Inner Models from Generalized Logics

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Ronald Jensen 80th birth day conference

A tribute to Ronald Jensen

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This talk is dedicated to Ronald with great admiration for his work and a great gratitude for his friendship.

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The present talk is about joint work with J. Kennedy and J. Vaananen (A work in progress)

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2. Completeness: Canonical definable objects should be included.
   Litmus test: Closure under sharps or other canonical operations.

Universe constructed from Generalized Logic

Generalized Logic $\mathcal{L}$ has two components (S, T) where S is the set of formulas (which may have free variables) and T is the truth predicate relation, between a model M, a formula Φ and an assignment to the free variables $\vec{a}$

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Definition

For a logic $\mathcal{L}$ and a set M we denote by $Def_{\mathcal{L}}(M)$ the collection of all subsets of M definable in the logic $\mathcal{L}$ in the structure $< M, \varepsilon >$ in the logic $\mathcal{L}$ using parameters from M.

Inner constructed by the Logic $\mathcal{L}$

Definition

Given the logic $\mathcal{L}$. The sequence of sets $L_\alpha^\mathcal{L}$ is defined by induction on the ordinal α:

1. $L_0^\mathcal{L} = \emptyset$
2. For α limit $L_\alpha^\mathcal{L} = \bigcup_{\beta<\alpha} L_\beta^\mathcal{L}$
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Definition

The inner model constructed by the logic $\mathcal{L}$ is $\mathcal{C}(\mathcal{L}) = \bigcup_{\alpha\in On} L_\alpha^{\mathcal{L}}$

Two extreme examples

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The class of hereditarily ordinal definable sets (a.k.a. HOD) is exactly $\mathcal{C}(\mathcal{L})$ *where* $\mathcal{L}$ *is second order logic.*

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HOD has maximal completeness , canonical objects are ordinal definable. It is somewhat robust under changes in the definition, but non robust across universes of Set Theory.

Some extensions of first order logic

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4. $\mathcal{L}(Q^{aa})$ is the logic ("stationary logic") is second order logic with only unary second order variables . The only second order quantifier is Q^{aa} where $Q^{aa} X\Phi(X)$ (X is a second order variable) meaning in a model M $\{X \in P_{\omega_1}(M) | \Phi(X)\}$ is a stationary subset of $P_{\omega_1}(M)$. ($P_{\omega_1}(M)$ is the collection of countable subsets of M .)

The four examples are nice logics

The four logics in the above examples we have a completeness theorem , (For the Magidor -Malitz quantifier assuming $\diamondsuit_{\omega_1}$.)

The four examples

The Logic of "there are uncountably many "

The Magidor-Malitz logic

The countable cofinality Logic

Stationary Logic

$\mathcal{C}(Q_1)$ is L (though "being uncountable set" is not absolute between models os Set Theory) because if $X \in L$ then if $\kappa = \omega_1^V$ then X is uncountable in V iff $L \models |X| \geq \kappa$.

$\mathcal{C}(Q_1)$ is L (though "being uncountable set" is not absolute between models os Set Theory) because if $X \in L$ then if $\kappa = \omega_1^V$ then X is uncountable in V iff $L \models |X| \geq \kappa$. Hence in any universe of set theory the steps of the construction of $\mathcal{L}(Q_1)$ can be defined in L.

$\mathcal{C}(Q_1^{MM})$ can be changed by forcing over L.

Usinging the ideas of Jensen We can define in L a sequence $\langle T_\alpha | \alpha < \omega_2 \rangle$ of Souslin trees on ω_1 which are independent in the sense that we can destroy the Souslinity of some without destroying the Soulinity of others.

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Using that We can code a non constructible subset of ω_2 as the set $B = \{\alpha < \omega_2 | T_\alpha \text{is Soulin}\}$. Since one can express in $\mathcal{L}(Q_1^{MM})$ that $(T, \prec)$ is a Souslin tree then one gets $B \in \mathcal{C}(Q_1^{MM})$. So we can have models in which $\mathcal{C}(Q_1^{MM} \models 2^{\aleph_0} = \aleph_1$ as well as models in which $\mathcal{C}(Q_1^{MM} \models 2^{\aleph_0} = \aleph_2$.

The effect of $0^\sharp$ on $\mathcal{C}(Q_1^{MM})$

Theorem

If $0^\sharp$ exists then $\mathcal{C}(Q_1^{MM}) = L$

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Sketch of proof: The main lemma is:

Lemma

Assume $0^\sharp$ exists . Let A be a subset of unordered pairs such that $A \in L$. Then there is a set B such that $|B| \geq \omega_1$ and $[B]^2 \subseteq A$ iff $L \models \exists B(|B| \geq \omega_1 \wedge [B]^2 \subseteq A)$

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The proof is based on the following lemma:

Lemma

X is a set of ordinals such that $X^\sharp$ exists. Let λ be an ordinal above $\sup(X)$ which has uncountable cofinality and which is a regular cardinal in $L[X]$ then λ is one of the canonical indescernibles for X.

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Since in $X \in \mathcal{C}(Q^{cf}_\omega)$ then we can find in $\mathcal{C}(Q^{cf}_\omega)$ arbitrarily long sequences of λ's satisfying the conditions in the lemma, so we can find enough indescernibles for X to define $X^\sharp$.

Some more closure of $\mathcal{C}(Q^{cf}_{\omega})$

$\mathcal{C}(Q^{cf}_{\omega})$ is closed under a large variety of definable operations for instance:

Theorem

- *If C is a set of ordinals then the Dodd-Jensen core models over X is included in* $\mathcal{C}(Q^{cf}_{\omega})$*.*
- *Let X be a set of ordinals in* $\mathcal{C}(Q^{aa})$*. Suppose that in V there is an inner model M with a measurable cardinal* κ *such* $X \subseteq \kappa$ *and* $X \in M$ *. Then there is such a model in* $\mathcal{C}(Q^{aa})$*.*

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- *If $X \in \mathcal{C}(Q^{cf}_{\omega})$ and $X^{\dagger}$ exists then $X^{\dagger} \in \mathcal{C}(Q^{cf}_{\omega})$*

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- *If* $X \in \mathcal{C}(Q^{cf}_\omega)$ *and* $X^\dagger$ *exists then* $X^\dagger \in \mathcal{C}(Q^{cf}_\omega)$

Robustness of $\mathcal{C}(Q_\omega^{cf})$

By forcing over L one can change $\mathcal{C}(Q_\omega^{cf})$. In particular make it violate the Continuum Hypothesis.

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Theorem (Assuming Woodin cardinals)

The sets of reals y such that $\mathcal{C}(Q^{cf}_{\omega})(y) \models CH, \diamondsuit_{\omega_1}$ contains a Turing cone.

Limiting the completeness of $\mathcal{C}(Q^{cf}_{\omega})$

Theorem (M.-Schindler)

Suppose there is a Woodin cardinal and that $M_1^\sharp$ exists. ($M_1^\sharp$ is a countable canonical model for Woodin cardinal with its sharp.) then every real of $\mathcal{C}(Q^{cf}_{\omega})$ appears in $M_1^\sharp$.

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Theorem

Assume $V = L_\mu$. Let M_α be the α's iterate of V by the (unique) normal measure on the unique measurable cardinal. Then $\mathcal{C}(Q^{cf}_\omega) = M_{\omega^2}[P]$ where P is a Prikry sequence of the unique measurable cardinal of M_{ω^2}. In particular $\mathcal{C}(Q^{cf}_\omega)$ has no measurable cardinal.

Club Determinacy

Theorem (A proper class of measurable Woodins or **MM**$^{++}$)

Let $\Phi(P)$ be a formula of $\mathcal{L}(Q^{aa})$ with the second order quantifier P. Then for every ordinal α if we let $M = L_{\alpha}^{Q^a a}$ then either

$$\{P \in P_{\omega_1}(M) | M \models \Phi(P)\}$$

contains a club in $P_{\omega_1}(M)$ or

$$\{P \in P_{\omega_1}(M) | M \models \neg\Phi(P)\}$$

Contains a club in $P_{\omega_1}(M)$.
(The formula $\Phi(P)$ may contain more second order unary variables which are replaced by parameters from $P_{\omega_1}(M)$.

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Assume a proper class of measurable Woodins, then $\mathcal{C}(Q^{aa})$ satisfies CH and $\diamondsuit_{\omega_1}$.

The Q^{aa}- extender

Let $M = \mathcal{L}_\alpha(Q^{aa})$. We define an extender on M by cosidering all functions $f : P_{\omega_1}(M) \to M$ such that there is a Q^{aa} formula $\Phi(P, x)$ such that for all $P \in P_{\omega_1}$
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Consider the equivalence relation on these functions $F \equiv G$ iff $\{P \in P_{\omega_1}(M) | F(P) = G(P)\}$ contains a club in $P_{\omega_1}(M)$.
Similarly $[F]_\equiv$ is member of $[G]_\equiv$ iff $\{P \in P_{\omega_1}(M) | F(P) \epsilon G(P)\}$ contains a club in $P_{\omega_1}(M)$.

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Lemma

$E(Q^{aa})$ is definable in $\mathcal{C}(Q^{aa})$. and it is iterable.

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Every countable Q^{aa} mouse can be iterated to structure of the form $\mathcal{L}_\beta(Q^{aa})$. Hence any two countable mice can be compared.

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Lemma

If a real $x \in \mathcal{L}_{\alpha+1}(Q^{aa}) - \mathcal{L}_\alpha(Q^{aa})$ then

$$\mathcal{C}(Q^{aa}) \models P(\omega) \cap \mathcal{L}_{\alpha+1}(Q^{aa}) \text{ is countable}$$

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It is a fascinating combination of Set Theory and Generalized Model Theory.

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Ronald, thank you !