Ralf Schindler: Talks#3 on Logic Summer School of Fudan University, 2020

Today:

- discuss some aspect of stationary sets;
- $\mathbf{MM} \Rightarrow 2^{\aleph_1} = \aleph_2$;
- effective counterexample to **CH**.

As for the third point: If **CH** is false, we have a surjective map $\exists f : \mathbb{R} \to \omega_2$. We may, for instance, discuss the complexity of $R_f = \{(x, y) \in \mathbb{R}^2 : f(x) \leq f(y)\}$: Under **MM**, we can pick such an f such that R_f is projective. Thus in models like $L(\mathbb{R})$, they already contains such a counterexample for **CH**.

1 MM $\implies$ ¬CH

Theorem 1. $\mathbf{MM} \Rightarrow 2^{\aleph_1} = \aleph_2$.

Claim. There is $(T_i : i < \omega_1)$ such that

1. $\omega_1 = \bigsqcup_{i<\omega_1} T_i$;
2. each T_i is stationary, and
3. for each $S \subset \omega_1$ stationary, $\exists i < \omega_1$, $S \cap T_i$ is stationary.

Proof. Pick $\alpha_n^\xi \nearrow \xi$, for any ordinal $\xi < \omega_1$ and $n < \omega$.

Subclaim. $\exists n \forall \alpha \{\xi < \omega_1 : \alpha_n^\xi \geq \alpha\}$ is stationary. □

Fix such n. Define a strictly increasing sequence $(\beta_i : i < \omega_1)$ discontinuous[1] such that

$$T_i = \{\xi < \omega_1 : \alpha_n^\xi \in [\sup_{j<i} \beta_j, \beta_i)\} \tag{1}$$

is stationary for all i. This follows from a simple recursion: Suppose $(\beta_j : j < i)$ are already chosen. By the **Subclaim**, $\{\xi < \omega_1 : \alpha_n^\xi > \sup_{j<i} \beta_j\}$ is stationary. By Fodor: $\exists \beta > \sup_{j<i} \beta_j$, $\{\xi < \omega_1 : \alpha_n^\xi = \beta\}$ is stationary. Pick $\beta_i = \beta + 1$ for such β. So T_i defined as (1), which is a superset, is also stationary.

This defines $(T_i : i < \omega_1)$. Let $S \subset \omega_1$ be stationary, then the map $S \ni \xi \mapsto \alpha_n^\xi < \xi$ is regressive. By Fodor: $\exists \alpha$ such that

$$\bar{S} = \{\xi \in S : \alpha_n^\xi = \alpha\} \text{ is stationary.}$$

$\alpha \in [\sup_{j<i} \beta_j, \beta_i)$, for some i. This gives $\bar{S} \subset T_i \cap S$. So $T_i \cap S$ is stationary. □

Proof. of the theorem:

[1] $\beta_i > (\sup_{j<i} \beta_j) + 1$.

- Fix $\omega_1 = \bigsqcup_{i<\omega_1} T_i$ each T_i stationary. $\forall T \subset \omega_1$ stationary, $\exists i, T \cap T_i$ stationary.
- $S^{\omega}_{\omega_2} = \{\xi < \omega_2 : \mathrm{cf}(\xi) = \omega\} = \bigsqcup_{i<\omega_1} S_i$ such that each S_i is stationary.[2]

Let $X \subset \omega_1$ nonempty. Our goal: Find $\alpha \in S^{\omega_1}_{\omega_2} = \{\xi < \omega_2 : \mathrm{cf}(\xi) = \omega_1\}$ such that

$$S_i \cap \alpha \text{ is stationary in } \alpha \iff i \in X.$$

So $X \mapsto$ "the least such α" is injective, so $2^{\aleph_1} \leq \aleph_2$.

Claim (MM). $X \subset \omega_1, X \neq \emptyset \implies \exists \alpha \in S^{\omega_1}_{\omega_2}[S_i \cap \alpha \text{ stationary} \Leftrightarrow i \in X]$.

Define $p \in \mathbb{P}$ iff $\exists \alpha < \omega_1 (p : \alpha + 1 \to S^{\omega}_{\omega_2})$ and p is normal[3] and

$$\forall \eta \leq \alpha \forall i[\eta \in T_i \implies p(\eta) \in S_{f(i)}],$$

here $f : \omega_1 \to \omega_1, \mathrm{ran}(f) = X$. For any $p, q \in \mathbb{P}$, $p \leq q \iff p \supseteq q$. If G is V-generic for $\mathbb{P}$ then $\bigcup G : \omega_1 \to S^{\omega}_{\omega_2}$ satisfies

$$\forall \eta < \omega_1 \forall i[\eta \in T_i \implies \left(\bigcup G\right)(\eta) \in S_{f(i)}].$$

This implies that if $j \notin X = \mathrm{ran}(f)$, then $\mathrm{ran}\,(\bigcup G) \cap S_j = \emptyset$. Thus S_j is no longer stationary in $V^{\mathbb{P}}$. On the other hand, if $j \in X$, S_j is still stationary in $V^{\mathbb{P}}$. Thus, to apply **MM**, we need to prove:

Subclaim. $\mathbb{P}$ preserves stationary subsets of ω_1.

Proof. Let $T \subset \omega_1$ be stationary and let $p \Vdash$ "$\dot{c} \subset \check{\omega_1}$ is club". Hence there is $i < \omega_1$ such that $T \cap T_i$ is stationary.

Let us look at stationary subset $S_{f(i)}$. We pick $x \prec H_\theta$ for some θ that is large enough, and let $|x| = \aleph_1$. Furthermore, we can let $x \cap \omega_2 \in S_{f(i)}$[4]. Then we pick countable $y \prec x \prec H_\theta$, which satisfies $\sup(y \cap \omega_2) = x \cap \omega_2 \in S_{f(i)}$, and $y \cap \omega_1 \in T \cap T_i$. Next, we construct a sequence

$$p = p_0 \geq p_1 \geq p_2 \geq \ldots \geq p_n \geq \ldots, \quad n < \omega,$$

for all $p_n \in y \cap \mathbb{P}$. F.a. dense $D \subseteq \mathbb{P}$ and $D \in y$, there exists n such that $p_n \in D$. Thus we have:

- $\bigcup_n \mathrm{dom}\, p_n = y \cap \omega_1 \in T \cap T_i$;
- $\sup(\mathrm{ran}\, p_n) = \sup(y \cap \omega_2) \in S_{f(i)}$.

This gives the fact that $q = \bigcup_n p_n \cup \{(y \cap \omega_1, \sup(y \cap \omega_2))\} \in \mathbb{P}$, and $q \vDash T \cap \dot{c} \neq \emptyset$. $\square$

Now we apply **MM**. $D^\alpha = \{p : \alpha \in \mathrm{dom}(p)\}$ is dense f.a. $\alpha < \omega_1$. Let G be a filter which meets all D^α. Then $\bigcup G : \omega_1 \to S^{\omega}_{\omega_1}$ will be normal, i.e. $C = \mathrm{ran}(\bigcup G)$ is a club. Since $G \in V$ by **MM**, $\sup(C) = \alpha < \omega_2$, and actually $\alpha \in S^{\omega_1}_{\omega_2}$. F.a. $\xi < \omega_1$ and all $i < \omega_1$,

[2] Repeat the argument above, and note that $S^{\omega}_{\omega_2}$ could be splitted into ω_2 many stationary sets.

[3] $p(\lambda) = \sup_{\beta<\lambda} p(\beta)$ for all $\lambda \leq \alpha$

[4] We may consider a continuous tower $(x_i : i < \omega_2)$, with each $x_i \prec H_\theta$, $|x_i| = \aleph_1$ and $x_i \cap \omega_2 \in \omega_2$ for all i. Then $\{x_i \cap \omega_2 : i < \omega_2\}$ forms a club of ω_2. Since $S_{f(i)}$ is stationary, we can choose such x.

$\xi \in T_i \implies (\bigcup G)(\xi) \in S_{f(i)}$. This implies that $(\bigcup G)"T_i$ is thus a stationary subset of $S_{f(i)} \cap \alpha$, so that $S_{f(i)}$ is stationary. And if $i \notin \mathrm{ran}(f)$, then $\mathrm{ran}(\bigcup G) \cap S_i = \emptyset$, so that then $S_i \cap \alpha$ is not stationary. This ends the proof of the **Claim**, as well as the **Theorem 1**. □

Now we aim to move on to showing **MM** implies that there are effective counterexamples to **CH**. This was first verified by Hugh Woodin. The above theorem is originally discovered by Foreman-Magidor-Shelah([1], [2]).

2 Iterations of V

Starting point: U, a measure on some $\kappa > \omega$. i.e. non-principal, $< \kappa$-complete(normal) ultrafilter on κ.

Given an ultrapower, we can define the corresponding canonical embedding:

$$j : V \to \mathrm{Ult}(V, U); \quad x \mapsto [\langle x \rangle]_U.$$

Good references of this process would be [3] and [4]. We may iterate this process:

$$V = M_0 \xrightarrow{j_{01}} M_1 = \mathrm{Ult}(M_0, U) \xrightarrow{j_{12}} ... \xrightarrow{j_{n\omega}} M_\omega = \lim_{i<\omega} M_i \xrightarrow{j_{\omega,\omega+1}} ...$$

Let this process ends at some λ such that M_λ is not well-founded. So λ is a limit ordinal, since every ultrapower is taken internally, every successor stage before λ is well-founded. Since M_λ is ill-founded, there are ordinals $\{\alpha_n : n < \omega\}$ of M_λ such that $\alpha_n > \alpha_{n+1}$ for all n. Let α be the least ordinal in V such that $j_{0\lambda}(\alpha) > \alpha_0$. Let M_γ be an iteration before M_λ which contains a preimage of α_0, that is:

$$\bar{\alpha}_0 = j^{-1}_{\gamma\lambda}(\alpha_0) \in M_\gamma; \ \bar{\alpha} = j_{0\gamma}(\alpha) \in M_\gamma.$$

Since the ultrapower is taken internally, every iteration can see all the iterations after it.[5] Now, the pair (λ, α) is the lexicographically minimal pair that satisfies the following conditions:

- M_λ is the least ill-founded iterated ultrapower;
- α is the least ordinal of V such that the ordinals below $j_{0\lambda}(\alpha)$ in M_λ contain a infinite descending sequence.

Thus, by elementarity, M_γ sees that $j_{0\gamma}(\lambda, \alpha) = (j_{0\gamma}(\lambda), \bar{\alpha})$ is the lexicographically minimal pair. However, it is clear that lexicographically, $(\lambda - \gamma, \bar{\alpha}_0) < (j_{0\gamma}(\lambda), \bar{\alpha})$ in M_γ, with ordinals below $j_{\gamma\lambda}(\bar{\alpha}_0)$ contain a infinite descending chain in M_λ. Contradiction.

To introduce $\mathbb{P}_{\max}$ forcing, we now move on to introduce external iteration.

Discussion of $\#'s$: Let $x \in \mathbb{R}$(subset of ω). In model $L[x]$, we cannot have a non-trivial embedding $j : L[x] \to L[x]$. However, it is possible to have such embedding in V:

Definition. $x^\#$ exists iff there is a non-trivial elementary embedding $j : L[x] \to L[x]$.

[5] This is the Factor Lemma, see [3].

Say $\kappa = \text{crit}(j)$. Define

$$U = \{X \in P(\kappa) \cap L[x] : \kappa \in j(X)\} \notin L[x].$$

It is easy to verify that U is a κ-complete, non-principal, normal ultrafilter of κ of $L[x]$. If U is obtained in that way, an iteration process can be carried on:

$$(L[x], U) \xrightarrow{j_{01}} (L[x], U_1) \xrightarrow{j_{12}} ... \to (L[x], U_i) \xrightarrow{j_{i,i+1}} ...$$

where

$$U_1 = \bigcup_{y \in L[x], |y| \leq \kappa \text{ in } \kappa} j_{01}(U \cap y).$$

Let $\tau = \kappa^{+L[x]}$, then $(L_\tau[x], U)$ is a premouse. This premouse is iterable, in the sense that we just discussed. A more detailed discussion is contained in [4].

So $x^\#$ exists iff there is a non-trivial embedding $j : L[x] \to L[x]$. This implies that we have some iterable $(L_\tau[x], U)$. The converse is still true.

We will move on to discussing ideals on ω_1 and generically iterating via such ideals.

Definition. $I \subset P(\kappa)$ is an ideal iff

- $\emptyset \in I$;
- $X, Y \in I \implies X \cup Y \in I$;
- $X \in I, Y \subset X \implies Y \in I$.
- non-triviality: $\kappa \notin I$.

Write $I^+ = \{X \subset \kappa : X \notin I\}$ the positive sets.

Definition. I is uniform iff $\forall X \in I^+, |X| = \kappa$.

Definition. I is normal iff $\nabla_{i<\kappa} X_i \in I$[6] f.a. $\{X_i : i < \kappa\} \subset I$.

Example: Non-stationary ideal $\mathbf{NS}_\kappa$ is normal, since the club filter of κ is normal. We shall denote $\mathbf{NS}_\kappa^+$ as the collection of all stationary subsets of κ.

Definition. I an ideal on κ. I is saturated iff every antichain in I^+ is small, that means: if $\{X_i : i < \theta\} \subset I^+$ is antichain[i.e. $X_i \cap X_j \in I, \forall i, j < \theta, i \neq j$], then $\theta \leq \kappa$.

The construction of $(T_i : i < \omega_1)$ in the proof of **Theorem 1** produced a maximal antichain in $\mathbf{NS}_{\omega_1}^+$, i.e., one which cannot be extended properly to a bigger antichain.

Let $I = \mathbf{NS}_{\omega_1}$. Say $\vec{S} = \{S_i : i < \omega_1\} \subset \mathbf{NS}_{\omega_1}^+$ is a maximal antichain, then $\vec{S}$ is sealed in the following way: Suppose $\kappa - \nabla S_i \in I^+$, then $\exists i_0[(\kappa - \nabla S_i) \cap S_{i_0} \in I^+]$. Pick $\xi \in S_{i_0}$ sufficiently big($\xi > i_0$), and $\xi \in \kappa - \nabla S_i$(in particular $\xi \notin S_{i_0}$). So $\kappa - \nabla S_i \in I$, and so there is a club $C \subset \omega_1$, $C \cap (\kappa - \nabla S_i) = \emptyset$, and so $\xi \in C \implies \exists i < \xi(\xi \in S_i)$.

We shall still focus on $I = \mathbf{NS}_{\omega_1}$.

[6] ∇ means the diagonal union, that is:

$$\nabla_{i<\kappa} X_i = \{\xi < \kappa : \xi \in \bigcup_{i<\xi} X_i\}.$$

Theorem 2. *The following are equivalent:*

1. $\mathbf{NS}_{\omega_1}$ *is saturated;*
2. *F.a. countable* $x \prec H_\theta$*, for* θ *sufficiently large,* $\alpha = x \cap \omega_1$*. If* $\mathcal{A} \subset \mathbf{NS}^+_{\omega_1}$ *is a maximal antichain,* $\mathcal{A} \in x$*, then* $\alpha \in S \in \mathcal{A} \cap x$*.*

Proof. " $\Rightarrow$ " Let $\mathcal{A} = (S_i : i < \omega_1) \in x$ and $C \subset \nabla_{i<\omega_1} S_i$, where C is a club, and $C \in x$ for some S. Since for every $\beta < \alpha$ in x, there is some γ in x such that $x \vDash \gamma > \beta$, and γ is still less than α. This gives that α is limit in x, which means $\alpha \in C$, and $\alpha \in S_i$ for some $i < \alpha$. We can take $S = S_i$.
" $\Longleftarrow$ " We are given an antichain $\mathcal{A}$. Let $(x_i : i < \omega_1)$ be a continuous tower[7] of countable substructure of H_θ, and $\mathcal{A} \in x_0$. Choose $f : \omega_1 \to \mathcal{A}$ s.t. there is a club C s.t.

$$\forall i \forall S \in C (S \in \mathcal{A} \cap x_i \implies S = f(j), \text{ for some } j < i).$$

In short, we enumerate the elements of $\mathcal{A} \cap x_i$ for each $i < \omega_1$, and those i satisfies the above condition forms a club. This gives a sealing of the antichain. □

Now we move on to precipitous ideals. Let $I \subset P(\kappa)$ be a normal uniform ideal on κ. Construe I^+ as poset by stipulating:

$$X \leq Y \text{ iff } X - Y \in I.$$

Thus $(I^+; \leq)$ is a poset. Let G be V-generic for this poset. Then $U = \{X \in P(\kappa) \cap V : \exists Y \in G(Y \subset X)\}$ is a V-ultrafilter:

- Let $X \subset \kappa$. Then either $X \in U$ or $\kappa - X \in U$: For all $Y \in I^+$, either $X \cap Y \in I^+$ or $Y - X \in I^+$ is true. Assume the first is true, then $X \cap Y$ is a stronger condition than Y, and $X \cap Y \Vdash \hat{X} \in \dot{U}$. Otherwise, $Y - X \Vdash \kappa - \hat{X} \in \dot{U}$.
- Let $X, Y \in U$, then $X \cap Y \in U$.

Let I be a normal uniform ideal on κ, and $G \subseteq I^+$ be V-generic. This give rise to a V-ultrafilter, and we can define

$$j : V \to \text{Ult}(V; U) (\text{or to write } \text{Ult}(V; G)),$$

where G, j only exist in $V[G]$.

Definition. I is precipitous iff f.a. V-generic G, $\text{Ult}(V; G)$ is well-founded.

We shall prove this in next lecture:

Theorem 3. *The followings are equivalent:*

- $\mathbf{NS}_{\omega_1}$ *is precipitous;*

[7] $x_{i+1} \supset x_i$ and $x_\delta = \bigcup_{i<\delta} x_i$ for limit δ.

- *Let $x \prec H_\theta$ be countable, and let M_x be its transitive collapse via embedding σ. Define*

$$G_x = \{X \in P(\omega_1^{M_x}) \cap M_x : \omega_1^{M_x} \in \sigma(X)\}.$$

Then the collection

$$\mathcal{S} = \{x \prec H_\theta : |x| = \omega \wedge G_x \text{ is } \sigma^{-1}(\mathbf{NS}_{\omega_1}^+)\text{-generic over } M_x\}$$

is projective stationary.

Definition. $\mathcal{S}$ is projective stationary iff f.a. $T \subset \omega_1$ stationary, $\{X \in \mathcal{S} : X \cap \omega_1 \in T\}$ is stationary.

For example, any club is projective stationary. This gives that $\mathbf{NS}_{\omega_1}$ is saturated implies $\mathbf{NS}_{\omega_1}$ is precipitous. We shall prove this in next lecture.

References

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