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Varsovian models, II, cont'd June 2017

We address some issues that should have been addressed before.

$M = M_{swsw}$.

Claim 9. $\delta_0^{\mathcal{M}_\infty^0} \leq \kappa_0^{+M}$.

Proof: Fix $\eta < \delta_0$. We aim to show that $\pi_{M,\mathcal{M}_\infty^0}(\eta) < \kappa_0^{+M}$.

Let $k < \omega$ be s.t.

$$\eta \in \mathrm{Hull}^{M \mid \aleph_{k+1}^V}(\aleph_1^V, \aleph_2^V, \dots, \aleph_k^V),$$

so that setting $s = \{\aleph_1^V, \dots, \aleph_{k+1}^V\}$, M is s-iterable and $\eta < \gamma_M^s$. Moreover, if $\mathcal{P}$ is from the $\mathcal{M}_\infty^0$-system (cf. p.3), then $\mathcal{P}$ is s-iterable and $\eta < \gamma_s^{\mathcal{P}}$, so that if both $\mathcal{P}$ and $\mathcal{P}'$ are from the $\mathcal{M}_\infty^0$-system s.t. $\pi_{\mathcal{P}\mathcal{P}'}\downarrow$, $\pi_{\mathcal{P}\mathcal{P}'}^s = \pi_{\mathcal{P}\mathcal{P}'} \restriction H_s^{\mathcal{P}} \in M$ and

$\pi_{M,\mathcal{P}}(\eta) \in H_s^{\mathcal{P}}$. In other words,

$$\pi_{M,\mathcal{M}_\infty^0}(\eta) = \text{dir lim}\left(\pi^s_{M,\mathcal{P}}(\eta),\ \pi^s_{\mathcal{P}\mathcal{P}'} \restriction \pi^s_{M,\mathcal{P}}(\eta) \ : \ \mathcal{P}, \mathcal{P}' \in \mathcal{M}_\infty^0\text{-system},\ \pi_{\mathcal{P}\mathcal{P}'}\downarrow \right)$$

can be computed inside M (s is fixed here!).

For each $\mathcal{P}$ from the $\mathcal{M}_\infty^0$-system, $\mathcal{P}$ is in V a normal iterate of M, say via the tree $\mathcal{T}^{\mathcal{P}}$. $\mathcal{T}^{\mathcal{P}} \restriction \mathrm{lh}(\mathcal{T}^{\mathcal{P}})-1 \in M$, and we may pick some α s.t. $\alpha \in [0, \mathrm{lh}(\mathcal{T}^{\mathcal{P}})-1)_{\mathcal{T}^{\mathcal{P}}}$ and $\pi^{\mathcal{T}^{\mathcal{P}}}_{\alpha\, \mathrm{lh}(\mathcal{T}^{\mathcal{P}})-1} \restriction \pi^{\mathcal{T}^{\mathcal{P}}}_{0\alpha}(\eta)+1 = \mathrm{id}$. Then $\pi^s_{M,\mathcal{P}}(\eta) = \pi_{M,\mathcal{P}}(\eta) = \pi^{\mathcal{T}^{\mathcal{P}}}_{0\alpha}(\eta)$ and M knows that $\mathrm{Card}(\pi^{\mathcal{T}^{\mathcal{P}}}_{0\alpha}(\eta)) < \kappa_0$ (by how $\mathcal{P}$ is obtained).

M also knows that there are κ_0 elements $\mathcal{P}$ from the $\mathcal{M}_\infty^0$-system.

Putting everything together gives that M knows that $\mathrm{Card}(\pi_{M,\mathcal{M}_\infty^0}(\eta)) \leq \kappa_0$. $\dashv$ (Claim 9)

<u>Claim 10.</u> $\delta_0^{\mathcal{M}^0_\infty} = \kappa_0^{+M}$.

Proof: By Claim 1 (p. 4), $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ is fully iterable inside M via $\Sigma_{\mathcal{M}^0_\infty}$.

Let's suppose that $\delta_0^{\mathcal{M}^0_\infty} < \kappa_0^{+M}$. Working inside M, we may then find some iterate M^* of $\mathcal{M}^0_\infty$ s.t. $\mathcal{M}^0_\infty$-to-M^* does not drop, $M^* \cap OR < \kappa_0^{+M}$, and $M | M^* \cap OR$ is generic over M^* for the extender algebra. Writing

$$\rho = \rho^{\mathrm{ult}(M;\,F)}(M^*) = \rho^M(M^*),$$

where F is the least total M-extender with critical point κ_0, ρ is (in V) fully iterable above $M^* \cap OR$, and it can't reach a # for an inner model looking like M_{swsw}.

M^* is an initial segment of a non-dropping iterate of $\mathcal{M}^0_\infty$ (hence of M). Therefore, ρ can't be set sized and define a counterexample to the Woodinness of $M^* \cap OR$, so that

$\mathcal{P}$ is class sized and $M^* \cap OR < \kappa_0^{+M}$ is a cardinal of $\mathcal{P}$. But now

$ult(M; F) = \mathcal{P}[M | M^* \cap OR]$ is a generic extension via a forcing with the $(M^* \cap OR)$–c.c.

Contradiction! $\dashv$ (Claim 10)

<u>Claim 11</u>. $\kappa_0^{+\mathcal{M}_\infty^0} < \kappa_0^{++M}$.

Proof. Using the Warsaw ghetto argument pp. 9–12.

Let us write $Q = \mathcal{M}_\infty^0 | \kappa_0^{+\mathcal{M}_\infty^0}$, as on p. 9,

and let i be as defined on p. 8: i comes from an iteration of $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ to make

$\mathcal{M}_\infty^0 | \kappa_0^{+\mathcal{M}_\infty^0}$ generic, $i \upharpoonright \delta_0^{\mathcal{M}_\infty^0} : \delta_0^{\mathcal{M}_\infty^0} \to \kappa_0^{+\mathcal{M}_\infty^0}$

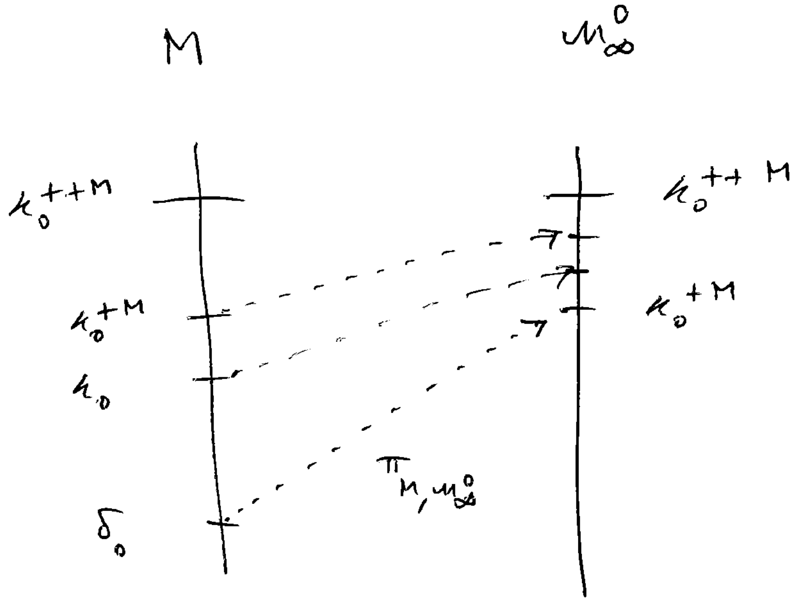
is cofinal, $i \in M$. Claim 2 shows that

$Q = \text{Hull}^Q(i)$. But i has cardinality

$\delta_0^{\mathcal{M}_\infty^0} = \kappa_0^{+M}$ (by Claim 10) inside M, so that

Q must have cardinality κ_0^{+M} in M.

This verifies Claim 11. $\dashv$ (Claim 11) the picture thus is :



As κ_0^{++M} must be a cardinal in $\mathcal{M}^0_\infty \subset M$, we must have that $\pi_{M,\mathcal{M}^0_\infty}(\kappa_0^{++M}) = \kappa_0^{++M}$, in other words :

<u>Claim 12.</u> $\kappa_0^{++\mathcal{M}^0_\infty} = \kappa_0^{++M}$.

An argument exactly as in [VMI] shows that

$$L[\mathcal{M}^0_\infty, \rho \mapsto \rho^*]$$

is a ground for M, as being witnessed by a forcing which is a subset of $\mathcal{M}^0_\infty | \kappa_0^{++M}$, has the $\delta_0^{\mathcal{M}^0_\infty}$-c.c., and adds $M | \kappa_0^{++M}$.

In particular, M and $L[\mathcal{M}_\infty^0, p \mapsto p^*]$ have the same cardinals $\geq \delta_0^{\mathcal{M}_\infty^0} = \kappa_0^{+M}$; in particular, $L[\mathcal{M}_\infty^0, p \mapsto p^*]$ doesn't have any cardinals $\in (\kappa_0^{+M}, \kappa_0^{++M})$ (whereas $\mathcal{M}_\infty^0$ has a strong limit of Woodins in this interval, namely $\pi_{M,\mathcal{M}_\infty^0}(\kappa_0)$).

As in [VMI], one may show:

<u>Claim 13.</u> $L[\mathcal{M}_\infty^0, p \mapsto p^*] = HOD^{M^{Col(\omega, <\kappa_0)}}$.

Proof: "$\supset$" is shown exactly as in [VMI]. Let us now work towards showing "$\subset$".

~~Let us fix~~ Let us fix g $Col(\omega, <\kappa_0)$-generic over M.

<u>Subclaim 1.</u> $\mathcal{M}_\infty^0 | \delta_\infty^{\mathcal{M}_\infty^0} \in HOD^{M[g]}$.

To show this, let N be any inner model of $M[g]$ with the same properties as M (to the extent that $M[g]$ can identify them); in particular, there is some g' $Col(\omega, <\kappa_0)$-generic over N s.t. $N[g'] = M[g]$, N is an iterable (to some extent) fine structural model, etc.

Let F be the least total M-measure with critical point κ_0, and let $\bar{j} : M \longrightarrow \mathrm{ult}(M;F)$ be the ultrapower embedding. Let g^* be $\mathrm{Col}(\omega, [\kappa_0, \bar{j}(\kappa_0)))$-generic over M s.t. $\bar{j}$ lifts to

$$j : M[g] \longrightarrow \underset{\underset{\mathrm{ult}(M;F)}{\shortparallel}}{\bar{j}(M)}[g^\frown g^*] .$$

We then also have

$$j : N[g'] \longrightarrow j(N)[j(g')] .$$

Notice that $\bar{j}(M)|\kappa_0^+[g] = M|\kappa_0^+[g] = H_{\kappa_0^+}^{M[g]} = H_{\kappa_0^+}^{N[g']} = N|\kappa_0^+[g'] = j(N)|\kappa_0^+[g']$. By reflection, there is hence some $\kappa < \kappa_0$ such that

$$M|\kappa^{+M}[g\restriction\kappa] = N|\kappa^{+N}[g'\restriction\kappa] .$$

In particular, $N|\kappa \in M[g\restriction\kappa]$, and $N|\kappa^{+N}$ is definable inside $M[g\restriction\kappa]$ from the parameter $N|\kappa$ as the stack of all sound premice

$Q \trianglerighteq N|\kappa$ s.t. $\rho_\omega(Q) = \kappa$ and Q satisfies condensation (i.e., if $\bar{Q} \longrightarrow Q$ ~~...~~ maps its critical point to κ and is sufficiently elementary, then $\bar{Q} \trianglelefteq N|\kappa$). Symmetrically, $M|\kappa \in N[g \restriction \kappa]$, and $M|\kappa^{+M}$ is definable inside $N[g \restriction \kappa]$ from the parameter $M|\kappa$.

We may now work inside $H^M_{\kappa^+} \cap H^N_{\kappa^+}$ and simultaneously compare all candidates for $M|\delta_0^M$ and $N|\delta_0^N$ in a fixed homogeneous forcing extension, simultaneously making all candidates for $M|\kappa^{+M}$, $N|\kappa^{+N}$ generic over the common pseudo-iterate.

This will produce two models P, P' s.t. P is in the $\mathcal{M}^0_\infty$-system based on M, P' is in the $\mathcal{M}^0_\infty$-system based on N, and $P|\delta_0^P = P|\kappa^{+M} = P'|\kappa^{+N} = P'|\delta_0^{P'}$.

This argument verifies Subclaim 1.

Subclaim 2. $\Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}$ is amenable to $\mathrm{HOD}^{M[g]}$ and definable over $\mathrm{HOD}^{M[g]}$.

This follows from the fact that $\delta_0^{\mathcal{M}^0_\infty} = \kappa_0^{+M} = \lambda_2^{M[g]}$, so that if $\mathcal{T}$ is a "maximal" tree on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$, then $M[g]$ has exactly one cofinal branch thru $\mathcal{T}$.

Subclaim 3. $\mathcal{M}^0_\infty | \delta_1^M \in \mathrm{HOD}^{M[g]}$.

This follows from Claim 3, p. 12. For any measurable cardinal Ω, $\delta_0^{\mathcal{M}^0_\infty} < \Omega < \delta_1^M = \delta_1^{\mathcal{M}^0_\infty}$, of M, $\mathcal{M}^0_\infty | \Omega$ is ~~the unique~~ inside $M[g] | \Omega$ the unique universal fully iterable weasel which end-extends $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ and embeds into any other such weasel.

Subclaim 2 and Schlutzenberg type arguments then yield that $\mathcal{M}^0_\infty \subset \mathrm{HOD}^{M[g]}$ and $\mathcal{V}_0^M = L[\mathcal{M}^0_\infty, p \mapsto p^*] \subset \mathrm{HOD}^{M[g]}$. ⊣ (Claim 13)

As in [VMI] this now gives

<u>Claim 14.</u> $H^{\mathcal{V}_0}_{\delta_0^{\mathcal{M}^0_\infty}} = \mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$.

The arguments of [VMI] also show:

<u>Claim 15.</u> $\delta_0^{\mathcal{M}^0_\infty}$ is a Woodin cardinal in $\mathcal{V}_0 = L[\mathcal{M}^0_\infty, \rho \mapsto \rho^*]$.

We know now that $\mathcal{V}_0 = L[\mathcal{M}^0_\infty, \rho \mapsto \rho^*] = \mathrm{HOD}^{M^{Col(\omega, <\kappa_0)}}$, and by $L[\mathcal{M}^0_\infty, \Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}] \subset \mathrm{HOD}^{M^{Col(\omega, <\kappa_0)}}$ and $L[\mathcal{M}^0_\infty, \rho \mapsto \rho^*] \subset L[\mathcal{M}^0_\infty, \Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}]$, we get that $L[\mathcal{M}^0_\infty, \rho \mapsto \rho^*] = L[\mathcal{M}^0_\infty, \Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}]$.

We now want to see yet another representation of this model.

By Bukowský, M is $\mathbb{P}$-generic over $\mathcal{V}_0 = L[\mathcal{M}^0_\infty, \rho \mapsto \rho^*]$ for some $\mathbb{P} \in \mathcal{V}_0$, $\mathbb{P} \subset \mathcal{V}_0 | \kappa_0^{++}$ (recall $\kappa_0^{++\mathcal{V}_0} = \kappa_0^{++M}$, which we denote by κ_0^{++}).

Let us do a revised $\mathcal{P}$-construction above

$\mathcal{V}_0 | \kappa_0^{+3}$, inside M, as follows. Let $\mathcal{P}|\nu$, $\mathcal{P}||\nu$ denote the passive, active levels of this $\mathcal{P}$-construction.

If ν indices an extender with critical point $> \kappa_0^{+3}$, E_ν^M of M, then we let $E_\nu^M \cap \mathcal{P}|\nu$ be the top extender of $\mathcal{P}||\nu$. If we reach a designated stage $\mathcal{P}|\nu$ s.t. there is a tree $\mathcal{I}$ on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ s.t. $\Sigma_{\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}}(\mathcal{I})$ was not yet added to the sequence, then we do it for the least such $\mathcal{I}$.

<u>Claim 16.</u> $\rho_\omega(\mathcal{P}||\nu) \geq \kappa_0^{+3}$ for all $\nu \geq \kappa_0^{+3}$.

Otherwise let ν be the least counterexample, and let $a = \{\xi < \rho : \mathcal{P}||\nu \models \varphi(\xi, \vec{p})\} \notin \mathcal{P}|\nu$ (i.e., $\notin \mathcal{V}_0 | \kappa_0^{+3}$) for some $\rho \leq \kappa_0^{+2}$, φ a fmla, $\vec{p} \in \mathcal{P}|\nu$. Writing $g = M|\kappa_0^{+2}$, g is $\mathbb{P}$-generic over $\mathcal{P}||\nu$ (where $\mathbb{P}$ is as above), and $M|\kappa_0^{+3} \subset \mathcal{P}|\nu[g]$, in fact,

$M | \kappa_0^{+3} = \mathcal{L}_0[g] | \kappa_0^{+3}$. Let $\tau \in (\mathcal{L}_0 | \kappa_0^{+3})^{\mathbb{P}}$ be such that $\tau^g = a$. Pick $p \in g$ s.t.

$$(*) \quad p \Vdash^{\mathbb{P}}_{\mathcal{P}\|\nu} \forall \xi < \rho \, (\xi \in \tau \longleftrightarrow \varphi^{\mathcal{P}\|\nu}(\xi, \vec{p}))\, .$$

We must then have that $\xi \in a$ iff $\mathcal{P}\|\nu \models \varphi(\xi, \vec{p})$ iff $p \Vdash^{\mathbb{P}}_{\mathcal{P}\|\nu} \varphi^{\mathcal{P}\|\nu}(\check{\xi}, \vec{p})$ iff $p \Vdash^{\mathbb{P}}_{\mathcal{P}\|\nu} \check{\xi} \in \tau$ (as there can't be a $q \leq p$ $q \Vdash^{\mathbb{P}}_{\mathcal{P}\|\nu} \check{\xi} \notin \tau$ by (*) above) iff $p \Vdash^{\mathbb{P}}_{\mathcal{P}\|\kappa_0^{+3}} \check{\xi} \in \tau$, which buys us

$$\mathcal{P}\|\kappa_0^{+3} = \mathcal{L}_0 | \kappa_0^{+3}$$

that $a \in \mathcal{L}_0 | \kappa_0^{+3}$. Contradiction!

Hence $\mathcal{L}_0 | \kappa_0^{+3} = H^{\mathcal{P}}_{\kappa_0^{+3}}$, when $\mathcal{P}$ is the final model. It is also not hard to verify that $\mathcal{P}[g] = M$: the iteration strategy $\Sigma_{\mathcal{M}^0_\infty | \delta_\gamma^{\mathcal{M}^0_\infty}}$ which we feed into $\mathcal{P}$ will extend to $\mathcal{P}[g]$ and it will code the extenders with critical point κ_0, etc. Let us add some more details.

Claim 17. $\mathcal{P}\|\nu[g] = M\|\nu$ for all ν.

It suffices to verify that if $\mathcal{P}|\nu[g] = M|\nu$, then $\mathcal{P}\|\nu[g] = M\|\nu$.

1st case. $E_\nu^M = \emptyset$.

Then nothing is activated at stage ν of the $\mathcal{P}$-construction either.

2nd case. $E_\nu^M \neq \emptyset$ and $\mathrm{crit}(E_\nu^M) > \kappa_0^{+3}$.

Then $E_\nu^{\mathcal{P}} = E_\nu^M \cap \mathcal{P}|\nu$, and $\mathcal{P}\|\nu[g] = M\|\nu$ follows by standard arguments.

3rd case. $E_\nu^M \neq \emptyset$ and $\mathrm{crit}(E_\nu^M) \leq \kappa_0^{+3}$.

In this case, $\mathrm{crit}(E_\nu^M) = \kappa_0$. Let us write $F = E_\nu^M$. We aim to show that we may define F over $\mathcal{P}\|\nu[g]$ and $\Sigma_{M_\infty^0|\delta_0, M_\infty^0}(\mathcal{T}^*)$ over $M\|\nu$ when $\mathcal{T}^*$ is the tree which will be taken care of at this stage.

First of all, which tree should $\mathcal{I}^*$ be?

Let $\mathcal{I}$ be the normal tree on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ which produces $\pi^M_F(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}) = (\mathcal{M}_\infty | \delta_0^{\mathcal{M}^0_\infty})^{\mathrm{ult}(M;F)}$.

By our hypothesis, $\mathcal{P}|\nu[g] = M|\nu$, and we may actually define $(\mathcal{M}_\infty|\delta_0^{\mathcal{M}^0_\infty})^{\mathrm{ult}(M;F)}$ over $M|\nu$ (we here use that ind., $\Sigma_{\mathcal{M}^0_\infty|\delta_0^{\mathcal{M}^0_\infty}} \restriction \mathcal{P}|\nu$ naturally extends to $\Sigma_{\mathcal{M}^0_\infty|\delta_0^{\mathcal{M}^0_\infty}} \restriction \mathcal{P}|\nu[g]$ by an argument which we are about to give); therefore $\mathcal{I} \restriction \nu = \mathcal{I} \restriction \delta_0^{(\mathcal{M}^0_\infty)^{\mathrm{ult}(M;F)}}$ can be defined over $M|\nu = \mathcal{P}|\nu[g]$.

Let us absorb g by some h which is $\mathrm{Col}(\omega, \kappa_0^{+2})$-generic over $\mathcal{P}|\nu$, i.e., $g \in \mathcal{P}|\nu[h]$. Let $\dot{\mathcal{I}}$ "$\in$" $\mathcal{P}|\nu$ s.t. $\dot{\mathcal{I}}^h = \mathcal{I}$. In M, let $\mathcal{I}^*$ be the tree on $\mathcal{M}^0_\infty|\delta_0^{\mathcal{M}^0_\infty}$ resulting from comparing $\mathcal{M}^0_\infty|\delta_0^{\mathcal{M}^0_\infty}$ with all

$$\mathcal{M}(\dot{\mathcal{I}}^{h_s}),$$

$s \in {}^{<\omega}(\kappa_0^{+2})$. We get that $\mathcal{I}^* \restriction lh(\mathcal{I}^*)-1$ can be defined on $\mathcal{P}|\nu$. Let

$\Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}} (\mathcal{J}^* \restriction lh(\mathcal{J}^*)-1) = [0, lh(\mathcal{J}^*)-1)_{\mathcal{J}^*}$ be the information which is added to ~~P~~ $P|\nu$ to give $P\|\nu$ at this stage.

If $b = \Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}} (\mathcal{J} \restriction \nu) = [0, \nu)_{\mathcal{J}}$, then b is identified as the unique b s.t. there is some k with $\mathcal{M}^{\mathcal{J}}_b \xrightarrow{k} \mathcal{M}^{\mathcal{J}^*}_{lh(\mathcal{J}^*)-1}$ s.t. $k \circ \pi^{\mathcal{J}}_{0,b} = \pi^{\mathcal{J}}_{0, lh(\mathcal{J}^*)-1}$. So b may be read off from $P\|\nu[g]$. But then F may be read off over $P\|\nu[g]$, as F is the unique extender s.t. $(M|\nu; F)$ is a premouse and

$$\pi^M_F \restriction \kappa_0^{+M} = \pi^{\mathcal{J}}_{0,b} \restriction \underset{\delta_0^{\mathcal{M}^0_\infty}}{\overset{\|}{\kappa_0^{+M}}} .$$

We gave a sketch of the proof of Claim 17.

We now have that $P[g] = M$. By a theorem of Hamkins-Laver-Woodin, this implies that

<u>Claim 18.</u> $P = \nu_0$.

In particular, the extenders of $\mathcal{V}_0$ are therefore the ones below $\delta_0^{\mathcal{M}^0_\infty}$ plus the ones with critical point above κ_0+1, and the latter ones are restrictions of M-extenders.

We now aim to show that $\mathcal{M}^0_\infty$ is iterable enough so that the system giving rise to $\mathcal{M}^1_\infty$ makes sense. The issue is the following. On p. 32 we say that "in V, all such $\mathcal{P}$ are iterates of $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$." In order for this to be true we need to verify that $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$ is iterable via the extenders F with $\kappa_0^{++M} <$ $\mathrm{crit}(F) < \mathrm{lh}(F) < \delta_1^M$. But which ones are those extenders of $L[\overline{\mathcal{M}^0_\infty}, \bar{\Sigma}]$?

Let $\kappa_0^{++M} < \lambda < \delta_1^M$, assume λ to be measurable in M, and let $\mathcal{U}$ be a total measure of M with $\lambda = \mathrm{crit}(\mathcal{U})$.

<u>Claim 19.</u> $\pi_{M,\mathcal{M}^0_\infty}(\lambda) = \lambda$.

Proof of Claim 19. Let M^* be the result of iterating $\mathcal{U}$ and its images out of the universe, cutting the resulting model off at OR. Let I denote the club class of critical points. I is then a generating class of indiscernibles for M^*. We may define $(\mathcal{M}_\infty^0)^{M^*}$, which agrees with $\mathcal{M}_\infty^0$ up thru λ, and $\pi_{M,\mathcal{M}_\infty^0} \restriction \lambda = \pi_{M^*,(\mathcal{M}_\infty^0)^{M^*}} \restriction \lambda$. We get that I is a class of indiscernibles for $(\mathcal{M}_\infty^0)^{M^*}$, as $(\mathcal{M}_\infty^0)^{M^*}$ is an inner model of M^* which is definable from no parameters. We may define

$$\sigma : M^* \longrightarrow (\mathcal{M}_\infty^0)^{M^*}$$

$$\tau^{M^*}(i_0, \dots, i_{k-1}) \mapsto \tau^{(\mathcal{M}_\infty^0)^{M^*}}(i_0, \dots, i_{k-1}),$$

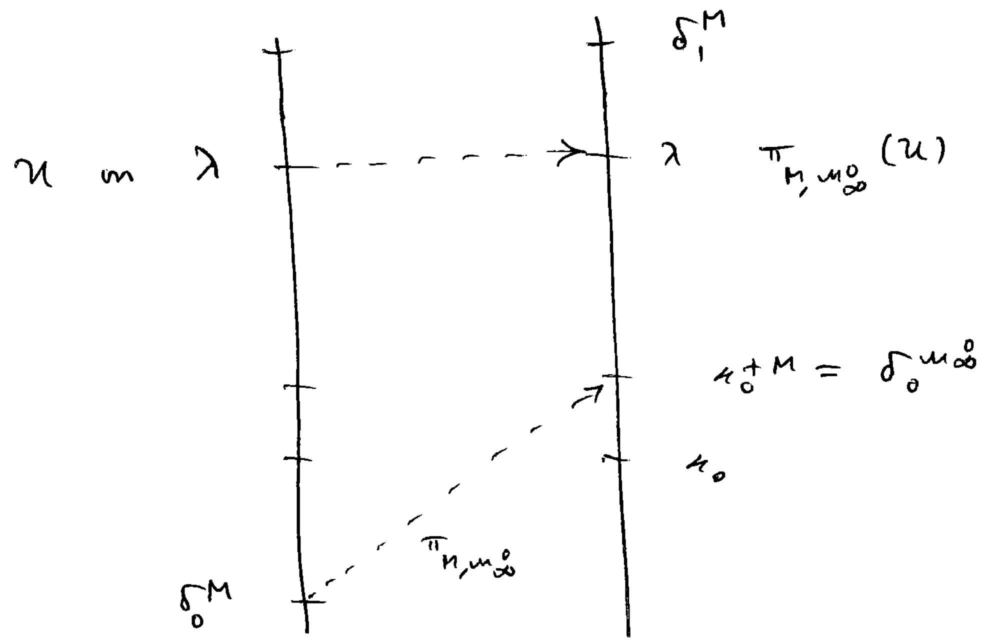
when τ is a Skolem term, and $i_0 < \dots < i_{k-1}$ is from I. This is well-defined and elementary. By the minimality of iteration maps (using Dodd-Jensen), $\sigma(\xi) \geq \pi_{M^*,(\mathcal{M}_\infty^0)^{M^*}}(\xi)$ for all ξ. But then by continuity at λ,

$$\pi_{M, m^0_\infty}(\lambda) = \sup \pi_{M, m^0_\infty} {}'' \lambda$$

$$= \sup \pi_{M^*, (m^0_\infty)^{M^*}} {}'' \lambda$$

$$\leq \sup \sigma {}'' \lambda \leq \sigma(\lambda) = \lambda,$$

so that indeed $\pi_{M, m^0_\infty}(\lambda) = \lambda$. $\dashv$ (Claim 19)



<u>Claim 20.</u> $\pi_{M, m^0_\infty}(\mathcal{U}) = \mathcal{U} \cap m^0_\infty$.

Proof of Claim 20. Let us write $\pi = \pi_{M, m^0_\infty}$.

Let $\pi(f)(a) \in \pi(\mathcal{U})$, where $a \in [\delta_0^{m^0_\infty}]^{<\omega}$,

$f : [\eta]^{\bar{\bar{a}}} \longrightarrow M$, $a \in \pi([\eta]^{<\omega})$, $f \in M$.

Hence $a \in \pi(Z)$, where $Z = \{u \in [\eta]^{\bar{\bar{a}}} : f(u) \in \mathcal{U}\}$.

Set $Y = \bigcap \{ f(u) : u \in Z \}$, so that $Y \in M$ and $Y \subset f(u) \in \mathcal{U}$ for all $u \in Z$.

Hence $\pi(Y) \subset \pi(f)(a)$. Because $\mathcal{U}$ is $<\lambda$-closed in M, $Y \in \mathcal{U}$.

We have shown that for every $X \in \pi(\mathcal{U})$ there is some $Y \in \mathcal{U}$ with $\pi(Y) \subset X$, i.e., $\pi''\mathcal{U}$ generates $\pi(\mathcal{U})$.

Now let $Y \in \mathcal{U}$. We claim that $\pi(Y) \in \mathcal{U}$. To verify this, it suffices to see that $\lambda \in \pi^M_{\mathcal{U}}(\pi(Y))$. However,

$$\pi^M_{\mathcal{U}}(\pi(Y)) = \pi^M_{\mathcal{U}}(\pi^S_{M, m^0_\infty}(Y)), \quad S = \{\lambda_1^{IV}, \ldots, \lambda_k^{IV}\},$$

some k

$$= \pi^M_{\mathcal{U}}(\pi^S_{M, m^0_\infty})(\pi^M_{\mathcal{U}}(Y))$$

$$= \pi_{ult(M;\mathcal{U}), (m^0_\infty)^{ult(M;\mathcal{U})}}(\pi^M_{\mathcal{U}}(Y)).$$

By an argument as for Claim 19,

$\pi_{ult(M;\mathcal{U}), (m^0_\infty)^{ult(M;\mathcal{U})}}(\lambda) = \lambda$. Hence $\lambda \in \pi^M_{\mathcal{U}}(Y)$ implies $\lambda \in \pi^M_{\mathcal{U}}(\pi(Y))$.

We have verified that if $\pi(f)(a) \in \pi(\mathcal{U})$, then there is $Y \in \mathcal{U}$ with $\pi(Y) \subset \pi(f)(a)$, and $\pi(Y) \in \mathcal{U}$. Hence $\pi(f)(a) \in \mathcal{U}$.

This shows Claim 20. $\dashv$ (Claim 20)

We have that $\mathcal{V}_0^M = L[\bar{\mathcal{M}}_\infty^0, \bar{\Sigma}] = L[\mathcal{M}_\infty^0, \rho \mapsto \rho^*]$, and $M = \mathcal{V}_0^M[M | \kappa_0^{++M}]$, where $M | \kappa_0^{++M}$ is $\mathbb{P}$-generic over $\mathcal{V}_0^M$ for some forcing of size κ_0^{++M}.

Therefore, all ~~xxx~~ $\mathcal{V}_0^M$-measures above κ_0^{++M} are restrictions of M-measures, and by Claim 20, those $\mathcal{V}_0^M$-measures are lifts of $\mathcal{M}_\infty^0$-measures.

The following is relevant for the $\mathcal{M}_\infty^1$ system (cf. pages 26 ff.) to make sense.

<u>Claim 21.</u> In V, $\mathcal{V}_0 = L[\mathcal{M}_\infty^0, \rho \mapsto \rho^*]$ is iterable w.r.t. trees which live on $[0, \delta_0^{\mathcal{M}_\infty^0}) \cup (\kappa_0^{++M}, \infty)$.

Proof: We follow the proof of [VMI, Lemma 2.15].

We let

$$\bar{\mathcal{V}}_0 \overset{\sigma}{\cong} \mathrm{Hull}^{\mathcal{V}_0}(\mathrm{ran}(\pi_{M,\mathcal{M}^0_\infty}))\ .$$

By [VMI, Lemma 2.13 and Cor. 2.14], $\bar{\mathcal{V}}_0 =$ $L[M, \pi_{M,\mathcal{M}^0_\infty} \restriction \mathrm{OR}]$. Of course it suffices to verify that $\bar{\mathcal{V}}_0$ is iterable w.r.t. trees which live on $[0, \delta_0) \cup (\kappa_0^{++M}, \infty)$.

Let $\mathcal{T}$ be a tree on $\bar{\mathcal{V}}_0$, say $\mathcal{T}$ is normal.

Let $\mathcal{T} = \mathcal{T}_0 {}^\frown \mathcal{T}_1$, where $\mathcal{T}_0$ lives on $L[M, \pi_{M,\mathcal{M}^0_\infty} \restriction \mathrm{OR}] | \delta_0 = M | \delta_0$, and $\mathcal{T}_1$ uses extenders with critical point $> \kappa_0^{++ M^{\mathcal{T}_0}_{lh(\mathcal{T}_0)-1}}$.

For notational convenience, let's assume $\mathcal{T}_0$ doesn't involve any drop.

$\mathcal{T}_0$ induces a tree, $\bar{\mathcal{T}}_0$, on M, with the same tree structure and extenders used. For each $\alpha < lh(\mathcal{T}_0)$, we aim to construct π_α, $\tilde{\pi}_\alpha$, M^*_α, π^*_α s.t.

$$\mathcal{M}_\alpha^{\bar{\mathcal{I}}_0} \subset \mathcal{M}_\alpha^{\mathcal{I}_0} = L[\mathcal{M}_\alpha^{\bar{\mathcal{I}}_0}, \pi_\alpha] \xrightarrow{\pi_\alpha \subset \tilde{\pi}_\alpha} \underbrace{L[\mathcal{M}_\alpha^*, \pi_\alpha^*]}_{= L_0^{\mathcal{M}_\alpha^{\bar{\mathcal{I}}_0}}} \subset \mathcal{M}_\alpha^{\bar{\mathcal{I}}_0}$$

and $\tilde{\pi}_\beta \restriction \mathrm{lh}(E_\gamma^{\mathcal{I}_0}) = \tilde{\pi}_\alpha \restriction \mathrm{lh}(E_\gamma^{\mathcal{I}_0})$ for $\gamma+1 \le \beta < \mathrm{lh}(\mathcal{I}_0)$.

We set $\pi_0 = \pi_{M,\mathcal{M}_\infty^0} \restriction \mathrm{OR}$, $\tilde{\pi}_0 = \sigma$ (as on the previous page), $M_0^* = \mathcal{M}_\infty^0$, and $\pi_\alpha^* = (p \mapsto p^*)$.

Let us do the successor step of the construction, say we aim to define $\pi_{\beta+1}$, $\tilde{\pi}_{\beta+1}$, $M^*_{\beta+1}$, $\pi^*_{\beta+1}$, and $\mathcal{M}^{\mathcal{I}_0}_{\beta+1} = \mathrm{ult}(\mathcal{M}_\alpha^{\mathcal{I}_0}; F)$, $F = E_\beta^{\mathcal{I}_0}$, so that also $\mathcal{M}^{\bar{\mathcal{I}}_0}_{\beta+1} = \mathrm{ult}(\mathcal{M}_\alpha^{\bar{\mathcal{I}}_0}; F)$, $F = E_\beta^{\bar{\mathcal{I}}_0}$.

We have that

$$\mathcal{M}^{\mathcal{I}_0}_{\beta+1} \models \varphi([a,f]_F^{\mathcal{M}_\alpha^{\mathcal{I}}}) \iff$$

$$\{u \in [\mathrm{crit}(F)]^{\bar{\bar{a}}} : \mathcal{M}_\alpha^{\mathcal{I}_0} \models \varphi(f(u))\} \in F_a$$

$$\iff \{u \in [\mathrm{crit}(F)]^{\bar{\bar{a}}} : L[\mathcal{M}_\alpha^*, \pi_\alpha^*] \models$$

$$\varphi(\pi_\alpha^*(f)(\underbrace{\pi_\alpha^*(u)}_{= \pi_\alpha \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}}}))\} \in F_a$$

$$\pi^{\overline{\overline{J}}_0}_{\alpha\beta+1}(\ldots)$$

$$\Longleftrightarrow \quad a \in \pi^{J_0}_{\alpha\beta+1}\Big(\{u \in [\mathrm{crit}(F)]^{\bar{\bar{a}}} :$$

$$L[\mathcal{M}^*_\alpha, \pi^*_\alpha] \models \varphi\,(\pi^*_\alpha(f)(\pi_\alpha \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}}(u)))\}\Big)$$

$$\Longleftrightarrow \quad L[\pi^{\overline{J}_0}_{\alpha\beta+1}(\mathcal{M}^*_\alpha), \pi^{\overline{J}_0}_{\alpha\beta+1}(\pi^*_\alpha)] \models$$

$$\varphi\Big(\pi^{\overline{J}_0}_{\alpha\beta+1}(\pi^*_\alpha(f))(\pi^{\overline{J}_0}_{\alpha\beta+1}(\pi_\alpha \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}})(a))\Big).$$

Notice that this makes sense, as $L[\mathcal{M}^*_\alpha, \pi^*_\alpha] \subset \mathcal{M}^{\overline{J}_0}_\alpha$ and $\pi_\alpha \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}} \in \mathcal{M}^{\overline{J}_0}_\alpha$: the latter is true, as $\pi_{M,\mathcal{M}^0_\infty} \restriction \eta = \sigma \restriction \eta = \pi_0 \restriction \eta \in M$ for all $\eta < \delta_0^M$, $\pi^{\overline{J}_0}_{0\alpha}{}''\,\delta_0^M$ is cofinal in $\pi^{\overline{J}_0}_{0\alpha}(\delta_0^M) = \delta_0^{\mathcal{M}^{\overline{J}_0}_\alpha}$, so that $\pi_\alpha \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}} = \pi^{J_0}_{0\alpha}(\sigma \restriction \eta) \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}}$ for some sufficiently big $\eta < \delta_0^{M_0}$.

Let $\mathcal{M}^*_{\beta+1} = \pi^{\overline{J}_0}_{\alpha\beta+1}(\mathcal{M}^*_\alpha)$, $\pi^*_{\beta+1} = \pi^{\overline{J}_0}_{\alpha\beta+1}(\pi^*_\alpha)$,

and

$$\tilde{\pi}_{\beta+1}\left([a,f]_F^{\mathcal{M}^{J_0}_\alpha}\right) = \left[a, u \mapsto \pi^*_\alpha(f)\Big(\big(\pi_\alpha \restriction [\mathrm{crit}(F)]^{\bar{\bar{a}}}\big)(u)\Big)\right]_F^{\mathcal{M}^{\overline{J}_0}_\alpha}.$$

By the above computations,

$$\mathcal{M}^{\mathcal{I}_0}_{\beta+1} \models \varphi\left([a,f]^{\mathcal{M}^{\mathcal{I}_0}_\alpha}_F\right) \iff$$

$$L[\mathcal{M}^*_{\beta+1}, \pi^*_{\beta+1}] \models \varphi\left(\tilde{\pi}_{\beta+1}\left([a,f]^{\mathcal{M}^{\mathcal{I}_0}_\alpha}_F\right)\right) .$$

If $\xi < lh(F)$, then $\tilde{\pi}_{\beta+1}(\xi) = \pi^{\bar{\mathcal{I}}_0}_{\alpha\beta+1}(\pi_\alpha \restriction [crit(F)]^{\bar{\bar{a}}})(\xi)$

$= \pi_\alpha(\xi) = \tilde{\pi}_\alpha(\xi)$, as $\pi^{\bar{\mathcal{I}}_0}_{\alpha\beta+1}(\pi_\alpha \restriction [crit(F)]^{\bar{\bar{a}}}) =$

the canonical map from $ult(M_\alpha; F)$ to ~~ult~~

$ult(M^*_\alpha; \pi^*_\beta(F))$, restricted to $\pi_F([crit(F)]^{\bar{\bar{a}}})$, which

agrees with π_α below $lh(F)$ (cf. [VMI, (37)]).

We also set $\pi_\alpha = \pi^*_\alpha \restriction OR$.

As in [VMI, (34)],

$$\mathcal{M}^{\mathcal{I}_0}_{\beta+1} = L[\mathcal{M}^{\bar{\mathcal{I}}_0}_{\beta+1}, \tilde{\pi}_{\beta+1} \restriction OR] .$$

Writing $\theta = lh(\mathcal{I}_0) - 1$, we get

$$\mathcal{M}^{\bar{\mathcal{I}}_0}_\theta \subset \mathcal{M}^{\mathcal{I}_0}_\theta = L[\mathcal{M}^{\bar{\mathcal{I}}_0}_\theta, \pi_\theta] \xrightarrow{\tilde{\pi}_\theta} \underbrace{L[\mathcal{M}^*_\theta, \pi^*_\theta]}_{\substack{\parallel \\ L_0^{\mathcal{M}^{\bar{\mathcal{I}}_0}_\theta}}} \subset \mathcal{M}^{\bar{\mathcal{I}}_0}_\theta ,$$

$\tilde{\pi}_\theta \supset \pi_\theta$.

It remains to be verified that $\mathcal{I}_1$, a tree on $\mathcal{M}_\theta^{\overline{\mathcal{I}_0}}$ which uses extenders with critical points $> \pi_{0\theta}^{\mathcal{I}_0}(\kappa_0^{++M})$, is well-behaved.

For this it suffices to verify that $\tilde{\pi}_\theta \mathcal{I}_1$, i.e., $\mathcal{I}_1$ copied onto $\mathcal{L}_0^{\mathcal{M}_\theta^{\overline{\mathcal{I}_0}}} = L[\mathcal{M}_\theta^*, \pi_\theta^*]$ via $\tilde{\pi}_\theta$, is well-behaved.

By the argument for Claim 18 (cf. page 59), $\mathcal{L}_0^{\mathcal{M}_\theta^{\overline{\mathcal{I}_0}}} = \mathcal{P}$, where $\mathcal{P}$ is the revised $\mathcal{P}$-construction inside $\mathcal{M}_\theta^{\overline{\mathcal{I}_0}}$ (in fact, this follows from Claim 18 using the elementarity of $\pi_{0\theta}^{\mathcal{I}_0}$).

We thus need to see that $\tilde{\pi}_\theta \mathcal{I}_1$ on $\mathcal{P}$ is well-behaved. But $\mathcal{P}[\mathcal{M}_\theta^{\overline{\mathcal{I}_0}} | \pi_{0\theta}^{\mathcal{I}_0}(\kappa_0^{+3M})] = \mathcal{M}_\theta^{\overline{\mathcal{I}_0}}$ via a poset of size $\pi_{0\theta}^{\mathcal{I}_0}(\kappa_0^{+2M})$ which has the $\pi_{0\theta}^{\mathcal{I}_0}(\kappa_0^{+M})$-c.c., so that in fact $\tilde{\pi}_\theta \mathcal{I}_1$ may be construed as a tree on $\mathcal{M}_\theta^{\overline{\mathcal{I}_0}}$.

the fact that $\overline{\mathcal{T}}_0 {}^\frown \underbrace{\tilde{\pi}_\theta \mathcal{T}_1}_{\substack{\text{construed as} \\ \text{a tree on} \\ M_\theta^{\overline{\mathcal{T}}_0}}}$ is well-behaved

thus establishes that $\mathcal{T} = \mathcal{T}_0 {}^\frown \mathcal{T}_1$ is well-behaved. $\dashv$ (Claim 21)

To be cont'd.