Ralf Schindler: Talks#4 on Logic Summer School of Fudan University, 2020

TODAY:

- Show a characterization of precitiousness;
- V is generically iterable with respect to precitious ideals;
- Discussion of effecitive counterexamples to **CH**.
- Illustrations of Admissible Club Guessing(ACG) $\Longrightarrow$ $\mathfrak{u}_2 = \omega_2$.
- Prove ACG follows from **MM**.

Theorem 1. *The followings are equivalent:*

- $\mathbf{NS}_{\omega_1}$ *is precipitous;*
- *Let $x \prec H_\theta$ be countable, and let M_x be its transitive collapse via embedding σ. Define*

$$G_x = \{X \in P(\omega_1^{M_x}) \cap M_x : \omega_1^{M_x} \in \sigma(X)\}.$$

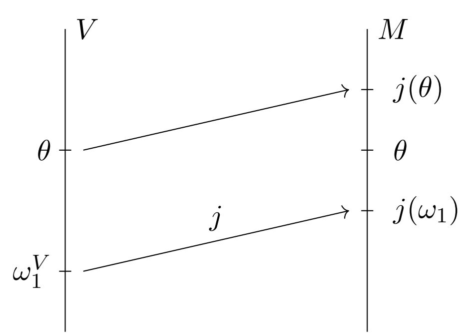
Then the collection

$$\mathcal{S} = \{x \prec H_\theta : |x| = \omega \wedge G_x \text{ is } \sigma^{-1}(\mathbf{NS}^+_{\omega_1})\text{-generic over } M_x\}$$

is projective stationary.

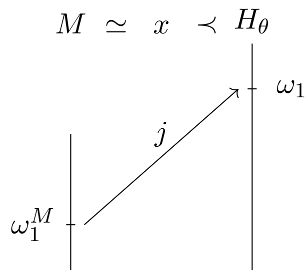
Definition. $\mathcal{S}$ is projective stationary iff f.a. $T \subset \omega_1$ stationary, $\{x \in \mathcal{S} : x \cap \omega_1 \in T\}$ is stationary.

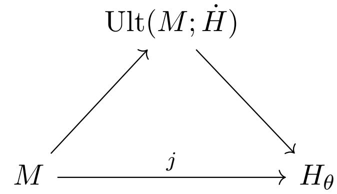
Proof. (Sketch) "$\Longrightarrow$": Fix a stationary set $T \subset \omega_1$ and let $C \subset [H_\theta]^\omega = \{x \in [H_\theta]^\omega : f\text{"}x \subset x, f : H_\theta^{<\omega} \to H_\theta\}$ be a club. Let G be V-generic for $\mathbf{NS}^+_{\omega_1}$ such that $T \in G$. This implies the existence of an elementary embedding $j : V \to \mathrm{Ult}(V; G) = M$.



Since $T \in G$, $\omega_1^V \in j(T)$. Now we consider the structure $j\text{"}H_\theta^V$. It is a substructure of $H^M_{j(\theta)}$, $\omega_1^M \cap j\text{"}H_\theta^V = \omega_1^V$, and is closed under $j(f)$. By reflection and absoluteness between $V[G]$ and M, it will be true that in M there is a countable $x \prec H^M_{j(\theta)}$ such that $x \cap j(\omega_1^V) = \omega_1^V \in j(T)$, and x is closed under $j(f)$. By pulling back the appropriate statement via j, we get that in V there is a countable $x \prec H_\theta$ with $x \cap \omega_1 \in T$ and closed under f. Since θ is chosen as large as we want, we now have that $\mathcal{S}$ is in fact projective stationary.

" $\Longleftarrow$ " Assume that $\mathbf{NS}_{\omega_1}$ is not precipitous and let $x \in \mathcal{S}$. Then there is a stationary set $T \in \mathbf{NS}^+_{\omega_1}$, such that $T \Vdash_V$ " $\text{Ult}(V; \dot{G})$ is ill-founded".





Then by definition $\omega_1^M = \omega_1 \cap M \in T$. Let $\bar{T} = T \cap \omega_1^M \in M$, then the condition $\omega_1^M \in j(\bar{T})$ gives that $\bar{T} \in H$, where H is the generic filter derived from j. Thus by downstairs elementarity, $\bar{T} \Vdash_M$ " $\text{Ult}(M; \dot{H})$ is ill-founded". This leads to a contradiction since by factoring j by H, the diagram on the right commutes, therefore $\text{Ult}(M; \dot{H})$ embeds elementarily into H_θ, which is well-founded. $\square$

Corollary 2. $\mathbf{NS}_{\omega_1}$ *saturated* $\Longrightarrow$ $\mathbf{NS}_{\omega_1}$ *precipitous.*

How do you obtain models in which $\mathbf{NS}_{\omega_1}$ is saturated/precipitous? We can first pick a measurable cardinal and collapse it to ω_1. Then there is a precipitous ideal of ω_1. Then we can shoot a club through the stationary complements of members of the ideal and make this precipitous ideal to be $\mathbf{NS}_{\omega_1}$. However, this does not work for $\mathbf{NS}_{\omega_1}$ to be saturated. To get it one may need the existence of a Woodin cardinal δ, and some δ-c.c. semi-proper forcing to do that. A proof of this theorem can be found on here*.

It is also true that $\mathbf{MM} \Longrightarrow \mathbf{NS}_{\omega_1}$ is saturated. This theorem is obtained by Foreman-Magidor-Shelah in their original $\mathbf{MM}$ paper [2], [3].

1 Generic iterablity

Assume $I \subset P(\omega_1)$ is a precitious normal uniform ideal on ω_1. Work in $V^{Col(\omega,\theta)}$, where $\theta > \omega_1$ is large enough. We may do the following iteration process:

$$V = M_0 \xrightarrow{j_{01}} M_1 = \text{Ult}(M_0; G_0) \to ... \to M_\omega = \lim \text{dir}_{n \to \omega}(M_n; G_n) \to ...$$

For every $n < \omega$, we let G_n to be the generic filter of $(\mathbf{NS}^+_{\omega_1})^{M_n}$. By elementarity, $(\mathbf{NS}^+_{\omega_1})^{M_n}$ is percipitous for every $n < \omega$, so every successor stages are well-founded. We now show that whenever we can construct (M_α, G_α), it is always well-founded.

Remark. Notice that here, the use of $Col(\omega, \theta)$ is to collape H_θ to countable size, so we can always pick G_n(clearly not unique) in $V^{Col(\omega,\theta)}$. Because of this, we cannot do this iteration to any ordinal stages, but only as much as we want.

*`https://ivv5hpp.uni-muenster.de/u/rds/sat_ideal_better_version.pdf`

Theorem 3. *Let $I \subset P(\omega_1)$ is a precipitous normal uniform ideal on ω_1, then V is generically iterable via I^+ and its images.*

Here, being generically iterable means that V can be iterated along the sequence without being ill-founded at limit stages.

Proof. A variant of the argument of the iterablity of V via a measure in V and its images. Suppose the statement is false. Then we pick the least triple $(\theta, \lambda, \alpha)$ with respect to lexicographic order, such that

- θ is the least ordinal such that in $V^{Col(\omega,\theta)}$, there is a $\lambda < \theta$ such that the λ-th generic iteration taken inside $V^{Col(\omega,\theta)}$ is ill-founded.
- λ is the least ordinal such that the λ-th generic iteration contains an ill-founded sequence of ordinals;
- α is the least ordinal such that there is a ill-founded sequence of ordinals below $j_{0\lambda}(\alpha)$.

Suppose $\gamma < \lambda$ such that there is an $\bar{\alpha}_1$, $j_{\gamma\lambda}(\bar{\alpha}_1)$ is the first element of the infinite descending sequence in M_λ. By elementarity, M_γ sees that $(j_{0\gamma}\theta, j_{0\gamma}\lambda, j_{0\gamma}\alpha)$ is the lexicographically least triple, however, $(\theta, \lambda - \gamma, \bar{\alpha}_1)$ is lexicographically smaller, and it satisfies our requirements listed above. Contradiction. □

2 Effective counterexamples to CH

In the last lecture, we have proved that **MM** $\implies 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$. This implies a surjection $f : \mathbb{R} \to \omega_2$. We now look at the set

$$R_f = \{(x, y) \in \mathbb{R}^2 : f(x) \leq f(y)\}.$$

What are possible levels of definability of R_f? And can we have f such that $R_f \in L(\mathbb{R})^\dagger$? Or even: Can(in the presence of large cardinals, or under **MM**) such an R_f be projective?

Definition. $R \subset \mathbb{R}^n$ is projecive iff R is definable(with parameters) over $(H_{\omega_1}; \in)$.

This is not the usual definition for projectiveness; however, since every element in H_{ω_1} is coded by a real, $H_{\omega_1}^{L(\mathbb{R})} = H_{\omega_1}^V$. So if something is definable over $(H_{\omega_1}; \in)$, then it is certainly inside $L(\mathbb{R})$. Equivalently, R is projective iff we can write $\vec{x} \in R$ iff

$$\exists x_0 \in \mathbb{R} \forall x_1 \in \mathbb{R} ... Q x_k (\vec{x}, x_0, ..., x_k) \in C,$$

where C is a Borel set of $\mathbb{R}^{n+k+1}$.

Let us look at H_{ω_2}. A formula ϕ is $\Pi_2^{H_{\omega_2}}$ if it is equivalent(in **ZFC**) to a function of the form:

$$\forall A \in H_{\omega_2} \exists B \in H_{\omega_2} \psi(A, B),$$

where ψ is Σ_0. It turns out that **MM** is complete with respect to $\Pi_2^{H_{\omega_2}}$ statements. Important example of $\Pi_2^{H_{\omega_2}}$ statements:

$^\dagger L(\mathbb{R})$ = the least transitive model of **ZF** which contains $\mathbb{R} \cup$ ORD.

- $\mathfrak{u}_2$‡$= \omega_2^V$.
- Admissible Club Guessing(ACG).
- $\varphi_{\mathbf{AC}}$ and $\psi_{\mathbf{AC}}$, etc..

$\mathfrak{u}_2 = \omega_2$ is a $\Pi_2^{H_{\omega_2}}$ statement: Since $\mathfrak{u}_2 = \sup\{(\omega_1^V)^{+L[x]} : x \in \mathbb{R}\}$, and $\mathfrak{u}_2 \leq \omega_2^V$, we have

- $\mathfrak{u}_2 \geq \omega_2^V \iff \forall\alpha < \omega_2 \exists x \in \mathbb{R}[(\omega_1^V)^{+L[x]} \geq \alpha]$;
- $(\omega_1^V)^{+L[x]} \geq \alpha \iff \exists\beta[L_\beta[x] \vDash \alpha \leq (\omega_1^V)^+] \iff \exists\beta\exists N[N$is a transitive structure of hight $\beta \wedge N \vDash$"Everything is at most countable"$\wedge L_\beta[x] \vDash$ "$\alpha \leq N \cap \mathrm{ORD}$"].

Under the hypothesis $\forall x \in \mathbb{R}(\exists x^\#)$(given by **MM**) and $\mathfrak{u}_2 = \omega_2$, we have

$$f : \mathbb{R} \to \omega_2; \quad \omega \supset x \mapsto \omega_1^{+L[x]}.$$

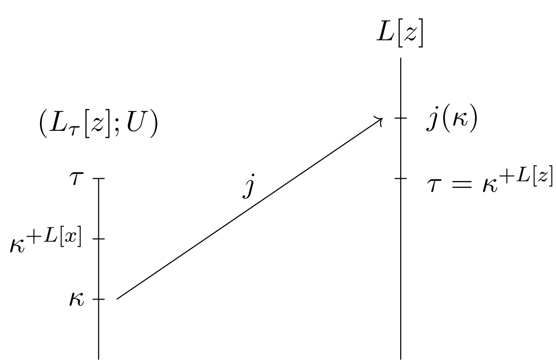
Since $\mathfrak{u}_2 = \omega_2$, f is cofinal. Now look at R_f we defined above and we want to claim this is projective, actually Δ_3^1. For any $x, y \in \mathbb{R}$, we have

$$\begin{aligned}(\omega_1^V)^{+L[x]} \leq& (\omega_1^V)^{+L[y]} \\ &\iff \exists z \subset \omega \exists (L_\tau[z]; U) \text{ iterable } [\kappa = \mathrm{crit}(j) \wedge (L_\tau[z]; U) \vDash \kappa^{+L[x]} \leq \kappa^{+L[y]}].\end{aligned}$$

Proof. Note that here, $\kappa^{+L[x]}$ and $\kappa^{+L[y]}$ is actually the interpretation inside $L_\tau[z]$, that is: $(\kappa^{+L[x]})^{L\tau[z]}$ and $(\kappa^{+L[y]})^{L_\tau[z]}$. Thus to get " $\Longleftarrow$ " direction, we may need to assume x, y are Turing reducible to z(or other canonical way), and we want to prove that $(\kappa^{+L[x]})^{L\tau[z]} = \kappa^{+L[x]}$. Let z, τ be chosen to satisfy:

$$(L_\tau[z]; U) \vDash \kappa^{+L[x]} \leq \kappa^{+L[y]},$$

for $\kappa = \mathrm{crit}(j)$. Then by the amenability of $(L_\tau[z], U)$, we may iterate this structure and see that $\tau = \kappa^{+L[z]}$. Thus there are no more subset of κ beyond τ and by elementarity, $\kappa^{+L[x]} = (\kappa^{+L[x]})^{L\tau[z]}$.



□

‡$\mathfrak{u}_2$ is the second uniform indiscernible ordinal: Suppose $x^\#$ exists for all $x \in \mathbb{R}$. Since the x-indiscernible ordinal class C_x is a club for every $x \in \mathbb{R}$, $\bigcap_{x\in\mathbb{R}} C_x = (\mathfrak{u}_i : i \geq 1)$ is another class of indiscernibles called the uniform indiscernibles. Clearly countable ordinals can never be uniform indiscernible, and ω_1 is uniform indiscernible, $\mathfrak{u}_1 = \omega_1$.

So the complexity of this statement is Σ_2 over H_{ω_1}. Thus, this statement is Δ^1_3.

Definition. ACG is the following statement:

$$\forall C \subset \omega_1 \text{ club } \exists D \subset C \text{ club } \exists x \subset \omega_1 [D \text{ is the set of all countable } x\text{-admissible}].$$

Here, x-admissible are ordinals τ such that $L_\tau[x]$ are model of **KP** set theory. Or just: $D \in L[z]$ for some $z \subset \omega$.

Remark. It can be proved that ACG implies the existence of $x^\#$ for all $x \subset \omega$.

Definition. $\psi_{\mathbf{AC}}$ is the following statement:

$\forall S, T \subset \omega_1$ stationary and co-stationary $\exists \eta < \omega_2 \exists C \subset \omega_1$ club $\forall \xi \in C[\xi \in T \iff f_\eta(\xi) \in S]$.

Here, f_η is the function defined by some surjection $g : \omega_1 \to \eta$ such that $f_\eta(\xi)$ is the ordertype of $g"\xi$. It is also called the canonical function of η.

All the listed $\Pi_2^{H_{\omega_2}}$ statements are implied by $(*)$, and $\mathbf{MM}^{++}$ implies $(*)$. Next we want to show ACG implies $\mathfrak{u}_2 = \omega_2$ and $\mathbf{MM}^{++}$ implies ACG.

Theorem 4. *ACG* $\implies \mathfrak{u}_2 = \omega_2$.

Proof. (Sketch) Let $\alpha < \omega_2$ and some bijection $f : \omega_1 \to \alpha$. Moreover, fix a continuous tower $(X_i : i < \omega_1)$ of countable substructure of H_θ. Let N_i be the transitive collapse of X_i. We then let $f \in X_0$, which gives that there is $\alpha_i > \omega_1^{N_i}$ in N_i such that α_i would be maped to α in H_θ. We can then modify the tower such that $\alpha_i < \omega_1^{N_{i+1}}$ for every $i < \omega_1$. Thus ACG gives a club $D \subset \{\omega_1^{N_i} : i = \omega_1^{N_i}, i < \omega_1\}$, and $D \in L[x]$. We may then assume that D is definable with ω_1^V as the only parameter. Thus,

$$\xi \in D \iff L[x] \vDash \phi(\xi, x, \omega_1^V);$$

Assuming that $\eta < \omega_1^V$ is x-indiscernible, we have

$$\xi \in D \cap \eta \iff L[x] \vDash \phi(\xi, x, \eta).$$

So now $\eta \mapsto \omega_1^V$, and $D \cap \eta \mapsto D$ by the elementary embedding from $L[x]$ to itself. This gives every x-indiscernible in D is a limit point of D. Thus if $\xi \in D$, then $\xi = \omega_1^{N_\xi}$ and the next x-indiscernible $> \xi$ is bigger than $\omega_1^{N_{\xi+1}}$, thus bigger than α_ξ.
Now we pick another tower $(Y_i : i < \omega_1)$ such that $x^\# \in Y_0$(in particular, $D \in Y_0$), and $f \in Y_0$. So there is a club $E \subset D$ such that $Y_i \cap \alpha = X_i \cap \alpha$ for all $i \in E$. Now if $i \in E$, we denote the transitive collapse of Y_i as M_i, and thus

$$M_i \vDash \text{"the next } x\text{-indiscernible} > i \text{ is} > \alpha_i\text{"}.$$

By elementarity, this gives

$$H_\theta \vDash \text{"the next } x\text{-indiscernible} > \omega_1 \text{ is} > \alpha\text{"}.$$

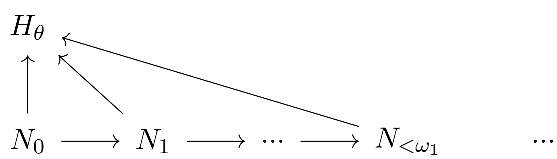
which gives $\mathfrak{u}_2 = \omega_2$. □

Theorem 5. $\mathbf{MM}^{++} \implies$ *ACG.*

Proof. (Sketch, Easier) Let $C \subset \omega_1$ be a club. Now we are going to construct a tower $(X_i : i < \omega_1)$ of countable substructures of H_θ, where $\theta \geq \omega_2$. Let X_0 be some countable transitive substructure of H_θ, and N_0 its transitive collapse. Let X_0 satisfy:

- $\omega_1 \in X_0$ and $\omega_1^{N_0} = \alpha_0$;
- There is some $\alpha_0 \in N_0$ such that the elementary embedding $j_0 : N_0 \to H_\theta$ maps α_0 to ω_1;
- $C \in X_0$ and $C_0 \cap \alpha_0 \in N_0$.

Let $G_0 = \{s \in P(\alpha_0) \cap N_0 : \alpha_0 \in j_0(x)\}$. Then this filter is N_0-generic, since $\mathbf{NS}_{\omega_1}^{+N_0}$ is saturated(by $\mathbf{MM}$, $\mathbf{NS}_{\omega_1}$ is saturated). Now by the precitiousness, we can do the generic iteration:



where $N_{i+1} = \mathrm{Ult}(N_i, G_i)$. By elementarity, the derived G_i is always saturated, so this iteration process can keep on going before we meet ω_1. Now let $X_i = \mathrm{ran}(j_i)$. Since N_0 is countable, we can find some countable $x \subset \omega$ such that x codes N_0. Now by the following unproved claim:

Claim. Suppose $\alpha < \omega_1$ is x-admissible, then α is the limit point of $\{\omega_1^{N_i} : i < \omega_1\}$.[§]

We have that every x-admissible ordinal α is inside C since $\omega_1^{N_i}$ is the limit point of C for every $i < \omega_1$. Let D be the set of all limit point of α and hence ACG is proved. $\square$

Now we would like to present a harder proof which can be further motified into a way to prove $\mathbf{MM}^{++} \implies (*)$.

Proof. (Sketch, Harder, [1]) We would like to force the existence of some iterable countable structure $(M; I)$, together with its generic iteration $(M_i; I_i : i \leq \omega_1)$ such that $M_{\omega_1} = H_{\omega_2}^V$[¶]. We do it via a forcing which preserves stationary subsets of ω_1.

We aim to find a transitive model N in the generic extension such that

$$N \vDash \text{"}\exists \text{generic iteration } (M_i, G_i : i < \omega_1), |M_i| = \omega \text{ s.t.}$$
$$M_0 \text{ iterable } \wedge M_{\omega_1} = \lim \mathrm{dir}_{i \to \omega_1} M_i = (H_{\omega_2}^V; \in, \mathbf{NS}_{\omega_1}^V)\text{"}.$$

[§]It seems that we only need the $\mathbf{MM}^{++}$ to make $\mathbf{NS}_{\omega_1}$ saturated until this claim. However, to prove this claim we may need a little bit more, say there is a measurable cardinal in V, or $P(\omega_1)^\#$ exists. This follows from descriptive set theory, where one can draw the conclusion from the existence of such a ω_1-iterable structure N_0.

[¶]Clearly, the iteration cannot be performed in V, since $|M_{\omega_1}| = \aleph_1$. Moreover, since the generic iteration embedding is cofinal, M_{ω_1} adds a ω-cofinal sequence of ω_2.

Think of N as a term model. The forcing will consist of finite sets of sentences in a language describing the full theory of such a model + starting to prove that this model is well-founded by ranking the constants:

$$\phi(c_{i_0}, ..., c_{i_k}),\ \ f : c_{i_0} \mapsto \xi \in \mathrm{ORD}$$

such that in some outer model, this finite piece of information can be extended to a maximal consistenct theory + a proof that the model which arises is well-founded.

Our forcing notion will actually have size $2^{\omega_2} \geq \omega_3$. We will need to assume $2^{\omega_2} = \omega_3$, which follows from $\Diamond_{\omega_3}$[‖]. We will finish this proof in our next lecture. □

References

[1] Benjamin Claverie, Ralf Schindler, et al. Increasing $\mathfrak{u}_2$ by a stationary set preserving forcing. *Journal of Symbolic Logic*, 74(1):187–200, 2009.

[2] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals, and non-regular ultrafilters. part i. *Annals of Mathematics*, pages 1–47, 1988.

[3] Matthew Foreman, Menachem Magidor, and Saharon Shelah. Martin's maximum, saturated ideals and non-regular ultrafilters. part ii. *Annals of Mathematics*, pages 521–545, 1988.

[‖] There is a sequence $((Q_\alpha, A_\alpha) : \alpha < \omega_3)$ such that $(Q_\alpha : \alpha < \omega_3)$ is a tower of transitive substructures of H_{ω_3} of size $\aleph_2$ with $\bigcup_\alpha Q_\alpha = H_{\omega_3}$; Moreover, for all $A \subset H_{\omega_3}$, $\{\alpha : (Q_\alpha, A_\alpha) \prec (H_{\omega_3, A})\}$ is stationary.