WHEN IS A GIVEN REAL GENERIC OVER L?

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Abstract. In this paper we isolate a new criterion for when a given real x is generic over L in terms of x's capability of lifting elementary embeddings of initial segments of L.

1. Introduction

The results established in [2] show that the property of closure under sharps for reals is preserved by certain tree forcing notions such as Sacks, Silver, Mathias, Miller and Laver by proving that enough ground-model embeddings $j : L[x] \to L[x]$, $x \in {}^\omega\omega$, lift to the generic extension. Here we turn our attention to a related problem:

Question 1.1. *Suppose that $0^\#$ exists. Assume that $x \in {}^\omega\omega$ is such that every elementary embedding $j : L \to L$ lifts to $j^* : L[x] \to L[x]$. Is x set generic over L?*

In this paper we will characterize the reals $x \in V$ which are generic by set forcing over L by means of their capability to lift partial elementary embeddings of L (see theorem 3.2). In order to prove our result, we present a version of Woodin's extender algebra for (partial) extenders which exist in L, and also we introduce the notion of "weak Woodiness," which turns out not to be a large cardinal concept at all.

2. Woodin's extender algebra, modified, and Bukowský's Theorem

Definition 2.1. For a regular cardinal $\delta \geq \omega_1$ and an ordinal $\mu \leq \delta$ let $\mathscr{L}_{\mu,\delta}$ be the least infinitary language which has constants $\check{\xi}$, all $\xi < \mu$, as well as $\dot{a}$, and which has atomic formulae "$\check{\xi} \in \dot{a}$", $\xi < \mu$, and is closed under negation and disjunction of length $< \delta$, i.e.,

(i) if $\phi \in \mathscr{L}_{\mu,\delta}$, then $\neg\phi \in \mathscr{L}_{\mu,\delta}$, and
(ii) if $\theta < \delta$ and $\phi_\alpha \in \mathscr{L}_{\mu,\delta}$ for all $\alpha < \theta$, then $\bigvee\!\!\!\bigvee_{\alpha<\theta} \phi_\alpha \in \mathscr{L}_{\mu,\delta}$.

Each $x \subseteq \mu$, x not necessarily in V, defines a model for the logic $\mathscr{L}_{\mu,\delta}$. Given $\varphi \in \mathscr{L}_{\mu,\delta}$ we may define the meaning of $x \models \varphi$ recursively:

(1) $x \models$ "$\dot{\xi} \in \dot{x}$" if and only if $\xi \in x$,
(2) $x \models \neg\varphi$ if and only if $x \not\models \varphi$, and
(3) $x \models \bigvee\!\!\!\bigvee \Gamma$ if and only if $x \models \varphi$ for some $\varphi \in \Gamma$, where Γ is an enumeration of formulas in $\mathscr{L}_{\mu,\delta}$ of length $< \delta$.

In this setting, notice that the statement "$x \models \varphi$" is absolute between transitive models of ZFC containing x and φ.

Definition 2.2. Let $\mathbb{P} \in V$ be a forcing notion and suppose that g is $\mathbb{P}$-generic over V. For $\varphi \in \mathscr{L}_{\mu,\delta}$, let

$$A^g_\varphi = \{x \in \wp(\mu) \cap V[g] : x \models \varphi\}$$

Further, if $\Gamma \subseteq \mathscr{L}_{\mu,\delta}$ is a theory we set $A^g_\Gamma = \{x \in \wp(\mu) \cap V[g] :$ for all $\varphi \in \Gamma$, $x \models \varphi\}$.

Definition 2.3. Let $\mathbb{P} \in V$ be a forcing notion. We say that a theory Γ in $\mathscr{L}_{\mu,\delta}$ is *consistent* if and only if $A^g_\Gamma \neq \varnothing$ for some $\mathbb{P}$-generic filter g. If $\Gamma \cup \{\phi\} \subset \mathscr{L}_{\mu,\delta}$, then we write $\Gamma \vdash \varphi$ iff $\Gamma \cup \{\neg\phi\}$ is inconsistent.

Lemma 2.4. *For every theory $\Gamma \subseteq \mathscr{L}_{\mu,\delta}$ and every $\varphi \in \mathscr{L}_{\mu,\delta}$ the following are equivalent:*

(1) $\Gamma \vdash \varphi$

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(2) $A^g_{\Gamma\cup\{\varphi\}} = A^g_\Gamma$ *for any g which is $\mathbb{P}$-generic for a forcing notion $\mathbb{P}$ which makes $\delta^{<\delta}$ countable.*
(3) $A^g_{\Gamma\cup\{\varphi\}} = A^g_\Gamma$ *for any g which is $\mathrm{Coll}(\omega, \delta^{<\delta})$-generic over V.*

Proof. Easy. See [3, Lemma 2.2] or [4, Lemma 1.2]. □

Definition 2.5. Let $\delta \geq \omega_1$ be a regular cardinal such that $\delta = \delta^{<\delta}$ and let $\mu \leq \delta$. Let $\mathcal{E}$ be a collection of elementary embeddings $j : M \to N$ with critical point $\kappa = \kappa_j$ such that M, N are transitive, and $M \models \mathsf{ZFC}^-$, $|N| < \delta$, together with a map $\nu_\mathcal{E} : \mathcal{E} \to \mathsf{OR}$ satisfying $\kappa_j + 1 \leq \nu_\mathcal{E}(j) \leq j(\kappa_j)$ for each j. We associate to $\mathcal{E}$, in addition to the axioms and rules for the infinitary logic $\mathscr{L}_{\mu,\delta}$, a set of axioms $A_\mathcal{E}$ as follows:

$$A_\mathcal{E}: \quad \text{Whenever } j \in \mathcal{E},\ j : M \to N \text{ and } \vec{\phi} = (\phi_i : i < \kappa_j) \in M \text{ then}$$
$$\bigvee\!\!\!\bigvee j(\vec{\phi}) \restriction \nu_\mathcal{E}(j) \to \bigvee\!\!\!\bigvee \vec{\phi}.$$

Definition 2.6. Given $\mathcal{E}$ as above and $\phi, \psi \in \mathscr{L}_{\mu,\delta}$, we define

$$\phi \sim_\mathcal{E} \psi \iff A_\mathcal{E} \vdash \psi \leftrightarrow \phi,$$

and we let $[\phi] = \{\psi : \psi \sim_\mathcal{E} \phi\}$. We write $\mathbb{P}_\mathcal{E} = \{[\phi] : \phi \in \mathscr{L}_{\mu,\delta}, \phi \text{ consistent with } A_\mathcal{E}\}$. In $\mathbb{P}_\mathcal{E}$ we stipulate

$$[\phi] \leq [\psi] \iff A_\mathcal{E} \vdash \phi \to \psi.$$

Definition 2.7. We say that δ is *weakly Woodin* as being witnessed by $\mathcal{E}$ iff for all $A \subset \delta$ there is some $\kappa < \delta$ such that for all $\alpha < \delta$ there is some $j : M \to N$, $j \in \mathcal{E}$ with $\kappa_j = \kappa$, $A \cap \kappa \in M$ and $j(A \cap \kappa) \cap \alpha = A \cap \alpha$.

By taking hulls, it is easy to see that every regular cardinal $\delta \geq \omega_1$ is weakly Woodin, so "weak Woodinness" is not a large cardinal concept.

Lemma 2.8. *Let $\delta \geq \omega_1$ be regular. Let $\theta > \delta$, $p \in H_\delta$, and let $\mathcal{E}$ be the collection of all $j : M \to N$ such that M and N are transitive, $M \models \mathsf{ZFC}^-$, $|N| < \delta$ and $p \in H^M_{\kappa_j}$. Then $\mathcal{E}$ witness that δ is weakly Woodin.*

Proof. Let $A \subset \delta$, and let $X \prec H_{\delta^+}$ with $X \cap \delta \in \delta$, $p \in X$, $|X| < \delta$ and $A \in X$. Let $i : M \cong X$ where M is transitive. Write $\kappa = \kappa(i) = X \cap \delta < \delta$. Of course, $A \cap \kappa = i^{-1}(A)$.

Let $\alpha < \delta$ and let $Y = \mathrm{Hull}^{H_{\delta^+}}(X \cup \{\kappa\} \cup (\alpha+1)) \prec H_{\delta^+}$. Suppose $k : N \cong Y$, where N is transitive. Setting $j = k^{-1} \circ i$, $j : M \to N$ has critical point κ, and $j(A \cap \alpha) = k^{-1}(A)$ and $k^{-1}(A) \cap \alpha = A \cap \alpha$ as $k \restriction \alpha + 1 = \mathrm{id}$. So $j(A \cap \kappa) \cap \alpha = A \cap \alpha$. □

Lemma 2.9. *Suppose $\delta \geq \omega_1$ is a regular cardinal with $\delta^{<\delta} = \delta$. Let $\mathcal{E}$ witness that δ is weakly Woodin, and let $\nu : \mathcal{E} \to \mathsf{OR}$ be a map with $\kappa_j + 1 \leq \nu_\mathcal{E}(j) \leq j(\kappa_j)$ for each $j \in \mathcal{E}$. Then $\mathbb{P}_\mathcal{E}$ as being defined in 2.6 with the theory $A_\mathcal{E}$, has the δ-c.c.*

Proof. Let $\vec{\phi} = \langle \phi_i : i < \delta \rangle$ be such that $\langle [\phi_i] : i < \delta \rangle$ is an antichain in $\mathbb{P}_\mathcal{E}$. As $\delta^{<\delta} = \delta$ and every ϕ_i is a formula in $\mathcal{L}_{\mu,\delta}$, we may code $\vec{\phi}$ by a subset A of δ (so $\vec{\phi}$ may be identified with A). As δ is weakly Woodin, we may pick κ as in 2.7. We have $\kappa + 1 < \delta$, and we may pick $j : M \to N$ in $\mathcal{E}$ such that if $\kappa = \kappa_j$, then

$$j(\vec{\phi} \restriction \kappa)(\kappa) = \phi_\kappa$$

Since $\nu(j) \geq \kappa + 1$, the axioms of $A_\mathcal{E}$ tell us that:

$$A_\mathcal{E} \vdash \phi_\kappa \to \bigvee\!\!\!\bigvee j(\vec{\phi} \restriction \kappa) \restriction \nu(j) \to \bigvee\!\!\!\bigvee \vec{\phi} \restriction \kappa$$

Theferore, $\{[\phi_i] : i \leq \kappa\}$ is not an antichain, which leads to a contradiction. □

For $x \subset \mu$ such that $x \models A_\mathcal{E}$, let $G_x = \{[\phi] \in \mathbb{P}_\mathcal{E} : x \models \phi\}$. It is easy to see that $G_x \subset \mathbb{P}_\mathcal{E}$ is an ultrafilter. Now, given a $\mathbb{P}_\mathcal{E}$-generic filter G over V, notice that for $x := \{\xi < \mu : \text{"}\dot{\xi} \in \dot{x}\text{"} \in G\}$ we have $G_x = G$. In such case, we say that x is $\mathbb{P}_\mathcal{E}$-*generic* over V.

Lemma 2.10. *Let $\mathcal{E}$ witness that δ is weakly Woodin, where $\delta^{<\delta} = \delta$. Let $x \subset \mu$, x not necessarily in V but in a transitive outer model $V[x]$ of ZFC, and assume that $x \models A_\mathcal{E}$. Then x is $\mathbb{P}_\mathcal{E}$-generic over V.*

Proof. We show that $G_x = \{[\varphi] \in \mathbb{P}_{\mathcal{E}} : x \models \varphi\}$ is $\mathbb{P}_{\mathcal{E}}$-generic over V. Let $A = \{[\phi_i] : i < \theta\}$ be a maximal antichain of $\mathbb{P}_{\mathcal{E}}$ in V. By 2.9 we have that $\mathbb{P}_{\mathcal{E}}$ is δ-c.c. so $\theta < \delta$ and hence $\bigvee\!\!\!\bigvee_{i<\theta} \phi_i$ is in $\mathscr{L}_{\mu,\delta}$ and in fact $[\bigvee\!\!\!\bigvee_{i<\theta} \phi_i] \in \mathbb{P}_{\mathcal{E}}$. Since A is maximal, we have that $A_{\mathcal{E}} \vdash \bigvee\!\!\!\bigvee_{i<\theta} \phi_i$.

We claim that $x \models A_{\mathcal{E}}$ yields that

$$x \models \bigvee\!\!\!\bigvee_{i<\theta} \phi_i.$$

Suppose that $x \models A_{\mathcal{E}} \cup \{\neg \bigvee\!\!\!\bigvee_{i<\theta} \phi_i\}$. Then in $V[x]$ and hence in $V[x]^{\mathrm{Coll}(\omega,\delta)}$ there is some x' with $x' \models A_{\mathcal{E}} \cup \{\neg \bigvee\!\!\!\bigvee_{i<\theta} \phi_i\}$, and by Shoenfield absoluteness there will be an x' in $V^{\mathrm{Coll}(\omega,\delta)}$ with $x' \models A_{\mathcal{E}} \cup \{\neg \bigvee\!\!\!\bigvee_{i<\theta} \phi_i\}$. But this contradicts $A_{\mathcal{E}} \vdash \bigvee\!\!\!\bigvee_{i<\theta} \phi_i$.

Thus $x \models \phi_i$ for some $i < \theta$, so $G_x \cap A \neq \varnothing$ and hence G_x is $\mathbb{P}_{\mathcal{E}}$-generic. □

Lemma 2.11. *Let δ, p, $\mathcal{E}$ be as in the statement of lemma 2.8. Let $x \subset \mu$, x not necessarily in V, be such that for every $j : M \to N$ in $\mathcal{E}$, there is some elementary $\hat{j} : M[x] \to N[x]$ with $\hat{j} \supset j$ and $M[x] \models \mathsf{ZFC}^-$. Then x is $\mathbb{P}_{\mathcal{E}}$-generic over V.*

Proof. By the previous lemmas, it suffices to show that $x \models A_{\mathcal{E}}$. So let $j : M \to N$ in $\mathcal{E}$ and let $\vec{\phi} = (\phi_i : i < \kappa(j)) \in M$. Let us assume that $x \models \bigvee\!\!\!\bigvee j(\vec{\phi}) \restriction \nu_{\mathcal{E}}(j)$. Then

$$N[x] \models \text{“}x \models \bigvee\!\!\!\bigvee j(\vec{\phi}) \restriction \nu_{\mathcal{E}}(j)\text{”}$$

hence by elementarity of $\hat{j}$, $M[x] \models$ “$x \models \bigvee\!\!\!\bigvee \vec{\phi} \restriction \kappa$” and so $x \models \bigvee\!\!\!\bigvee \vec{\phi}$. □

The arguments given so far allow us to reprove a theorem of Bukowský's, see [1], which we state as follows:

Theorem 2.12. *Let $X \subset \mu$, X not necessarily in V but in a transitive outer model $V[X]$ of ZFC, and let δ be a regular uncountable cardinal. The following are equivalent:*

1. *There is some $\mathbb{P}$ such that $\mathbb{P}$ has the δ-c.c. and X is $\mathbb{P}$-generic over V.*
2. *There is some $\theta > \mu$ and some club C of $Y \prec \mathrm{H}(\theta)$ with $Y \cap \delta \in \delta$, $|Y| < \delta$ such that if $j : M \cong Y$, where M is transitive, then there is some elementary $\hat{j} : M[\bar{X}] \to \mathrm{H}(\theta)[X]$ for some $\bar{X}$ and $\mathrm{H}(\theta)[X] \models \mathsf{ZFC}^-$.*
3. *If $f : \theta \to \mathrm{Ord}$, some θ, $f \in V[x]$, then there is $g : \theta \to \wp(\mathrm{Ord})$, $g \in V$ such that $|g(\xi)| < \delta$ in V and $f(\xi) \in g(\xi)$ for all $\xi < \theta$.*

Proof. “(2)⇒(1)”: This is by the proof of Lemma 2.11.
“(3)⇒(2)”: Let $\theta > \mu$ be sufficiently big. Notice that for all X, $\mathrm{H}(\theta) \subset \mathrm{H}(\theta)^{V[X]} = \mathrm{H}(\theta)[X]$. Doing a suitable book-keeping, let $\tilde{h} : \omega \times (\mathrm{H}(\theta)^{V[G]})^{<\omega} \to \mathrm{H}(\theta)^{V[G]} \in V[G]$ be a Skolem function. Thus, for all $A \subset \mathrm{H}(\theta)$, $\tilde{h}\text{''}A \prec \mathrm{H}(\theta)^{V[G]}$.
Let us define $h : \theta \to \mathrm{Ord}$ by using $\tilde{h}$ as follows:

$$h(x) = \begin{cases} \tilde{h}(x) & \text{if } \tilde{h}(x) \in \mathrm{H}(\theta) \\ \varnothing & \text{otherwise} \end{cases}$$

By using (3), we may find some $g \in V$ such that for all $\xi < \theta$, $|g(\xi)| < \delta$ and $h(\xi) \in g(\xi)$. In particular, for every $A \subset \mathrm{H}(\theta)$ of size $< \delta$ in V, $g\text{''}A \cap \delta \in \delta$ and $h\text{''}A \subset g\text{''}A$.
Notice that if $\tilde{h}(x) \in V$, $x \in A$, then $\tilde{h}(x) = h(x) \in g(x) \subset A$, so $\tilde{h}\text{''}A \cap V \subset A$. On the other side, as $\tilde{h}$ is a Skolem function we have that $A \subset \tilde{h}\text{''}A \cap V$ for $A \in V$. Thus if $A \in V$ then $\tilde{h}\text{''}A \cap V = A$. This shows that for every $A \in V$, $A \subset \mathrm{H}(\theta)$, $A = g\text{''}A$ holds. Therefore, as $\tilde{h}\text{''}A \prec \mathrm{H}(\theta)^{V[G]}$ and $\tilde{h}\text{''}A \cap V = A$, we have that $A \prec \mathrm{H}(\theta)^V$.
According to this, let us define C as the collection of all g-closed $Y \prec \mathrm{H}(\theta)$ with $Y \cap \delta \in \delta$ and $|Y| < \delta$. By construction, we have that the collection C satisfy (2).
“(1)⇒(3)”: This is a standard argument. □

Question 2.13. *Suppose that $\mathcal{E} \subset V_\delta$ is a collection of extenders. Define $A_{\mathcal{E}}$ as in section 2, that is for each extender $E \in \mathcal{E}$ and for every sequence of formulae $\vec{\varphi} = (\varphi_i : i < \mathrm{crit}(E))$ we associate the axiom*

$$\bigvee\!\!\!\bigvee i_E(\vec{\varphi}) \restriction \nu(E) \to \bigvee\!\!\!\bigvee \vec{\varphi}$$

where $i_E : M \to Ult(M, E)$ and $\nu(E)$ is the strength of the extender. Assume that $\mathbb{P}_{\mathcal{E}}$ has the δ-c.c. Is δ Woodin?

3. A criterion for set-genericity over L

Proposition 3.1 (Kunen)**.** *Let $j : L \to L$ be an elementary embedding with critical point κ. Then, for all $\alpha < \kappa^{+L}$, $j \restriction L_\alpha \in L$.*

Proof. This is by the "ancient Kunen argument." Let $\alpha < \kappa^{+L}$ and pick $f : \kappa \to L_\alpha$ onto, $f \in L$. Note that for every $x \in L_\alpha$, $j(x) = y$ if and only if

$$\exists \xi < \kappa (x = f(\xi) \wedge y = j(f)(\xi))$$

Since $j(f) \in L$, for every $x \in L_\alpha$ we can compute $j(x)$ in L from f and $j(f)$. Therefore, $j \restriction L_\alpha \in L$ as required. □

Theorem 3.2. *The following are equivalent for a given real $x \in V$:*

(1) *x is set-generic over L*
(2) *There is some $p \in L$ such that for all elementary $j : L_\alpha \to L_\beta$ with critical point κ, where $j \in L$, $L_\alpha \models \mathsf{ZFC}^-$ and $p \in L_\kappa (\subsetneq L_\alpha)$, there is some $\hat{j} : L_\alpha[x] \to L_\beta[x]$ with $\hat{j} \supset j$ and $L_\alpha[x] \models \mathsf{ZFC}^-$.*

Proof. "$\Leftarrow$": Given p, let $\delta \geq \omega_1$ be regular with $p \in L_\delta$. Let $\mathcal{E}$ be defined in L as in the statement of lemma 2.8. Then if $\mathbb{P}_{\mathcal{E}}$ is defined as in definition 2.6 inside L, x is $\mathbb{P}_{\mathcal{E}}$-generic over L.
"$\Rightarrow$": Let $\mathbb{P} \in L$ be such that x is $\mathbb{P}$-generic over L. Write $\mu = \mathrm{Card}^L(\mathbb{P})$ and let $p = L_{\mu^+}$, where $\mu^+ = \mu^{+L}$. Let $j : L_\alpha \to L_\beta$ with $\mathrm{crit}(j) = \kappa$, $\mu^+ < \kappa$, $L_\alpha \models \mathsf{ZFC}^-$.
Without loss of generality, let us assume that $\mathbb{P} \in p \in L_\kappa$, so that x is $\mathbb{P}$-generic over L_α. This gives $L_\alpha[x] \models \mathsf{ZFC}^-$. The real x is also $\mathbb{P}$-generic over L_β, and by writing $X = \mathrm{rng}(j)$, $\mathbb{P} \in p \in X$. In order to see that there is $\hat{j} : L_\alpha[x] \to L_\beta[x]$ with $\hat{j} \supset j$ it suffices to verify that $X[x] \cap \beta = X \cap \beta$[1]. Let $\tau \in X \cap L^{\mathbb{P}}$ be a name for an ordinal. Note that $B = \{\gamma : \exists p \in \mathbb{P}\, (p \Vdash^{\mathbb{P}}_{L_\beta} \tau = \hat{\gamma})\} \in X$ and $\mathrm{otp}\,(B) < \mu^+$. Thus, the order isomorphism $\pi : \mathrm{otp}\,(B) \cong B$ is in X, and since $\mu^+ < \kappa \subset X$, we have $\mathrm{otp}\,(B) \cup \{\mathrm{otp}\,(B)\} \subseteq X$. Hence $B \subseteq X$ and, in particular, $\tau^x \in X$. □

The paramater p as in the statement (2) above is necessary. Suppose that $j : L_\alpha \to L_\beta$ is an elementary embedding with critical point κ, and let x be $\mathrm{Coll}(\omega, \kappa)$-generic over L. In this case, such an embedding cannot be lifted to a $\hat{j} : L_\alpha[x] \to L_\beta[x]$ because $\mathrm{crit}(j)^{L_\alpha[x]}$ is countable.

By Theorem 3.2, if $x \in \mathbb{R} \cap V$ is such that (2) of the statement of Theorem 3.2 holds true and $0^\#$ exists, then $x^\#$ exists.

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[1] We pretend x is the $\mathbb{P}$-generic filter.