Ralf Schindler: Talks#1 on Logic Summer School of Fudan University, 2020

- How many real numbers are there?
- More specifically: We want to discuss 2 sets of prominent axioms which decides the size of $2^{\aleph_0}$ the same way;
  - Forcing Axioms: **MA**, **PFA**, **SPFA** = **MM**;
  - Woodin's Axiom $(*)$
- $\mathbf{MM}^{++} \Rightarrow (*)$

Richness: If you have a set of axioms which has a transitive model, then you have a transitive model inside L.
"Maximize": If an object can be imagined to exist, then it exists.

TODAY:

- Stationary sets;
- Forcing revisited;
- Forcing Axioms: **MA**;
- Proper forcing; semi-proper forcing; stationary set preserved forcing;
- **PFA**, **SPFA**, **MM**.

1 Stationary Sets

Definition. $C \subset [X]^\omega$ is a *club* iff:

$$\exists f : X^{<\omega} \to X, \quad C = \{x \in [X]^\omega : f"x^{<\omega} \subset x\}.$$

Remark. We may think f as a set of relations on X, and consider $(X; f)$ as a model. Then C is just the collection of every countable substructures of $(X; f)$.

Definition. $S \subset [X]^\omega$ is *stationary* iff $S \cap C \neq \emptyset$ for all club $C \subseteq [X]^\omega$

Remark. Hence, S is stationary iff for all models $(X; f)$, there is some $x \in S$, $x \prec (X; f)$.

Lemma 1 (Fodor). *Let $S \subseteq [X]^\omega$ be stationary, let $f : S \to V$, $f(x) \in x$ for all $x \in S$(regressive), then there is a stationary $T \subseteq S$, $f \restriction T$ is constant.*

Proof. o.w.(otherwise) f.a.(for all) $a \in X$, $S_a = \{x \in S : f(x) = a\}$ is nonstationary. Thus there is a club $C_a = \{x \in [X]^\omega : f_a"x^{<\omega} \subset x\}$ with some function $f_a : X^{<\omega} \to X$ and $C_a \cap S_a = \emptyset$.
Define $f^*(a, u) = f_a(u)$, for $u \in [X]^{<\omega}$. Let $C = \{x \in [X]^\omega : f^*"x^{<\omega} \subset x\}$. For all $a \in x \in C$, $x \in C_a$. Pick $x \in S \cap C$. Let $a = f(x) \in x$, then $x \in C_a$. Contradicts to the choice of C_a. □

Observation. If $S \subset [\omega_1]^\omega$ is stationary, then so is $\{\xi \in S : \xi \in \omega_1\}$.

[Hint. if $C \subset [\omega_1]^\omega$ is a club, then so is $\{\xi \in S : \xi \in \omega_1\}$.]

1.1 Splitting stationary sets

Theorem 2 (Solovay)**.** *$S \subset \omega_1$ stationary, then we may split $S = \bigsqcup_{i<\omega_1} S_i$, while all S_i are stationary.*

Proof. Let $a_n^\xi \nearrow \xi < \omega_1, n < \omega$.

Claim. $\exists n \forall \alpha < \omega_1 \{\xi \in S : \alpha_n^\xi \geq \alpha\}$ is stationary.

Otherwise, $\forall n \exists \alpha_n \exists$ club $C_n : C_n \cap \{\xi \in S : \alpha_n^\xi \geq \alpha_n\} = \emptyset$. Therefore we can pick $\xi \in (\bigcap_{n<\omega} C_n) \cap S$, thus $\xi > \sup_n \alpha_n$. However, $\alpha_n^\xi < \sup_m \alpha_m$ for all n. Contradiction.

Remark. Improve: Fix n as in the **Claim.**. As a immediate consequence of Fodor's Lemma, we have

Claim. $\forall \alpha < \omega_1 \exists \beta \geq \alpha \{\xi \in S : \alpha_n^\xi = \beta\}$ stationary.

Now we only need the pairwise disjoint property. Construct $(S_i, \beta_i : i < \omega_1)$ as the above **Claim.**: Assume $(S_i, \beta_i : i < j)$ are defined, let $\alpha = \sup_{i<j} \beta_i + 1$ and $\beta_j = \beta$ as in the **Claim.**, and let S_j be the corresponding set defined in the **Claim.**. □

Comment. (Shi.) This statement may be credited to Ulam, since the technique of Ulam matrix proves the statement for all successor ordinal instead of just ω_1. This procedure is described in [3], **Theorem 6.11**.

Comment. In fact Solovay has proved that the above statement works for any weakly inaccessible cardinal. See [4]

2 Forcing

$V \ni \mathbb{P}$, $\mathbb{P} = (\mathbb{P}; \leq_{\mathbb{P}})$ a partial order. $D \subset \mathbb{P}$ is dense iff

$$\forall p \in \mathbb{P} \exists q \in D : q \leq_{\mathbb{P}} p (q \text{ is stronger than } p)$$

$G \subseteq \mathbb{P}$ is $V-$generic iff $G \cap D \neq \emptyset$ f.a. $D \subset \mathbb{P}$, $D \in V$ dense.
$V[G] = \{\tau^G : \tau \in V^{\mathbb{P}}\}$ where τ is a $\mathbb{P}-$name.

Theorem 3 (Forcing Theorem)**.** *If $V[G] \vDash \phi(\tau^G, ...)$, then $\exists p \in G$, $p \Vdash \phi(\tau, ...)$. If $p \Vdash \phi(\tau, ...)$, then $V[G] \vDash \phi(\tau^G, ...)$ f.a. $G \ni p$.* □

3 Forcing Axiom

Definition. $\mathbb{P}$ has the c.c.c.(countable chain condition) iff $\mathbb{P}$ does not have any uncountable antichain.
$A \subseteq \mathbb{P}$ is an antichain iff $\forall p, q \in A, p \neq q \to p \perp q (p, q$ incompatible = no common extension).
$\mathbb{C}$ = Cohen forcing = $\omega^{<\omega}, p \leq_{\mathbb{C}} q$ iff $p \supset q$.

Definition. $\mathbf{MA}_\kappa$(Martin's Axiom for κ): $\mathbb{P}$ has the c.c.c., $\mathcal{D} = \{D_i : i < \kappa\}$ a collection of dense sets; then there is a filter $G \subseteq \mathbb{P}$, $G \cap D_i \neq \emptyset$ for all $i < \kappa$.

$\mathbf{MA}_\omega$ is always true: define ω-sequence

$$p_1 \leq p_2 \leq ... \leq p_i \leq ..., \quad i < \omega$$

while $p_i \in D_i$. Thus the filter $G = \{q \in \mathbb{P} : \exists n \in \omega(q \geq p_n)\}$ is V-generic.

Remark. This is exactly the diagonal argument, known as the *Rasiowa-Sikorski Lemma.*

$\mathbf{MA}_{2^{\aleph_0}}$ is false: $\mathbb{C}$ Cohen forcing: Let $(x_i : i < 2^{\aleph_0})$ enumerate all sets of ω^ω. $D_i = \{p \in \mathbb{C} : \exists n \in dom(p), p(n) \neq x_i(n)\}$. $\{D_i : i < 2^{\aleph_0}\}$ is a collection of dense sets. If $G \cap D_i \neq \emptyset$ f.a. $i < 2^{\aleph_0}$, then $\bigcup G : \omega \to \omega$, so $\bigcup G = x_i$ for some $i < 2^{\aleph_0}$. However,

$$\exists p \in G \exists n[p(n) \neq x_i(n) \implies x_i(n) \neq \bigcup G(n)].$$

Contradiction. □

Using a.d.(almost disjoint) coding, we can prove the *Souslin Hypothesis*:

$$\mathbf{MA}_{\omega_1} \implies 2^{\aleph_0} = 2^{\aleph_1}$$

Claim. $\exists$ a.d. sequence $(a_\xi : \xi < \omega_1)$ of subsets ω, i.e., f.a. $\xi, \eta < \omega_1$, $\xi \neq \eta$, $a_\xi \cap a_\eta$ is finite.

Proof. Look at $2^{<\omega}$. Let $e : 2^{<\omega} \to \omega$ be bijection. Let $(b_\xi : \xi < \omega_1)$ be a sequence of pairwise different branches of the tree $2^{<\omega}$. Let $a_\xi = \{e(b_\xi \upharpoonright n) : n < \omega\}$. Then a_ξ proves the statement. □

Theorem 4. $\mathbf{MA}_{\omega_1} \implies 2^{\aleph_0} = 2^{\aleph_1}$.

Proof. Let $(a_\xi : \xi < \omega_1)$ be a sequence of pairewise a.d. subsets of ω. Let $X \subset \omega_1$. $p \in \mathbb{P}$ iff $p = (f, x)$:

- $f : n \to 2$, for some $n < \omega$;
- $x \subset X$ finite.

$(f', x') \leq_{\mathbb{P}} (f, x)$ iff $f' \supset f, x' \supset x$, and $\{m \in \mathrm{dom}(f') - \mathrm{dom}(f) : f'(m) = 1\} \cap a_\xi = \emptyset$ for all $\xi \in x$.

One can check that this forcing satisfies c.c.c. since every pair of conditions that shares a common f is compatible. $\{(f, x) : n \in \mathrm{dom}(f)\}$ is dense for all n; $\{(f, x) : \xi \in x\}$ is dense for all $\xi \in X$.$\Rightarrow$ the generic gives rise to a function $F : \omega \to \omega$ such that f.a. $\xi \in X$, $\{n \in \omega : F(n) = 1\} \cap a_\xi$ is finite. And if $\xi \notin X$, $\{(f, x) : \exists m \geq n(m \in a_\xi \wedge f(m) = 1)\}$ is dense f.a. $n < \omega$. Thus f.a. $\xi \notin X, \{n \in \omega : F(n) = 1\}$ is infinite.

In sum, the generic filter G gives rise to $F : \omega \to \omega$ such that if $a \subset \omega$ such that F is the characteristic function of a, then $[a \cap a_\xi$ of finite $\Leftrightarrow \xi \in X]$ f.a. $\xi < \omega_1$. So a codes $X \subseteq \omega_1$ modulo $(a_\xi : \xi < \omega_1)$ in that sense. Thus,

$$\mathbf{MA}_{\omega_1} \to \forall X \subset \omega_1 \exists a \subset \omega \forall \xi < \omega(\xi \in X \Leftrightarrow a \cap a_\xi \text{ is finite.}) \tag{1}$$

Define $T : \mathcal{P}(\omega_1) \to \mathcal{P}(\omega), \quad X \mapsto a$ where a satisfies (1). Clearly T is injective. □

Thus, $\mathbf{MA}_{\omega_1} \implies \neg\mathbf{CH}$ since $2^{\aleph_0} = 2^{\aleph_1} \leq \aleph_2$.One can show $2^{\aleph_0} > \aleph_1$ and $\mathbf{MA}_\kappa$ for all $\kappa < 2^{\aleph_0}$ is consistent.

We will go ahead and discuss more profound forcing axiom.

Rest of today:

- Proper forcing; (**PFA**)
- Semi-proper forcing; (**SPFA**)
- Stationary set preserving forcing. (**MM**)

4 Proper forcing

Definition. $\mathbb{P}$ is proper iff for all X, if $S \subset [X]^\omega$ is stationary, then S is still stationary in $V^{\mathbb{P}}$.

Remark. Here $V^{\mathbb{P}}$ means all generic extension.

Examples of forcing notions that are NOT proper:

- $Col(\omega, \omega_1)$; [1]
- (Shoot a club) Let $S \subset \omega_1$, S stationary and $\omega_1 - S$ is stationary. There is a forcing which adds $C \subset S$ club, every stationary subset of S remains stationary (In consequence, ω_1 is not collapsed). But C witness the fact that $\omega_1 - S$ is no longer stationary.

Definition. Let $x \prec H_\theta$, x countable, $p \in \mathbb{P} \cap x$. $q \leq_{\mathbb{P}} p$ is $x-$generic iff f.a. $\tau \in V^{\mathbb{P}} \cap x$ such that $\Vdash \tau \in \check{H}_\theta$, we have $q \Vdash \tau \in \check{x}$.(E.g. There is no x for $Col(\omega, \omega_1)$ to be $x-$generic.)

Lemma 5. *The following statements are equivalent:*

(1) $\mathbb{P}$ is proper;

(2) F.a. $x \prec H_\theta$, (x countable, θ sufficiently large,) f.a. $p \in \mathbb{P} \cap x$, $\exists q \leq p$ $x-$generic.

Proof. ([2], **Theorem 31.7**.)

(2) $\implies$ (1): Let $S \subset [X]^\omega$ be stationary. $p \Vdash$"$\dot{C}$ is a club in$[X]^\omega, \dot{C} = \{x \in [X]^\omega : \dot{f}"x^{<\omega} \subset x\}$". Let $x \prec H_\theta$, x countable, and $p, \dot{C}, \dot{f} \in x$, $x \cap X \in S$(possible, as S is stationary).

Let $q \leq p$ be x-generic.

Claim. $q \Vdash \dot{C} \cap \check{S} \neq \emptyset$; in fact, $q \Vdash (x \cap X)\check{} \in \dot{C}$.

This follows from the definition of x-genericity.

(1) $\implies$ (2): We may not prove that for all substructures x (2) holds but, the countable

[1] Proper forcing does not collapse $\aleph_1$. See [2], **Lemma. 31.4**.

substructures satisfying (2) form a club of $[H_\theta]^\omega$.[2] Towards a contradiction, let

$$S = \{x \prec H_\theta : |x| \leq \omega, \exists p \in x \cap \mathcal{P}(\nexists f \leq p \ x\text{-generic})\}$$

is stationary. By Fodor's Lemma, let $g : S \to V$ maps x to some $p \in x$ where p does not have any x-generic extension. g is regressive and thus there is a stationary $T \subset S$ such that $\exists p \forall x \in T(p \in x \wedge \nexists f \leq p[f \ x\text{-generic}])$. Pick a filter G that is $V-$generic, $p \in G$. T is still stationary in $V[G]$. This implies we may pick countable $x \prec H_\Omega[G]$ so that $x \cap H_\theta \in T$. This implies a contradiction since if $\tau \in V^{\mathcal{P}} \cap x \cap H_\theta$, $\Vdash \tau \in H_\theta$, then $\tau^G \in x \cap H_\theta$. This is forced by some $q \leq p$. □

Definition. $x \prec H_\theta$, x countable, $p \in \mathbb{P} \cap x$, $q \leq p$ is $x-$semigeneric iff f.a. $\tau \in V^{\mathbb{P}} \cap x$, $\Vdash \tau \in \check{\omega}_1$, we have $q \Vdash \tau \in \check{x} (\Leftrightarrow \tau \in (x \cap \omega_1)\check{})$. That is, $q \Vdash \tau \in \check{\alpha}$, where $\alpha = x \cap \omega_1 \in \omega_1$, since $x \cap \omega_1$ is transitive.

Definition. $\mathbb{P}$ is semi-proper iff f.a. $x \prec H_\theta$, countable, $\mathbb{P} \in x$, f.a. $p \in x \cap \mathbb{P}$ there is $q \leq p$ such that q is $x-$semigeneric.

Observation. $\mathbb{P}$ is proper, then $\mathbb{P}$ is semiproper.

Definition. $\mathbb{P}$ preserves stationary subsets (of ω_1) iff

$$\forall S \subset \omega_1 (S \text{ stationary in } V \implies S \text{ stationary in } V^{\mathbb{P}}).$$

Lemma 6.
- $\mathbb{P}$ *is semi-proper* $\implies$ $\mathbb{P}$ *preserves stationary subsets of* ω_1*;*
- $\mathbb{P}$ *has the c.c.c., then* $\mathbb{P}$ *is proper.* □

Definition.
- **PFA**: Every ω_1 family of every proper forcing notion has a generic filter;
- **SPFA**: Every ω_1 family of every semiproper forcing notion has a generic filter;
- **MM**: Every ω_1 family of every stationary preserving forcing notion has a generic filter.

Remark. One cannot extend those axioms to κ families like what we do in **MA**, since these axioms implies(as we shall later show,) that $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$.

Theorem 7. *The followings are equivalent:*
- **MM***;*
- *f.a. models* $M \in V$ *(signature* $\leq \omega_1$*) f.a.* $\mathbb{P}$ *stationary set preserving f.a.* ϕ Σ_1-*formula, if* $V^{\mathbb{P}} \vDash \phi(M)$*, then* $\exists j : \overline{M} \to M$ *elementary,* $|\overline{M}| \leq \omega_1$*,* $V \vDash \phi(\overline{M})$*.*

Proof given by: [1], **Theorem 1.3**.

[2] Suppose $\mathcal{C}$ is a club of countable $x \in [H_\theta]^\omega$ such that every $p \in \mathcal{P} \cap x$ has an x-generic extension. Let $[H_\theta]^\omega \in H_\Omega$, with Ω sufficiently large, and let some $x \prec H_\Omega$ be countable with $\mathcal{P} \in x$. Then some θ and $\mathcal{C}$ are elements of x, but then $x \cap H_\theta \in \mathcal{C}$, from which it follows that every $p \in \mathcal{P} \cap x$ can be extended to an x-generic condition. So if f.a. sufficiently large θ there is a club of countable $x \in [H_\theta]^\omega$ s.t. every $p \in \mathcal{P} \cap x$ can be extended to an x-generic condition, then for all sufficiently large θ and for every $x \in [H_\theta]^\omega$ with $\mathcal{P} \in x$, every $p \in \mathcal{P} \cap x$ can be extended to an x-generic condition.

References

[1] Benjamin Claverie and Ralf Schindler. Woodin's axiom (*), bounded forcing axioms, and precipitous ideals on ω 1. *The Journal of Symbolic Logic*, 77(2):475–498, 2012.

[2] Thomas Jech. *Set theory*. Springer Science & Business Media, 2013.

[3] Kenneth Kunen. *Set theory: An introduction to independence proofs*. North-Holland, 1980.

[4] Robert M Solovay. Real-valued measurable cardinals. In *Axiomatic set theory*, volume 13, pages 397–428, 1971.