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Varsovian models, II

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Let us write $M = M_{swsw}$ for the "least" inner model in which there are $\delta_0 < \kappa_0 < \delta_1 < \kappa_1$ s.t. $M \models$ "δ_n is Woodin" and $M \models$ "κ_n is strong" for $n \in \{0,1\}$. In this note, we aim to sketch the analysis which leads to identifying the Varsovian model associated with M; this model is also the mantle of M.

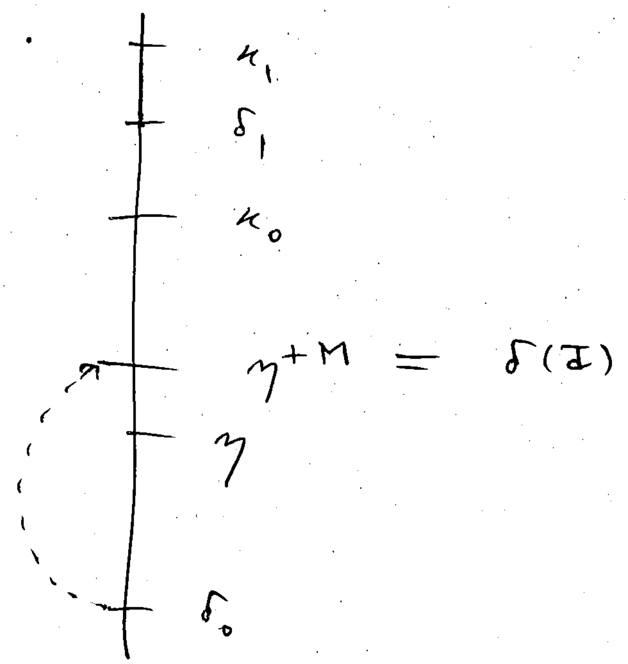
We build upon our joint paper "Varsovian models, I," submitted to the JSL, abbreviated by [VMI].

We assume that M has a nice iteration strategy, Σ, cf. [VMI]. We denote by δ_0 the least Woodin cardinal of M.

We build a first $\mathcal{M}^0_\infty$ in much the same way as $\mathcal{M}_\infty$ was constructed in [VMI].

Let η be a cutpoint, $\delta_0 < \eta < \kappa_0$, and let $\mathcal{T}$ be a tree on M which lives on $M|\delta_0$ s.t.

- $lh(\mathcal{I}) = \eta^{+M}$,
- $\mathcal{I}$ is certified by P-constructions, i.e. for all limit $\lambda < \eta^{+M}$, $P^M(\mathcal{M}(\mathcal{I} \restriction \lambda))$ gives a Q-structure and $P^M(\mathcal{M}(\mathcal{I}\restriction\lambda)) \trianglelefteq \mathcal{M}^{\mathcal{I}}_\lambda$ or more gen., a Q-str. $Q \trianglelefteq \mathcal{M}^{\mathcal{I}}_\lambda$ may be obtained by pulling back a Q-str. obtained by a P constr. as in Sch-St, "The self-iterability of $L[E]$," [SiLE] (in particular, $\mathcal{I}$ is according to Σ),
- on a tail end, $\mathcal{I}$ results from hitting the least extender violating an axiom of the appropriate extender algebra, with a non-dropping iterate of M as a starting model, and
- $M|\eta^{+M}$ is generic over $\mathcal{M}(\mathcal{I})$ for the extender algebra.



In this situation,

$$\mathcal{P} = \mathcal{P}^M(\mathcal{M}(\mathcal{J})) = \mathcal{M}_b^{\mathcal{J}},$$

where $b = \Sigma(\mathcal{J})$. The points in the $\mathcal{M}_\infty^0$ - system consist of all such $\mathcal{P}$.

We let

$(\mathcal{M}_\infty^0, \pi_{\mathcal{P}\infty}^0 : \mathcal{P} \in \text{system}) =$

dir lim of all $\mathcal{P} \in$ system under the (unique) maps $\pi_{\mathcal{P}\mathcal{P}'}$, when $\mathcal{P}'$ is a Σ-iterate of $\mathcal{P}$.

A priori, $\mathcal{M}_\infty^0$ may not be in M, but exactly as in [VMI] we may actually show that it is. (Define the concept of s-<u>iterability</u>, s a finite set of ordinals,

$$\gamma_s^{\mathcal{P}} = \sup\Big(\mathrm{Hull}^{\mathcal{P}|\max(s)}(\underbrace{s^-}_{s\setminus\{\max(s)\}}) \cap \overbrace{\delta_0^{\mathcal{P}}}^{\text{least Wdn. of } \mathcal{P}}\Big)$$

$$H_s^{\mathcal{P}} = \mathrm{Hull}^{\mathcal{P}|\max(s)}(\gamma_s^{\mathcal{P}} \cup s^-), \text{ etc.}$$

The maps $\pi_{\mathcal{P}\mathcal{P}'}^s = \pi_{\mathcal{P}\mathcal{P}'} \restriction H_s^{\mathcal{P}}$ are in M, when $\mathcal{P}, \mathcal{P}'$ are s-iterable, $\mathcal{P}'$ is an iterate of $\mathcal{P}$, etc.)

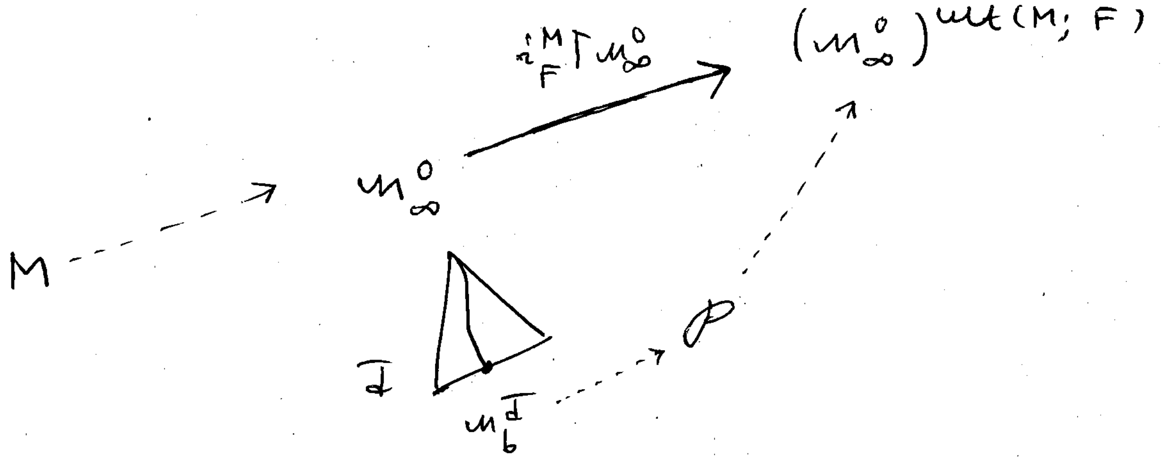
In V, $\mathcal{M}_\infty^0$ is a Σ-iterate of M.

In much the same way as before we have:

<u>Claim 1.</u> M knows $\Sigma_{\mathcal{M}^0_\infty}$, restricted to trees which live on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$.

$\delta_0^{\mathcal{M}^0_\infty}$ = least Wdn. of $\mathcal{M}^0_\infty$

Proof: Let $\mathcal{T}$ be a tree on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ of limit length which is acc. to $\Sigma_{\mathcal{M}^0_\infty}$, $\mathcal{T} \in M$. Let us assume that $b = \Sigma_{\mathcal{M}^0_\infty}(\mathcal{T})$ doesn't drop and $\delta(\mathcal{T}) = \pi^{\mathcal{T}}_{0b}(\delta_0^{\mathcal{M}^0_\infty})$.



Let F be a total extender with crit(F) = κ_0 from the M-sequence s.t. $\mathcal{T} \in ult(M;F)$.

By simultaneously comparing with $\mathcal{M}(\mathcal{T})$ and making some $ult(M;F) | \eta^{+ult(M;F)}$ generic, where η is a cutpoint of $\mathrm{ult}(M;F)$, η sufficiently large $< i_F^M(\mathrm{crit}(F))$, we may find some $\mathcal{P} \in$ the $\mathcal{M}_\infty^0$-system of $\mathrm{ult}(M;F)$ s.t. in V, there is some canonical embedding $k : \mathcal{M}(\mathcal{I}) \longrightarrow \mathcal{P} | \delta_0^{\mathcal{P}}$, $\delta_0^{\mathcal{P}} = \eta^{+\mathrm{ult}(M;F)} = \delta(\mathcal{U})$, where $\mathcal{U}$ is the tree producing $\mathcal{P}$. (k is actually an iteration map.) Also $\pi_{\mathcal{P},(\mathcal{M}_\infty^0)^{\mathrm{ult}(M;F)}}$ maps $\mathcal{P}$ into $(\mathcal{M}_\infty^0)^{\mathrm{ult}(M;F)}$.

We claim that in any generic extension of V, b is the unique cofinal branch c thru $\mathcal{I}$ s.t. there is $\ell : \mathcal{M}(\mathcal{I}) \longrightarrow (\mathcal{M}_\infty^0)^{\mathrm{ult}(M;F)} | \delta_0^{(\mathcal{M}_\infty^0)^{\mathrm{ult}(M;F)}}$ with

$$\ell \circ i_b^{\mathcal{I}} = i_F^M \restriction \mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0} .$$

First, b is such a branch c : Let $\xi < \delta_0^{\mathcal{M}_\infty^0}$. $\xi = (\pi_{\bar{\mathcal{P}}\infty}^0)^S(\bar{\xi})$, some $\bar{\xi}$ and $\bar{\mathcal{P}}$ and S. $\bar{\xi} < \kappa_0$,

$$\begin{gathered} (\pi^0_{\bar{\mathcal{P}}\infty})^S \\ \| \\ \pi^0_{\bar{\mathcal{P}}\infty} \restriction H_S^{\bar{\mathcal{P}}} \in M \end{gathered}$$

so $i_F^M(\xi) = i_F^M((\pi_{\bar{\mathcal{P}}\infty}^0)^S)(\bar{\xi}) =$

$$\left(\pi^{S}_{\bar{P}(\mathcal{M}^0_\infty)\,\mathrm{ult}(M;F)}\right)(\bar{\xi}) =$$

$$\left(\pi^{S}_{\mathcal{M}^0_\infty\;\mathcal{M}^0_\infty\,\mathrm{ult}}\right)\circ\left(\pi^{S}_{\bar{P}\;\mathcal{M}^0_\infty}\right)(\bar{\xi}) = \pi^{S}_{\mathcal{M}^0_\infty\;\mathcal{M}^0_\infty\,\mathrm{ult}}(\xi)$$

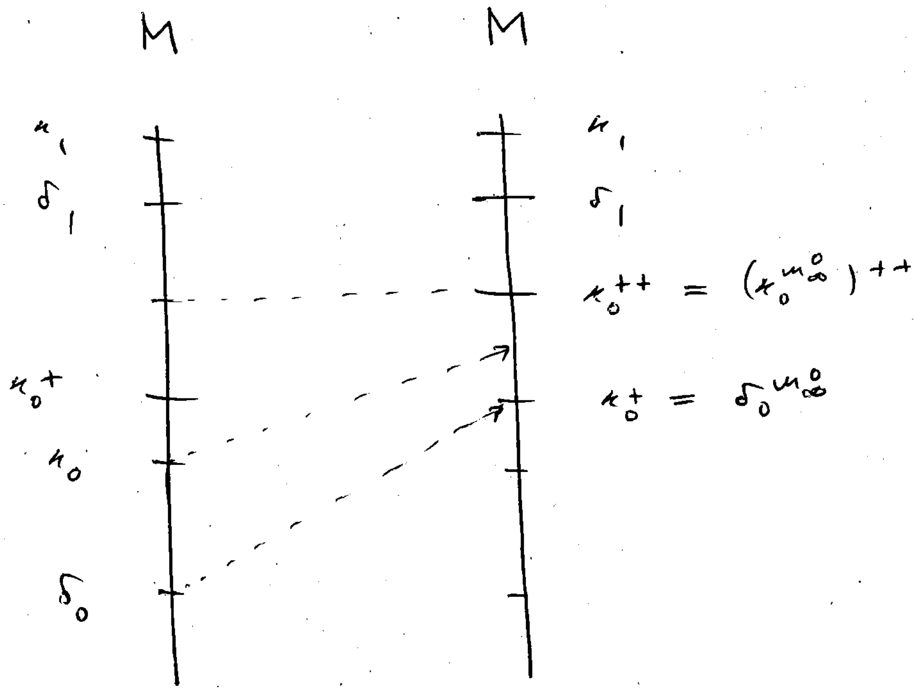
$$= \pi_{P,\,\mathcal{M}^0_\infty\,\mathrm{ult}} \circ k \circ \hat{i}^{I}_{b}(\xi).$$

[Here, we confuse $\bar{P}$ with $\mathrm{ult}(\bar{P}; F \upharpoonright \bar{P})$, and we use $\pi^{S}_{\mathcal{M}^0_\infty\;\mathcal{M}^0_\infty\,\mathrm{ult}}$ for the canonical map which sends ~~H~~ $H^{\mathcal{M}^0_\infty}_{S}$ to $H^{(\mathcal{M}^0_\infty)\,\mathrm{ult}(M;F)}_{S}$; we may assume S consists of V-cardinals which are fixed points under all maps.]

A similar argument shows that in V, $i^M_F \upharpoonright \mathcal{M}^0_\infty$ is actually an iteration map of $\mathcal{M}^0_\infty$ obtained by $\Sigma_{\mathcal{M}^0_\infty}$.

But then b is the unique such c by branch condensation.

This shows that $b \in M$, and in fact that Claim 1 is true. $\dashv$ (Claim 1)



As in [VMI], let us write $\rho \mapsto \rho^*$ for the map

$$\rho \mapsto \min\{\pi^0_{\mathcal{P}\infty}(\rho) : \mathcal{P} \in \text{system}\}$$
$$= \min\{(\pi^0_{\mathcal{P}\infty})^S(\rho) : \mathcal{P} \in \text{system}\}$$

which is also equal to the map sending M^0_∞ into its M^0_∞.

Exactly as in [VMI], M is a generic extension of $L[M^0_\infty, \rho \mapsto \rho^*]$ via a forcing $\mathbb{P}$ which has size κ_0^{++M} and has the $\cancel{(\kappa_0^+)}$ (κ_0^{++M})-chain condition.

We have that M and $L[M^0_\infty, \rho \mapsto \rho^*]$ have the same cardinals $\geq (\kappa_0^+)^M = \delta_0^{M^0_\infty}$

and $$V^{\mathcal{M}_\infty^0}_{\delta_0^{\mathcal{M}_\infty^0}} = V^{L[\mathcal{M}_\infty^0; p \mapsto p^*]}_{\delta_0^{\mathcal{M}_\infty^0}}, \quad \text{etc.}$$

We now aim to verify that M knows how to iterate $\mathcal{M}_\infty^0$ w.r.t. trees which act on $\mathcal{M}_\infty^0 | (\kappa_0^+)^{\mathcal{M}_\infty^0}$.

$$\overset{\shortparallel}{(\kappa_0^{\mathcal{M}_\infty^0})^{+\mathcal{M}_\infty^0}}$$

Let $\mathcal{U}$ on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ arise in the following fashion. First ~~...~~ iterate the least measurable and its images $\kappa_0^{\mathcal{M}_\infty^0}$ times, then start making $\mathcal{M}_\infty^0 | \kappa_0^{+\mathcal{M}_\infty^0}$ generic over an iterate, i.e. always hit the least extender violating an axiom of the relevant extender algebra.

Straightforward arguments give that $lh(\mathcal{U}) = \kappa_0^{+\mathcal{M}_\infty^0} + 1$, ~~...~~ $\delta(\mathcal{U} \upharpoonright lh(\mathcal{U})-1) = \kappa_0^{+\mathcal{M}_\infty^0} = \delta_0^{\mathcal{M}_\infty^{\mathcal{U}}}$. Let us write

$$i = \pi^{\mathcal{U}}_{0\infty} \upharpoonright \mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0},$$

so that i goes cofinally from $\delta_0^{\mathcal{M}_\infty^0}$ into $\kappa_0^{+\mathcal{M}_\infty^0}$.

A variant of the above argument for Claim 1 shows that $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ is iterable in every generic extension of M, and that $L[\mathcal{M}^0_\infty, p \mapsto p^*]$ knows how to iterate $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$. (Cf. [VMI].)

Hence $i \in L[\mathcal{M}^0_\infty, p \mapsto p^*]$.

Let us write $Q = \mathcal{M}^0_\infty | \kappa_0^{+\mathcal{M}^0_\infty}$.

<u>Claim 2.</u> $Q = \mathrm{Hull}^Q(i)$

[Note: We can write $\mathrm{Hull}^Q(i)$ also as $\mathrm{Hull}^Q(\delta_0^{\mathcal{M}^0_\infty} \cup \mathrm{ran}(i))$.]

Proof: Let

$$\bar{Q} \overset{\sigma}{\cong} \mathrm{Hull}^Q(i) \prec Q .$$

We aim to prove that $\sigma = \mathrm{id}$.

We have that $\mathcal{U} \upharpoonright \mathrm{lh}(\mathcal{U})-1$ is definable over Q (from no parameters). We may then define $b = [0, \mathrm{lh}(\mathcal{U})-1)$ over Q from the predicate i. Let $\bar{\mathcal{U}}$ be the tree on

$\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty^0}$ which is defined over $\bar{Q}$ as $\mathcal{U} \upharpoonright lh(\mathcal{U})-1$ is defined over Q, and let $\bar{b}$ be defined over $\bar{Q}$ from $\bar{\imath}$, as b is defined over Q from i, where

$$\begin{aligned} \bar{\imath} &= \sigma^{-1}{}''\, i \\ &= \{ (\xi, \sigma^{-1}(i(\xi))) : \xi < \delta_0^{\mathcal{M}_\infty^0} \}. \end{aligned}$$

Construing $\mathcal{U}$ as a tree on $\mathcal{M}_\infty^0$, we may define

$$\sigma' : \mathcal{M}_{\bar{b}}^{\bar{\mathcal{U}}} \longrightarrow \mathcal{M}_b^{\mathcal{U}} = \mathcal{M}_\infty^{\mathcal{U}}$$

by $\sigma'(\pi_{\alpha \bar{b}}^{\bar{\mathcal{U}}}(x)) = \pi_{\sigma(\alpha)\, b}^{\mathcal{U}}(\sigma(x))$. Notice $\sigma' \upharpoonright \bar{Q} \cap OR = \sigma \upharpoonright \bar{Q} \cap OR$.

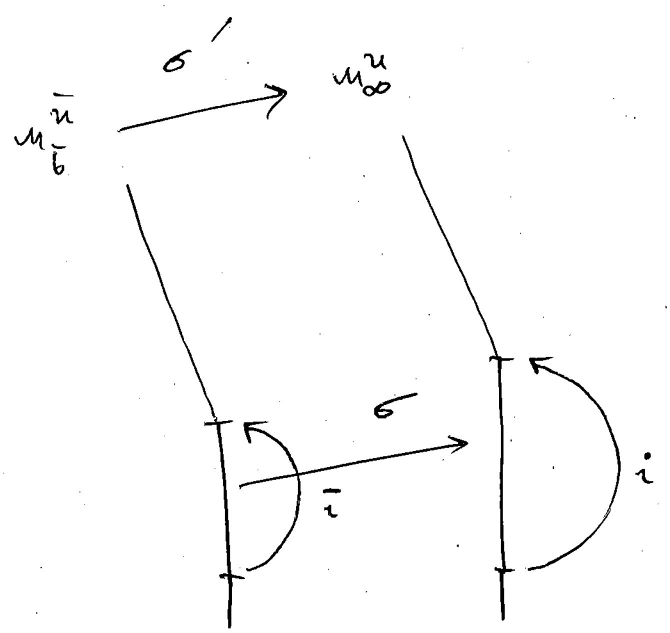
It is straightforward to verify that

$$\mathcal{M}_\infty^{\mathcal{U}} = \mathcal{P}^{ult(\mathcal{M}_\infty^0; F)}(m(\mathcal{U} \upharpoonright lh(\mathcal{U})-1)),$$

where F = the least total $\mathcal{M}_\infty^0$-extender with critical point $\kappa_0^{\mathcal{M}_\infty^0}$.

~~[illegible]~~

Q is generic over $\mathcal{M}_\infty^{\mathcal{U}}$, and $\bar{Q}$ is generic over $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{U}}}$ for the respective extender



algebras. We may extend σ' to

$$\sigma^*: \mathcal{M}_{\bar b}^{\bar u}[\bar a] \longrightarrow \mathcal{M}_\infty^{u}[Q] = ult(\mathcal{M}_\infty^0; F)$$

$$\| \qquad\qquad //$$

$$p^{ult(\mathcal{M}_\infty^0;F)}(\mathcal{M}(u\restriction lh(u)-1))[Q]$$

by setting $\sigma^*(\tau^{\bar a}) = \sigma'(\tau)^Q$ for any $\mathcal{M}_{\bar b}^{\bar u}$-term τ. This is well-defined, as

$$p \Vdash_{\mathcal{M}_{\bar b}^{\bar u}} \varphi(\tau, -) \quad \text{iff}$$

$$\sigma'(p) \Vdash_{\mathcal{M}_\infty^{u}} \varphi(\sigma'(\tau))$$

and $\sigma'(p) = \sigma(p)$ and so $p \in \bar Q$ iff $\sigma'(p) \in Q$.

Write Γ for the class of all M-indiscernibles. Every element of Γ is a fixed point under all embeddings, and

$$ult(\mathcal{M}^0_\infty; F) = Hull^{ult(\mathcal{M}^0_\infty; F)}(\delta_0^{\mathcal{M}^0_\infty} \cup \{\kappa_0^{\mathcal{M}^0_\infty}\} \cup \Gamma)$$

by the construction of $\mathcal{M}^0_\infty$.

But $\delta_0^{\mathcal{M}^0_\infty} \cup \{\kappa_0^{\mathcal{M}^0_\infty}\} \cup \Gamma \subset ran(\sigma^*)$, so that $\sigma^* = id$. However, $\sigma^* \upharpoonright \bar{Q} \cap OR = \sigma' \upharpoonright \bar{Q} \cap OR = \sigma \upharpoonright Q \cap OR$, so that $\sigma = id$, as desired.

$\dashv$ (Claim 2)

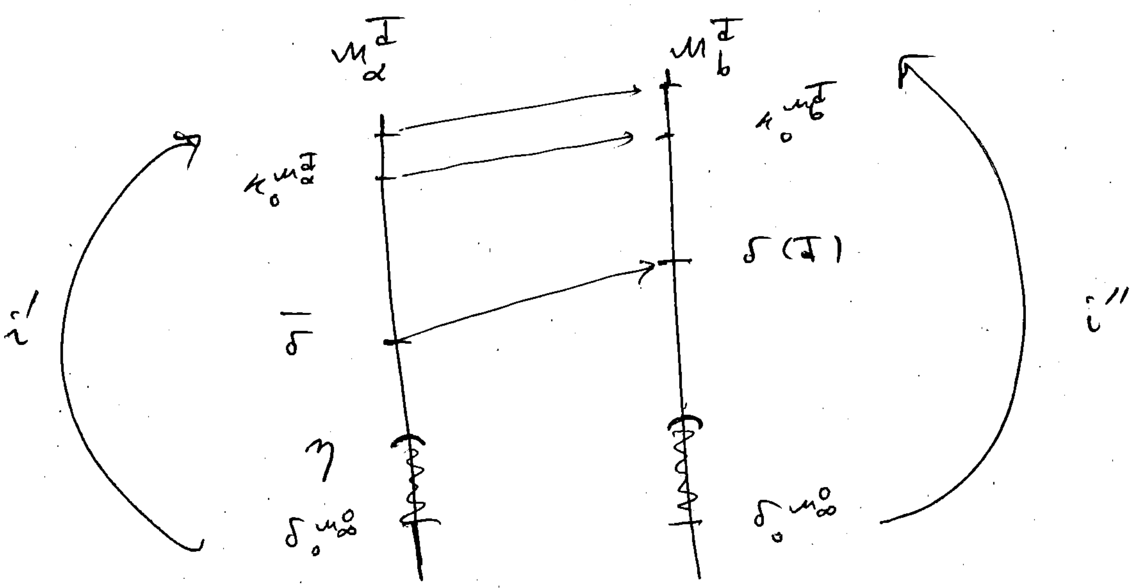
<u>Claim 3.</u> $\Sigma_{\mathcal{M}^0_\infty} \upharpoonright t$ "$\in$" M, where t are all normal trees $\mathcal{T}$ on $\mathcal{M}^0_\infty$ which are in M and <u>act on</u> $\mathcal{M}^0_\infty | \kappa_0^{+\mathcal{M}^0_\infty}$ by which we mean that $crit(E_\alpha^{\mathcal{T}}) < \pi^{\mathcal{T}}_{0\alpha^*}(\kappa_0^{+\mathcal{M}^0_\infty})$.

$$\alpha^* = \mathcal{T}\text{-pred}(\alpha+1)$$

Proof: Let us fix such a tree $\mathcal{T}$ of limit length s.t. $\mathcal{I}$ is acc. to $\Sigma_{\mathcal{M}_\infty^0}$.

Let $b = \Sigma_{\mathcal{M}_\infty^0}(\mathcal{I})$, and let us assume that $\delta(\mathcal{I})$ is Woodin in $\mathcal{M}_b^{\mathcal{I}}$.

Let α be least in b s.t. $\delta(\mathcal{I}) \in \operatorname{ran} \pi_{\alpha b}^{\mathcal{I}}$, and let us write $\bar\delta = (\pi_{\alpha b}^{\mathcal{I}})^{-1}(\delta(\mathcal{I}))$.



Let us assume that $\mathcal{I}$ did not touch $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$. [Otherwise replace $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ by the final image thereof, etc.]

Recall i from p. 8. ~~By [illegible]~~

$(\mathcal{M}_\infty^0 \| \kappa_0^{+\mathcal{M}_\infty^0}\ ;\ i)$ is an amenable structure, so we may write

$$i' = \pi^{\mathcal{I}}_{0\alpha}(i) = \bigcup_{\xi < \delta_0^{M_\infty}} \pi^{\mathcal{I}}_{0\alpha}(i \restriction \xi) \quad \text{and}$$

$$i'' = \pi^{\mathcal{I}}_{0b}(i) = \bigcup_{\xi < \delta_0^{M_\infty}} \pi^{\mathcal{I}}_{0b}(i \restriction \xi) .$$

By branch condensation, i' and i'' are the ~~___~~ iteration maps coming from the true cofinal ~~___~~ branches thru $\pi^{\mathcal{I}}_{0\alpha}(\mathcal{U} \restriction lh(\mathcal{U})-1)$ and $\pi^{\mathcal{I}}_{0b}(\mathcal{U} \restriction lh(\mathcal{U})-1)$, resp. (where $\mathcal{U}$ is as on p. 8)

Let η = sup of the generators of the extenders used along $[0,\alpha]_{\mathcal{I}}$.

As $\alpha < lh(\mathcal{I})$, $\eta < \delta$, and by Claim 2,

$$\pi^{\mathcal{I}}_{0\alpha}(Q) = \mathrm{Hull}^{\pi^{\mathcal{I}}_{0\alpha}(Q)}(i') .$$

<u>Subclaim 1.</u> b is the unique cofinal w.f. branch c thru $\mathcal{I}$ s.t.

$$\pi^{\mathcal{I}}_{\alpha c}{}'' \mathrm{ran}(i') \cap \delta = \mathrm{ran}(i'') \cap \delta(\mathcal{I}) .$$

This holds true by the zipper argument.

$\dashv$ (Subclaim 1)

Let us split the remaining argument into two cases.

<u>Case 1.</u> $\bar{\delta}$ is not overlapped by any extender in $\mathcal{M}_\alpha^{\mathcal{I}}$.

We may then work in some $ult(M; F)$ and construct a version of $\mathcal{M}_b^{\mathcal{I}}$ and of i'' so as to identify b, as follows.

Let $W = L[E](\mathcal{M}(\mathcal{I}))^{ult(M; F)}$, where F is a suff. long extender with $crit(F) = \kappa_0$.

W is iterable above $\delta(\mathcal{I})$ in V, so that we may successfully compare W, $\mathcal{M}_b^{\mathcal{I}}$ in V:

$$\begin{tikzcd}
i'' \arrow[d, mapsto] & \mathcal{M}_b^{\mathcal{I}} \arrow[dr, "\pi_{\mathcal{M}_b^{\mathcal{I}},\mathcal{R}}"'] & & W \arrow[dl, "\pi_{W,\mathcal{R}}"] & i^* \arrow[d, mapsto] \\
\tilde{\imath} & & \mathcal{R} & & \tilde{\imath}
\end{tikzcd}$$

Let κ_0^{+W} be the W-succ. of the least strong of W, and let i^* arise from iterating $\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty^0}$ so as to make $W | \kappa_0^{+W}$ gen. over the iterate (i.e., producing a version of $\mathcal{U}$ for W).

By branch condensation, using the notation from p.14,

$$\pi_{\mathcal{M}_b^{\mathcal{I}},\mathcal{R}}(i'') = \pi_{W,\mathcal{R}}(i^*)$$

call it $\tilde{i}$.

But then for $\xi < \delta_0^{\mathcal{M}_\infty^0}$ s.t. $i'(\xi) < \bar{\delta}$:

$$\pi^{\mathcal{I}}_{\alpha b}(i'(\xi)) = i''(\xi) = \tilde{i}(\xi) = i^*(\xi),$$

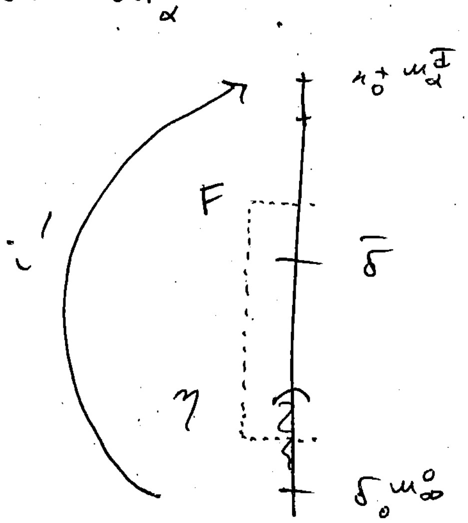
as $\pi_{\mathcal{M}_b^{\mathcal{I}},\mathcal{R}} \restriction \delta(\mathcal{I}) = \pi_{W,\mathcal{R}} \restriction \delta(\mathcal{I}) = \mathrm{id}$,

and b is the unique cof. w.f. branch c thru $\mathcal{I}$ s.t.

$$\pi^{\mathcal{I}}_{\alpha c}{}''\,\mathrm{ran}(i') \cap \bar{\delta} = \mathrm{ran}(i^*) \cap \delta(\mathcal{I}).$$

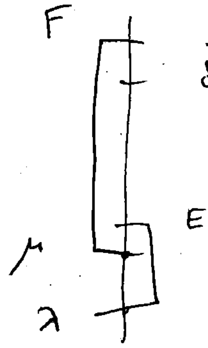
Therefore b can be computed and exists in M.

<u>Case 2.</u> $\bar{\delta}$ is overlapped by an extender in $\mathcal{M}_\alpha^{\mathcal{I}}$.



Subclaim 2. Let F overlap $\bar{\delta}$ in $M_\alpha^{\mathcal{I}}$. Then $\mathrm{crit}(F) \geq \eta$.

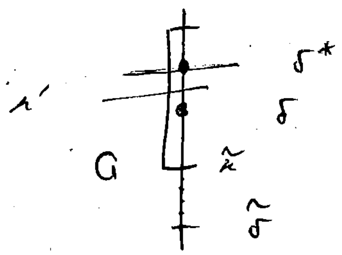
Proof: Assume $\mathrm{crit}(F) < \eta$. Let $E = E_\beta^{\mathcal{I}}$ be the least extender such that $\beta+1 \in (0, \alpha]$ and E has a generator $\geq \mathrm{crit}(F)$.



Write $\mu = \mathrm{crit}(F)$ and $\lambda = \mathrm{crit}(E)$.

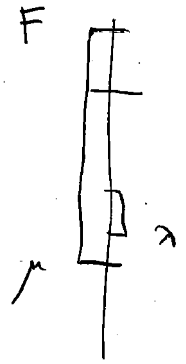
If $\lambda < \mu$, then λ is strong in $M_\alpha^{\mathcal{I}} | \mu$. But μ is a limit of Woodins in $M_\alpha^{\mathcal{I}}$, and hence λ is a limit of Woodins. This contradicts the choice of M.

By the choice of M, any extender in any iterate of M can overlap at most one Woodin cardinal.

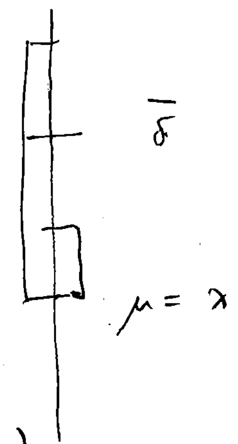


Otherwise if $\delta < \delta^*$ are both Woodins overlapped by G, then the model cut off at δ^* has $\tilde{\delta} < \hat{\kappa} < \delta < \kappa'$ s.t. $\tilde{\delta}, \delta$ are Woodin and $\hat{\kappa}, \kappa'$ are strong.

Hence if $\lambda > \mu$, then $\bar{\delta} \in \operatorname{ran}(\pi^{\mathcal{I}}_{\beta^* \alpha})$, where $\beta^* = \mathcal{I}\text{-pred}(\beta+1)$, in contradiction with the choice of α.

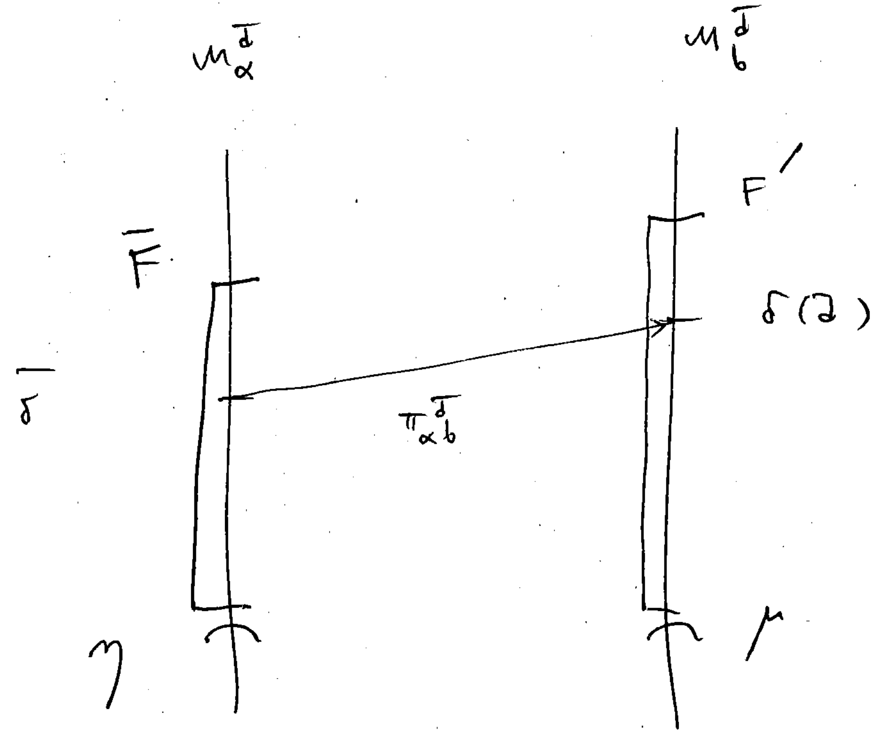


The case $\lambda = \mu$ is similar to the case $\lambda < \mu$. If $\lambda = \mu$, then in $M^{\mathcal{I}}_{\alpha}$ is a limit of Woodins as well as a limit of strongs. Contradiction!



$\dashv$ (Subclaim 2)

The situation is therefore as follows.



In order to find a version of M_b^J inside M, we'll do a revised $L[E](M(J))$ construction which allows adding extenders overlapping $\delta(J)$, similar to a construction that was given in Hauser, Schindler, "Projective uniformization revisited," APAL, abbr. by [PUR].

Let F' be the least extender of M_b^J overlapping $\delta(J)$. Write $\lambda = \mathrm{crit}(F')$. Then <u>all</u> extenders of M_b^J which overlap $\delta(J)$ have the same critical point, λ. In particular, μ is a cutpoint.

Let F be a suff. long extender with critical point κ_0. We do a revised $L[E](M(J))$ construction $(M_\xi, W_\xi : \xi \le OR)$ as follows. We add extenders with critical point $> \delta(J)$ if they are fully backgrounded in $ult(M; F)$.

But we also add extenders G with ~~[illegible]~~ $\mathrm{crit}(G) = \lambda$ provided that the following is true. ~~[illegible]~~

Suppose ~~...~~ $\mathcal{M}_\xi = \mathcal{W}_\xi$ has been constructed, a model of ZFC^-, and there is an extender G s.t.

(a) $(\mathcal{M}_\xi ; G)$ is a premouse,

(b) $\operatorname{crit}(G) = \lambda$, and

(c) for all suff. long extenders F with $\operatorname{crit}(F) = \kappa_0$,
$$\operatorname{ult}(\, L[E](\mathfrak{m}(\mathcal{I})) \,/\, \lambda^{+\mathfrak{m}(\mathcal{I})}\,)^{\operatorname{ult}(M;F)} \; ; \; G)$$
is iterable (in V).

A priori, condition (c) cannot be checked inside M; however, having the class of all indiscernibles replace thick classes, arguments exactly as in [PUR] show the following.

<u>Subclaim 3</u> The construction of the $(\mathcal{M}_\xi, \mathcal{W}_\xi : \xi \leq \mathrm{OR})$ never breaks down, and if W ~~...~~ denotes the ~~...~~ last model, then W, $\mathcal{M}_b^{\mathcal{I}}$ compare to a common model via comparisons which only use extenders with critical points $> \delta(\mathcal{I})$ on both sides.

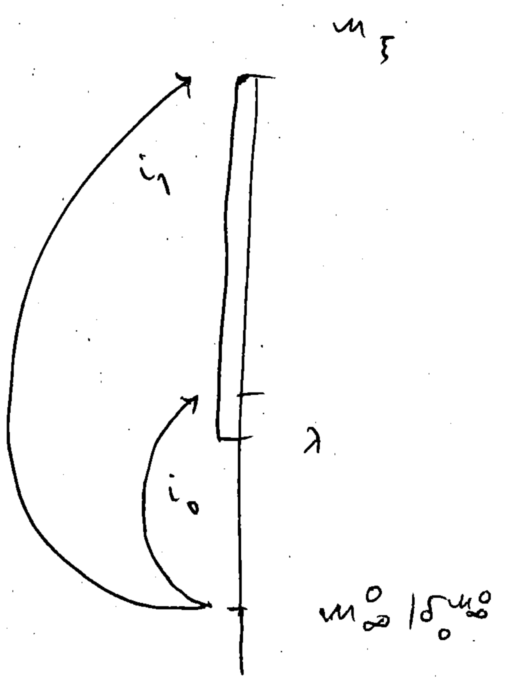
$\dashv$ (Subclaim 3)

The following shows that $(M_\xi, N_\xi : \xi < OR)$ may be defined inside M after all, so that the last model, W, of this construction exist in M.

<u>Subclaim 4.</u> TFAE.

(1) G satisfies (c) on p. 20, and

(2) if i_0, i_1 result from iterating $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ to make $\mathcal{M}_\xi | \lambda^{+\mathcal{M}_\xi}$, $\mathcal{M}_\xi$ (resp.) generic in a canonical fashion in much the same way as i was produced on p. 8, then



$$\pi_G{}'' \operatorname{ran}(i_0) = \operatorname{ran}(i_1).$$

Proof: (1) $\Longrightarrow$ (2). This is by an argument ~~as~~ which has also been used in the proof of Claim 2. Let $\mathcal{U}_0, \mathcal{U}_1$ resp., be the trees which give rise to i_0, i_1.

$\mathcal{U}_0 \upharpoonright lh(\mathcal{U}_0)-1$, $\mathcal{U}_1 \upharpoonright lh(\mathcal{U}_1)-1$ may be defined over $\mathcal{M}_\xi | \lambda^{+\mathcal{M}_\xi}$, $\mathcal{M}_\xi$ resp., by the same formula. But then

$$[0, lh(\mathcal{U}_1)-1)_{\mathcal{U}_1} = \pi_G([0, lh(\mathcal{U}_0)-1)_{\mathcal{U}_0})$$

$$= \bigcup_{\alpha \in [0, lh(\mathcal{U}_0)-1)_{\mathcal{U}_0}} \pi_G([0,\alpha)_{\mathcal{U}_0})$$

by branch condensation.

(2) $\Rightarrow$ (1) : Again by an argument which has been used in the proof of Claim 2.

If $\mathcal{U}_0, \mathcal{U}_1$ are as in the proof of (1) $\Rightarrow$ (2), then $\mathcal{M}_\infty^{\mathcal{U}_1}$ is the "ultrapower" of $\mathcal{M}_\infty^{\mathcal{U}_1}$ via the map π_G. This extends to a canonical map

$$\tilde{\pi} : \mathcal{M}_\infty^{\mathcal{U}_0}[\mathcal{M}_\xi | \lambda^{+\mathcal{M}_\xi}] \longrightarrow \mathcal{M}_\infty^{\mathcal{U}_1}[\mathcal{M}_\xi].$$

But any $L[E](\mathcal{M}(I) | \lambda^{+\mathcal{M}(I)})$ as in
$$\overset{\shortparallel}{\mathcal{M}_\xi | \lambda^{+\mathcal{M}_\xi}}$$

(c) compares with $\mathcal{M}_\infty^{\kappa_0}[\mu_\xi \,|\, \lambda^{+\mu_\xi}]$ above λ, so that the following diagram shows that $ult(\, L[E](m(J) \,|\, \lambda^{+m(J)}) \,;\, G)$ is iterable :

$$\begin{tikzcd}
 & \mathcal{M}_\infty^{\kappa_1}[\mu_\xi] \arrow[drr, "\tilde{\pi} S_1"] & & \\
\mathcal{M}_\infty^{\kappa_0}[\mu_\xi \,|\, \lambda^{+\mu_\xi}] \arrow[ur, "\tilde{\pi}"] \arrow[drr, "S_1"] & & & Q' \\
 & & Q \arrow[ur] \arrow[r, "G"'] & Q^* \arrow[u] \\
L[E](m(J) \,|\, \lambda^{+m(J)}) \arrow[urr, "S_0"'] \arrow[dr, "G"'] & & & \\
 & ult(L[E](\ldots) \,;\, G) \arrow[uurr] & &
\end{tikzcd}$$

⊣ (Subclaim 4)

But now we are in a position as on p. 15 f. replacing the W there by the output of any ~~the~~ revised $L[E](m(J))$ as being defined on p. 19 f. By Subclaim 4, this revised $L[E](m(J))$ is in M, and b may be computed in M as on p. 16.

⊣ (Claim 3)

Before proceeding further, let us reorganize the model $L[\mathcal{M}_\infty^0, \rho \mapsto \rho^*]$.

Let us write $\bar{\Sigma}$ for $\Sigma_{\mathcal{M}_\infty^0}$, restricted to trees which live on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$. We have that $\bar{\Sigma}$ is amenable to $L[\mathcal{M}_\infty^0, \rho \mapsto \rho^*]$ and definable over $L[\mathcal{M}_\infty^0, \rho \mapsto \rho^*]$.

Let us write $L[\overline{\mathcal{M}_\infty^0}, \bar{\Sigma}]$ for the model which is obtained as follows.

We construct over $\mathcal{M}_\infty^0 | \underset{\substack{\shortparallel \\ (\kappa_0^{\mathcal{M}_\infty^0})^{+\mathcal{M}_\infty^0}}}{\kappa_0^{+\mathcal{M}_\infty^0}}$; if we reach a stage ν where $\mathcal{M}_\infty^0$ has an extender F with critical point $> \kappa_0^{+\mathcal{M}_\infty^0}$, then we add F (we will see that $L[\overline{\mathcal{M}_\infty^0}, \bar{\Sigma}] | \nu = L[\mathcal{M}_\infty, \rho \mapsto \rho^*] | \nu$, as far as the universes go, so this makes sense); otherwise, if $L[\overline{\mathcal{M}_\infty^0}, \bar{\Sigma}] | \nu$ has been constructed, then we let $\mathcal{T} \in L[\overline{\mathcal{M}_\infty^0}, \bar{\Sigma}] | \nu$ be the least tree on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ which is according to $\bar{\Sigma}$, $\mathcal{T}$ of limit length, s.t. we didn't tell the model yet $\bar{\Sigma}(\mathcal{T})$ — we then add

$\bar{\Sigma}(\bar{T})$ in an amenable way.

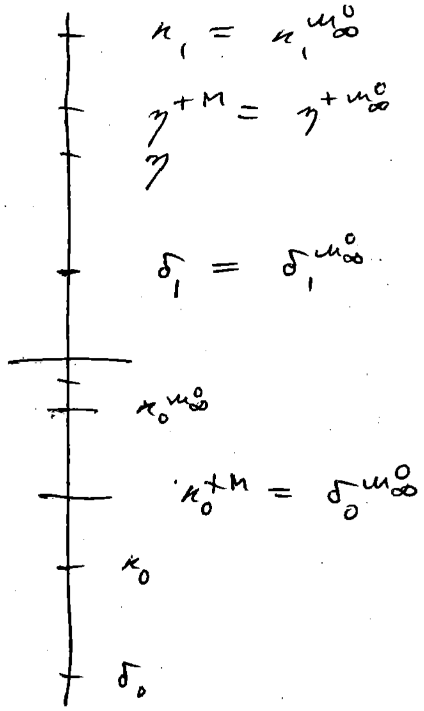
Notice that we don't add any extenders with critical point $\kappa_0^{M_\infty^0}$. However, as was shown above, any such extender $F = E_\nu^{M_\infty^0}$ from the M_∞^0-sequence is coded by a pair (i, i') (or, equivalently, by a pair (b, b')) where $\mathcal{U}, \mathcal{U}'$ are trees on $M_\infty^0 | \delta_0^{M_\infty^0}$ making $M_\infty^0 | \kappa_0^{+M_\infty^0}$, $M_\infty^0 | \nu$, resp., generic over an iterate of $M_\infty^0 | \delta_0^{M_\infty^0}$, ~~and~~ $lh(\mathcal{U}) = \kappa_0^{+M_\infty^0} + 1$, $lh(\mathcal{U}') = \nu + 1$, and $i = \pi_{0 \kappa_0^{+M_\infty^0}}^{\mathcal{U}}$, $i' = \pi_{0\nu}^{\mathcal{U}'}$ (and $b = [0, \kappa_0^{+M_\infty^0})_{\mathcal{U}}$, $b' = [0, \nu)_{\mathcal{U}'}$). As $b = \bar{\Sigma}(\mathcal{U})$ and $b' = \Sigma'(\mathcal{U}')$, F will be in $L[\overline{M_\infty^0}, \bar{\Sigma}] | \nu^*$, where $\nu^* - \nu$ is very small. ~~But what really~~ Also, $\rho \mapsto \rho^*$ clearly gets into $L[\overline{M_\infty^0}, \bar{\Sigma}] | \nu^*$, where $\nu^* - \kappa_0^{+M_\infty^0}$ is very small.

But then if $F = E_\nu^{M_\infty^0}$ is an M_∞^0-extender with critical point $> \kappa_0^{+M_\infty^0}$, then

$L[\overline{M_\infty^0}, \bar{\Sigma}] | \nu = L[M_\infty^0, \rho \mapsto \rho^*] | \nu$.

In particular, $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}] = L[\mathcal{M}^0_\infty, \rho H \rho^*]$, as far as the universes go.

Let us now describe the system which will produce $\mathcal{M}^1_\infty$.



Let η be a cutpoint in $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$ between δ_1 and κ_1, i.e., there is no extender F on the $L[\mathcal{M}^0_\infty, \bar{\Sigma}]$-sequence with $\mathrm{crit}(F) \leq \eta$ and $\mathrm{lh}(F) > \eta$. (This means that in $\mathcal{M}^0_\infty$, the only extenders which overlap η are ones with critical point $\kappa_0^{\mathcal{M}^0_\infty}$.)

Let $\mathcal{U}$ be a tree on $\mathcal{M}^0_\infty$ which arises in the following fashion. Recall that

$$V_{\delta_0^{\mathcal{M}_\infty^0}}^{L[\mathcal{M}_\infty^0, \bar{\Sigma}]} = V_{\delta_0^{\mathcal{M}_\infty^0}}^{L[\mathcal{M}_\infty^0, p \mapsto p^*]} = V_{\delta_0^{\mathcal{M}_\infty^0}}^{\mathcal{M}_\infty^0},$$

$\delta_0^{\mathcal{M}_\infty^0}$ is Woodin in $L[\mathcal{M}_\infty^0, p \mapsto p^*]$, and $L[\mathcal{M}_\infty^0, p \mapsto p^*]$ is iterable via $\bar{\Sigma}$ with respect to trees which live on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$.

Let $\mathcal{U} = \mathcal{U}_0 ^\frown \mathcal{U}_1$, whe $\mathcal{U}_0$ has succ. length $< \eta^{+M}$, $\mathcal{U}_0$ lives on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ and is acc. to $\bar{\Sigma}$, $\mathcal{U}_0 \in L[\mathcal{M}_\infty^0, \bar{\Sigma}] | \eta^{+M}$, $[0, lh(\mathcal{U}_0)-1]_{\mathcal{U}_0}$ does not drop, and $\mathcal{U}_1$ uses extenders with critical point $> \kappa_0^{\mathcal{M}_\infty^{\mathcal{U}_0}}$ and below $\delta_1^{\mathcal{M}_\infty^{\mathcal{U}_0}}$ and eventually starts making an initial segment of $L[\mathcal{M}_\infty^0, \bar{\Sigma}] | \eta^{+M}$ generic at the "image" of $\delta_1^{\mathcal{M}_\infty^{\mathcal{U}_0}}$.

In order to specify $\mathcal{U}_1$ more precisely, we need revised $\mathcal{P}$-constructions.

Suppose $\mathcal{U}_1 \restriction \lambda$ (λ a limit ordinal $\leq \eta^{+M}$) is guided by $\mathfrak{M}$-small Q-structures (and hence according to $\overline{\Sigma}$), $u_1 \restriction \lambda \in L[\overline{\mathcal{M}^0_\infty}, \overline{\Sigma}] \mid \delta(u_1 \restriction \lambda) + \omega$, and $L[\overline{\mathcal{M}^0_\infty}, \overline{\Sigma}] \mid \delta(u_1 \restriction \lambda)$ is generic over $\mathcal{M}(u_1 \restriction \lambda) + \omega$. We then perform a revised $\mathcal{P}$-construction as follows.

We construct over $\mathcal{M}(u_1 \restriction \lambda)$, ~~.....~~ maintaining inductively that $L[\overline{\mathcal{M}^0_\infty}, \overline{\Sigma}] \mid \nu$ is generic over $\mathcal{P} \mid \nu$. Suppose we constructed $\mathcal{P} \mid \nu$. If E_ν is an extender from the $L[\overline{\mathcal{M}^0_\infty}, \overline{\Sigma}]$-sequence, then we add $E_\nu \cap \mathcal{P} \mid \nu$. Otherwise we add the next piece of information concerning

$$\overline{\Sigma}_{\pi_0^{u_0} \, \mathrm{lh}(u_0)-1} (\mathcal{M}^0_\infty \mid \delta_0^{\mathcal{M}^0_\infty})$$

to $\mathcal{P}$, i.e., we add the branch acc. to this strategy for the least tree $\mathcal{T}$ of lim length on $\pi_0^{u_0} \, \mathrm{lh}(u_0)-1 (\mathcal{M}^0_\infty \mid \delta_0^{\mathcal{M}^0_\infty})$ which is acc. to this strategy but which was not yet added.

We claim that if E_ν is as above, then

$$\mathcal{P}|\nu\,[L[\overline{\mathcal{M}}_\infty^0,\bar\Sigma]\,|\,\delta(u_1\restriction\lambda)] = L[\overline{\mathcal{M}}_\infty^0,\bar\Sigma]\,|\,\nu ,$$

as far as the universes go.

This is true by the following reasoning. We have (w.l.o.g.)

$$\pi^{u_0}_{0\,lh(u_0)-1}\restriction \mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} \in \mathcal{P}|\nu\,[L[\overline{\mathcal{M}}_\infty^0,\bar\Sigma]\,|\,\delta(u_1\restriction\lambda)] .$$

Let us write

$$\pi : \mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} \longrightarrow \mathcal{M}^*$$

rather than

$$\pi^{u_0}_{0\,lh(u_0)-1}\restriction \mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} : \mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} \longrightarrow \pi^{u_0}_{0\,lh(u_0)-1}(\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0}) .$$

Let $\bar{\mathcal{T}} \in$ ~~$\mathcal{P}$~~ $\mathcal{P}|\nu\,[L[\overline{\mathcal{M}}_\infty^0,\bar\Sigma]\,|\,\delta(u_1\restriction\lambda)]$ be a tree on $\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0}$ which is according to $\bar\Sigma$; Let $\mathcal{T} = \pi\bar{\mathcal{T}}$ be the result of copying $\bar{\mathcal{T}}$ onto $\mathcal{M}^*$ using π; this works and gives a tree according to $\bar\Sigma_{\mathcal{M}^*}$, as $\bar\Sigma$ is also the strategy obtained by pulling back $\bar\Sigma_{\mathcal{M}^*}$ via π (hull condensation).

In order to see that $\bar{\Sigma}(\bar{I}) \in$ $\mathcal{P}|\nu[L[\bar{m}^0_\infty, \bar{\Sigma}]|\delta(u_1 \restriction \lambda)]$, it thus suffices to show that $\bar{\Sigma}_{m^*}(I)$ can be computed inside $\mathcal{P}|\nu[L[\bar{m}^0_\infty, \bar{\Sigma}]|\delta(u_1 \restriction \lambda)]$, in other words, that this generic extension of $\mathcal{P}|\nu$ can figure out how to extend $\bar{\Sigma}_{m^*}$.

Let g be $Col(\omega, \delta(u_1 \restriction \lambda))$-generic over $\mathcal{P}|\nu$ such that $L[\bar{m}^0_\infty, \bar{\Sigma}]|\delta(u_1 \restriction \lambda)] \in \mathcal{P}|\nu[g]$. We aim to verify that $\bar{\Sigma}_{m^*}(\bar{I})$ is uniformly definable in $\mathcal{P}|\nu[g]$ from parameters in $\mathcal{P}|\nu[L[\bar{m}^0_\infty, \bar{\Sigma}]|\delta(u_1 \restriction \lambda)]$.

Let $I = \dot{I}^g$, and let, for $s \in {}^{<\omega}\delta(u_1 \restriction \lambda)$, g_s result from g by replacing $g \restriction lh(s)$ by s. In $\mathcal{P}|\nu[g]$ let's start a pseudo comparison of all $m(\dot{I}^{g_s})$ (where $\dot{I}^{g_s}$ is according to $\bar{\Sigma}_{m^*}$) and also with m^* itself.

The relevant Q-structures for the trees on $\mathcal{M}(\dot{\mathcal{I}}^{g_s})$ may be taken from the tree on $\mathcal{M}^*$. Notice that this tree on $\mathcal{M}^*$ is definable from $\{\dot{\mathcal{I}}^{g_s} : s \in {}^{<\omega}\delta(\mathcal{M}_1|\lambda)\}$, and is hence in $\mathcal{P}|\nu$, so that $\mathcal{P}|\nu$ can handle this tree. We get

$$\begin{tikzcd}
 & \mathcal{M}(\dot{\mathcal{I}}^{g^s}) \arrow[dr, dashed] \arrow[d, phantom, "\vdots"] & \\
\mathcal{M}^* \arrow[r, dashed, "\mathcal{I}"] & \mathcal{M}(\mathcal{I}) = \mathcal{M}(\dot{\mathcal{I}}^{g}) \arrow[r, dashed] \arrow[d, phantom, "\vdots"] & \mathcal{M}^{**} \\
 & \mathcal{M}^* \arrow[ur, "\sigma"'] &
\end{tikzcd}$$

the map $\sigma : \mathcal{M}^* \to \mathcal{M}^{**}$ exists in $\mathcal{P}|\nu$, all the other maps only exist in V, as far as we know at this point.

But now by branch condensation $\Sigma_{\mathcal{M}^*}(\mathcal{I})$ is the unique cofinal branch b thru $\mathcal{I}$ such that there is a map $k : \mathcal{M}(\mathcal{I}) \longrightarrow \mathcal{M}^{**}$

with $$k \circ \pi^{d}_{0b} = \sigma .$$

(We use $\mathcal{P}|\nu$ is a ZFC^- model.)

We verified that

$$\mathcal{P}|\nu[L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]|\delta(\mathcal{U}, \Gamma)] = L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]|\nu ,$$

as desired.

The points in the system which will give $\mathcal{M}^1_\infty$ now consist of all $\mathcal{P}$ which are obtained by $\mathcal{P}$-constructions over such $\mathcal{M}(\mathcal{U})$, where $lh(\mathcal{U}) = \eta^{+M}$, η a cutpoint of $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$, $\delta_1 < \eta < \kappa_1$, and such that

$$\mathcal{P}[L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]|\eta^{+M}] = L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}] .$$

In V, all such $\mathcal{P}$ are iterates of $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$. We let $\mathcal{M}^1_\infty$ be the direct limit of the system given by all these $\mathcal{P}$ and iteration maps.

For $s \in {}^{<\omega}OR$ we define

$$\gamma_s^{\mathcal{P}} = \sup\Big(\mathrm{Hull}^{\mathcal{P}|\max(s)}(\underset{\substack{\parallel \\ s|\{\max(s)\}}}{s^-}) \cap \delta_1^{\mathcal{P}}\Big)$$

$$H_s^{\mathcal{P}} = \mathrm{Hull}^{\mathcal{P}|\max(s)}(\gamma_s^{\mathcal{P}} \cup s^-),$$

and we say that $\mathcal{P}$ is s-iterable iff for all relevant trees $\mathcal{T}$ on $\mathcal{P}$ there is a branch b thru $\mathcal{T}$ which maps

$$\mathrm{Th}^{\mathcal{P}|\max(s)}(\delta_1^{\mathcal{P}} \cup s^-) \text{ to } \mathrm{Th}^{\mathcal{M}_b^{\mathcal{T}}|\max(s)}(\delta_1^{\mathcal{M}_b^{\mathcal{T}}} \cup s^-).$$

For any s, there are cof. many $\mathcal{P}$ in the system which are s-iterable and the maps

$$\pi^s_{\mathcal{P}\mathcal{P}'} : H^{\mathcal{P}}_s \longrightarrow H^{\mathcal{P}'}_s$$

are in M if $\mathcal{P}$ is s-iterable and $\mathcal{P}'$ is an iterate of $\mathcal{P}$.

We ~~may~~ then see in much the same way as before that $\mathcal{M}_\infty^1$ is definable in

$L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$ (and hence in M).

Let us write

$$\rho^{**} = \min\{\pi^s_{\rho,\rho'}(\rho) : \rho \text{ is a system}, \rho' \text{ is a i.r. of } \rho\},$$

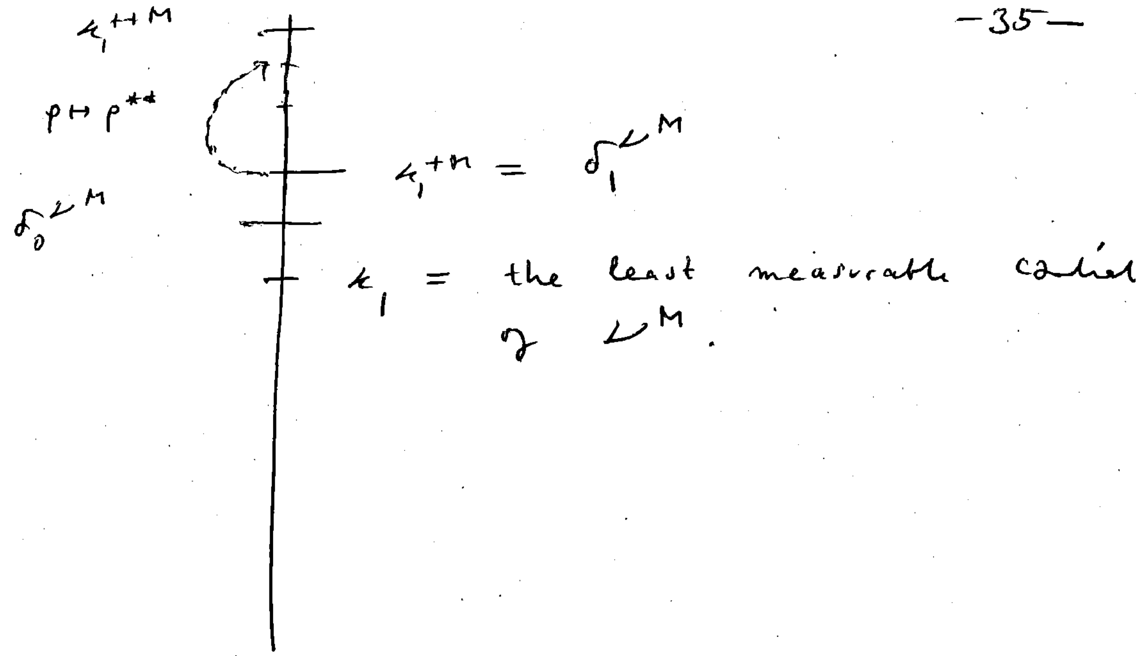
so that $\rho \mapsto \rho^{**}$ is also in $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$, and as before we can also think of $\rho \mapsto \rho^{**}$ as the map which sends ρ to the image of ρ under the map obtained by forming $\mathcal{M}'_\infty$ inside $\mathcal{M}'_\infty$.

An argument as in [VMI] shows that

$$L[\mathcal{M}'_\infty, \rho \mapsto \rho^{**}]$$

is a ground for $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$ (as every ρ from the system is) and hence for M.

We call $L[\mathcal{M}'_\infty, \rho \mapsto \rho^{**}]$ the <u>Varsovian model</u> derived from M, denoted by $\mathcal{V}M$.



The proof of Claim 1 shows that M knows $\Sigma_{\mathcal{M}_\infty^1}$, restricted to trees which live between $\kappa_0^{\mathcal{M}_\infty^1}$ and $\delta_1^{\mathcal{M}_\infty^1}$,*) so that in fact M and then also L^M knows how to iterate $\mathcal{M}_\infty^1 | \delta_1^{\mathcal{M}_\infty^1}$ for trees with critical points above $\kappa_0^{\mathcal{M}_\infty^1}$.

Let us write Σ^* for the relevant fragment of $\Sigma_{\mathcal{M}_\infty^1}$.

<u>Claim 4.</u> $L[\mathcal{M}_\infty^1, p \mapsto p^{**}] = L[\mathcal{M}_\infty^1 | \delta_1^{\mathcal{M}_\infty^1}, \Sigma^*]$.

Proof: Let us show that $\mathcal{M}_\infty^1 \subset L[\mathcal{M}_\infty^1 | \delta_1^{\mathcal{M}_\infty^1}, \Sigma^*]$.

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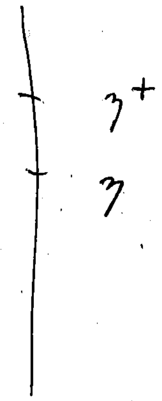
*) There is a slight abuse of notation here, see also footnote p. 37

Notice that $\mathcal{M}^1_\infty$ is Π-small and has one strong cardinal above $\delta_1^{\mathcal{M}^1_\infty}$. We may thus consider $K(\mathcal{M}^1_\infty | \delta_1^{\mathcal{M}^1_\infty})$ inside $L[\mathcal{M}^1_\infty | \delta_1^{\mathcal{M}^1_\infty}]$, let us write K for it.



Suppose that K doesn't have a strong cardinal $> \delta_1^{\mathcal{M}^1_\infty}$. Let η be a cutpoint of K, where $\eta^{+K} = \eta^{+L[\mathcal{M}^1_\infty | \delta_1^{\mathcal{M}^1_\infty}]} = \eta^+$.

Inside $L[\mathcal{M}^1_\infty | \delta_1^{\mathcal{M}^1_\infty}, \Sigma^*_?]$, we may iterate $\mathcal{M}^1_\infty | \delta_1^{\mathcal{M}^1_\infty}$ in a way so as to make an initial segment of $K | \eta^+$ generic over the iterate. Due to the presence of Σ^*, this has to terminate with the final image δ of $\delta_1^{\mathcal{M}^1_\infty}$ being between η and η^+. Let $\mathcal{M}^*$ be the final iterate. Then $\mathcal{M}^*[K | \delta] = K$, but the forcing has the δ-c.c. Contradiction!

Suppose that $\mathcal{M}_\infty^1 \neq K$. There is then in $L[\mathcal{M}_\infty^1, p \mapsto p^*]$ an iteration map

$$i : \mathcal{M}_\infty^1 \longrightarrow K .$$

However, the map $i \restriction \mathrm{crit}(i)^{+\mathcal{M}_\infty^1}$ may be recovered inside $L[\mathcal{M}_\infty^1 | \delta_1^{\mathcal{M}_\infty^1}]$ using $\mathcal{M}_\infty^1 | \delta_1^{\mathcal{M}_\infty^1}$ and Σ^* in a fashion as in the proof of Subclaim 4. $\dashv$

<u>Claim 5.</u> M knows how to iterate $\mathcal{L}^M$ with respect to trees using extenders from the intervals $(0, \delta_0^{\mathcal{L}^M})$ and $(\kappa_0^{\mathcal{L}^M}, \delta_1^{\mathcal{L}^M})$ and their images, in fact $\Sigma_{\mathcal{L}^M}$ *) restricted to those trees is amenable to M and definable on M.

---

*) This is a slight abuse of notation, as $\mathcal{L}^M$ is not an iterate of M, but the relevant extenders are (fattenings of) extenders of an iterate of M.

Proof: This is again by the proof of Claim 1. ⊣ (Claim 5)

Recall that M knows $\Sigma_{\mathcal{M}^0_\infty}$, restricted to trees which live on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ (cf. Claim 1). We write $\bar{\Sigma}$ for the relevant iteration strategy, cf. p. 24, and we reorganized $L[\mathcal{M}^0_\infty, p \mapsto p^*]$ as $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$, cf. p. 24 f. $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$ is the base for the ~~direct~~ directed system which gives $\mathcal{M}^1_\infty$.
Both $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$ and $\mathcal{M}^1_\infty$ (as well as $\Sigma^* = \Sigma_{\mathcal{M}^1_\infty}$, restricted to trees which live between $\kappa_0^{\mathcal{M}^1_\infty}$ and $\delta_1^{\mathcal{M}^1_\infty}$, cf. p. 35) are in M.
In V, $\mathcal{M}^1_\infty | \delta_0^{\mathcal{M}^1_\infty}$ is a $\bar{\Sigma}$-iterate of $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$.*)
Let us write $\mathcal{M}^1_\infty = L[\overline{\mathcal{M}^0_\infty}^*, \bar{\Sigma}^*]$.

<u>Claim 6.</u> $\bar{\Sigma}^* = \Sigma_{\overline{\mathcal{M}^0_\infty}^*}$, restricted to trees which live on $\overline{\mathcal{M}^0_\infty}^* | \delta_0^{\overline{\mathcal{M}^0_\infty}^*}$ and which exist inside $\mathcal{M}^1_\infty$.

---

*) which in turn is a Σ-iterate of $M | \delta_0$.

Proof: As M can see $\overline{\Sigma}$, it can also see $\Sigma_{\overline{\mathcal{M}_\infty^0}^*}$, restricted to trees which live on $\overline{\mathcal{M}_\infty^0}^* | \delta_0^{\overline{\mathcal{M}_\infty^0}^*}$.

But M can also, of course, see $\overline{\Sigma}^*$.

Now let $\mathcal{T}$ be a tree on $\overline{\mathcal{M}_\infty^0}^* | \delta_0^{\overline{\mathcal{M}_\infty^0}^*}$, $\mathcal{T} \in \mathcal{M}_\infty^1$. Suppose $\mathcal{T}$ is acc. to $\Sigma_{\overline{\mathcal{M}_\infty^0}^*}$.

<u>1st case.</u> $\delta(\mathcal{T}) = \pi_{0b}^{\mathcal{T}}(\delta_0^{\overline{\mathcal{M}_\infty^0}^*})$, where $b = \Sigma_{\overline{\mathcal{M}_\infty^0}^*}(\mathcal{T})$.

Notice that $cf(\delta(\mathcal{T})) = cf(\delta_0^{\overline{\mathcal{M}_\infty^0}^*}) = cf(\delta_0) = \delta_0$ in M in this case. So M can have at most one cofinal branch thru $\mathcal{T}$, so that $b = \overline{\Sigma}(\mathcal{T})$.

<u>2nd case.</u> $\delta(\mathcal{T}) < \pi_{0b}^{\mathcal{T}}(\delta_0^{\overline{\mathcal{M}_\infty^0}^*})$, where again $b = \Sigma_{\overline{\mathcal{M}_\infty^0}^*}(\mathcal{T})$.

In this case, b is the unique branch s.t. there is some Q-structure $Q \trianglelefteq \mathcal{M}_b^{\mathcal{T}}$. However, this true Q-structure may be found inside $\mathcal{M}_\infty^1$, as follows. We may also think of $\mathcal{M}_\infty^1$ as being gotten by a system with base model $L[\mathcal{M}_\infty^0, p \mapsto p^*]$, so that we may write $\mathcal{M}_\infty^1 = L[(\mathcal{M}_\infty^0)^*; (p \mapsto p^*)^*]$.

$(\mathcal{M}_\infty^0)^*$ is a tree iterate of $\mathcal{M}_\infty^0$, and it can be seen inside $\mathcal{M}_\infty^1$; hence Q may be obtained by an $L[E](\mathcal{M}(\mathcal{I}))$ construction inside $\mathrm{ult}((\mathcal{M}_\infty^0)^*; F)$ for an appropriate $(\mathcal{M}_\infty^0)^*$-extender F.

$\dashv$ (Claim 6)

Instead of $(\mathcal{M}_\infty^0)^*$, let us write K in what follows, i.e., let K denote the image of $\mathcal{M}_\infty^0$ under the direct limit which produces $\mathcal{M}_\infty^1$. K may be thought of as the underlying $L[E]$ inside $\mathcal{L}M$, and the notation will be justified by Claim 8 below.

The proofs of Claims 3 ~~[illegible]~~ and Claim 5 show:

<u>Claim 7.</u> Inside $\mathcal{L}M = L[\mathcal{M}_\infty^1, p \mapsto p^{**}]$, K is iterable with respect to trees which act on $K | \kappa_0^{+K}$ as well as with respect to trees which use critical points $> \kappa_0^K$.

We don't know (yet) that $\mathcal{L}M$ knows how to fully iterate K, though. But:

<u>Claim 8.</u> $K \cong \mathrm{Hull}^K(\Gamma)$ for all thick classes Γ which exist in M.

Proof: As usual, for every thick class Γ' of M there is a thick class $\Gamma \subset \Gamma'$ of L^M; this uses that M is generic over L^M. It hence suffices to prove that $K \cong \mathrm{Hull}^K(\Gamma)$, where Γ is a thick class of L^M.

Let Γ be a thick class of L^M, and write

$$\bar{K} \overset{\sigma}{\cong} \mathrm{Hull}^K(\Gamma) \prec K,$$

where $\bar{K}$ is transitive. By the minimality of $M = M_{swsw}$ and because in V, K is a Σ-iterate of M, $\bar{K}$ and K in V coiterate to a common model, call it Q:

$$\begin{tikzcd} \bar{K} \arrow[r, "\sigma"] \arrow[dr] & K \arrow[d] \\ & Q \end{tikzcd}$$

In particular, σ is continuous at regular non-measurables.

Assume that $\sigma \neq \mathrm{id}$, and let $\lambda = \mathrm{crit}(\sigma)$.

1st case. $\sigma(\lambda) < \delta_0^K$.

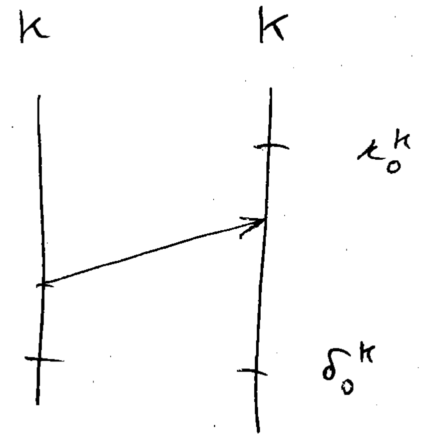
But then $\underset{\lambda^{+K}}{\overset{\|}{\sigma \restriction \lambda^{+\bar{K}}}}$ is cofinal from λ^{+K} to $\sigma(\lambda)^{+K}$;

but $\sigma \restriction \lambda^{+\bar{K}} \in V^{<M}_{\delta_0^K} = V^K_{\delta_0^K}$. So $\sigma(\lambda)^{+K}$ is singular in K which is nonsense.

2nd case. $\sigma(\lambda) = \delta_0^K$.

Then K provides a counterexample to λ being Woodin, but $\bar{K}$ doesn't have it. Contradiction.

3rd case. $\sigma(\lambda) \in (\delta_0^K, \kappa_0^K]$.



$K|\kappa_0^K$ is generated from indiscernibles together with points below δ_0^K. This is true by the way

$\mathfrak{M}'_\infty$ is formed, cf. p. 27, which gives that

$$K|\kappa_0^K \subset \mathrm{Hull}^K(\delta_0^K \cup \Gamma^*),$$

for any infinite set or class Γ^* of M-indiscernibles. There is certainly an infinite set of V-cardinals which is contained in $\mathrm{ran}(\sigma)$. But then $\lambda \in \mathrm{ran}(\sigma)$. Contradiction!

4th case. $\sigma(\lambda) \in (\kappa_0^K, \delta_1^K)$.

This is similar to case 1. $\sigma \restriction \lambda^{+\bar{K}}$ would be

$$\overset{\lambda^{+\bar{K}}}{\underset{\lambda^{+K}}{\shortparallel}}$$

cofinal from λ^{+K} to $\sigma(\lambda)^{+K}$, but $\sigma \restriction \lambda^{+\bar{K}} \in V_{\delta_1^K}^{\mathcal{L}M} = V_{\delta_1^K}^{\mathfrak{M}'_\infty}$; so $\sigma(\lambda)^{+K}$ is singular in $\mathfrak{M}'_\infty$. But $\sigma(\lambda)^{+K} = \sigma(\lambda)^{+\mathfrak{M}'_\infty}$, so this is nonsense.

5th case. $\sigma(\lambda) = \delta_1^K$.

This is like case 2.

6th case. $\sigma(\lambda) > \delta_1^K$.

~~this is like in [?]. Suppose that~~

Then $\sigma \upharpoonright \lambda^{+\bar{\kappa}}$ would be cofinal in $\sigma(\lambda)^{+\kappa} = \sigma(\lambda)^{+L^M}$, which is absurd as $\sigma \upharpoonright \lambda^{+\bar{\kappa}} \in L^M$.

$\dashv$ (Claim 8)

To be cont'd.