Ralf Schindler March, 2018

A model with everything except for a well-ordering of the reals

This is joint work with J. Brendle, F. Castiblanco, L. Wu, and L. Yu.

A *Sierpiński set* is an uncountable set $S \subset \mathbb{R}$ s.t. every unctble. $\bar{S} \subset S$ is non-null; a *Luzin set* is an uncountable $\Lambda \subset \mathbb{R}$ s.t. every unctble. $\bar{\Lambda} \subset \Lambda$ is non-meager.

Lemma 1. Assume CH. There is a Sierpiński set as well as a Luzin set.

Proof: Let $(N_i : i < \omega_1)$ be an enumeration of all G_δ null sets. Recursively define $(x_i : i < \omega_1)$ s.t. $x_i \notin \bigcup \{N_j : j < i\} \cup \{x_j : j < i\}$. Then $S = \{x_i : i < \omega_1\}$ is Sierpiński. The same proof produces a Luzin set, starting out with an enumeration $(M_i : i < \omega_1)$ of all F_σ meager sets.

$\dashv$

As we may write $\mathbb{R} = N \cup M$, where N is null and M is meager, no set can be both Sierpiński as well as Luzin.

<u>Definition.</u> $\mathbb{S}$ = Sacks forcing = the set of all perfect trees $T \subset {}^{<\omega}2$, ordered by inclusion (i.e., $T \leq S$ iff $T \subset S$).

Let us write $\mathbb{S}(\omega_1)$ for the countable support product of ω_1 copies of $\mathbb{S}$.

<u>Lemma 2.</u> Let $\mathbb{S}(\omega_1) \ni p \Vdash \tau \in {}^{\omega}\omega$. There is some $q \leq p$ and some $f \in {}^{\omega}[\omega]^{<\omega} \cap V$ s.t.

$q \Vdash \forall n \; \tau(n) \in f(n)$ and $\overline{\overline{f(n)}} \leq 2^{2n}$ f.a. n.

Let's refer to this as the "Sacks property." It is shown by a simple fusion argument.

Proof: Let $X \prec V_\theta$, $p, \tau \in X$, $\bar{\bar{X}} = \aleph_0$, $\theta \gg \omega_1$. Write $\alpha = X \cap \omega_1$. The support $\mathrm{supp}(p)$ of p is an element of X, hence also a subset of X. We shall construct f and q

as in the statement of Lemma 2 with $\mathrm{supp}(q) \subset \alpha$.

Let $e : \omega \leftrightarrow \alpha$. We aim to produce a sequence $(p_n : n<\omega)$ s.t. $p_0 = p$, and $p_{n+1} \leq p_n$, $p_n \in X$ for all $n<\omega$. (Then also $\mathrm{supp}(p_n) \subset \alpha$, all $n<\omega$.)

Let p_n be given. Working in X, we shall produce $p_{n+1} \leq p_n$ s.t. for all $k<n$, the n^{th} level of $p_{n+1}(e(k))$ *) is equal to the n^{th} level of $p_n(e(k))$ and there is some $a \in [\omega]^{\leq 2^{2n}}$ s.t. $p_{n+1} \Vdash \tau(\check{n}) \in \check{a}$. The "intersection" of all p_n and the function given by the associated a's then gives q and f as in Lemma 2.

We may produce p_{n+1} by some sequence $(q_m : m \leq 2^{2n})$ defined as follows inside X. $q_0 = p_n$.

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*) The n^{th} level of $T \in \mathbb{S}$ is the set of all $s \in T$ which are $(n+1)^{\text{st}}$ splitting nodes of T.

Fix some enumeration $(\vec{s}_m : m < 2^{2n})$ of all tuples $\vec{s} = (s_{e(0)}, \ldots, s_{e(n-1)})$ s.t. $s_{e(k)}$ is an element of the n^{th} level of $p_n(e(k))$.

Suppose q_m has been chosen, we aim to define q_{m+1}. For each $k < n$, let $\bar{m}_k \leq m$ be maximal s.t. $s_{e(k)} \in q_{\bar{m}_k}(e(k))$, and define $\bar{q}$ with the same support as q_m by :

$$\bar{q}(\xi) = \begin{cases} \left(q_{\bar{m}_k}(e(k))\right)_{s_{e(k)}} & \text{if } \xi = e(k) \quad {}^{*)} \\ q_m(\xi) & \text{if } \xi \neq e(k), \text{ all } k < n \end{cases}$$

(By construction, we will have $\left(q_{\bar{m}_k}(e(k))\right)_{s_{e(k)}} = q_{\bar{m}_k}(e(h))$ or $= p_n(e(k))_{s_{e(k)}}$.)

Let $q_{m+1} \leq \bar{q}$ decide $\tau(\check{n})$, and put

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*) $T_s = \{t \in T : t > s \text{ or } t < s\}$.

the $l < \omega$ with $q_{m+1} \Vdash \tau(\check{n}) = \check{l}$ into a.

This defines $(q_m : m \le 2^{2n})$.

Let us then define p_{n+1} as follows.

For each $k < n$ and $s \in n^{\text{th}}$ level of $p_n(e(k))$ let $\bar{m}_{k,s} \le m$ be maximal s.t. $s \in q_{\bar{m}_{k,s}}(e(k))$.

(Then $\left(q_{\bar{m}_{k,s}}(e(k))\right)_s = q_{\bar{m}_{k,s}}(e(k))$.) Let p_n have the same support as $q_{2^{2n}}$ and

$$p_n(\xi) = \begin{cases} \bigcup q_{\bar{m}_{k,s}}(e(k)) & : s \in n^{\text{th}} \text{ level of } p_n(e(k)), \\ & \quad \xi = e(k) \\ q_{2^{2n}}(\xi) & : \xi \neq e(k), \text{ all } k < n. \end{cases}$$

It is easy to see that this works. $\dashv$

<u>Lema 3.</u> Let g be $\mathbb{S}(\omega_1)$-generic over V.

If N is a null set of $V[g]$, then there is a G_δ null set $\bar{N}$ in V s.t. $N \subset \bar{N}^{V[g]}$, where $\bar{N}^{V[g]}$ is the version of $\bar{N}$ in $V[g]$.

If M is a meager set of $V[g]$, then there is an F_σ meager set $\bar{M}$ in V s.t. $M \subset \bar{M}^{V[g]}$, where $\bar{M}^{V[g]}$ is the version of $\bar{M}$ in $V[g]$.

Proof for "null": We may assume that $N = \bigcap \{ \mathcal{O}_n : n < \omega \}$, where each $\mathcal{O}_n$ is an open set, $\mu(\mathcal{O}_n) \le \frac{1}{n+1} \cdot \frac{1}{2^{2n}}$. Say each $\mathcal{O}_n$ is a (countable) union of open intervals with rational endpoints. We may then use Lemma 2 to "guess" those intervals by 2^{2n} ground model intervals of the same length. We leave the details to the reader. $\dashv$

<u>Corollary 4.</u> If g is $\mathbb{S}(\omega_1)$-generic over V, and if S, Λ are Sierpiński / Luzin sets of V, then S, Λ are also Sierpiński/Luzin sets of $V[g]$.

$B \subset \mathbb{R}$ is called a <u>Burstin basis</u> *) iff B is a basis for $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$, and $B \cap P \neq \emptyset$ for every perfect set P (equivalently, $B \cap D \neq \emptyset$ for every uncountable Borel set).

If B is Burstin, then also $P \setminus B \neq \emptyset$ for every perfect P. E.g., let P be perfect, $x, y, z \in B$, pairwise different. The shift $P + x + y + z = \{u + x + y + z : u \in P\}$ is perfect also, let $u \in B \cap (P + x + y + z)$. Then $u - x - y - z \in P \setminus B$. Hence every Burstin basis is automatically a Bernstein set.

It is easy to construct a Burstin set: Let $(P_i : i < 2^{\aleph_0})$ be an enumeration of all perfect sets and $(x_i : i < 2^{\aleph_0})$ be an enumeration of all reals. Construct sets $(b_i : i \leq 2^{\aleph_0})$ with $b_i \supset b_j$ for

*) Celestyn Burstin, Die Spaltung des Kontinuums in c im L. Sinne nichtmeßbare Mengen, Sitz. ber. K. Akad. Wiss., MNW Klasse 1916, pp. 1525 – 1551.

$i \geq j$ recursively. $b_0 = \emptyset$, $b_\lambda = \bigcup\{b_i : i < \lambda\}$ for λ limit. Given b_i, pick

$$x \in P_i \setminus \mathrm{span}(b_i),$$

and let

$$b_{i+1} = \begin{cases} b_i \cup \{x\} & \text{if } x_i \in \mathrm{span}(b_i \cup \{x\}) \\ b_i \cup \{x, x_i\} & \text{otherwise} \end{cases}$$

then $b = b_{2^{\aleph_0}}$ is a Burstin basis.

We define a forcing adding a generic Burstin basis:

Definition. $p \in \mathbb{P}_B$ iff there is some real x such that $p \in L[x]$ and $L[x] \models$ "p is a Burstin basis." $p \leq \bar{p}$ iff $p \supset \bar{p}$.

Lemma 5. Let $b \in L[x]$ be linearly independent, ~~again~~ $x \in \mathbb{R}$. Let $y \in \mathbb{R} \setminus L[x]$. There is then some $p \supset b$, $p \in L[x, y]$, $L[x, y] \models$ "p is a Burstin set."

Proof: We are going to make use of a highly non-trivial result of Groszek–Slaman *) which says that every perfect $P \in L[x,y]$ has a perfect subset $\bar{P} \subset P$, $\bar{P} \in L[x,y]$, such that $\bar{P} \subset L[x,y] \setminus L[x]$.

This immediately implies that if $P \in L[x,y]$ is perfect and $z \in L[x,y]$, then there is some perfect $\bar{P} \subset P$, $\bar{P} \in L[x,y]$, such that $\bar{P} \cap$ $(\mathbb{R} \cap L[x]) + z = \{ u+z : z \in \mathbb{R} \cap L[x] \} = \emptyset$: given P, let $\tilde{P} \subset P - z$ be perfect s.t. $\tilde{P} \subset L[x,y] \setminus L[x]$. Then $\bar{P} = \tilde{P} + z \subset P$ is perfect, and if $u \in \tilde{P}$ (equivalently, $u + z \in \tilde{P} + z$), then $u \notin L[x]$, so $u + z \notin (\mathbb{R} \cap L[x]) + z$.

A fusion argument then gives that if $P \in L[x,y]$ is perfect and $\{z_0, z_1, \dots\} \in L[x,y] \cap [\mathbb{R}]^{\aleph_0}$,

*) "A basis theorem for perfect sets", Bull. Symb. Logic $\underline{4}$ (2), 1998, pp. 204 — 209. See also my handwritten notes "Groszek and Slaman on Prikry's problem."

then there is some perfect $\bar{P} \subset P$, $\bar{P} \in L[x,y]$ s.t. $\bar{P} \cap \operatorname{span}((\mathbb{R} \cap L[x]) \cup \{z_0, z_1, \ldots\}) = \emptyset$.

We may assume of course that if $z \in \operatorname{span}((\mathbb{R} \cap L[x]) \cup \{z_0, \ldots\})$, then $z \in (\mathbb{R} \cap L[x]) + z_n$, some $n < \omega$.

~~We~~ Given P, we may use the previous observation to construct a sequence of perfect sets, $P = P_0 \supset P_1 \supset \ldots$ s.t. the n^{th} level of P_{n+1} (construed as a perfect tree) $=$ the n^{th} level of P_n, and $P_{n+1} \cap \cancel{\text{spa}}((\mathbb{R} \cap L[x]) + z_n) = \emptyset$.

Setting $\bar{P} = \bigcap \{P_n : n < \omega\}$, $\bar{P}$ is then perfect, and $\bar{P} \cap \operatorname{span}((\mathbb{R} \cap L[x]) \cup \{z_0, \ldots\}) = \emptyset$.

To show Lemma 5, let $(P_i : i < \omega_1)$ be a list of all perfect sets of $L[x,y]$. Let us work in $L[x,y]$ and recursively define $(b_i : i < \omega_1)$. Let $(y_i : i < \omega_1) \in L[x,y]$ enumerate the reals of $L[x,y]$.

given $(b_j : j<i)$, we will have that $\bar{b} = \bigcup\{b_j : j<i\}$ is at most countable. Let $\bar{P} \subset P_i$ be perfect such that $\bar{P} \cap \operatorname{span}((\mathbb{R} \cap L[x]) \cup \bar{b}) = \emptyset$, and pick $\bar{x} \in \bar{P}$. Let

$$b_i = \begin{cases} \bar{b} \cup \{\bar{x}\} & \text{if } y_i \in \operatorname{span}((\mathbb{R} \cap L[x]) \cup b) \\ \bar{b} \cup \{\bar{x}, y_i\} & \text{otherwise.} \end{cases}$$

Then if $c \in L[x]$, $L[x] \models$ "c is a Hamel basis," $c \supset b$, we get that

$$p = c \cup \bigcup\{b_i : i<\omega_1\}$$

is as desired. $\dashv$

Lemma 5 shows extendability: If $p \in \mathbb{P}_B$ and if y is a real not in $\operatorname{span}(p)$, then there is some $q \leq p$, q being a Burstin basis of $\mathbb{R} \cap L[x,y]$, where $L[x] \models$ "p is a Burstin basis."

Also, Lemma 5 shows that $\mathbb{P}_B$ is countably closed.

Notice that $p \in \mathbb{P}_B$ iff $\exists \vec{x} \in p^{<\omega} \exists \vec{q} \in Q^{<\omega}\, \varphi(\vec{x}, \vec{q}, p)$, where φ is Π^1_2.

Now let g be $\mathbb{S}(\omega_1)$-generic over L, and let b be $\mathbb{P}_B$-generic over $L[g]$. Let

$$N = L(\mathbb{R}, b)^{L[g, b]} .$$

As $\mathbb{P}_B$ is ω-closed, $\mathbb{R} \cap N = \mathbb{R} \cap L[g]$, so that $N \models$ "b is a Burstin basis." By Corollary 4, N has a Luzin as well as a Sierpiński set. Also, $N \models ZF + DC$.

<u>Lemma 6</u>. $N \models$ "There is no well-ordering of the reals."

Proof: Let us assume that

$(\ast)$ $L[g, b] \models$ "$\varphi(-, -, \vec{x}, \vec{\alpha}, b)$ defines a well-order of ${}^{\omega}2$," where $\vec{x} \in \mathbb{R} \cap L[g,b] = \mathbb{R} \cap L[g]$ and $\vec{\alpha} \in OR$. Say

$$b \ni p \Vdash^{\mathbb{P}_B}_{L[g]} \text{"}\varphi(-,-,\check{\vec{x}},\check{\vec{\alpha}},\dot{b}) \text{ defines a w.o. of } {}^{\omega}2\text{,"}$$

where $\dot{b}$ is the canonical name for the $\mathbb{P}_B$-generic Burstin base. As $\bar{p} \in \mathbb{P}_B$ iff $\exists \vec{y} \in \bar{p}^{<\omega} \exists \vec{q} \in \mathbb{Q}^{<\omega} \psi(\vec{y},\vec{q},\bar{p})$, where ψ is Π^1_2, and $\dot{b} = \{(\bar{p},\check{\bar{p}}) : \bar{p} \in \mathbb{P}_B\}$, we may think of $\dot{b}$ as being replaced by a Σ^1_3 formula with no parameters ~~[illegible]~~ in a way that $(\bar{p},\check{\bar{p}}) \in \dot{b}$ is absolute between transitive class sized models of set theory.

$\mathbb{S}(\omega_1)$ is proper (by an argument as for Lemma 2*)), so we may pick some $\xi < \omega_1$ with $p, \vec{x} \in L[g \restriction \xi]$. By homogeneity,

$$p \Vdash^{\mathbb{P}_B}_{L[g\restriction\xi][g\restriction[\xi,\omega_1)]} \text{"}\varphi(-,-,\check{\vec{x}},\check{\vec{\alpha}},\dot{b}) \text{ defines a w.o. of } {}^{\omega}2\text{"}$$

$$\underbrace{}_{\mathbb{S}(\omega_1)\text{-gen.}/L[g\restriction\xi]}$$

gives that

*) The argument in fact shows $\mathbb{S}(\omega_1)$ so axiom A. See p. 17.

$$\mathbb{1} \Vdash_{L[g\upharpoonright\xi]}^{\mathbb{S}(\omega_1)} \check{p} \Vdash_{L[g\upharpoonright\xi][\dot{g}]}^{\mathbb{P}_B} \text{“}\varphi(-,-,\check{\vec{x}},\check{\vec{\alpha}},\dot{b}) \text{ def. a w.o. of } {}^{\omega}2\text{”},$$

$\dot{g}$ name f. the $\mathbb{S}(\omega_1)$-gen.

where still "$\dot{b}$" is translated away via the above Σ^1_3 formula.

Let g^* be $\mathbb{S}(\omega_1)$-generic over $L[g]$ (so that $g\upharpoonright[\xi,\omega_1)$, g^* are mutually $\mathbb{S}(\omega_1)$-generic over $L[g\upharpoonright\xi]$), and let b^* be $\mathbb{P}_B$-generic over $L[g\upharpoonright\xi, g^*]$, $p \in b^*$.

We also have

$L[g\upharpoonright\xi, g^*][b^*] \models$ "$\varphi(-,-,\vec{x},\vec{\alpha}, b^*)$ defines a w.o. of ${}^{\omega}2$"

As $\mathbb{R} \cap L[g\upharpoonright\xi, g^*][b^*] = \mathbb{R} \cap L[g\upharpoonright\xi, g^*] \neq \mathbb{R} \cap L[g] = \mathbb{R} \cap L[g][b]$, there is then some β and some $n<\omega$ s.t., say,

$L[g,b] \models$ "(the β^{th} elt. of ${}^{\omega}2$ given by $\varphi(-,-,\vec{x},\vec{\alpha},\dot{b})$)$(n) = 0$," and

$L[g\upharpoonright\xi, g^*][b^*] \models$ "(the β^{th} elt. of ${}^{\omega}2$ given by $\varphi(-,-,\vec{x},\alpha,\dot{b})$)$(n) = 1$."

Let $p_0 \in b$, $p_0 \leq p$, and $p_1 \in b^*$, $p_1 \leq p$, be such that

$$p_0 \Vdash^{\mathbb{P}_B}_{L[g]} \text{"(the } \beta^{th} \text{ elt of } {}^{\omega}2 \text{ given by } \varphi(-,-,\vec{\check{x}},\vec{\check{\alpha}},\dot{b}))(\check{n}) = \check{0}\text{", and}$$

$$p_1 \Vdash^{\mathbb{P}_B}_{L[g\restriction\xi,\, g^*]} \text{"(the } \beta^{th} \text{ elt. of } {}^{\omega}2 \text{ given by } \varphi(-,-,\vec{\check{x}},\vec{\check{\alpha}},\dot{b}))(\check{n}) = \check{1}\text{."}$$

Pick $\eta \geq \xi$, $\eta < \omega_1$, s.t. $p_0 \in L[g\restriction\eta]$ and $p_1 \in L[g\restriction\xi,\, g^*\restriction\eta]$, say $\xi + \eta = \eta$. Then

$$\mathbb{1} \Vdash^{\mathbb{S}(\omega_1)}_{L[g\restriction\eta]} \check{p}_0 \Vdash^{\mathbb{P}_B}_{L[g\restriction\eta][\dot{g}]} \text{"(the } \beta^{th} \text{ elt. [...] given by } \varphi(-,-,\vec{\check{x}},\vec{\check{\alpha}},\dot{b}))(\check{n}) = \check{0}\text{",}$$

$$\mathbb{1} \Vdash^{\mathbb{S}(\omega_1)}_{L[g\restriction\xi,\, g^*\restriction\eta]} \check{p}_1 \Vdash^{\mathbb{P}_B}_{L[g\restriction\xi,\, g^*\restriction\eta][\dot{g}]} \text{"(——— '' ———)}(\check{n}) = \check{1}\text{."}$$

<u>Key Claim.</u> $p_0 \cup p_1$ is linearly independent.

Proof: We may assume w.l.o.g. that $L[g\restriction\xi] \models$ "p is a Burstin basis, in particular, a Hamel basis."

If $p_0 \cup p_1$ were dependent, we had $\vec{y} \in p$, $\vec{z} \in p_0 \setminus p$, $\vec{u} \in p_1 \setminus p$ and some rationals $\vec{q}_0, \vec{q}_1, \vec{q}_2$ s.t.

$$\Sigma\, \vec{q}_0\, \vec{y} + \Sigma\, \vec{q}_1\, \vec{z} + \Sigma\, \vec{q}_2\, \vec{u} = 0 .$$

But then $\Sigma \vec{q}_0 \vec{y} + \Sigma \vec{q}_1 \vec{z} = -\Sigma \vec{q}_2 \vec{u} \in L[g \restriction \eta] \cap L[g \restriction \xi, g^* \restriction \eta]$

$= L[g \restriction \xi]$ by mutual genericity, so

that $\vec{q}_2 = \vec{0} = \vec{q}_1$, as p is a Hamel basis for

the reals of $L[g \restriction \xi]$, and hence also $\vec{q}_0 = \vec{0}$. $\dashv$

We may construe $g \restriction [\eta, \omega_1) \frown g^*$ as $\mathbb{S}(\omega_1)$-generic over

$L[g \restriction \eta]$ and $g \restriction [\xi, \omega_1) \frown g^* \restriction [\eta, \omega_1)$ as $\mathbb{S}(\omega_1)$-generic

over $L[g \restriction \xi, g^* \restriction \eta]$. Then

$$p_0 \Vdash^{\mathbb{P}_B}_{L[g][g^*]} \text{“(the } \beta^{\text{th}} \text{ elt. of } {}^{\omega}2 \text{ given by } \varphi(-,-,\check{\vec{x}},\check{\vec{\alpha}},\check{b}))(\check{n}) = \check{0}\text{,” and}$$

$$p_1 \Vdash^{\mathbb{P}_B}_{L[g][g^*]} \text{“(} \quad \text{—”—} \quad)(\check{n}) = \check{1}\text{.”}$$

By the key Claim and by Lemma 5, there is

$q \le p_0, p_1$, $q \in (\mathbb{P}_B)^{L[g][g^*]}$. But then q

forces two contradictory statements. $\dashv$

One can also simultaneously force a Mazurkiewicz set

to exist.

The proof of Lemma 2 actually yields the following which readily implies the statement of Lemma 2 as well as the properness of $\mathbb{S}(\omega_1)$.

<u>Lemma 7.</u> Let $\mathbb{S}(\omega_1) \ni p$, $X \prec V_\theta$ countable (with $\theta \gg \omega_1$), $p \in X$, and let $(\tau_n : n<\omega)$ be a sequence of terms for ordinals, $\{\tau_n : n<\omega\} \subset X$ (possibly, but not necessarily, $(\tau_n : n<\omega) \in X$). There is then some $q \leq p$ and some $f \in {}^\omega([X \cap OR]^{<\omega}) \cap V$ s.t.

$$\forall n \qquad q \Vdash \tau_n \in f(\check{n}) \quad \text{and}$$

$$\overline{\overline{f(n)}} \leq 2^{2n} .$$

Proof: Almost literally the same as for Lemma 2, just replacing $\tau(\check{n})$ by τ_n. The proof of Lemma 2 did not make use of the fact that $(\tau(\check{n}) : n<\omega) \in X$. $\dashv$