Grigor Sargsyan, Ralf Schindler

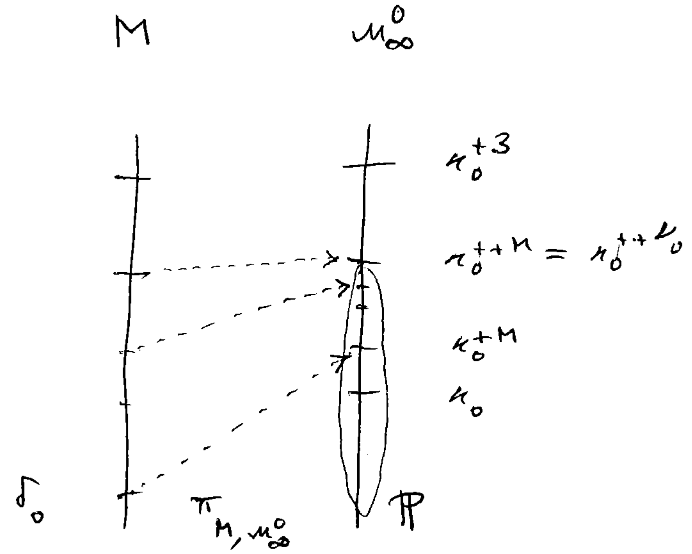
Varsovian models, II, cont'd   Sept 2017

Correction: the argument for Claim 16 on p.55 f. is bogus. The argument is supposed to use the forcing theorem to get a $p \in g$ with $p \Vdash^{\mathbb{P}}_{\mathcal{P}||\nu} \forall \xi < \rho (\xi \in \tau \leftrightarrow \varphi^{\mathcal{P}||\nu}(\xi, \vec{p}))$, cf. p. 56 lines 1-2. To prove the relevant instance of the forcing theorem, though, we'd need dense sets to exist in $\mathcal{P}||\nu$ which are definable over $\mathcal{P}||\nu$ by formulae whose complexity is at least the complexity of φ, but φ witnesses $\rho_\omega(\mathcal{P}||\nu) \leq \rho < \kappa_0^{+3}$. The argument is thus circular at best. The revised $\mathcal{P}$-construction which is proposed on pp. 54 ff. is suspicious for other reasons also.

The ~~proof~~ of Claim 21 is not affected by this, though.

We now discard pp. 55 — 59 and go back to the task of reorganizing $\mathcal{V}_0 = L[\mathcal{M}_\infty^0, p \mapsto p^*]$ $= \mathrm{HOD}^{M^{\mathrm{Col}(\omega, <\kappa_0)}}$.

We know that M is $\mathbb{P}$-generic over $\mathcal{V}_0$ for some $\mathbb{P} \in \mathcal{V}_0$, $\mathbb{P} \subset \mathcal{V}_0 | \kappa_0^{++}$, $\mathcal{V}_0 \models$ "$\mathbb{P}$ has the κ_0^{+M}-c.c." (and $\kappa_0^{+M} = \delta_0^{\mathcal{M}_\infty^0}$, $\kappa_0^{++M} = \kappa_0^{++\mathcal{V}_0}$).



Let us do a revised $\mathcal{P}$-construction above $\mathcal{V}_0 | \kappa_0^{+3}$, inside M, as follows. Let $\mathcal{P}|\nu$, $\mathcal{P}||\nu$ denote the passive, active levels of this $\mathcal{P}$-construction, respectively.

While this construction is performed inside M, we aim to inductively maintain that the following are satisfied.

(*) $\mathcal{P}|\nu$, $\mathcal{P}||\nu \in \mathcal{L}_0$ for all ν,

(**) $\mathcal{P}|\nu[g] = M|\nu$, $\mathcal{P}||\nu[g] = M||\nu$ for all ν, where $g = M|\kappa_0^{+2}$ is $\mathbb{P}$-generic over $\mathcal{P}|\nu$ ($\mathbb{P}$ as above).

Notice that by (*), $\mathcal{P}|\kappa_0^{+3} = \mathcal{L}|\kappa_0^{+3}$, which automatically gives that g is $\mathbb{P}$-generic over $\mathcal{P}|\nu$. (Claim 16 on p. 55 will thus be true for the new $\mathcal{P}$; (**) is a restatement of Claim 17; Claim 18 is then given by Hamkins-Laver-Woodin, cf. Lemma 2 of the appendix.)

To describe our new revised $\mathcal{P}$-construction, we only have to say how to get from $\mathcal{P}|\nu$ to $\mathcal{P}||\nu$.

Thus let $\nu \geq \kappa_0^{+3}$, and let $\mathcal{P}|\nu$ be given such that $\mathcal{P}|\nu \in \mathcal{L}_0$ and $\mathcal{P}|\nu[g] = M|\nu$.

If ν indexes an extender with critical point $> \kappa_0^{+3}$ (which is equivalent to the fact that E_ν^M has critical point κ_0), then we let $E_\nu^M \cap \mathcal{P}|\nu$ be the top predicate of $\mathcal{P}\|\nu$.

In order to get (*), we need that

(A) the extension of E_ν^M to $M^{\mathrm{Col}(\omega, <\kappa_0)}$ is OD in $M^{\mathrm{Col}(\omega, <\kappa_0)}$.

This is true because if (A) holds, then by $\mathcal{P}|\nu \in \mathcal{V}_0 = \mathrm{HOD}^{M^{\mathrm{Col}(\omega,<\kappa_0)}}$, $E_\nu^M \cap \mathcal{P}|\nu$ = (the extension of E_ν^M to $M^{\mathrm{Col}(\omega,<\kappa_0)}$) $\cap \mathcal{P}|\nu \in \mathrm{HOD}^{M^{\mathrm{Col}(\omega,<\kappa_0)}} = \mathcal{V}_0$.

On the other hand, suppose that $E_\nu^M \cap \mathcal{P}|\nu \in \mathcal{V}_0$. Let $a \in [\nu]^{<\omega}$, and let X be in the a^{th} component of the extension of E_ν^M to $M^{\mathrm{Col}(\omega,<\kappa_0)}$, $X \in M^{\mathrm{Col}(\omega,<\kappa_0)}$, say $X = \tau^{g*h}$, where h is $\mathrm{Col}(\omega,<\kappa_0)$-generic over $M = \mathcal{V}_0[g]$. By closure, there is one $p \in g*h$ s.t. $\{u \in [\mathrm{crit}(E_\nu^M)]^{\bar{\bar{a}}} : p \Vdash_{\mathcal{V}_0}^{\mathbb{P}*\mathrm{Col}(\omega,<\kappa_0)} \check{u} \in \tau\}$

$\in E_\nu^M$. In other words, there is some $\bar{X} \in (E_\nu^M \cap \mathcal{P}|\nu)_a$, $\bar{X} \in \mathcal{P}|\nu$, which "forces" X to be in the a^{th} component of the extension of E_ν^M to $M^{\mathrm{Col}(\omega, <\kappa_0)}$ in the sense that $\bar{X} \subset X$, and hence this extension is definable in $M^{\mathrm{Col}(\omega,<\kappa_0)}$ from the parameter $E_\nu^M \cap \mathcal{P}|\nu$. Hence if ~~...~~ $E_\nu^M \cap \mathcal{P}|\nu \in \mathcal{V}_0 = \mathrm{HOD}^{M^{\mathrm{Col}(\omega,<\kappa_0)}}$, then (A) holds true. (A) is thus equivalent to $E_\nu^M \cap \mathcal{P}|\nu \in \mathcal{V}_0$.

We defer the proof of (A).

Let us now suppose that ν indexes an extender with critical point $= \kappa_0$. We then let the top predicate of $\mathcal{P}||\nu$ code

$$\left\{ \left(\mathcal{T}, \Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}(\mathcal{T})\right) : \mathcal{T} \in \mathcal{P}|\lambda^{\mathcal{P}|\nu}, \text{ a tree of limit length on } \mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty} \right\}$$

in an amenable way (this is possible as $\mathrm{cf}(\nu) = \mathrm{cf}(\kappa_0^+) = \delta_0^{\mathcal{M}_\infty^0}$ in M). Here, $\lambda^{\mathcal{P}|\nu}$ is the largest cardinal of $\mathcal{P}|\nu$.

By Subclaim 2 on p. 53, we certainly have (*).

Let us argue that $\mathcal{P}\|\nu[g] = M\|\nu$. We have to see that E_ν^M is definable over $\mathcal{P}\|\nu[g]$.

Let $\mathcal{T} \in M|\lambda^{\mathcal{P}|\nu}$ be a tree of limit length on $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty^0}$, say $\mathcal{T} = \tau^g$, $\tau \in (\mathcal{P}|\nu)^{\mathbb{P}}$. Let $\mathcal{U}$ on $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty^0}$ arise from comparing all $\mathcal{M}(\tau^{g_s})$ with $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty^0}$, where g_s is the finite variant of g given by s. Then $\mathcal{U} \in \mathcal{P}|\nu$, and hence $b = \Sigma_{\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0}}(\mathcal{U})$ is given by the top predicate of $\mathcal{P}\|\nu$. But then $c = \Sigma_{\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0}}(\mathcal{T})$ is the unique branch c' thru $\mathcal{T}$ s.t. there is an embedding $k : \mathcal{M}_{c'}^{\mathcal{T}} \longrightarrow \mathcal{M}_b^{\mathcal{U}}$ with $k \circ \pi_{0,c'}^{\mathcal{T}} = \pi_{0,b}^{\mathcal{U}}$.

(Cf. Lemma 2.1 of [VMI].)

Let α be the next admissible after ν, i.e., $M|\alpha \models KP$. Then $\mathcal{P}|\alpha{+}\omega$ can identify a direct limit system whose direct limit is exactly $\pi_{E^M_\nu}(M^0_\infty | \delta_0^{M^0_\infty})$. But then

$$\pi_{E^M_\nu} \restriction M^0_\infty | \delta_0^{M^0_\infty} : M^0_\infty | \delta_0^{M^0_\infty} \longrightarrow \pi_{E^M_\nu}(M^0_\infty | \delta_0^{M^0_\infty})$$

is in $\mathcal{P}|\alpha{+}\omega$. Let $e : \kappa_0^+ \longleftrightarrow M|\kappa_0^+$ be the canonical enumeration which is given by the can. w.o. of $M|\kappa_0^+$, so that e is $\Sigma_1^{M|\kappa_0^+}$. Then $\pi_{E^M_\nu}(e) : \nu \longleftrightarrow M|\nu$ is the can. enumeration given by the can. w.o. of $M|\nu$. $Y = \pi_{E^M_\nu}(X)$ iff there is some $\xi < \kappa_0^+$ s.t. $X = e(\xi)$ and $Y = \pi_{E^M_\nu}(e)(\pi_{E^M_\nu}(\xi))$.

As e, $\pi_{E^M_\nu}(e)$, and $\pi_{E^M_\nu} \restriction \kappa_0^+$ are in $\mathcal{P}|\alpha{+}\omega[g]$, E^M_ν is in $\mathcal{P}|\alpha[g]$ also. I.e., $\mathcal{P}|\alpha{+}\omega[g] = M|\alpha{+}\omega$.

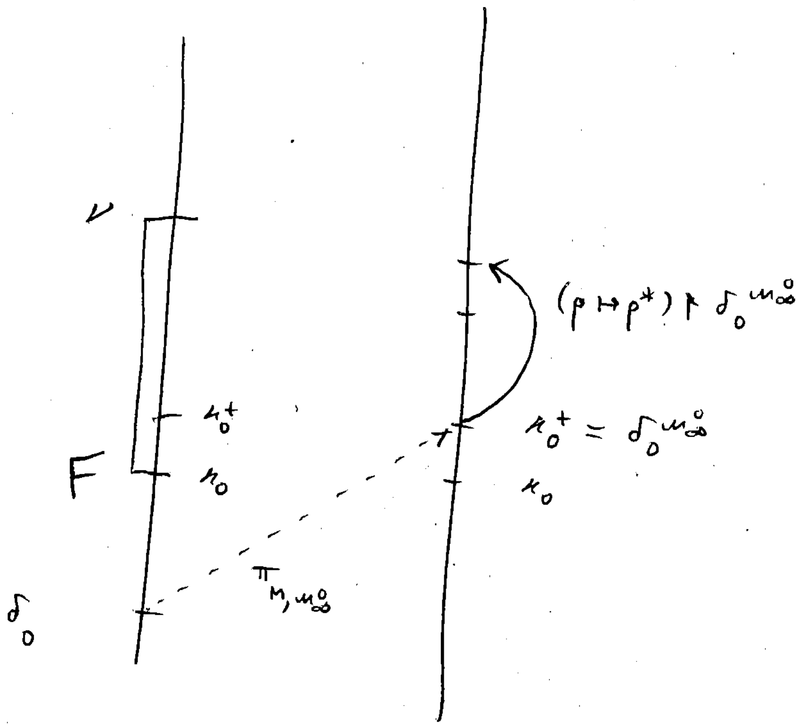
This basically shows (**) if we reinterpret $\mathcal{P}||\nu[g] = M||\nu$ as meaning that $\mathcal{P}|\alpha{+}\omega[g] = M|\alpha{+}\omega$.

If $E_\nu^M = \emptyset$, we simply construct one step further.

Modulo (A) on p. 74, we thus showed how to reorganize $\mathcal{V}_0$; (A) was shown to hold true by Farmer Schlutzenberg by generalizing arguments from his Ph.D. thesis.

The revised $\mathbb{P}$-construction we presented on pp. 72 – 78 is a construction above $\mathcal{V}_0 \mid \kappa_0^{+3}$. It does not give any information as to how $\mathcal{V}_0$ could be organized as some kind of useful "premouse" below κ_0^{+3}. We're now going to do exactly that.

Recall that $\mathcal{V}_0 = L[\mathcal{M}_\infty^0, \rho \mapsto \rho^*]$, and $\rho \mapsto \rho^*$ may be computed from $\mathcal{M}_\infty^0$ and $(\rho \mapsto \rho^*) \restriction \delta_0^{\mathcal{M}_\infty^0}$. $\rho \mapsto \rho^*$ is the map from $\mathcal{M}_\infty^0$ into its own $\mathcal{M}_\infty^0$ (restricted to the ordinals).



Let F be the least M-measure on κ_0.

Let $\mathcal{M}^\infty_\infty = (\mathcal{M}^0_\infty)^{\mathcal{M}^0_\infty}$, and let $\kappa_0^{+\mathcal{M}^0_\infty}$ be the $\mathcal{M}^0_\infty$-successor of the least strong of $\mathcal{M}^0_\infty$, so that

$(\rho\mapsto\rho^*)\restriction\delta_0^{\mathcal{M}_\infty} : \delta_0^{\mathcal{M}_\infty} = \kappa_0^+ \longrightarrow \kappa_0^{+\mathcal{M}^0_\infty}$ is cofinal.

<u>Claim 22.</u> $\kappa_0^{+\mathcal{M}^0_\infty} < \kappa_0^{++\,ult(M;F)}$.

Proof: Let α be least s.t. $J_\alpha[\mathcal{M}^0_\infty|\kappa_0^{+\mathcal{M}^0_\infty}, \rho\mapsto\rho^*\restriction\delta_0^{\mathcal{M}^0_\infty}]$ is a ZFC^- model. We show that $\alpha < \kappa_0^{++\,ult(M;F)}$.

$ult(M;F)$ can compute $\mathcal{M}^0_\infty|\delta_0^{\mathcal{M}^0_\infty}$ as well as $(\rho\mapsto\rho^*)\restriction\delta_0^{\mathcal{M}^0_\infty}$. By the argument pp. 9 – 12,

$J_\alpha[m^0_\infty | \kappa_0^{+m^0_\infty}, \rho \mapsto \rho^* \upharpoonright \delta_0^{m^0_\infty}]$ knows that every ordinal $\in [\kappa_0^+, \kappa_0^{m^0_\infty})$ has size $\kappa_0^+ = \delta_0^{m^0_\infty}$, and of course it knows that $\kappa_0^{+m^0_\infty}$ has cofinality $\kappa_0^+ = \delta_0^{m^0_\infty}$.

Suppose that $\kappa_0^{m^0_\infty}$ were a cardinal in $J_\alpha[-] = J_\alpha[m^0_\infty | \kappa_0^{+m^0_\infty}, \rho \mapsto \rho^+ \upharpoonright \delta_0^{m^0_\infty}]$, i.e. $\kappa_0^{m^0_\infty} = (\delta_0^{m^0_\infty})^{+J_\alpha[-]}$. Let $\mathcal{I}$ on $m^0_\infty / \delta_0^{m^0_\infty}$ be the canonical tree of length $\kappa_0^{+m^0_\infty}$ to make $m^0_\infty | \kappa_0^{+m^0_\infty}$ generic, so that $\mathcal{I}$ is definable over $m^0_\infty | \kappa_0^{+m^0_\infty}$.

Let $b = \Sigma_{m^0_\infty | \delta_0^{m^0_\infty}}(\mathcal{I})$.

I claim that $b \in J_\alpha[-]$. As $\kappa_0^{m^0_\infty} = (\delta_0^{m^0_\infty})^{+J_\alpha[-]}$, we may take, inside $J_\alpha[-]$,

$$N \overset{\sigma}{\cong} X \prec_{\Sigma_{1000}} J_\alpha[-]$$

s.t. $X \cap \kappa_0^{m^0_\infty} \in \kappa_0^{m^0_\infty}$, $\mathcal{I} \in X$, etc., and N is transitive. Let $\sigma(\bar{\mathcal{I}}) = \mathcal{I}$.

$J_\alpha[-]$ can see $\Sigma_{M^0_\infty | \delta_0^{M^0_\infty}}(\bar{J})$ (call it c), by searching for c' s.t. there is $M^{\bar{J}}_{c'} \to M^\infty_\infty | \kappa_0^{+M^0_\infty}$.

As we may assume $X \cap OR$ is cofinal in $\kappa_0^{+M^0_\infty}$ (recall $cf(\kappa_0^{+M^0_\infty}) = \delta_0^{M^0_\infty} < \kappa_0^{M^0_\infty}$ in $J_\alpha[-]$), ~~[illegible]~~ $\sigma'' c \in J_\alpha[-]$ induces a cofinal branch thru $\bar{J}$. By branch condensation, this branch is equal to b.

But now the argument pp. 9–12 again gives that $J_\alpha[-]$ knows that $\mathrm{Card}(\kappa_0^{+M^0_\infty}) = \kappa_0^+$.

As everything can be done in $ult(M; F)$, $\alpha < \kappa_0^{++ ult(M;F)}$ as desired. $\dashv$ (Claim 22)

In particular, $i_F \restriction M^0_\infty | \delta_0^{M^0_\infty} \neq (p \mapsto p^*) \restriction M^0_\infty | \delta_0^{M^0_\infty}$.

Let's still write α for the least α s.t. $J_\alpha[M^0_\infty | \kappa_0^{+M^0_\infty}, (p \mapsto p^*) \restriction \delta_0^{M^0_\infty}]$ is a ZFC$^-$ model; so $\alpha < \kappa_0^{++ ult(M;F)}$, and hence α is smaller than the index of F (no matter if we use

Jensen or Mitchell-Steel indexing.

<u>Claim 23.</u> Let's write $Q = J_\alpha[m_\infty^0 | \kappa_0^{+m_\infty^0}, (\rho \Vdash \rho^*) \restriction \delta_0^{m_\infty^0}]$.

$M|\alpha$ is ~~...~~ a generic extension of Q.

Proof: It is not hard to show that Q is a definable inner model of $M|\alpha$. We claim that for each function $f : \theta \longrightarrow OR$, $f \in M|\kappa$ (θ arbitrary), there is some $g \in Q$, $\mathrm{dom}(g) = \theta$, $f(\xi) \in g(\xi)$ and $\overline{\overline{g(\xi)}} < \delta_0^{m_\infty^0}$ for all $\xi < \theta$. Otherwise let f be the least counterexample, so that $f(\xi) = \eta$ iff $M|\alpha \models \varphi(\xi, \eta)$. But then we may let

$$g(\xi) = \{\eta : \exists q \Vdash^{\text{ext. alg.}}_{m_\infty^0|\alpha} \varphi(-,-) \text{ defines a fctn. and } \varphi(\xi^*, \eta^*)\}.$$

This gives a function g as desired and a contradiction.

The result now follows by Bukowský.

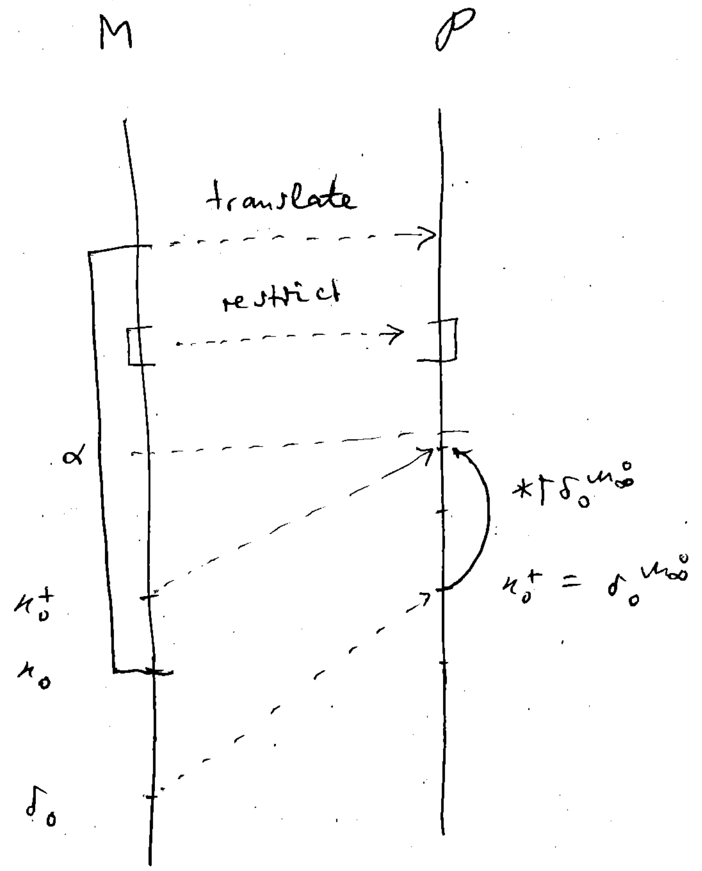
$\dashv$ (Claim 23)

According to Schlutzenberg, $\kappa_0^{+M_\infty^0} <$ the least α s.t. $J_\alpha[M|\kappa_0^+]$ is admissible.

We now perform a revised $\mathcal{P}$-construction above

$$J_\alpha[\, M_\infty^0 | \kappa_0^{+M_\infty^0}, (p \mapsto p^*) \restriction \delta_0^{M_\infty^0} \,]$$

in much the same way as on pp. 72 — 78.



We set $\mathcal{P}|\alpha = J_\alpha[\, M_\infty^0 | \kappa_0^{+M_\infty^0}, (p \mapsto p^*) \restriction \delta_0^{M_\infty^0} \,]$.

Having constructed $\mathcal{P}|\nu$, we proceed as follows.

1st case. $E_\nu^M = \emptyset$.

Then we just construct one step further.

2nd case. $E_\nu^M \neq \emptyset$ has critical point $\neq \kappa_0$.

Then $\mathrm{crit}(E_\nu^M) > \alpha$, and we let

$$P\|\nu = (P|\nu;\ E_\nu^M \cap P|\nu).$$

3rd case. $E_\nu^M \neq \emptyset$ has critical point $= \kappa_0$.

Then we let

$$P\|\nu = (P|\nu\ ;\ S_\nu^P),$$

where $S_\nu^P = \{ (\mathcal{T}, \Sigma_{\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0}}(\mathcal{T})) : \mathcal{T} \in P|\lambda^{P|\nu},$ a tree of limit length on $\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} \}$,

$\lambda^{P|\nu}$ = the largest cardinal of $P|\nu$.

We inductively maintain that all $P\|\nu$ are in $\mathrm{HOD}^{M^{\mathrm{Col}(\omega, <\kappa_0)}} = \mathcal{V}_0$ and that $M|\nu = P|\nu[g]$ for some g which is $\mathbb{P}^\nu$ gen. over $P|\nu$, $\mathbb{P}^\nu$ given by Bukowský.

For the relevant ν, $\mathbb{P}^\nu \subset M|(\kappa_0^+)^{+P|\nu}$, and $\mathbb{P}^\nu \subset \mathbb{P}^{\nu'}$ for $\nu \leq \nu'$ (so that the $\mathbb{P}^\nu$ stabilize).

We have $\nu_0 = P$. We leave the details to the reader.

The proof of Claim 21, p. 64, shows:

<u>Claim 24.</u> In V, $\nu_0 = P$ is iterable w.r.t. trees which use extenders from the P-sequence.

There is a more general notion of iterability of ν_0, though.

~~these~~ Over $(\mathcal{M}_\infty^0 | \kappa_0^{+\mathcal{M}_\infty^0}; (\rho \mapsto \rho^*) \restriction \delta_0^{\mathcal{M}_\infty^0})$, the map $(\rho \mapsto \rho^*) \restriction \delta_0^{\mathcal{M}_\infty^0}$ is intertranslatable with

$$\{ (\mathcal{T}, \Sigma_{\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}}(\mathcal{T})) : \mathcal{T} \in \mathcal{M}_\infty^0 | \kappa_0^{+\mathcal{M}_\infty^0} \text{ is a tree of limit length on } \mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0} \},$$

so that we reorganized ν_0 as a "premouse" of the form $L[\vec{E}, \vec{S}]$, where

- $\vec{E}$ is a sequence of extenders,
- $\vec{E} \upharpoonright \kappa_0^{+\mathcal{M}_\infty^0} = \vec{E}^{\mathcal{M}_\infty^0} \upharpoonright \kappa_0^{+\mathcal{M}_\infty^0}$,
- $\vec{E} \upharpoonright [\kappa_0^{+\mathcal{M}_\infty^0}, \infty) = \{ E_\nu^M \cap P|\nu : \nu > \kappa_0^{+\mathcal{M}_\infty^0}, \mathrm{crit}(E_\nu^M) > \kappa_0 \}$, and
- $\vec{S} = (S_\nu : \nu \in \{ \kappa_0^{+\mathcal{M}_\infty^0} \} \cup \{ \nu' : \mathrm{crit}(E_{\nu'}^M) = \kappa_0 \})$,

  $S_\nu = \{ (\mathcal{T}, \Sigma_{\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0}}(\mathcal{T})) : \mathcal{T} \in P|\lambda^{P|\nu}$ is a tree of limit length on $\mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} \}$.

Here, $P|\nu = J_\nu[\vec{E}, \vec{S} \upharpoonright \nu]$, $P||\nu = (P|\nu; (\vec{E}, \vec{S})_\nu)$.

Claim 24 says that $\mathcal{L}_0$ is iterable w.r.t. $\vec{E}$ and its images.

However, each S_ν as above is intertranslatable over $P||\nu$ with a (long) extender, namely the canonical map

$$\pi_\nu : \mathcal{M}_\infty^0|\delta_0^{\mathcal{M}_\infty^0} \longrightarrow (\mathcal{M}_\infty)^\nu,$$

where $(\mathcal{M}_\infty)^\nu$ is the direct limit of all non-dropping (on the main branch) $\Sigma_{\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}}$-iterates of $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ which $P|\lambda^{P|\nu}$ can see. By normalization (Schlutzenberg-Steel), π_ν as above is given by a normal iteration of $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ according to $\Sigma_{\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}}$.

We may then "take the ultrapower" of $\mathcal{V}_0$ by S_ν as $ult(\mathcal{V}_0; \pi_\nu)$. As π_ν is given by a normal tree on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$ acc. to $\Sigma_{\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}}$ $\cong \Sigma_{\mathcal{M}_\infty^0} \restriction$ trees living on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0}$, $ult(\mathcal{V}_0; \pi_\nu)$ is given by a normal iteration of $\mathcal{V}_0$ via a tree living on $\mathcal{M}_\infty^0 | \delta_0^{\mathcal{M}_\infty^0} = \mathcal{V}_0 | \delta_0^{\mathcal{M}_\infty^0}$.

The more general notion of an iteration of $\mathcal{V}_0$ is then given by allowing $ult(\mathcal{V}_0; \pi_\nu)$ — of course not only for $\mathcal{V}_0$ itself but also for iterates of $\mathcal{V}_0$.

The key fact is the following. Let $\mathcal{I}$ be an iteration of $\mathcal{L}_0$ (in the generalized sense) with last model $\mathcal{M}^{\mathcal{I}}_\alpha$. Say $[0,\alpha]_{\mathcal{I}}$ doesn't drop. We claim that if $S_\nu^{\mathcal{M}^{\mathcal{I}}_\alpha} \neq \emptyset$, then

$$S_\nu^{\mathcal{M}^{\mathcal{I}}_\alpha} = \{ (\mathcal{T}, \Sigma_{\pi^{\mathcal{I}}_{0\alpha}(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty})}(\mathcal{T})) \;:\; \mathcal{T} \in \mathcal{M}^{\mathcal{I}}_\alpha | \lambda^{\mathcal{M}^{\mathcal{I}}_\alpha | \nu}$$
$$\text{is a tree of limit length on } \pi^{\mathcal{I}}_{0\alpha}(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}) \}.$$

By internal coherence, which is 1^{st} order, it suffices to prove this for $\nu \in \operatorname{ran}(\pi^{\mathcal{I}}_{0\alpha})$, say $\nu = \pi^{\mathcal{I}}_{0\alpha}(\bar\nu)$. Let $\mathcal{U}^{\bar\nu}$ be the tree on $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$ which gives rise to $\pi_{\bar\nu}$, $lh(\mathcal{U}^{\bar\nu}) = \bar\nu$, $\mathcal{U}^{\bar\nu}$ being definable over $\mathcal{L}_0 \| \bar\nu$; $S_{\bar\nu}^{\mathcal{L}_0}$ can be read off from $\Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}(\mathcal{U}^{\bar\nu})$.

Let $\mathcal{U}^\nu$ be the tree on $\pi^{\mathcal{I}}_{0\alpha}(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty})$ which gives rise to π_ν (π_ν the map induced by $S_\nu^{\mathcal{M}^{\mathcal{I}}_\alpha}$), $\mathcal{U}^\nu$ being definable on $\mathcal{M}^{\mathcal{I}}_\alpha \| \nu$ in the same manner as $\mathcal{U}^{\bar\nu}$ was definable on $\mathcal{L}_0 \| \bar\nu$; $lh(\mathcal{U}^\nu) = \nu$; $S_\nu^{\mathcal{M}^{\mathcal{I}}_\alpha}$ can be read off from $\Sigma_{\pi^{\mathcal{I}}_{0\alpha}(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty})}$. Inductively, $\pi^{\mathcal{I}}_{0\alpha}(\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty})$ is a $\Sigma_{\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}}$-iterate of $\mathcal{M}^0_\infty | \delta_0^{\mathcal{M}^0_\infty}$.

Notice that $\delta_0^{m^0_\infty}$ is a continuity point of all the relevant iteration maps, so that $\pi^{\mathcal{I}}_{0\alpha}$ is continuous at $\bar{\nu}, \nu$.

$$\begin{tikzcd}[column sep=large]
M \arrow[r] & m^0_\infty \arrow[r, "\substack{\text{inductively, a}\\ \Sigma_{m^0_\infty | \delta_0^{m^0_\infty}}\text{-iteration of } m^0_\infty}"] \arrow[dr, "\substack{\text{given by}\\ \Sigma_{m^0_\infty|\delta_0^{m^0_\infty}}(\mathcal{U}^{\bar{\nu}})}"'] & \mathrm{ult}(m^0_\infty ; \pi^{\mathcal{I}}_{0\alpha} \upharpoonright m^0_\infty | \delta_0^{m^0_\infty}) \arrow[r, "\substack{\text{given by}\\ \Sigma_{\pi^{\mathcal{I}}_{0\alpha}(m^0_\infty|\delta_0^{m^0_\infty})}(\mathcal{U}^{\nu})}"] & \mathcal{R}_0 \\
& & \mathcal{R}_1 \arrow[u, dashed, "\pi^{\mathcal{I}}_{0\alpha}"'] &
\end{tikzcd}$$

We may pull back $\Sigma_{\pi^{\mathcal{I}}_{0\alpha}(m^0_\infty|\delta_0^{m^0_\infty})}(\mathcal{U}^{\nu})$ to produce a cofinal branch c thru $\mathcal{U}^{\bar{\nu}}$ s.t. the limit model $m_c^{\mathcal{U}^{\bar{\nu}}}$ can be embedded into $\mathcal{R}_0$ ($\mathcal{R}_0$ as in the diagram). By branch condensation for Σ,

$$c = \Sigma_{m^0_\infty|\delta_0^{m^0_\infty}}(\mathcal{U}^{\bar{\nu}}).$$

But then $\Sigma_{\pi^{\mathcal{I}}_{0\alpha}(m^0_\infty|\delta_0^{m^0_\infty})}(\mathcal{U}^{\nu})$ is the cofinal branch thru $\mathcal{U}^2$ given by lifting up $\Sigma_{m^0_\infty / \delta_0, m^0_\infty}(u^-)$, which is also the branch given by $\vec{S}^{m^\mathcal{I}_\alpha}$. Therefore, $\pi^{\mathcal{I}}_{0\alpha}$ moves $\vec{S}$ correctly.

We have shown that V_0 is iterable in the generalized sense.

Another correction. Possibly, Claim 13 on p. 50 is not quite right, at least its proof seems suspicious.*) Let us replace Claim 13 by the following.

Claim 13'. $V_0 = L[m^0_\infty, p \mapsto p^*] = HOD^{M^{Col(\omega, <\kappa_0)}}_{\mathcal{E}}$,

where $\mathcal{E}$ is defined as follows.

Let $L[E]$ be an extender model, and let κ be a regular cardinal of $L[E]$. Let g be $Col(\omega, <\kappa)$-generic over $L[E]$.

Let us assume that κ is a "weak cutpoint"

*) We thank F. Schlutzenberg for pointing this out to us.

of $L[E]$ in the sense that there is no E_ν of $L[E]$'s extender sequence with $\nu > \kappa$ and $crit(E_\nu) < \kappa$ ($crit(E_\nu) = \kappa$ being allowed).

Let us define $\tilde{E}_g = (\tilde{E}_{g,\nu} : \nu > \kappa,\ E_\nu \text{ active})$, where $\tilde{E}_{g,\nu}$ is the extender or ideal extender (cf. Claverie's Ph.D. thesis) defined by

$$Y \in \tilde{E}_{g,\nu,a} \iff Y \in L[E][g] \wedge Y \in \mathcal{P}([crit(E_\nu)]^{\bar{\bar{a}}}) \wedge \exists X \subset Y \ X \in E_{\nu,a} .$$

(Then $\tilde{E}_{g,\nu}$ is an extender iff $crit(E_\nu) > \kappa$ and it is an ideal extender iff $crit(E_\nu) = \kappa$.)

Let $L[E]$, $L[E']$ be two extender models. We say that $L[E]$, $L[E']$ are <u>intertranslatable above κ</u> iff κ is a reg. cardinal and a weak cutpoint in both $L[E]$ and $L[E']$ and for some (all) g $Col(\omega, <\kappa)$-gen. / $L[E] \cup L[E']$,

$$\tilde{E}_g = \tilde{E}'_g .$$

(It then follows that if $L[E][g] = L[E'][g]$, E is definable from $E \upharpoonright \kappa$ and $\tilde{E}'_g$ inside $L[E][g] = L[E'][g]$, and conversely E' is definable from $E' \upharpoonright \kappa$ and $\tilde{E}_g$ inside the same model.)

We write $\mathcal{E}$ for the "set" of all E s.t. $L[E] \models$ "I'm the least inner model with $\delta_0' < \kappa_0' < \delta_1' < \kappa_1'$, δ_i Woodin, κ_i strong, $i \in \{0,1\}$, and I'm sufficiently iterable," and there is some $\eta < \kappa_0$ s.t. $L[E]$, $M = M_{swsw}$ are intertranslatable above κ (in particular, $\kappa_0' = \kappa_0$, $\delta_1' = \delta_1$, $\kappa_1' = \kappa_1$). (Notice that if $E \in \mathcal{E}$, then $\mathcal{E}^{L[E]}$ [i.e., $\mathcal{E}$ as being defined inside $L[E]$] is equal to $\mathcal{E} = \mathcal{E}^M$.)

Proof of Claim 13': the proof of [VMI, Claim 2.10] shows that if $E \in \mathcal{E}$, then the $\mathcal{M}^0_\infty$-system of $L[E]$ has cofinally many points in common with the $\mathcal{M}_\infty^0$ of M.

This shows that $L[\mathcal{M}_\infty^0, p \mapsto p^*] \subset \mathrm{HOD}_{\mathcal{E}}^{M^{\mathrm{Col}(\omega, <\kappa_0)}}$.

Now let X be a set of ordinals in $\mathrm{HOD}_{\mathcal{E}}^{M^{\mathrm{Col}(\omega, <\kappa_0)}}$. Say $\xi \in X$ iff

$M^{\mathrm{Col}(\omega, <\kappa_0)} \models \varphi(\xi, \vec{\alpha}, \mathcal{E})$, φ a formula, $\vec{\alpha}$ ordinals. Given ξ, let us pick P from the $\mathcal{M}_\infty^0$-system of M s.t. $\pi_{P,\infty}(\xi, \vec{\alpha}) = \xi^*, \vec{\alpha}^*$.

Then P (or rather, its extender sequence) is in $\mathcal{E}$, so that $\mathcal{E}^P = \mathcal{E}$. Hence $\xi \in X$ iff

$P^{\mathrm{Col}(\omega, <\kappa_0)} \models \varphi(\xi, \vec{\alpha}, \mathcal{E}^P)$ iff

$(\mathcal{M}_\infty^0)^{\mathrm{Col}(\omega, <\kappa_0^{\mathcal{M}_\infty^0})} \models \varphi(\xi^*, \vec{\alpha}^*, (\mathcal{E})^{\mathcal{M}_\infty^0})$.

We have shown that $X \in \mathcal{V}_0$.

$\dashv$ (Claim 13′)

Claim 13′ may be used in a nice way to show that the reorganization of $\mathcal{V}_0$ along the lines of pp. 83 – 85 works.

The point is just that now, writing $M = M_{swsw} = L[E]$, if g is any $\mathrm{Col}(\omega, <\kappa_0)$-generic filter of M, then $\tilde{E}_g$ (def. from E as on p. 91) is $OD_{\mathcal{E}}$ in $M[g]$, as $\tilde{E}_g$ is the common value of all $\tilde{E}'_g$ for $E' \in \mathcal{E}$. But then a straightforward induction gives that all the models of the construction ~~from~~ pp. 83-85 are in $\mathrm{HOD}_{\mathcal{E}}^{M^{\mathrm{Col}(\omega,<\kappa_0)}}$, hence by Claim 13′ they are all in $\mathcal{V}_0$. This construction thus gives a reorganization of $\mathcal{V}_0$.

To be cont'd.