_1(A) \quad \text{and} \quad \Sigma^1_{\lambda+1} = \Sigma^1_1(\Delta^1_{<\lambda}) = \Sigma^1_1(A)$$

for any $A = \bigoplus_i A_i$ with $\langle A_i : i \in \omega\rangle$ Wadge cofinal in $\Delta^1_{<\lambda}$, where $\Sigma^0_1(\Lambda)$ is the smallest non-selfdual pointclass containing Λ and closed under $\exists^\omega$, $\Sigma^1_1(\Lambda)$ is the smallest pointclass containing Λ and closed under $\exists^\mathbb{R}$ and $\forall^\omega$, and where $\Sigma^i_j(A) = \Sigma^i_j(\{A\})$. Σ^1_λ has the prewellordering property and so it is closed under arbitrary wellordered unions and, moreover, it is closed under finite intersections, but not $\forall^\omega$.

In [**Bec85**], Becker proved the following:

Lemma 8.2. For λ a limit of countable cofinality

$$\delta^1_{\lambda+1} = \sigma^1_\lambda = w^1_\lambda.$$

equivalently

$$\delta^1_1(A) = w(\Delta^0_1(A)) = \sigma_{\Sigma^0_1(A)}$$

where $A = \bigoplus_{i\in\omega} A_i$ for some $\langle A_i : i \in \omega\rangle$ Wadge cofinal in $\Delta^1_{<\lambda}$. $\dashv$

Of course we already knew that $w(\Sigma^0_1(A)) \leq \delta^1_1(A)$ and that $\sigma_{\Sigma^0_1(A)} \leq \sigma^1_1(A) = \delta^1_1(A)$. Becker argues

$$\sigma^1_1(A) \leq \sigma_{\Sigma^0_1(A)} \leq w(\Sigma^0_1(A)).$$

This shows that Δ^1_λ is quite a bit larger than $\Delta^1_{<\lambda}$ at least as far as the Wadge hierarchy is concerned (recall $\delta^1_{\lambda+1}$ is measurable).

In the case that $\mathrm{cf}(\lambda) > \omega$, then, as mentioned above, there is a Levy class Γ so that $\Delta_\Gamma = \Delta^1_{<\lambda}$; if chosen to be on the side that separation fails, Γ is closed under $\forall^\mathbb{R}$ and satisfies the prewellordering property. It follows that $\Gamma = \Pi^1_\lambda$, if Γ is not closed under $\exists^\mathbb{R}$, and $\Gamma = \Sigma^1_\lambda$, otherwise. So for λ a limit of uncountable cofinality, $\Delta^1_{<\lambda} = \Delta^1_\lambda$ and hence $\delta^1_\lambda = w^1_\lambda = w^1_{<\lambda} = \delta^1_{<\lambda}$ and $\mathrm{cf}(\lambda) = \mathrm{cf}(w^1_{<\lambda}) = \mathrm{cf}(w^1_\lambda)$. There are three subcases for the hierarchy when $\mathrm{cf}(\lambda) > \omega$.

If Π^1_λ is not closed under $\exists^\omega$, and hence not under $\exists^\mathbb{R}$, then Proj_λ is called *type II.* In [**Ste81b**], Steel showed that Π^1_λ is closed under $\exists^\omega$ iff Π^1_λ is closed under finite unions. In fact, Steel showed that is separation fails for Γ, then $\Gamma \vee \Gamma \subseteq \Gamma \iff \bigvee_\omega \Gamma \subseteq \Gamma$.

If w^1_λ is singular, then $\Pi^1_\lambda \vee \Delta^1_\lambda \not\subseteq \Pi^1_\lambda$ and thus Proj_λ is type II [**Ste81a**, pg 150]. Thus in the non-type II case w^1_λ is regular and $\lambda \geq \mathrm{cf}(\lambda) = \mathrm{cf}(w^1_\lambda) = w^1_\lambda \geq \lambda$, so $\lambda = w^1_\lambda$. Also $w^1_\lambda = \delta^1_\lambda$, the last equality having already been discussed.

The projective hierarchy, Proj_λ is called *type III* if Π^1_λ is not closed under $\exists^\mathbb{R}$, but is closed under finite unions, equivalently, countable unions. In the type III case, $\lambda = \delta^1_\lambda = w^1_\lambda$ is regular, even measurable.

It is open whether or not w^1_λ regular implies that $\Pi^1_\lambda \vee \Pi^1_\lambda \subseteq \Pi^1_\lambda$. If so, then whether or not w^1_λ is regular or not would determine the type of Proj_λ.

Finally, if $\mathrm{cf}(\lambda) > \omega$ and Π^1_λ is closed under both real quantifiers, then prewellordering holds on the Σ-side. In this case, Proj_λ is a *type IV* hierarchy. As in the type III case, δ^1_λ is measurable. In a type IV hierarchy, it is the pair $(\Sigma^1_{\lambda+1}, \Pi^1_{\lambda+1})$ that has weak closure properties. $\Pi^1_{\lambda+1} = \Sigma^1_\lambda \wedge \Pi^1_\lambda$. This pointclass is closed under $\forall^{\mathbb{R}}$, but not finite unions, it has the prewellordering property.

For A a set of reals let $\Pi(A)$ be whichever of $\Pi^1_1(A)$ or $\Pi^1_2(A)$ has the prewellordering property. Let $\Sigma(A)$, $\Delta(A)$, $\sigma(A)$, and $\delta(A)$ have the obvious definitions. The following lemma, a corollary of Lemma 8.1, summarizes what we will need.

Lemma 8.3. Let A be a set of reals,

(1) $\delta(A)$ is the rank of any $\Pi(A)$-norm on a complete $\Pi(A)$ set.
(2) $\sigma(A) = \delta(A)$ is measurable.
(3) $\Delta(A)$ is closed under $<\delta(A)$-wellordered unions/intersections. $\dashv$

8.2. Coding Lemma. One version of the Moschovakis Coding Lemma, see [**Mos09, Jac10**], is as follows:

Theorem 8.4 (Moschovakis)**.** Suppose Γ is a Σ-Levy class closed under finite intersection and let $\prec$ be a wellfounded relation in Γ of rank γ. Let $U \subseteq \mathbb{R}\times\mathbb{R}$ be universal for Γ. For any $f : \gamma \to \mathcal{P}(\mathbb{R})$, there is ε so that for all $\beta < \gamma$:

(1) If $f(\beta) \neq \emptyset$, then $\exists x, y\, ||x||_\prec = \beta \,\&\, U_\varepsilon(x, y)$.
(2) For all x, y, $U_\varepsilon(x, y)$ implies $x \in \text{field}(\prec)$ and $y \in f(||x||_\prec)$.

ε is called a *code* of f. $\dashv$

If Γ is a Σ-Levy class closed under finite intersections and $\prec$ is the strict part of a prewellordering $\preceq$ of length γ with both $\prec$ and $\preceq$ in Γ, then for any $S \subseteq \gamma$ there is a $\preceq$ invariant code for $C \in \Gamma$ for S, i.e., for $x \in \text{field}(\preceq)$:

$$C(x) \Longleftrightarrow S(||x||_\prec)$$

Since the same is true for $\gamma \setminus S$ we have S is Δ in $\preceq$. The following variant of the coding lemma is discussed in [**Jac10**, §2.2].

Theorem 8.5. If Γ is a Π-Levy class closed under $\vee$ with the prewellordering property, e.g., $\Pi(A)$ for any set A, then any subset of δ_Γ has a Δ_Γ-code (rather than a $\Delta_{\exists^{\mathbb{R}}\Gamma}$-code) with respect to a fixed Γ-norm on a complete Γ-set. $\dashv$

There are several other variants of the coding lemma. I will use the terminology $S \in \Gamma$ to mean that there is a prewellordering $\leq$ in Γ so that $\{x : ||x||_\leq \in S\} \in \Gamma$. So for example, the preceding theorem can be stated as $\mathcal{P}(\delta_\Gamma) \subseteq \Delta_\Gamma$.

One way the coding lemma will be used is given by the following corollary.

Corollary 8.6. If M and N are models of $\mathrm{ZF} + \mathrm{AD}$, then for all $\gamma < \Theta^N \cap \Theta^M$, $\mathcal{P}(\gamma) \cap M = \mathcal{P}(\gamma) \cap N$. $\dashv$

In [**KKMW81**] it is shown that AD implies its own strengthening to $<\boldsymbol{\kappa}_\infty$-ordinal determinacy. What is actually shown is that if $\gamma < \Theta$ and $A \subseteq \gamma^\omega$ is κ-Suslin/κ-co-Suslin for some $\kappa < \Theta$, then the game on γ, $G^\gamma(A)$ is determined. The determinacy of of $G^\gamma(A)$ is absolute between any M and N modeling AD and having the same reals provided $\gamma < \Theta^M \cap \Theta^N$ and the set $A \in M \cap N$, since the Coding Lemma will guarantee that these two models will have the same strategies, since a strategy "is" a subset of γ. This gives the downward absoluteness of ordinal determinacy mentioned in the introduction.

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Department of Mathematics, Boise State University, 1910 University Drive, Boise, ID 83725-1555

URL: http://sites.google.com/site/richardketchersid/

E-mail address: ketchers@math.muohio.edu