Ralf Schindler

<u>Bukowský and Varsovian models, revisited</u>

Dec 2017

In the past, we abstractly cited a theorem of Bukowský's to show that any reasonable $M = L[E]$ is a generic extension of its first Varsovian model, $\mathcal{V}_0$. We now isolate a natural forcing $\mathbb{B}$ for which M is generic over $\mathcal{V}_0$.

$\mathbb{B}$ is not the extender algebra, but it is defined in terms of the extender algebra.

Let us fix $M = M_{sw}$ or M_{swsw} or any other reasonable model. Let $\mathcal{V}_0$ be the first Varsovian model of M, i.e. $\mathcal{V}_0 = L[\mathcal{M}^0_\infty, p \mapsto p^*]$, where $\mathcal{M}^0_\infty$ is the direct limit of a collection of grounds $\mathcal{P}$, and $p^* = \min\{\pi_{\mathcal{P}\mathcal{M}^0_\infty}(p) : \mathcal{P} \in \text{System}\}$ $= \pi_{\mathcal{M}^0_\infty(\mathcal{M}^0_\infty)^{\mathcal{M}^0_\infty}}(p)$. See "Varsovian models I" (with G. Sargsyan) or "Varsovian models II" (with G. Sargsyan and F. Schlutzenberg) for details.

We aim to make $M | \kappa_0$ generic over L_0.

It can be shown that $M = L_0[M | \kappa_0]$.

We think of $M | \kappa_0$ as a subset of κ_0.

Let us first work in L_0 and define an infinitary language $\mathcal{L}^+$ as follows.

Atomic formulae: "$\check{\xi} \in \dot{a}$" for all $\xi < \kappa_0$, $\dot{a}$ being a fixed name for $M | \kappa_0$. We close under negation and conjunctions and disjunctions of length $< \delta_0^{L_0} = \kappa_0^{+M}$.

If g is generic over $\mathcal{M}_\infty^0$ for the extender algebra at $\delta_0^{\mathcal{M}_\infty^0}$, then it may be that g is a premouse of height $\delta_0^{\mathcal{M}_\infty^0}$, and we write $\mathcal{M}_\infty^0[g] | \kappa_0^{\mathcal{M}_\infty^0}$ for the premouse $W \rhd g$ (if it ex.) which arises from g by flattening the $\mathcal{M}_\infty^0$-extenders above $\delta_0^{\mathcal{M}_\infty^0}$, W of height $\kappa_0^{\mathcal{M}_\infty^0}$. If $\mathcal{P} \in \mathcal{M}_\infty^0$-system, we shall also use the notation $\mathcal{P}[g] | \kappa_0$. Notice that by the construction of $\mathcal{P}$, there is always a g (namely, $M | \delta_0^{\mathcal{P}}$) s.t. $M | \kappa_0 = \mathcal{P}[g] | \kappa_0$.

Let us define $\mathcal{L} \subset \mathcal{L}^+$ by $\varphi \in \mathcal{L}$ iff there is some $p \in \mathbb{B}^{\nu_0}_{\delta_0^{\nu_0}}$ (= the extender algebra of ν_0 at $\delta_0^{\nu_0}$) s.t.

$$p \Vdash^{\mathbb{B}^{\nu_0}_{\delta_0^{\nu_0}}}_{\nu_0} \mathcal{M}^0_\infty[g] | \kappa_0^{\mathcal{M}^0_\infty} \models \varphi^* .$$

Here $(\ )^*$ is the image of $(\ )$ under $\rho \mapsto \rho^*$, extended to all objects in $\mathcal{M}^0_\infty$.

We let, for $\varphi, \psi \in \mathcal{L}$, $\varphi \leq \psi$ iff for all $p \in \mathbb{B}^{\nu_0}_{\delta_0^{\nu_0}}$, $p \Vdash \mathcal{M}^0_\infty[g] | \kappa_0^{\mathcal{M}^0_\infty} \models \varphi^* \longrightarrow \psi^*$.

$(\mathcal{L}; \leq)$ is a partial order which is an element of ν_0.

<u>Claim 1.</u> $\nu_0 \models (\mathcal{L}; \leq)$ has the $\delta_0^{\nu_0}$-c.c.

Proof: Let $(\varphi_i : i < \theta)$ be an antichain. For each i, pick $p_i \in \mathbb{B}^{\nu_0}_{\delta_0^{\nu_0}}$ s.t.

$p_i \Vdash \mathcal{M}^0_\infty[g] | \kappa_0^{\mathcal{M}^0_\infty} \models \varphi_i^*$. We must have $p_i \perp p_j$ for $i \neq j$, as otherwise $\varphi_i \wedge \varphi_j \in \mathcal{L}$

and $\varphi_i \wedge \varphi_j \leq \varphi_i, \varphi_j$. But then $\theta < \delta_0^{\nu_0}$, as $\mathbb{B}^{\nu_0}_{\delta_0^{\nu_0}}$ has the $\delta_0^{\nu_0}$-c.c. ⊣

<u>Claim 2.</u> $G_{M|\kappa_0} = \{\varphi \in \mathcal{L} : M|\kappa_0 \models \varphi\}$ is $(\mathcal{L}; \leq)$-generic over ν_0.

Proof: Let $(\varphi_i : i<\theta)$ be ~~an~~ a max. antichain. By Claim 1, $\theta < \delta_0^{\nu_0}$. If $G_{M|\kappa_0} \cap \{\varphi_i : i<\theta\} = \emptyset$, then $M|\kappa_0 \models \bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i$ (and $\bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i \in \mathcal{L}^+$).

But $\bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i \in \mathcal{L}$: We have that for $\mathcal{P} \in \mathcal{M}_\infty^0$-system and $g = M|\delta_0^{\mathcal{P}}$, $\mathcal{P}[g]|\kappa_0 \models \bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i$, so that $p \Vdash^{\mathbb{B}^{\mathcal{P}}_{\delta_0^{\mathcal{P}}}}_{\mathcal{P}} \mathcal{P}[\dot{g}]|\kappa_0 \models \bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i$, for some p. Here, $\mathbb{B}^{\mathcal{P}}_{\delta_0^{\mathcal{P}}}$ is the extender algebra of $\mathcal{P}$ at $\delta_0^{\mathcal{P}}$. Notice that $\nu_0 \subset \mathcal{P}$ for all $\mathcal{P}$ so that $\bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i \in \mathcal{P}$.

If $\mathcal{P}$ is sufficiently far out in the system, then $\pi_{\mathcal{P}\,\mathcal{M}_\infty^0}(\bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i) = (\bigwedge\!\!\!\bigwedge_{i<\theta} \neg\varphi_i)^*$, and then we get that

$$p \Vdash^{\mathbb{B}^{\mathcal{M}^0_\infty}_{\delta_0^{\kappa_0}}}_{\mathcal{M}^0_\infty} \mathcal{M}^0_\infty[j] \,|\, \kappa_0^{\mathcal{M}^0_\infty} \models \left(\bigwedge_{i<\theta} \neg \varphi_i \right)^*,$$

some p. Hence indeed $\bigwedge_{i<\theta} \neg\varphi_i \in \mathcal{L}$.

But $\bigwedge_{i<\theta} \neg\varphi_i \perp \varphi_i$ for each i, so that $(\varphi_i : i<\theta)$ was not maximal. $\dashv$

We now have that $M|\kappa_0 = \{ \xi < \kappa_0 :$ "$\check{\xi} \in \dot{a}$" $\in G_{M|\kappa_0} \}$, so that

$$V_0[G_{M|\kappa_0}] = V_0[M|\kappa_0],$$

and $G_{M|\kappa_0}$ is $(\mathcal{L}; \leq)$-generic over V_0.