Equiconsistencies at subcompact cardinals

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Outline and credits

Assuming iterability, we prove some equiconsistency results involving subcompact cardinals.

With newer hierarchies for long extenders, can get a bit further, to $+n$-subcompact. Iterability still assumed for short extenders only.

Joint work with John Steel.

Relies heavily on earlier work of Jensen, Schimmerling, Schindler, and Steel.

For long extenders, key insight that they need only be *coded* by extenders on the sequence is due to Woodin, structure of comparison was discovered by Steel and extended by Woodin, our hierarchy is similar to a hierarchy of Woodin.

Coherent sequences

Recall that $\vec{C} = \langle C_\alpha \mid \alpha \in Z \subseteq \delta \rangle$ is coherent on Z if C_α is club in α and $\alpha \in \mathrm{Lim}(C_\beta) \cap Z \to C_\alpha = C_\beta \cap \alpha$.

A thread through $\vec{C}$ is a club $C \subseteq \delta$ so that $\alpha \in \mathrm{Lim}(C) \cap Z \to C_\alpha = C \cap \alpha$.

Note that C_β is a thread through $\vec{C} \restriction \beta$.

δ is threadable if every coherent sequence on δ has a thread. (A compactness principle.)

$\square(\delta)$ is the statement that δ is *not* threadable.

$\square_\kappa$ is the statement that κ^+ is not threadable and this is witnessed by $\vec{C} = \langle C_\alpha \mid \alpha \in \delta \rangle$ with $\mathrm{Ordtype}(C_\alpha) \leq \kappa$. The ordertype restriction implies $\vec{C}$ is not threadable.

Threadability and large cardinals

δ is subcompact if for all $A \subseteq H(\delta^+)$ there is $\kappa < \kappa^+ < \delta$, $B \subseteq H(\kappa^+)$, so that $(H(\kappa^+); \kappa, B)$ embeds elem. in $(H(\delta^+); \delta, A)$ with critical point κ.

δ is weakly compact if for all $A \subseteq H(\delta)$ there is $\kappa < \delta$ so that $(H(\kappa); A \cap H(\kappa))$ is Π^1_1 elem. in $(H(\delta); A)$.

The following are well known:

1. If δ is subcompact, then it is weakly compact.
2. If δ is weakly compact, then it is threadable.
3. If δ is subcompact, then $\square_\delta$ fails.
4. If δ is subcompact, then it is a Woodin cardinal.

δ is Woodin if for every $A \subseteq \delta$ there is an $\kappa < \delta$ which is $< \delta$-strong relative to A.

Subcompactness is in fact much stronger, the beginning of a hierarchy interlaced with supercompactness.

First equiconsistency

Summarizing from last slide, if δ is subcompact, then δ is Woodin, threadable, and $\square_\delta$ fails.

Theorem (N.-Steel)

Assume SBH_δ*. Suppose* δ *is Woodin, threadable, and* $\square_\delta$ *fails. Then* δ *is subcompact in a class inner model. Moreover this inner model satisfies* SBH_δ*.*

So the following are equiconsistent:

1. SBH_δ, δ is Woodin, threadable, and $\square_\delta$ fails.
2. SBH_δ, δ is subcompact.

SBH_δ: good player has a winning strategy in the length $\omega_1 + 1$ iteration game (for linear comp. of +2 normal non-overlapping trees) on the transitive collapse of any ctbl $H \prec V_\delta$, with only strictly short extenders allowed.

Special case of the *Strategic Branches Hypothesis* of Martin-Steel.

An extender E is strictly short if $i_E(\mathrm{crit}(E)) > \mathrm{strength}(E)$.

Sketch of proof

Relies heavily on the *stacking* methods of Jensen-Schimmerling-Schindler-Steel.

Fix a regular uncountable δ. Let $\mathcal{W} \subseteq V_\delta$ be an inner model built using some backgrounding condition, with $\mathrm{Ord} \cap W = \delta$, and δ a limit of cardinals of $\mathcal{W}$.

Consider the collection $\mathcal{C}$ of mice $\mathcal{M}$ extending $\mathcal{W}$, projecting to δ, sound.

(Iterability required for countable substructures of $\mathcal{M}$.)

Any two mice in $\mathcal{C}$ are comparable, meaning one is an initial segment of the other. This is a consequence of condensation: appropriate hulls are initial segments of $\mathcal{W}$, hence comparable.

Can therefore *stack* the the elements of $\mathcal{C}$: Set $\mathcal{S}(\mathcal{W})$ to be the supremum of the mice in $\mathcal{C}$.

Sketch of the proof, cont.

Several results about the stack from J-S-S-S, clever consequences of condensation and lifting of embeddings:

1. $\mathrm{Ord} \cap \mathcal{S}(\mathcal{W})$ remains a cardinal in $L(\mathcal{S}(\mathcal{W}))$.
2. If $\eta := \mathrm{cof}(\mathrm{Ord} \cap \mathcal{S}(\mathcal{W})) < \delta$, then writing $\mathcal{S}(\mathcal{W})$ as an increasing continuous union $\bigcup_{\xi<\delta} H_\xi$ with $|H_\xi| < \delta$, have, for a club of ξ of cofinality $\neq \omega, \omega_1, \eta$,

$$\mathcal{R}_\xi = \mathcal{W}{\restriction}(\xi^+)^{\mathcal{W}}, \quad \mathrm{cof}(\mathrm{Ord} \cap \mathcal{R}_\xi) = \eta$$

where $\mathcal{R}_\xi$ is the transitive collapse of H_ξ.

Conclusion of (2) gives an extender of $\mathcal{W}$ indexed at some $(\zeta^+)^{\mathcal{W}}$, a contradiction. (More on this later.) So $\eta \not< \delta$.

Consider $\mathcal{Q} = L(\mathcal{S}(\mathcal{W}))$. Suppose δ is not subcompact in $\mathcal{Q}$. By Schimmerling-Zeman, $\square_\delta^{\mathcal{Q}}$ holds, by a sequence which remains non-threadable in V.

Then in V, using same sequence: if $\eta = \delta^+$ get $\square_\delta$; if $\eta = \delta$ get δ is not threadable.

Sketch of proof, cont.

If $\eta := \mathrm{cof}(\mathrm{Ord} \cap \mathcal{S}(\mathcal{W})) < \delta$, then writing $\mathcal{S}(\mathcal{W})$ as an increasing continuous union $\bigcup_{\xi<\delta} H_\xi$ with $|H_\xi| < \delta$, have, for a club of ξ of cofinality $\neq \omega, \omega_1, \eta$,

$$\mathcal{R}_\xi = \mathcal{W}\restriction(\xi^+)^{\mathcal{W}}, \quad \mathrm{cof}(\mathrm{Ord} \cap \mathcal{R}_\xi) = \eta$$

where $\mathcal{R}_\xi$ is the transitive collapse of H_ξ.

Suppose for contradiction the conclusion of (2) holds.

Let $c_\xi \colon \mathcal{R}_\xi \to H_\xi \subseteq \mathcal{S}(\mathcal{W})$ be the anti-collapse embedding
Let $c_{\xi,\zeta} \colon \mathcal{R}_\xi \to \mathcal{R}_\zeta$ be $c_\zeta^{-1} \circ c_\xi$. Note $c_{\zeta,\xi}$ is cofinal.

Let $E_{\xi,\zeta}$ be the extender derived from $c_{\xi,\zeta}$, a total extender over $\mathcal{W}\restriction(\zeta^+)^{\mathcal{W}}$, with critical point ξ.

Enough to show some $E_{\xi,\zeta}$ is on the sequence of $\mathcal{W}$ (or even just belongs to $\mathcal{W}$).

It would singularize $(\zeta^+)^{\mathcal{W}}$ in $\mathcal{W}$ to have cofinality $(\xi^+)^{\mathcal{W}}$, a contradiction.

Sketch of proof, cont.

Enough to show some $E_{\xi,\zeta}$ is on the sequence of $\mathcal{W}$.

J-S-S-S used a partially backgrounded construction for $\mathcal{W}$, argued some elementary hull relative to $\langle c_{\xi,\zeta} \mid \xi < \zeta < \delta \rangle$ provides an appropriate background for one of the extenders $E_{\xi,\zeta}$.

Cannot do this from SBH_δ, since backgrounds are not extenders in V.

We used a *fully* background construction (with one modification). SBH_δ is then enough for iterability of $\mathcal{W}$.

Our one modification for the background condition: Put E on the $\mathcal{W}$ sequence if it embeds into (rather than equals) the restriction of a full background to $\mathcal{W}$.

Woodiness of δ relative to $\langle c_{\xi,\zeta} \mid \xi < \zeta < \delta \rangle$ then provides a (short) background for one of the (superstrong type) extenders $E_{\xi,\zeta}$.

Another result

So far characterized threadability of δ + failure of $\square_\delta$. What about threadability of δ + threadability of δ^+?

Theorem (N.-Steel)

Assume SBH_δ*. Suppose* δ *is Woodin,* δ *is threadable, and* δ^+ *is threadable. Then* δ *is* Π^2_1 *subcompact in a class inner model. Moreover this inner model satisfies* SBH_δ*.*

Gives equiconsistency as before.

Π^2_1 subcompactness analogous to subcompactness, requiring the embedding of $(H(\kappa^+); \kappa, B)$ to $(H(\delta^+); \delta, A)$ to be Π^1_1-over-$H(\delta^+)$ (equivalently, Π^2_1-over-$H(\delta)$).

Involves characterization of threadability of δ^+ in inner models in terms of Π^2_1 subcompactness. Methods similar to ones in earlier characterization by Kypriotakis-Zeman in terms of simultaneous reflection.

$+n$ subcompactness

δ is $+\alpha$ subcompact if for every $A \subseteq H(\delta^{+(\alpha)})$ there is $\kappa < \nu < \delta$, $B \subseteq H(\nu)$, so that $(H(\nu); \kappa, B)$ embeds elem. in $(H(\delta^{+(\alpha)}); \delta, A)$ with critical point κ.

The case of $\alpha = 1$ gives subcompactness.

δ is supercompact iff it is $+\alpha$ subcompact for all α.

$+2$ subcompactness and higher witnessed by extenders E so that $\mathrm{strength}(E) > i_E(\mathrm{crit}(E))$.

Any such E has *generators above* $i_E(\mathrm{crit}(E))$. Called long exteners.

Iteration trees developed to avoid moving generators of E in comparison maps that apply E.

No way to avoid moving generators if E is long. For this and other reasons, earlier developments in inner model theory excluded long extenders.

Comparing with moving generators

One way to handle moving generators: Make sure to know where they move.

Liberalize definition of mice, to allow long extenders E, but require, when E_α is long, that $(\mathcal{M}\restriction\alpha; E_\alpha)$ projects to or below $i_{E_\alpha}(\mathrm{crit}(E_\alpha))$. (projectum requirement)

Then for any embedding $\sigma\colon (\mathcal{M}\restriction\alpha; E_\alpha) \to (\mathcal{N}, F)$ (which moves the standard parameter correctly), the restriction of σ to $i_{E_\alpha}(\mathrm{crit}(E_\alpha))$ determines σ completely.

Can use this to argue that comparison maps involving long extenders on their main branches, move the long generators the same way on the two sides of the comparison. (For extender in standard termination argument.)

This is enough to show that comparisons terminate.

How restrictive is the projectum requirement?

Coding long extenders

Consider a long extender F over $\mathcal{M}$, with $\mathrm{crit}(F) = \kappa$, $F(\kappa) = \tau$, $\mathrm{strength}(F) = \tau^+$, generators contained in τ^+.

No reason in general for F to project below τ^+. May violate projectum requirement.

Let $\mathcal{M}^* = \mathrm{Ult}(\mathcal{M}, F)$. Let $\lambda = i_F(\tau)$. Let E be the (τ, λ) extender derived from $i_F(F)$.

E is superstrong, but without long generators. Can be placed on sequence of $\mathcal{M}^*$. May assume $E \in \mathcal{M}^*$.

Let $F^* = i_F(F) \circ F$. This is a λ^+ strong extender over $\mathcal{M}^*$, with critical point κ, mapping κ to λ.

Note F can be recovered from E and F^*: for $\nu < \tau^+$, $A \subseteq \kappa^+$, have $\nu \in F(A)$ iff $i_E(\nu) \in F^*(A)$.

In particular F^* defines a subset of $\tau^+ < \lambda$, hence projects below $\lambda = i_{F^*}(\mathrm{crit}(F^*))$.

Projectum requirements holds for F^*, and F^* codes F.

Back to stacking

Stacking arguments can be adapted to long extender hierarchy, with repeated stacking, e.g. forming $\mathcal{S}(\mathcal{S}(\mathcal{W}))$.

Some additional arguments needed, since $\mathrm{Ord} \cap \mathcal{S}(\mathcal{W})$ is not a limit of cardinals of $\mathcal{S}(\mathcal{W})$.

Write $\mathcal{S}(\mathcal{S}(\mathcal{W}))$ as an increasing continuous union $\bigcup_{\xi<\delta} H_\xi$, $|H_\xi| < \delta$ (collapse δ^+ if needed).

As before, one of the keys to obtaining large cardinals in $L(\mathcal{S}(\mathcal{S}(\mathcal{W})))$ is to contradict the statement that for all ξ in some large enough set $C \subseteq \delta$,

$$\mathcal{R}_\xi = \mathcal{W}{\restriction}(\xi^{++})^{\mathcal{W}}, \quad \mathrm{cof}(\mathrm{Ord} \cap \mathcal{R}_\xi) = \eta < \delta$$

where $\mathcal{R}_\xi$ is the transitive collapse of H_ξ.

Suppose this statement holds.

Fix $\kappa < \tau < \lambda$ in C which are $< \delta$-strong relative to a predicate coding the maps $c_{\xi,\zeta}$.

Back to stacking, cont.

$$\mathcal{R}_\xi = \mathcal{W} \restriction (\xi^{++})^{\mathcal{W}}, \quad \mathrm{cof}(\mathrm{Ord} \cap \mathcal{R}_\xi) = \eta < \delta$$

Let F be the long extender derived from $c_{\kappa,\tau}$. $\mathrm{crit}(F) = \kappa$, $F(\kappa) = \tau$, F is $(\tau^+)^{\mathcal{W}}$ strong, generators $\subseteq (\tau^+)^{\mathcal{W}}$.

F singularizes $(\tau^{++})^{\mathcal{W}}$ to cofinality $(\kappa^{++})^{\mathcal{W}}$, so $F \notin \mathcal{W}$.

Define F^* similarly from $c_{\kappa,\lambda}$.

Let E be the superstrong type extender of $c_{\tau,\lambda} \restriction \mathcal{P}(\tau)^{\mathcal{W}}$.

Original stacking argument shows $E \in \mathcal{W}$.

Coding argument shows F can be recovered from F^*, E. So (since $F \notin \mathcal{W}$) F^* satisfies projectum requirement.

Original stacking argument adapts to show F^* is fully backgrounded (in embeddability sense) by a *short* extender.

So F^* is on the sequence of $\mathcal{W}$, hence $F \in \mathcal{W}$, a contradiction.

Equiconsistencies

Gives equiconsistencies. For example:

Theorem (N.-Steel)

Assume SBH_δ*. Suppose that for all* $n < \omega$*, it is forced in* $\mathrm{Col}(\delta, \delta^{+(n)})$ *that*

> δ *is weakly compact and that for every set* Z *in the Woodin filter for* δ *the weak compactness of* δ *can be witnessed using partial measures concentrating on* Z*.*

Then there is a class inner model in which GCH *holds and for all* $n < \omega$*,* δ *is* $+n$ *supercompact.*

The Woodin filter for δ: the filter generated by the sets $\{\kappa < \delta \mid \kappa \text{ is } < \delta \text{ strong wrt } A\}$ ranging over $A \subseteq \delta$.

Converse of theorem an easy forcing argument.

On square

Not as elegant as the equiconsistencies for subcompact.

What's missing for generalizing these equiconsistencies is a characterization of $\square$ in inner models with $+n$ subcompact cardinals.

Best result to date:

Theorem (Voellmer)

Suppose no long extender in $\mathcal{W}$ has infinitely many long generators. Then $\mathcal{W}$ satisfies $\square_{\lambda,2}$ whenever λ is not subcompact and not the successor of a $+2$ subcompact.

Questions

Work brings up some obvious questions.

1. Characterization of square in inner models with $+n$ subcompacts.

Ideally $\square_\lambda$ holds except for $\lambda \in [\delta, \delta^{+(n)})$ where δ is $+n$ subcompact.

Would allow generalizing the first equiconsistency to $+n$ subcompacts.

2. Inner models for $+\omega$ subcompactness and beyond.

To obtain the projectum requirement we code $+(n+1)$ subcompactness extenders using their $+n$ restrictions as parameters. This seems to limit us to finite subcompactness levels.

Thank you!