Ralf Schindler: Talks#2 on Logic Summer School of Fudan University, 2020

TODAY:

- Restate $\mathbf{PFA}, \mathbf{SPFA}, \mathbf{MM}$ as well as $\mathbf{PFA}^{++}, \mathbf{SPFA}^{++}, \mathbf{MM}^{++}$;
- A few words on *iterated* forcing
- Supercompact Cardinals, Laver functions;
- Forcing $\mathbf{SPFA}^{(++)}$
- Weak reflection principle;
- $\mathbf{MM} \Rightarrow 2^{\aleph_1} = \aleph_2$.

Recall our forcing axioms:

- $\mathbf{MA}_{\omega_1}$: If $\mathbb{P}$ has the c.c.c. and if $\mathcal{D} = \{D_i : i < \omega_1\}$ is a family of dense sets in $\mathbb{P}$ then there is a filter G such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$.
- **PFA** : Same with "c.c.c." replaced by "proper".
- **SPFA** : Same with "c.c.c." replaced by "semi-proper".
- **MM** : Same with "c.c.c." replaced by "stationary set preserving".
- $\mathbf{MA}_{\omega_1}^{++}$: If $\mathbb{P}$ has the c.c.c. and if $\mathcal{D} = \{D_i : i < \omega_1\}$ is a family of dense sets in $\mathbb{P}$ and if $\{\tau_i : i < \omega_1\}$ is s.t. $\Vdash_{\mathbb{P}}$"$\tau_i \subset \check{\omega}_1$ is stationary" then there is a filter G such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and

$$\tau_i^G = \{\xi < \omega_1 : \exists p \in G (p \Vdash \check{\xi} \in \tau_i)\}$$

is stationary for all $i < \omega_1$.

- $\mathbf{PFA}^{++}, \mathbf{SPFA}^{++}, \mathbf{MM}^{++}$: Just add the underlined part to the axioms.

A reformulation of $\mathbf{MM}^{++}$:

Theorem 1. *The following are equivalent:*

- $\mathbf{MM}^{++}$*;*
- *Let $\mathbb{P}$ be a stationary set preserving forcing, let M be a model such that M's signature has size at most $\aleph_1$, let ϕ be Σ_1, and suppose*

$$\Vdash \phi(M, \mathbf{NS}_{\omega_1});$$

where $\mathbf{NS}_{\omega_1}$ is the ideal of all non-stationary subset of ω_1. Then in V there is some elementary $j : \overline{M} \to M$ and $\phi(\overline{M}, \mathbf{NS}_{\omega_1})$.

Comment. Since $\mathbb{P}$ is stationary preserving, $\mathbf{NS}_{\omega_1}^{V^{\mathbb{P}}} \cap V = \mathbf{NS}_{\omega_1}^{V}$. Then in V their are the same.

Remark. If M has size $\leq \aleph_1$, then we can add all of M's elements as constant symbols, where we can let $j = id$, with $\bar{M} = M$. Thus the theorem implies that if $\phi(M, \mathbf{NS}_{\omega_1})$ holds in the generic extension, it holds in the ground model as well.

1 Iterated forcing

Say $\mathbb{P} \in V$, let G be V-generic for $\mathbb{P}$. Say $\mathbb{Q} \in V[G]$, let H be $V[G]$-generic for $\mathbb{Q}$.

We can also say it in another way. $\mathbb{P} \in V$, $\mathbb{P} \Vdash$"$\dot{\mathbb{Q}}$ is a poset". Then we may define

- $\mathbb{P} * \dot{\mathbb{Q}} \ni (p, \dot{q})$ such that $p \in \mathbb{P}$, $\Vdash_{\mathbb{P}} \dot{q} \in \dot{\mathbb{Q}}$.
- $(p', \dot{q}') \leq_{\mathbb{P}*\dot{\mathbb{Q}}} (p, \dot{q})$ iff $p' \leq p$ and $p' \Vdash \dot{q}' \leq \dot{q}$.

Longer iterations:

$$((\mathbb{P}_\alpha : \alpha \leq \theta), (\dot{\mathbb{Q}}_\alpha : \alpha < \theta)).$$

Given $\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha$, $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$. For limit stage $\mathbb{P}_\lambda$: $\vec{p} = (p_i : i < \lambda) \in \mathbb{P}_\lambda$ iff $\mathbb{P}_i \Vdash p_i \in \dot{\mathbb{Q}}_i$ for all i.

Countable support iteration: $\{i < \lambda : \mathbb{P}_i \not\Vdash p_i = 1_{\dot{\mathbb{Q}}}\}$ is at most countable.

Revised countable support iteration: $\{i < \lambda : \mathbb{P}_i \not\Vdash p_i = 1_{\dot{\mathbb{Q}}}\}$ is at most countable as being forced by some $\mathbb{P}_j, j < \lambda$.

Theorem 2 (Shelah's iteration theorems)**.**

1.

$$((\mathbb{P}_\alpha : \alpha \leq \theta), (\dot{\mathbb{Q}}_\alpha : \alpha < \theta)).$$

is a countable support iteration s.t. $\mathbb{P}_\alpha \Vdash$"$\dot{\mathbb{Q}}_\alpha$ *is proper" f.a.* $\alpha < \theta$*, then* $\mathbb{P}_\theta$ *is proper.*

2.

$$((\mathbb{P}_\alpha : \alpha \leq \theta), (\dot{\mathbb{Q}}_\alpha : \alpha < \theta)).$$

is a revised countable support iteration such that $\mathbb{P}_\alpha \Vdash$"$\dot{\mathbb{Q}}_\alpha$ *is semiproper" f.a.* $\alpha < \theta$*, then* $\mathbb{P}_\theta$ *is semiproper.*

Please note that everything we do today is covered by [1].

2 Supercompact cardinal and forcing

Definition. (Magidor Characterization) κ is a supercompact cardinal iff: f.a. cardinal $\lambda > \kappa$, and f.a. $X \in H_\lambda$, there is some $\bar{\lambda} < \kappa$ and some elementary embedding $j : H_{\bar{\lambda}} \to H_\lambda$[1] such that

- $j(\mathrm{crit}(j)) = \kappa$; and
- $X \in \mathrm{ran}(j)$.

Now we denote δ as some supercompact cardinal without further notice.

Definition. $f : \delta \to V_\delta$ is a Laver function iff f.a. $\lambda > \delta$, f.a. $x = (\mathbb{P}, M) \in H_\lambda$ there is some $\alpha < \delta$ and an elementary embedding

$$j : H_{f(\alpha)_0} \to H_\lambda, \quad f(\alpha)_1 \mapsto x.$$

[$f(\alpha)$ is an ordered pair $(\bar{\lambda}, \bar{x})$.]

Theorem 3 (Laver)**.** $\exists \delta$ *supercompact* $\implies$ *There is a Laver function.*

[1] It could also be V_λ by a matter of taste.

Proof. Recursively construct $f(\alpha), \alpha < \delta$. Suppose $f \upharpoonright \alpha$ is given. Let $f(\alpha) =$ some pair $(\bar{\lambda}, \bar{x})$ such that there is no $\beta < \alpha$ with a elementary embedding

$$j : H_{f(\beta)_0} \to H_{\bar{\lambda}}; \quad f(\beta)_1 \mapsto \bar{x}$$

while $\bar{\lambda}$ is the least possible cardinal. If there is no such pair, let $f(\alpha) = (0, 0)$. This works since otherwise let $\lambda > \delta, x \in H_\lambda$ be a counterexample. Let Ω be sufficiently large. Then there is $H_{\bar{\Omega}}$, together with elementary embedding j which sends $\bar{x}$ to x, $\bar{\lambda}$ to λ and $\bar{\delta}$ to δ. Since H_Ω can see that (λ, x) is a counterexample, by elementarity, $H_{\bar{\Omega}}$ can see $(\bar{\lambda}, \bar{x})$ is an counterexample and $f(\alpha)$ is defined. Let $(\lambda^*, x^*) = j(f(\alpha)_0, f(\alpha)_1)$. Then by elementarity,

$$H_\Omega \models \text{"There is no } k : H_{f(\beta)_0} \to H_{\lambda^*};\ f(\beta)_1 \mapsto x^{*}\text{"}$$

This is a contradiction since H_Ω can see j. □

Theorem 4 (Foreman-Magidor-Shelah)**.** *If δ is supercompact, then* **SPFA** *holds in a generic extension.*

Proof. Define a revised countable support iteration

$$((\mathbb{P}_\alpha : \alpha \leq \theta), (\dot{\mathbb{Q}}_\alpha : \alpha < \theta)).$$

of semiproper forcing. Let $f : \delta \to V_\delta$ be a Laver function. At stage α: Force with $\dot{\mathbb{Q}}_\alpha = f(\alpha)_{1,0}$ provided

$$\mathbb{P}_\alpha \Vdash \text{"}f(\alpha)_{1,0} \text{ is a semiproper forcing."}$$

Otherwise force with $Col(\omega_1, f(\alpha)_0)$. Verify that **SPFA** holds true in $V^{\mathbb{P}_\delta}$: Let G be V-generic for $\mathbb{P}_\delta$. Given $M \in V[G]$, a model with signature $\leq \aleph_1$, $\mathbb{P} \in V[G]$ a semiproper forcing, ϕ is Σ_1, $\mathbb{P} \Vdash \phi(M)$. Let λ be sufficiently large and note that δ would be collapsed to ω_2[2]. Take V-names $\dot{M}, \dot{\mathbb{P}}$ for $M, \mathbb{P}$, respectively. In V we may pick an elementary embedding j, which witnessing f is a Laver function with $j(f(\alpha)) = (\lambda, (\dot{\mathbb{P}}, \dot{M}))$. Then $\dot{\mathbb{P}}$ is the name for the forcing used at stage α of the iteration. So $\phi(j^{-1}(\dot{M}^{G \upharpoonright \alpha}))$ holds true in $H_{f(\alpha)_0}[G \upharpoonright (\alpha+1)]$, hence in $V[G \upharpoonright (\alpha+1)]$, hence in $V[G]$(as ϕ is Σ_1). But j lifts to $\hat{j}$ from $H_{f(\alpha)_0}[G \upharpoonright \alpha]$ to $H_\lambda[G]$, so in the end we get what we want. As the tail end of the iteration from $\alpha + 1$ to δ preserves stationary sets, ϕ could be Σ_1 in the language with a predicate for $\mathbf{NS}_{\omega_1}$, and then the argument actually produces $\mathbf{SPFA}^{++}$. □

Now what about **MM**? Foreman-Magidor-Shelah actually verified that $\mathbf{MM}^{++}$ holds true in the model $V[G] = V^{\mathbb{P}_\delta}$ which we constructed. In order to verify this:

Definition. $\mathbf{WRP}(\theta)$(Weak Reflection Principle at uncountable θ):

> If $S \subset [H_\theta]^\omega$ is stationary, then there is some $Y \subset H_\theta$, $|Y| = \aleph_1$, $\omega_1 + 1 \subset Y$ such that $S \cap [Y]^\omega$ is stationary.

[2]Since it is a revised countable support iteration, and each forcing notion is semi-proper which has size $< \delta$, $\aleph_1$ and cardinal above δ are preserved.

Theorem 5.

(A) If $\mathbf{WRP}(\theta)$ *holds true for all* θ*, then every stationary set preserving forcing is semiproper.*

(B) In the model $V[G]$ *constructed before,* $\mathbf{WRP}(\theta)$ *holds true for all* θ*.*

Corollary 6. $\mathbf{MM}^{++}$ *holds true in* $V[G]$*.* [3]

Proof. Let's first prove (B). Let $[H_\theta[G]]^\omega$ and $S \subset [H_\theta]^\omega$ be stationary. Then by forcing with $Col(\omega_1, H_\theta)$, since it is proper, we have in the extension that $f : \omega \to H_\theta$ surjective together with a stationary $T \subset \omega_1$ s.t. $f"\xi \in S$ for all $\xi \in T$. Now apply $\mathbf{SPFA}^{++}$[4], which gives $f : \omega_1 \to H_\theta$ and a stationary $T \subset \omega_1$ s.t. $f"\xi \in S$ for all $\xi \in T$.

For (A), let the sentence "every stationary set preserving forcing is semiproper" be denoted as (†). Fix $\mathbb{P}$ preserving stationary subsets of ω_1. If $\mathbb{P}$ is not semiproper, then

$$S = \{x \prec H_\theta : |x| = \omega \wedge \text{"there is } p \in x \cap \mathbb{P} \text{ with no } x\text{-semigeneric extension"}\}$$

is stationary.

- By Fodor: $\exists T \subset S$ stationary $\exists p \in \mathbb{P}$ such that $p \in x$ f.a. $x \in T$ and there is no $x-$generic extensions of p f.a. $x \in T$.
- By $\mathbf{WRP}(\theta)$: $\exists Y \subset H_\theta$ of size $\aleph_1$ such that $T \cap [Y]^\omega$ is stationary.

Pick G as V-generic for $\mathbb{P}$, with $p \in G$. $\mathbb{P}$ preserves stationary sets of $\omega_1 \implies$ we can find $x \in T \cap [Y]^\omega$ s.t. $x[G] \cap \omega_1 = x \cap \omega_1 = \alpha$. In $V[G] : \tau^G < \alpha$ f.a. $\tau \in x$, $\Vdash \tau < \omega_1^V$. This is forced by some $q \leq p$, thus q is an x-semigeneric extension. Contradiction. □

To summarize:

$$\begin{aligned}\delta \text{ supercompact } &\implies V^{\mathbb{P}_\delta} \Vdash \mathbf{SPFA}^{++} \wedge \mathbf{WRP}(\theta) \text{ f.a. } \theta \implies V^{\mathbb{P}_\delta} \Vdash \mathbf{SPFA}^{++} \wedge (\dagger) \\ &\implies V^{\mathbb{P}_\delta} \models \mathbf{MM}^{++}.\end{aligned}$$

It is called "Martin's Maximum" since if we allow forcing that is not stationary preserving, then the forcing may lead to an inconsistent result. Let $S \subset \omega_1$ be stationary and co-stationary set, we may force S to contain a club $C \subset S$ by a forcing which preserves the stationarity of all stationary subsets of S. Since "S contains a club set" is Σ_1, if we further assume that a forcing axiom for this kind of forcing holds true, then in the ground model one can still find the Σ_1 sentence holds true. Contradiction. Thus $\mathbf{MM}^{++}$ cannot be extended in this way. However, can we extend the $\mathbf{MM}^{++}$ in another direction to allow more dense sets to be met by the generic filter?

Theorem 7 ((FMS))**.** $\mathbf{MM} \implies 2^{\aleph_1} = \aleph_2$*.*

[Consequence: A version of **MM** when G is supposed to hit more than $\aleph_1$ dense sets is inconsistent.]

[3] Iterating stationary set preserving forcing notions can be problematic, as it can probably collapse ω_1 with even an ω-long iteration. This is proved by Shelah.

[4] In fact, $\mathbf{PFA}^+$ is enough.

Proof. Define $S^{\omega}_{\omega_2} = \{\xi < \omega_2 : \mathrm{cf}(\xi) = \omega\}$. Then $S^{\omega}_{\omega_2}$ is a stationary subset of ω_2. Solovay's theorem $\implies$ $S^{\omega}_{\omega_2} = \bigsqcup_{i<\omega_2} S_i$, where every S_i is stationary. Fix $\omega_1 = \bigsqcup_{i<\omega_1} T_i$, each T_i is stationary, and let $S^{\omega}_{\omega_2} = \bigsqcup_{i<\omega_1} S_i$. Fix $X \subset \omega_1$ nonempty. Our goal: find $\alpha \in S^{\omega_1}_{\omega_2}$ such that $S_i \cap \alpha$ is stationary in α iff $i \in X$. We propose the following forcing to prove this:

> $p \in \mathbb{P}$ iff $\exists \alpha < \omega_1$ and $p : \alpha + 1 \to S^{\omega}_{\omega_2}$ normal[5], $\forall \eta \leq \alpha \forall i < \omega_1 [\eta \in T_i \implies p(\eta) \in S_{f(i)}]$, for $X \subset \omega_1$ non-empty and $f : \omega_1 \to \omega_1$, with $X = \mathrm{ran}(f)$.

We shall complete this proof in the next lecture. □

References

[1] Thomas Jech. *Set theory.* Springer Science & Business Media, 2013.

[5] Strictly increasing and continuous.