

# Subcomplete forcing and its forcing principles

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Focus: subcomplete forcing

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- The Subcomplete Forcing Axiom,  $\text{SCFA}$  ( $\Gamma$  =the collection of all subcomplete forcings)

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- Every  **$\omega_2$ -distributive forcing** is equivalent to a subcomplete forcing. (F.)

# Forcing SCFA



## Forcing SCFA

- Assuming the existence of a supercompact cardinal  $\kappa$ , one can iterate proper forcings with countable support, with iterands given by a Laver function for the supercompactness of  $\kappa$ , producing a model in which  $\text{PFA} + \kappa = \omega_2 = 2^\omega$  holds. (Baumgartner)

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- This can be modified to work for SPFA, by iterating semi-proper forcings with rcs, inserting collapses to  $\omega_1$  after each step in the iteration. (Foreman-Magidor-Shelah)
- This can be modified to work for SCFA, by iterating subcomplete forcings. During the iteration, CH will be forced, and since no reals are added, the final model will satisfy  $\text{SCFA} + \kappa = \omega_2 + \text{CH}$ . (Jensen)

Lower bounds on the consistency strength of these forcing axioms can be proved by showing that these principles imply the failure of  $\square$  principles.

## Definition (Jensen)

Let  $\kappa$  be a cardinal.  $\square_\kappa$  says that there is a  $\square_\kappa$ -sequence, that is, a sequence  $\langle C_\alpha \mid \kappa < \alpha < \kappa^+, \alpha \text{ limit} \rangle$  such that each  $C_\alpha$  is club in  $\alpha$ ,  $\text{otp}(C_\alpha) \leq \kappa$  and for each  $\beta$  that is a limit point of  $C_\alpha$ ,  $C_\beta = C_\alpha \cap \beta$ .

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If  $\lambda$  is also a cardinal, then  $\square_{\kappa,\lambda}$  is the assertion that there is a  $\square_{\kappa,\lambda}$ -sequence, i.e., a sequence  $\langle C_\alpha \mid \kappa < \alpha < \kappa^+, \alpha \text{ limit} \rangle$  such that each  $C_\alpha$  has size at most  $\lambda$ , and each  $C \in C_\alpha$  is club in  $\alpha$ , has order-type at most  $\kappa$ , and satisfies the coherency condition that if  $\beta$  is a limit point of  $C$ , then  $C \cap \beta \in C_\beta$ .

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$\square_{\kappa,\kappa^+}$  holds trivially, so  $\square_\kappa^*$  is the weakest nontrivial principle here.



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# Stationary reflection

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## Definition

Let  $\lambda$  be an uncountable regular cardinal, and let  $S \subseteq \lambda$  be stationary.  $S$  **reflects** at an ordinal  $\alpha < \kappa$  of uncountable cofinality iff  $S \cap \alpha$  is stationary in  $\alpha$ . It reflects iff it reflects at some such  $\alpha$ .

## Stationary reflection and $\square$

### Observation

Suppose  $\square_\kappa$  holds. Then every stationary subset  $S \subseteq \kappa^+$  has a stationary subset  $T$  that does not reflect.

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If  $\vec{C}$  is a  $\square_\kappa$ -sequence, then by Fodor's Theorem, we can let  $T \subseteq S$  be stationary so that all  $C_\beta$ , for  $\beta \in T$ , have the same order type, say  $\gamma$ .



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Now suppose that  $\alpha < \kappa^+$  has uncountable cofinality and  $T \cap \alpha$  is stationary. Then  $C'_\alpha$ , the set of limit points of  $C_\alpha$ , is club in  $\alpha$ , and whenever  $\beta \in C'_\alpha \cap T$ ,  $C_\beta = C_\alpha \cap \beta$  has order type  $\gamma$ .

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This cannot be, since the  $C_\alpha \cap \beta$ s are longer and longer initial segments of  $C_\alpha$ . □

# Friedman's Principle

## Definition

Let  $\kappa > \omega_1$  be a regular cardinal.

For regular  $\tau < \kappa$ , write  $S_{\tau}^{\kappa}$  for the set of  $\alpha < \kappa$  with  $\text{cf}(\alpha) = \tau$ .

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In the context of this definition,  $S$  reflects to  $\alpha = \sup f''\omega_1$ , because  $S \cap \alpha$  contains the club  $C = f''\omega_1$ .

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## Observation

Let  $\kappa$  be a cardinal. Then  $\text{FP}_{\kappa^+}$  implies the failure of  $\square_\kappa$ .

Otherwise, the set of ordinals below  $\kappa^+$  of countable cofinality would have to have a stationary subset that does not reflect.

## The failure of $\square$ under SCFA

### Fact (Jensen)

If  $\kappa > \omega_1$  is a regular cardinal and  $A \subseteq \kappa$  is a stationary set consisting of ordinals of countable cofinality, then the forcing  $\mathbb{P}_A$  to shoot a club of order type  $\omega_1$  through  $A$  is subcomplete.



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## The strong Friedman property

So, for regular  $\kappa > \omega_1$ , SCFA implies  $FP_{\kappa}$ , which implies that every stationary subset of  $S_{\omega}^{\kappa}$  reflects, which implies that  $\square_{\bar{\kappa}}$  fails, if  $\kappa = \bar{\kappa}^+$ .

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Towards reaching the failure of weak square principles, stronger principles of stationary reflection will be useful.

### Theorem (Jensen)

Assume SCFA. Let  $\kappa > \omega_1$  be a regular cardinal. Then the **Strong Friedman Principle  $SFP_{\kappa}$**  holds at  $\kappa$ :

Let  $\langle A_i \mid i < \omega_1 \rangle$  be a sequence of stationary subsets of  $S_{\omega}^{\kappa}$ . Let  $\langle D_i \mid i < \omega_1 \rangle$  be a partition of  $\omega_1$  into stationary sets. Then there is a normal function  $f : \omega_1 \rightarrow \tau$  such that for every  $i < \omega_1$ ,  $f''D_i \subseteq A_i$ .

# Simultaneous stationary reflection

## Definition (Cummings-Magidor)

Let  $\mu$  be a cardinal, let  $\lambda$  be an uncountable regular cardinal, and let  $S \subseteq \lambda$  be stationary. The **simultaneous reflection principle**  $\text{Refl}(\mu, S)$  holds iff for every sequence  $\langle T_i \mid i < \mu \rangle$  of stationary subsets of  $S$ , there exists an  $\alpha < \kappa$  of uncountable cofinality such that for all  $i < \mu$ ,  $T_i$  reflects to  $\alpha$  (" $\vec{T}$  reflects simultaneously at  $\alpha$ ").

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## Observation

Let  $\kappa > \omega_1$  be a regular cardinal. Then  $\text{SFP}_\kappa$  implies  $\text{Refl}(\omega_1, \mathcal{S}_\omega^\kappa)$ .

## Simultaneous reflection and weak $\square$

### Lemma (Cummings, Magidor)

*If  $\kappa$  is singular and  $\square_{\kappa, \mu}$  holds for some  $\mu < \kappa$ , then every stationary subset of  $\kappa^+$  has a collection of  $\text{cf}(\kappa)$  many stationary subsets which do not reflect simultaneously at any point of uncountable cofinality.*



## Simultaneous reflection and weak $\square$

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*If  $\kappa$  is singular and  $\square_{\kappa, \mu}$  holds for some  $\mu < \kappa$ , then every stationary subset of  $\kappa^+$  has a collection of  $\text{cf}(\kappa)$  many stationary subsets which do not reflect simultaneously at any point of uncountable cofinality.*

This lemma, together with our observations on  $\text{SFP}_{\kappa^+}$ , shows that if SCFA holds and  $\kappa$  is singular with  $\text{cf}(\kappa) \leq \omega_1$ , then  $\square_{\kappa, \mu}$  fails for every  $\mu < \kappa$ .

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### Lemma (Cummings, Magidor)

*If  $\kappa$  is an uncountable cardinal and  $\square_{\kappa, \mu}$  holds for some  $\mu < \text{cf}(\kappa)$ , then every stationary subset of  $\kappa^+$  has a stationary subset which does not reflect at any point of uncountable cofinality.*

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This lemma shows that SCFA implies that for every uncountable cardinal  $\kappa$  and every  $\mu < \text{cf}(\kappa)$ ,  $\square_{\kappa, \mu}$  fails. 

## SCFA and CH

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### Fact

CH implies that  $\square_{\omega_1}^*$  holds.

(Because CH implies the existence of a special  $\omega_2$ -Aronszajn tree, and the existence of a special  $\kappa^+$ -Aronszajn tree is equivalent to  $\square_{\kappa}^*$ .)

## The extent of weak $\square$ under SCFA

### Theorem

*Assume SCFA, and let  $\lambda$  be an uncountable cardinal.*

- 1 *If  $\text{cf}(\lambda) \leq \omega_1$ , then  $\square_{\lambda, \mu}$  fails, for every  $\mu < \lambda$ .*
- 2 *If  $\text{cf}(\lambda) \geq \omega_2$ , then  $\square_{\lambda, \mu}$  fails for every  $\mu < \text{cf}(\lambda)$ .*
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The situation is as with MM, except that:

- MM implies that CH fails, and that  $\square_{\omega_1}^*$  fails, and
- MM implies that if  $\text{cf}(\lambda) = \omega$ , then  $\square_{\lambda}^*$  fails, while I don't know the status of  $\square_{\lambda}^*$  under SCFA.

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It can be shown (using an argument of Cummings-Magidor) that the above results are optimal, i.e., from a supercompact cardinal, one can produce a model of SCFA in which, if  $\text{cf}(\lambda) = \omega_1$ , then  $\square_{\lambda}^*$  holds, etc.



## Generalized stationary reflection

There is a principle of reflection of stationary subsets of  $[H_\lambda]^\omega$ , for regular  $\lambda \geq \omega_2$ , that follows from MM. If  $\text{cf}(\kappa) = \omega$ , and the reflection principle holds for stationary subsets of  $[H_{\kappa^+}]^\omega$ , then  $\square_{\kappa^*}$  fails.

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It's unclear how much of this type of stationary reflection can be derived from SCFA alone.

Todorćević's strong reflection principle is too much, since it implies that the nonstationary ideal on  $\omega_1$  is saturated, while SCFA is consistent with  $\diamond$ .

## Another kind of $\square$

### Definition (Todorcevic, Jensen (?))

A sequence  $\vec{C} = \langle C_\alpha \mid \alpha < \lambda, \alpha \text{ limit} \rangle$  is **coherent** if for every limit  $\alpha < \lambda$ ,  $C_\alpha \neq \emptyset$  and for every  $C \in C_\alpha$ ,  $C$  is club in  $\alpha$ , and for every limit point  $\beta$  of  $C$ ,  $C \cap \beta \in C_\beta$ .

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The principle  $\square(\lambda, 1)$  is denoted  $\square(\lambda)$ .

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### Theorem (Todorćević)

*PFA implies the failure of  $\square(\kappa)$ , for every regular cardinal  $\kappa$ .*

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But it turns out that there is a route using stationary reflection. The goal is to determine the extent of  $\square(\kappa, \lambda)$  under SCFA.

## Diagonal reflection

### Definition (P. Larson)

The principle  $\text{OSR}_{\omega_2}$  says that whenever  $\langle T_\alpha \mid \alpha < \omega_2 \rangle$  is a sequence of stationary subsets of  $\omega_2$ , each consisting of ordinals of countable cofinality, then there is a  $\gamma < \omega_2$  with  $\text{cf}(\gamma) = \omega_1$  at which  $T_\alpha$  reflects, for all  $\alpha < \gamma$ .

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### Definition (F.)

Let  $\lambda$  be a regular cardinal, let  $S \subseteq \lambda$  be stationary in  $\lambda$ , and let  $\kappa < \lambda$ . The **diagonal reflection principle**  $\text{DSR}(<\kappa, S)$  says that whenever  $\langle S_{\alpha,i} \mid \alpha < \lambda, i < j_\alpha \rangle$  is a sequence of stationary subsets of  $S$ , where  $j_\alpha < \kappa$  for every  $\alpha < \lambda$ , then there is a  $\gamma < \lambda$  of uncountable cofinality, and there is a club  $F \subseteq \gamma$  such that for every  $\alpha \in F$  and every  $i < j_\alpha$ ,  $S_{\alpha,i} \cap \gamma$  is stationary in  $\gamma$ . The version of the principle in which  $j_\alpha \leq \kappa$  is denoted  $\text{DSR}(\kappa, S)$ .



The point of diagonal stationary reflection in the present context are the following two theorems.

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### Theorem (F.)

*Let  $\lambda$  be regular,  $\kappa < \lambda$  a cardinal, and assume that  $\text{DSR}(<\kappa, S)$  holds, for some stationary  $S \subseteq \lambda$ . Then  $\square(\lambda, <\kappa)$  fails.*

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Fortunately, diagonal reflection follows from SCFA.

### Theorem (F.)

*SCFA implies that for every regular  $\lambda > \omega_1$ ,  $\text{DSR}(\omega_1, S_\omega^\lambda)$  holds.*

## Effects of SCFA

### Lemma (F.)

Assume SCFA.

- 1 The principle  $\square(\omega_2, \omega)$  fails, but it is consistent that  $\square(\omega_2, \omega_1)$  holds.
- 2 If  $\lambda > \omega_2$  is a regular cardinal, then  $\square(\lambda, \omega_1)$  fails.

## Maximizing $\square$

### Lemma (F.)

*If the existence of a supercompact cardinal is consistent, then so is the existence of a supercompact cardinal  $\kappa$  such that for every regular cardinal  $\lambda > \kappa$ , the principle  $\square(\lambda, \kappa)$  holds.*

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The point is that in a model in which  $\kappa$  is supercompact and its supercompactness is indestructible by  $\kappa$ -directed closed forcing and GCH holds above  $\kappa$ , one can iterate to add a version of *indexed square* sequences of width  $\kappa$  at every  $\lambda > \kappa$ , using a forcing, due to Lambie-Hanson, that's  $\kappa$ -directed closed and  $\lambda$ -strategically closed.

## The extent of $\square(\cdot, \cdot)$ under SCFA

### Theorem (F.)

*Assume the consistency of the existence of a supercompact cardinal. It is consistent that*

- 1 SCFA + CH +  $\diamond$  holds
- 2 for every regular  $\lambda > \omega_2$ ,  $\square(\lambda, \omega_2)$  holds.

*But in any model of SCFA + CH, necessarily,  $\square(\lambda, \omega_1)$  fails for all regular  $\lambda > \omega_2$ ,  $\square(\omega_2, \omega)$  fails, and  $\square(\omega_2, \omega_1)$  holds.*

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*But in any model of SCFA + CH, necessarily,  $\square(\lambda, \omega_1)$  fails for all regular  $\lambda > \omega_2$ ,  $\square(\omega_2, \omega)$  fails, and  $\square(\omega_2, \omega_1)$  holds.*

*Sketch:* Starting in a model where  $\kappa$  is supercompact and  $\square(\lambda, \kappa)$  holds, for every regular cardinal  $\lambda > \kappa$ , run the Baumgartner iteration. The resulting model will satisfy SCFA +  $\diamond$  +  $\kappa = \omega_2$ . The forcing is  $\kappa$ -c.c., so the  $\square(\lambda, \kappa)$  sequences will survive and become  $\square(\lambda, \omega_2)$  sequences.  $\square(\omega_2, \omega_1)$  follows from CH. The claimed failure of  $\square$  principles follows from the lemma from two slides earlier.



## Effects of PFA

PFA does not imply  $\text{Refl}(\omega_1, S_\omega^\lambda)$ , since PFA is compatible with  $\square_{\kappa, \omega_2}$ , for every  $\kappa \geq \omega_2$ ; compare with the effects of simultaneous stationary reflection on the failure of weak squares by Cummings-Magidor. In particular, it does not imply  $\text{DSR}(\omega_1, S_\omega^\lambda)$ . So the argument using PFA necessarily has to be different. But it turns out that the original Todorćević argument for  $\square(\lambda)$  generalizes.

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### Lemma

*Assume PFA. Then the principle  $\square(\lambda, \omega_1)$  fails for every regular  $\lambda > \omega_1$ .*

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### Lemma

*Assume PFA. Then the principle  $\square(\lambda, \omega_1)$  fails for every regular  $\lambda > \omega_1$ .*

Maximizing works exactly as before.

# The extent of $\square(\cdot, \cdot)$ under PFA or MM

## Theorem (F.)

*Assume the consistency of the existence of a supercompact cardinal. Then it is consistent that*

- 1 MM or PFA holds
- 2 for every regular  $\lambda \geq \omega_2$ ,  $\square(\lambda, \omega_2)$  holds.

*In a model of (1), necessarily,  $\square(\lambda, \omega_1)$  fails, for every  $\lambda \geq \omega_2$ .*

## A limitation

One might hope that diagonal stationary reflection can be used to settle the question about  $\square_{\aleph_\omega}^*$  under SCFA. This is not so.

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### Theorem (F.)

*Assuming the consistency of infinitely many supercompact cardinals, it is consistent that for every nonzero  $n < \omega$ ,  $\text{DSR}(\aleph_n, \mathcal{S}_{<\aleph_n}^{\aleph_{\omega+1}}, \aleph_n)$  holds, and moreover,  $\square_{\aleph_\omega}^*$  holds.*

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There is a model constructed by Foreman-Cummings-Magidor in which  $\square_{\aleph_\omega}^*$  holds and also  $\text{Refl}(\aleph_n, \mathcal{S}_{<\aleph_n}^{\aleph_{\omega+1}}, \aleph_n)$  holds. One can check that that model actually satisfies  $\text{DSR}(\aleph_n, \mathcal{S}_{<\aleph_n}^{\aleph_{\omega+1}}, \aleph_n)$ , for all  $n < \omega$ .

## Bounded forcing axioms

### Definition (Goldstern-Shelah)

Let  $\Gamma$  be a class of forcings, and  $\lambda$  be a cardinal. Then  $\text{BFA}(\Gamma, \leq \lambda)$  is the statement that if  $\mathbb{P}$  is a forcing in  $\Gamma$ ,  $\mathbb{B}$  is its complete Boolean algebra, and  $\mathcal{A}$  is a collection of at most  $\omega_1$  many maximal antichains in  $\mathbb{B}$ , each of which has size at most  $\lambda$ , then there is a filter in  $\mathbb{B}$  that meets each antichain in  $\mathcal{A}$ . If  $\Gamma$  is the class of proper, semi-proper, stationary set preserving or subcomplete forcings, I write  $\text{BPFA}$ ,  $\text{BSPFA}$ ,  $\text{BMM}$ ,  $\text{BSCFA}$  (respectively) for  $\text{BFA}(\Gamma, \leq \omega_1)$ . In general, for a cardinal  $\lambda$ ,  $\text{BPFA}(\leq \lambda)$ ,  $\text{BSPFA}(\leq \lambda)$ ,  $\text{BMM}(\leq \lambda)$ ,  $\text{BSCFA}(\leq \lambda)$ , then have the obvious meaning.



## Definition (Goldstern-Shelah)

A regular cardinal  $\kappa$  is *reflecting* if for every  $a \in H_\kappa$  and every formula  $\varphi(x)$ , the following holds: if there is a regular cardinal  $\theta \geq \kappa$  such that  $H_\theta \models \varphi(a)$ , then there is a cardinal  $\bar{\theta} < \kappa$  such that  $H_{\bar{\theta}} \models \varphi(a)$ .

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## Theorem (Goldstern-Shelah)

BPFA is equiconsistent with the existence of a reflecting cardinal.

There is a proof of one direction of this equiconsistency result (showing that  $\omega_2^V$  is reflecting in  $L$ ), due to Todorčević, the idea of which generalizes from proper forcing to subcomplete forcing. The other direction generalizes very easily, given the iterability of subcomplete forcing.

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## Lemma (F.)

BSCFA *implies that*  $\omega_2$  *is reflecting in*  $L$ .

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*Proof.* We may assume that  $0^\#$  does not exist, as otherwise, every Silver indiscernible is reflecting in  $L$ . Let  $\kappa = \omega_2$ , fix  $a \in L_\kappa = (H_\kappa)^L$ , a formula  $\varphi(x)$ , a singular cardinal  $\gamma > \kappa$ , and let  $\theta = \gamma^+ = (\gamma^+)^L$ , by covering. Assume that  $L_\theta \models \varphi(a)$ . It suffices to show that there is an  $L$ -cardinal  $\bar{\theta} < \kappa$  such that  $L_{\bar{\theta}} \models \varphi(a)$ .

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*Proof.* We may assume that  $0^\#$  does not exist, as otherwise, every Silver indiscernible is reflecting in  $L$ . Let  $\kappa = \omega_2$ , fix  $a \in L_\kappa = (H_\kappa)^L$ , a formula  $\varphi(x)$ , a singular cardinal  $\gamma > \kappa$ , and let  $\theta = \gamma^+ = (\gamma^+)^L$ , by covering. Assume that  $L_\theta \models \varphi(a)$ . It suffices to show that there is an  $L$ -cardinal  $\bar{\theta} < \kappa$  such that  $L_{\bar{\theta}} \models \varphi(a)$ .

Let  $\langle C_\xi \mid \xi \text{ is a singular ordinal in } L \rangle$  be the canonical global  $\square$  sequence for  $L$ . It is  $\Sigma_1$ -definable in  $L$  and has the properties that for every  $L$ -singular ordinal  $\xi$ , the order type of  $C_\xi$  is less than  $\xi$ , and if  $\zeta$  is a limit point of  $C_\xi$ , then  $\zeta$  is singular in  $L$  and  $C_\zeta = C_\xi \cap \zeta$ .

Let  $B = \{\xi < \theta \mid \kappa < \xi < \theta \text{ and } \text{cf}(\xi) = \omega\}$ . By covering, every  $\xi \in B$  is singular in  $L$ . So  $C_\xi$  is defined for every  $\xi \in B$ , and since the function  $\xi \mapsto \text{otp}(C_\xi)$  is regressive, there is a stationary subset  $A$  of  $B$  on which this function is constant.



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This is a  $\Sigma_1$  statement about the parameters  $\omega_1$  and  $a$ . So by BSCFA, the same statement is true in  $V$ . Let  $\bar{\theta}, \bar{C}$  witness this. Since  $\omega_1, a \in H_{\omega_2}$ , such witnesses for a  $\Sigma_1$  formula can be found in  $H_{\omega_2}$ , so we may take  $\bar{\theta} < \omega_2 = \kappa$ .

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The point is now that  $\bar{\theta}$  must be regular in  $L$ . The reason is that if  $\bar{\theta}$  were singular in  $L$ , then  $C_{\bar{\theta}}$  would be defined. Note that  $\text{cf}(\bar{\theta}) = \omega_1$ . So, letting  $C'_{\bar{\theta}}$  be the set of limit points of  $C_{\bar{\theta}}$ ,  $C'_{\bar{\theta}} \cap \bar{C}$  is club in  $\bar{\theta}$ . Now take  $\xi < \zeta$ , both in  $C'_{\bar{\theta}} \cap \bar{C}$ . Then, since  $\xi, \zeta \in \bar{C}$ ,  $C_\xi$  and  $C_\zeta$  have the same order type, but since both are limit points of  $C_{\bar{\theta}}$ ,  $C_\xi = C_{\bar{\theta}} \cap \xi$ , which is a proper initial segment of  $C_\zeta = C_{\bar{\theta}} \cap \zeta$ .

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So  $\bar{\theta}$  is a regular cardinal in  $L$ ,  $\bar{\theta} < \omega_2$ , and  $H_{\bar{\theta}}^L = L_{\bar{\theta}} \models \varphi(a)$ , showing that  $\omega_2$  is reflecting in  $L$ . □

Miyamoto has analyzed the strength of these principles for proper forcing and introduced the following large cardinal concept.

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### Definition (Miyamoto)

Let  $\kappa$  be a regular cardinal,  $\alpha$  an ordinal, and  $\lambda = \kappa^{+\alpha}$ . Then  $\kappa$  is  $H_\lambda$ -reflecting, or I will say  $+\alpha$ -reflecting, iff for every  $a \in H_\lambda$  and any formula  $\varphi(x)$ , the following holds: if there is a cardinal  $\theta$  such that  $H_\theta \models \varphi(a)$ , then the set of  $N < H_\lambda$  such that

- 1  $N$  has size less than  $\kappa$ ,
- 2  $a \in N$ ,
- 3 if  $\pi_N : N \rightarrow H$  is the Mostowski-collapse of  $N$ , then there is a cardinal  $\bar{\theta} < \kappa$  such that  $H_{\bar{\theta}} \models \varphi(\pi_N(a))$

is stationary in  $\mathcal{P}_\kappa(H_\lambda)$ .

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In the context of bounded forcing axioms, it seems to make the most sense to emphasize that they generalize reflecting cardinals, so I will stick to calling them  $+1$ -reflecting.

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### Theorem (Miyamoto)

*BPFA( $\leq\omega_2$ ) is equiconsistent with the existence of a  $+1$ -reflecting cardinal.*

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Miyamoto's proof generalizes the original Goldstern-Shelah argument for BPFA, but the idea of Todorćević's argument generalizes to the subcomplete context.

## Theorem (F.)

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## Lemma (F.)

*BSCFA( $\leq \omega_3$ ) implies  $AD^{L(\mathbb{R})}$ .*

## Proof.

BSCFA( $\leq \omega_3$ ) implies  $SFP_{\omega_2}$  and  $SFP_{\omega_3}$ , which implies the failure of  $\square(\omega_2)$  and  $\square(\omega_3)$ , and also  $2^\omega \leq \omega_2$ . This constellation implies that the axiom of determinacy holds in  $L(\mathbb{R})$ , by Schimmerling and Steel. □

## The weak hierarchy

So I'm looking for strengthenings of  $\text{BFA}_\Gamma(\leq\omega_2)$  that are weaker than  $\text{BFA}_\Gamma(\leq\omega_3)$ , in consistency strength.



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### Fact (Claverie-Schindler)

*$\text{BFA}(\{\mathbb{Q}\}, \leq\kappa)$  is equivalent to the following statement: if  $M = \langle |M|, \epsilon, \langle R_i \mid i < \omega_1 \rangle \rangle$  is a transitive model for the language of set theory with  $\omega_1$  many predicate symbols  $\langle \dot{R}_i \mid i < \omega_1 \rangle$ , of size  $\kappa$ , and  $\varphi(x)$  is a  $\Sigma_1$ -formula, such that  $\Vdash_{\mathbb{Q}} \varphi(\dot{M})$ , then there is in  $\mathbb{V}$  a transitive  $\bar{M} = \langle |\bar{M}|, \epsilon, \langle \bar{R}_i \mid i < \omega_1 \rangle \rangle$  and an elementary embedding  $j : \bar{M} \prec M$  such that  $\varphi(\bar{M})$  holds.*

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Inspired this characterization, Bagaria, Gitman and Schindler introduced the weak proper forcing axiom, wPFA.

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Inspired this characterization, Bagaria, Gitman and Schindler introduced the weak proper forcing axiom, wPFA. By keeping track of the size of the model in question, one arrives at a hierarchy of these weak forcing axioms.

## The weak hierarchy

### Definition

Let  $\Gamma$  be a class of forcings, and let  $\kappa$  be an uncountable cardinal. The **weak  $\kappa$ -bounded forcing axiom for  $\Gamma$** ,  $\text{wBFA}(\Gamma, \leq \kappa)$ , says that whenever  $M = \langle |M|, \epsilon, \dots, R_i, \dots \rangle_{i < \omega_1}$  is a transitive model of size  $\kappa$  for a language  $\mathcal{L}$  with  $\omega_1$  many predicates  $\langle \dot{R}_i \mid i < \omega_1 \rangle$  and the binary relation symbol  $\dot{\epsilon}$ , and if  $\varphi(x)$  is a  $\Sigma_1$ -formula and  $\mathbb{P}$  is a forcing in  $\Gamma$  that forces that  $\varphi(\check{M})$  holds, then there is (in  $V$ ) a transitive model  $\bar{M} = \langle |\bar{M}|, \epsilon, \langle \bar{R}_i \mid i < \omega_1 \rangle \rangle$  for  $\mathcal{L}$  such that  $\varphi(\bar{M})$  holds (in  $V$ ), and such that in  $V^{\text{Col}(\omega, |\bar{M}|)}$ , there is an elementary embedding  $j : \bar{M} < M$ .

If  $\Gamma$  is the class of subcomplete forcings, then **wBSCFA**( $\leq \kappa$ ) is  $\text{wBFA}(\Gamma, \leq \kappa)$ . Similarly, we abbreviate these axioms for the class of proper forcings by **wBPFA**( $\leq \kappa$ ).

$\text{wBFA}(\Gamma, < \kappa)$  says that  $\text{wBFA}(\Gamma, \leq \bar{\kappa})$  holds for every  $\bar{\kappa} < \kappa$ , and **wBSCFA**( $< \kappa$ ), **wBPFA**( $< \kappa$ ) have the obvious meaning.

The large cardinal for wPFA turns out to be:

### Definition (Schindler)

A regular cardinal  $\kappa$  is **remarkable** if for every regular  $\lambda > \kappa$ , there is a regular cardinal  $\bar{\lambda} < \kappa$  such that in  $V^{\text{Col}(\omega, H_{\bar{\lambda}}^V)}$ , there is an elementary embedding  $j : H_{\bar{\lambda}}^V < H_{\lambda}^V$  with  $j(\text{crit}(j)) = \kappa$ .

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### Theorem (F.)

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There is a hierarchy of large cardinals, growing from the reflecting ones to the remarkable ones, corresponding to the weak bounded forcing axioms.

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## Definition (F.)

Let  $\kappa$  be an inaccessible cardinal and let  $\lambda \geq \kappa$  be a cardinal.  $\kappa$  is **remarkably  $\leq \lambda$ -reflecting** if the following holds: for any  $X \subseteq H_\lambda$  and any formula  $\varphi(x)$ , if there is a regular cardinal  $\theta > \lambda$  such that  $\langle H_\theta, \epsilon \rangle \models \varphi(X)$ , then there are cardinals  $\bar{\kappa} \leq \bar{\lambda} < \bar{\theta} < \kappa$  such that  $\bar{\theta}$  is regular, and there is a set  $\bar{X} \subseteq H_{\bar{\lambda}}$  such that  $\langle H_{\bar{\theta}}, \epsilon \rangle \models \varphi(\bar{X})$ , and a generic embedding  $j : \langle H_{\bar{\lambda}}, \epsilon, \bar{X}, \bar{\kappa} \rangle \prec \langle H_\lambda, \epsilon, X, \kappa \rangle$  (meaning that  $j$  exists in  $V^{\text{Col}(\omega, H_{\bar{\lambda}})}$ ) such that  $j \upharpoonright \bar{\kappa} = \text{id}$ .

$\kappa$  is **remarkably  $< \lambda$ -reflecting** iff it is remarkably  $\leq \bar{\lambda}$ -reflecting, for every cardinal  $\bar{\lambda} < \lambda$  with  $\kappa \leq \bar{\lambda}$ .

# Equiconsistencies for the weak hierarchy

## Theorem (F.)

*Let  $\lambda$  be a cardinal.*

- 1 *If  $\lambda \geq \omega_2$  and  $\text{wBSCFA}(\leq \lambda)$  holds, then  $\omega_2$  is remarkably  $\leq \lambda$ -reflecting in  $L$ .*
- 2 *If  $\lambda \geq \omega_2$  and  $\text{wBSCFA}(< \lambda)$  holds, then  $\omega_2$  is remarkably  $< \lambda$ -reflecting in  $L$ .*
- 3 *If  $\kappa$  is remarkably  $\leq \lambda$ -reflecting, where  $\kappa \leq \lambda$ , then  $\text{wBSCFA}(\leq \lambda)$  holds in a  $\kappa$ -c.c. subcomplete forcing extension.*
- 4 *If  $\kappa$  is remarkably  $< \lambda$ -reflecting, where  $\lambda > \kappa$ , then  $\text{wBSCFA}(< \lambda)$  holds in a  $\kappa$ -c.c. subcomplete forcing extension.*

## Resurrection

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The **resurrection axiom** for  $\Gamma$ , introduced by Hamkins and Johnstone, strengthens this by saying that for every  $\mathbb{P} \in \Gamma$ , there is a  $\dot{\mathbb{Q}}$  such that  $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \Gamma$  and

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Actually, their formulation used  $2^\omega$  in place of  $\omega_2$ , which is not useful for the subcomplete context. This change doesn't cause a change in consistency strength, and yields a very similar principle.

These principles can be strengthened and generalized to  $H_\kappa$ , with  $\kappa > \omega_2$ , by using elementary embeddings rather than elementary substructures.



## Definition (after Hamkins, Johnstone, Tsaprounis)

Let  $\kappa \geq \omega_2$  be a cardinal, and let  $\Gamma$  be a class of forcings. The resurrection axiom for  $\Gamma$  at  $H_\kappa$ ,  $\text{RA}_\Gamma(H_\kappa)$ , says that whenever  $G$  is generic over  $V$  for some forcing  $\mathbb{P} \in \Gamma$ , then there is a  $\mathbb{Q} \in \Gamma^{V[G]}$  and a  $\lambda$  such that whenever  $H$  is  $\mathbb{Q}$ -generic over  $V[G]$ , then in  $V[G][H]$ ,  $\lambda$  is a cardinal and there is an elementary embedding

$$j : \langle H_\kappa^V, \epsilon \rangle < \langle H_\lambda^{V[G][H]}, \epsilon \rangle$$

The principle  $\text{RA}_\Gamma(H_\kappa)$  says that for every  $A \subseteq H_\kappa$  and every  $G$  as above, there is a  $\mathbb{Q}$  as above such that for every  $H$  as above, in  $V[G][H]$ , there are a  $B$  and a  $j$  such that

$$j : \langle H_\kappa^V, \epsilon, A \rangle < \langle H_\lambda^{V[G][H]}, \epsilon, B \rangle,$$

and such that if  $\kappa$  is regular, then  $\lambda$  is regular in  $V[G][H]$ .

## Equiconsistencies at $\omega_2$

### Definition (Hamkins-Johnstone)

An inaccessible cardinal  $\kappa$  is **uplifting** if there are arbitrarily large inaccessible cardinals  $\lambda$  such that  $\langle V_\kappa, \epsilon \rangle < \langle V_\lambda, \epsilon \rangle$ . It is **strongly uplifting** if for every  $A \subseteq \kappa$ , there are arbitrarily large inaccessible  $\lambda$  such that for some  $B \subseteq \lambda$ ,  $\langle V_\kappa, \epsilon, A \rangle < \langle V_\lambda, \epsilon, B \rangle$ .

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### Theorem (Hamkins-Johnstone)

*For  $\Gamma$  the class of proper forcing notions,  $\text{RA}_\Gamma(H_{\omega_2})$  is equiconsistent with an uplifting cardinal, and  $\widetilde{\text{RA}}_\Gamma(H_{\omega_2})$  is equiconsistent with a strongly uplifting cardinal.*

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### Theorem (Minden)

*The same is true for the class of subcomplete forcings, and for the class of countably closed forcings.*

At  $H_{\omega_3}$ , a leap in consistency strength occurs again.  
Hence, it is natural to consider the hierarchy of the “virtual” resurrection axioms, where the elementary embeddings are added by some further forcing.

## Virtual resurrection

Let  $\kappa \geq \omega_2$  be a cardinal, and let  $\Gamma$  be a class of forcings.

### Definition

The **virtual resurrection axiom for  $\Gamma$  at  $H_\kappa$** ,  $\text{vRA}_\Gamma(H_\kappa)$ , says that whenever  $G$  is generic over  $V$  for some forcing  $\mathbb{P} \in \Gamma$ , then there is a  $\mathbb{Q} \in \Gamma^{V[G]}$  and a  $\lambda$  such that whenever  $H$  is  $\mathbb{Q}$ -generic over  $V[G]$ , there is some further forcing  $\mathbb{R} \in V[G][H]$  such that if  $I$  is generic for  $\mathbb{R}$  over  $V[G][H]$ , then in  $V[G][H][I]$ , there is an elementary embedding

$$j : \langle H_\kappa^V, \epsilon \rangle < \langle H_\lambda^{V[G][H]}, \epsilon \rangle$$

I will call such an embedding *virtual*.

## Definition

The **boldface virtual resurrection axiom for  $\Gamma$  at  $H_\kappa$** ,  $\text{vRA}_\Gamma(H_\kappa)$ , says that for every  $A \subseteq \kappa$  and every  $G$  as before, there is a  $\mathbb{Q}$  as before such that for every  $H$  as before, there are a  $B \in \mathbb{V}[G][H]$  and an  $\mathbb{R}$  as before such that for every  $I$  as before, there is a  $j$  in  $\mathbb{V}[G][H][I]$  such that

$$j : \langle H_\kappa^{\mathbb{V}}, \in, A \rangle < \langle H_\lambda^{\mathbb{V}[G][H]}, \in, B \rangle$$

and such that, if  $\kappa$  is regular in  $\mathbb{V}$ , then  $\lambda$  is regular in  $\mathbb{V}[G][H]$ . Finally, the **virtual unbounded resurrection axiom**  $\text{vUR}_\Gamma$  says that  $\text{vRA}_\Gamma(H_\kappa)$  holds for every cardinal  $\kappa \geq \omega_2$ .

# Virtual super extendibility

## Definition

Let  $\kappa$  be an inaccessible cardinal and  $\alpha$  an ordinal. Then  $\kappa$  is **virtually super  $\alpha$ -extendible** if there are arbitrarily large inaccessible cardinals  $\gamma$  such that for some  $\beta$ , there is an elementary embedding  $j$  in  $V^{\text{Col}(\omega, H_{\kappa+\alpha})}$  such that

$$j : \langle H_{\kappa+\alpha}^V, \in, \kappa \rangle < \langle H_{\gamma+\beta}^V, \in, \gamma \rangle$$

where  $j \upharpoonright \kappa = \text{id}$  (equivalently,  $j \upharpoonright H_\kappa = \text{id}$ ). Here,  $\kappa$  and  $\gamma$  are used as predicates in these structures, and it follows that  $j(\kappa) = \gamma$  if  $\alpha > 0$ .



## Definition

$\kappa$  is **strongly virtually super  $\alpha$ -extendible** if for every  $A \subseteq \kappa^{+\alpha}$ , there are arbitrarily large inaccessible cardinals  $\gamma$  such that for some  $\beta$  and some  $B \subseteq H_{\gamma+\beta}$  (in  $V$ ), there is an elementary embedding  $j$  in  $V^{\text{Col}(\omega, \theta)}$ , for some large enough  $\theta$ , such that

$$j : \langle H_{\kappa^{+\alpha}}^V, \epsilon, A, \kappa \rangle < \langle H_{\gamma+\beta}^V, \epsilon, B, \gamma \rangle$$

with  $j \upharpoonright \kappa = \text{id}$ , and such that, if  $\kappa^{+\alpha}$  is regular, then  $\gamma^{+\beta}$  is regular.

$\kappa$  is **virtually super  $< \alpha$ -extendible** if it is virtually super  $\bar{\alpha}$ -extendible for every  $\bar{\alpha} < \alpha$ .

## Theorem (F.)

Let  $\Gamma$  be the class of semiproper, proper, countably closed or subcomplete forcings.

- 1 If  $\kappa$  is virtually super  $<\theta$ -extendible, then in a  $\kappa$ -c.c. forcing extension by a forcing in  $\Gamma$ ,  $\text{vRA}_\Gamma(H_{\omega_{2+\bar{\theta}}})$  holds, for every  $\bar{\theta} < \theta$ .
- 2 If  $\kappa$  is strongly virtually super  $<\theta$ -extendible, then in a  $\kappa$ -c.c. forcing extension by a forcing in  $\Gamma$ ,  $\text{vRA}_\Gamma(H_{\omega_{2+\bar{\theta}}})$  holds, for every  $\bar{\theta} < \theta$ .
- 3 If  $\kappa$  is virtually extendible, then  $\text{vUR}_\Gamma$  holds in a  $\kappa$ -c.c. forcing extension by a forcing in  $\Gamma$ .
- 4 If  $\text{vRA}_\Gamma(H_{\omega_{2+\theta}})$  holds, then  $\omega_2$  is virtually super  $\theta$ -extendible in  $L$ .
- 5 If  $\text{vRA}_\Gamma(H_{\omega_{2+\theta}})$  holds, where  $\text{cf}(\omega_{2+\theta}) > \omega$ , then  $\omega_2$  is strongly virtually super  $\theta$ -extendible in  $L$ .
- 6 The consistency strength of  $\text{vUR}_\Gamma$  is a virtually extendible cardinal.

# Maximality principles

## Definition (Väänänen-Stavi, Hamkins (indep.))

Let  $\Gamma$  be a class term for a class of forcings. Let  $\varphi(\vec{x})$  be a formula, and let  $\vec{p}$  be parameters. Then  $\varphi(\vec{p})$  is  **$\Gamma$ -forceably necessary** if there is a  $\mathbb{P} \in \Gamma$  such that for every  $\dot{\mathbb{Q}}$  in  $\Gamma^{\mathbb{P}}$ ,  $\mathbb{P} * \dot{\mathbb{Q}} \Vdash \varphi(\vec{p})$ .

For a set  $X$ , the **maximality principle** for  $\Gamma$ , with parameters in  $X$ ,  **$\text{MP}_{\Gamma}(X)$** , is the scheme expressing that for every  $\varphi(\vec{p})$ , with  $\vec{p} \in X$ , if  $\varphi(\vec{p})$  is  $\Gamma$ -forceably necessary, then it is true.

In the case  $\Gamma$  is the class of subcomplete forcings, I'll write  **$\text{MP}_{\text{SC}}(H_{\omega_2})$**  for the “boldface” principle  **$\text{MP}_{\Gamma}(H_{\omega_2})$** .

## Geology

A **ground** of the universe is an inner model  $W$  such that  $V$  is a set-generic extension of  $W$ .

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### Theorem (Usuba)

*If there is a hyper-huge cardinal, then there are only set-many grounds.*

If there are only set-many grounds, then there is a smallest ground, and this ground is the intersection of all the grounds, that is, the mantle.

## Definable well-orderings

There is a history of results on the existence of definable well-ordering of  $\mathbb{R}$ , and even of  $\mathcal{P}(\omega_1)$ , as a consequence of forcing axioms.

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# The extended Namba problem

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## Theorem (Jensen)

*Let  $\kappa$  be an inaccessible cardinal, and assume that GCH holds below  $\kappa$ . Let  $A \subseteq \kappa$  be a set of regular cardinals. Then there is a subcomplete,  $\kappa$ -c.c. forcing  $\mathbb{P}$  of size  $\kappa$  such that if  $G$  is  $\mathbb{P}$ -generic, then  $\kappa = \omega_2^{V[G]}$  and for every regular  $\tau \in (\omega_1, \kappa)$ ,*

$$\text{cf}^{V[G]}(\tau) = \begin{cases} \omega_1 & \text{if } \tau \in A \\ \omega & \text{otherwise} \end{cases}$$

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$$\text{cf}^{V[G]}(\tau) = \begin{cases} \omega_1 & \text{if } \tau \in A \\ \omega & \text{otherwise} \end{cases}$$

The idea is to use this in order to code  $A$  into the “ $\text{cof } \omega/\omega_1$ ” pattern.

## Theorem

*Assume  $\text{MP}_{\text{SC}}(H_{\omega_2})$ , and assume that there are only set-many grounds. Then there is a well-ordering of  $\mathcal{P}(\omega_1)$ , definable without parameters, of order type  $\omega_2$  (and in particular,  $2^{\omega_1} = \omega_2$ ).*

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*Proof.* If there are only set-many grounds, then  $V = \mathbb{M}[g]$ , for some  $g$  which is generic over  $\mathbb{M}$  for some forcing  $\mathbb{P}$  in  $\mathbb{M}$ . Since  $\mathbb{M}$  is forcing-absolute, letting  $\kappa = \omega_2$ , it follows from  $\text{MP}_{\text{SC}}(H_{\omega_2})$  that  $V_{\kappa}^{\mathbb{M}} < \mathbb{M}$ , by verifying the Tarski-Vaught criterion for elementary substructures. Since  $\kappa$  is regular, it follows that  $\kappa$  is inaccessible in  $\mathbb{M}$ . This, in turn, implies that there is a proper class of inaccessible cardinals in  $\mathbb{M}$ , and since  $V$  is a set forcing extension of  $\mathbb{M}$ , also in  $V$ .

Let  $\alpha \geq \overline{\mathbb{P}}^+$ , and let  $\langle \kappa_i \mid i \leq \omega_1 \rangle$  enumerate the next  $\omega_1 + 1$  inaccessible cardinals above  $\alpha$ . We can perform an Easton iteration of at least countably closed forcings in order to reach an extension in which GCH holds below  $\kappa_{\omega_1}$ , such that if  $h$  is generic, each  $\kappa_i$  is still inaccessible in  $V[h]$ . Now, given a subset  $A$  of  $\omega_1$  in  $V$ , there is a subcomplete forcing  $\mathbb{Q}$  in  $V[h]$  which is  $\kappa_{\omega_1}$ -c.c., and such that if  $G$  is  $\mathbb{Q}$ -generic over  $V[h]$ , then for every  $i < \omega_1$ ,  $V[h][G]$  thinks that  $\text{cf}(\kappa_i) = \omega_1$  if  $i \in A$  and  $\text{cf}(\kappa_i) = \omega$  if  $i \in \omega_1 \setminus A$ .

Now if  $V[h][G][I]$  is a further subcomplete forcing extension of  $V[h][G]$ , then the cofinalities of the  $\kappa_i$ s, for  $i < \omega_1$ , cannot change, since subcomplete forcing does not add reals. Moreover, the sequence  $\langle \kappa_i \mid i < \omega_1 \rangle$  is definable in  $V[h][G][I]$  from the parameter  $\alpha$ , as the enumeration of the next  $\omega_1$  many inaccessible cardinals in  $\mathbb{M}$  beyond  $\alpha$ . Hence,  $A$  is definable as the set of  $i < \omega_1$  such that  $\text{cf}(\kappa_i) = \omega_1$ .

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Let  $\psi(A, \alpha)$  be the statement that if  $\langle \lambda_i \mid i < \omega_1 \rangle$  enumerates the next  $\omega_1$  many inaccessible cardinals of  $\mathbb{M}$  beyond  $\alpha$ , then for all  $i < \omega_1$ ,  $i \in A$  iff  $\text{cf}(\lambda_i) = \omega_1$  and  $i \notin A$  iff  $\text{cf}(\lambda_i) = \omega$ .



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Then the statement  $\varphi(A)$ , stating that there is an  $\alpha$  such that  $\psi(A, \alpha)$  holds, is true in  $V[h][G][I]$ .

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Then the statement  $\varphi(A)$ , stating that there is an  $\alpha$  such that  $\psi(A, \alpha)$  holds, is true in  $V[h][G][I]$ .

Thus,  $\varphi(A)$  is subcomplete-forceably necessary, and hence true in  $V$ .

Given  $A \subseteq \omega_1$ , let  $\beta(A)$  be the least  $\beta$  that  $\psi(A, \beta)$  holds. It is then forceably necessary that  $\beta(A) < \omega_2$ , and we can define  $A <^* B$  iff  $\beta(A) < \beta(B)$ , for  $A, B \subseteq \omega_1$ , giving the desired well-ordering of  $\mathcal{P}(\omega_1)$ . □

Thank you, Ronald!