Manuscript on fine structure, inner model theory, and the core model below one Woodin cardinal

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Preface

Here are the first five chapters of a prospective book. It is intended to provide a detailed introduction to fine structure theory, ultimately leading up to a proof of the Covering Lemma for the Core Model under the assumption that there is no inner model with a Woodin cardinal. This will be in chapter 6, which has still to be written.

I am grateful to various colleagues who have helped and encouraged me in this project. I especially want to thank Ralf Schindler, who has provided steadfast support at every stage. I am completely dependent on technical typing and, therefore, owe a special debt to Ms. Martina Pfeiffer, who typed the initial sections, and to Dr. Fabiana Castiblanco, Shervin Sorouri, and Taichi Yasuda who very competently continued this work.

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Chapter 0

Preliminaries

(1) Throughout the book we assume ZFC. We use "virtual classes", writing $\{x|\varphi(x)\}$ for the class of x such that $\varphi(x)$. We also write:

$$\{t(x_1,\ldots,x_n)|\varphi(x_1,\ldots,x_n)\},\ \text{(where e.g.}\\ t(x_1,\ldots,x_n)=\{y|\psi(y,x_1,\ldots,x_n)\})$$

for:

$$\{y|\bigvee x_1,\ldots,x_n(y=t(x_1,\ldots,x_n)\wedge\varphi(x_1,\ldots,x_n))\}$$

We also write

$$\mathbb{P}(A)=\{z|z\subset A\}, A\cup B=\{z|z\in A\vee z\in B\}\\ A\cap B=\{z|z\in A\wedge z\in B\}, \neg A=\{z|\notin A\}$$

(2) Our notation for ordered n–tuples is $\langle x_1,\ldots,x_n\rangle$. This can be defined in many ways and we don't specify a definition.

(3) An *n–ary relation* is a class of n–tuples. The following operations are defined for all classes, but are mainly relevant for binary relations:

$$\begin{aligned}&\mathrm{dom}(R)=:\{x|\bigvee y\langle y,x\rangle\in R\}\\ &\mathrm{rng}(R)=:\{y|\bigvee x\langle y,x\rangle\in R\}\\ &R\circ P=\{\langle y,x\rangle|\bigvee z|\langle y,z\rangle\in R\wedge\langle z,x\rangle\in P\}\\ &R\restriction A=\{\langle y,x\rangle|\langle y,x\rangle\in R\wedge x\in A\}\\ &R^{-1}=\{\langle y,x\rangle|\langle x,y\rangle\in R\}\end{aligned}$$

We write $R(x_1,\ldots,x_n)$ for $\langle x_1,\ldots,x_n\rangle\in R$.

(4) A *function* is identified with its *extension* or *field* — i.e. an n–ary function is an $n+1$–ary relation F such that

$$\bigwedge x_1\ldots x_n\bigwedge z\bigwedge w((F(z,x_1,\ldots,x_n)\wedge F(w,x_1,\ldots,x_n))\\ \rightarrow z=w)$$

$F(x_1, \ldots, x_n)$ then denotes the value of F at $x_1, \ldots, x_n$.

(5) "*Functional abstraction*" $\langle t_{x_1,\ldots,x_n} | \varphi(x_1, \ldots, x_n) \rangle$ denotes the function which is defined and takes value $t_{x_1,\ldots,x_n}$ whenever $\varphi(x_1, \ldots, x_n)$ and $t_{x_1,\ldots,x_n}$ is a set:

$$\langle t_{x_1,\ldots,x_n} | \varphi(x_1, \ldots, x_n) \rangle =: \\ \{\langle y, x_1, \ldots, x_n \rangle | y = t_{x_1,\ldots,x_n} \wedge \varphi(x_1, \ldots, x_n)\},$$

where e.g. $t_{x_1,\ldots,x_n} = \{z | \psi(z, x_1, \ldots, x_n)\}$.

(6) *Ordinal numbers* are defined in the usual way, each ordinal being identified with the set of its predecessors: $\alpha = \{\nu | \nu < \alpha\}$. The *natural numbers* are then the finite ordinals: $0 = \emptyset, 1 = \{0\}, \ldots, n = \{0, \ldots, n-1\}$. On is the class of all ordinals. We shall often employ small greek letters as variables for ordinals. (Hence e.g. $\{\alpha | \varphi(\alpha)\}$ means $\{x | x \in \mathrm{On} \wedge \varphi(x)\}$.) We set:

$$\sup A =: \bigcup(A \cap \mathrm{On}),\ \inf A =: \bigcap(A \cap \mathrm{On})$$
$$\mathrm{lub}\, A =: \sup\{\alpha + 1 | \alpha \in A\}.$$

(7) *A note on ordered n–tuples.* A frequently used definition of ordered pairs is:

$$\langle x, y \rangle =: \{\{x\}, \{x, y\}\}.$$

One can then define n–tuples by:

$$\langle x \rangle =: x,\ \langle x_1, x_2, \ldots, x_n \rangle =: \langle x_1, \langle x_1, \ldots, x_n \rangle \rangle.$$

However, this has the disadvantage that every $n+1$–tuple is also an n–tuple. If we want each tuple to have a fixed length, we could instead identify the n–tuples with *vector of length n* — i.e. functions with domain n. This would be circular, of course, since we must have a notion of ordered pair in order to define the notion of "function". Thus, if we take this course, we must first make a "preliminary definition" of ordered pairs — for instance:

$$(x, y) =: \{\{x\}, \{x, y\}\}$$

and then define:

$$\langle x_0, \ldots, x_{n-1} \rangle = \{(x_0, 0), \ldots, (x_{n-1}, n-1)\}.$$

If we wanted to form n–tuples of proper classes, we could instead identify $\langle A_0, \ldots, A_{n-1} \rangle$ with:

$$\{\langle x, i \rangle | (i = 0 \wedge x \in A_0) \vee \ldots \vee (i = n-1 \wedge x \in A_{n-1})\}.$$

(8) *Overhead arrow notation.* The symbol $\vec{x}$ is often used to donate a vector $\langle x_1, \ldots, x_n \rangle$. It is not surprising that this usage shades into what I shall call the *informal mode* of overhead arrow notation. In this mode $\vec{x}$ simply stands for a string of symbols $x_1, \ldots, x_n$. Thus we write $f(\vec{x})$ for $f(x_1, \ldots, x_n)$, which is different from $f(\langle x_1, \ldots, x_n \rangle)$. (In informal mode we would write the latter as $f(\langle \vec{x} \rangle)$.) Similarly, $\vec{x} \in A$ means that each of $x_1, \ldots, x_n$ is an element of A, which is different from $\langle \vec{x} \rangle \in A$. We can, of course, combine several arrows in the same expression. For instance we can write $f(\vec{g}(\vec{x}))$ for $f(g_1(x_1, \ldots, x_n), \ldots, g_m(x_1, \ldots, x_n))$. Similarly we can write $f(\overrightarrow{g(\vec{x})})$ or $f(\vec{g}(\vec{x}))$ for

$$f(g_1(x_{1,1}, \ldots, x_{1,p_1}), \ldots, g_m(x_{m,1}, \ldots, x_{m,p_m})).$$

The precise meaning must be taken from the context. We shall often have recourse to such abbreviations. To avoid confusion, therefore, we shall use overhead arrow notation *only* in the informal mode.

(9) A *model* or *structure* will for us normally mean an $n+1$–tuple $\langle D, A_1, \ldots, A_n \rangle$ consisting of a domain D of individuals, followed by relations on that domain. If φ is a first order formula, we call a sequence $v_1, \ldots, v_n$ of distinct variables *good for* φ iff every free variable of φ occurs in the sequence. If M is a model, φ a formula, $v_1, \ldots, v_n$ a good sequence for φ and $x_1, \ldots, x_n \in M$, we write: $M \models \varphi(v_1, \ldots, v_n)[x_1, \ldots, x_n]$ to mean that φ becomes true in M if v_i is interpreted by x_i for $i = 1, \ldots, n$. This is the *satisfaction relation.* We assume that the reader knows how to define it. As usual, we often suppress the list of variables, writing only $M \models \varphi[x_1, \ldots, x_n]$. We may sometimes indicate the variables being used by writing e.g. $\varphi = \varphi(v_1, \ldots, v_n)$.

(10) $\in$*–models.* $M = \langle D, E, A_1, \ldots, A_n \rangle$ is an $\in$*–model* iff E is the restriction of the $\in$–relation to D^2. Most of the models we consider will be $\in$–models. We then write $\langle D, \in, A_1, \ldots, A_n \rangle$ or even $\langle D, A_1, \ldots, A_n \rangle$ for $\langle D, \in \cap D^2, A_1, \ldots, A_n \rangle$. M is *transitive* iff it is an $\in$–model and D is transitive.

(11) *The Levy hierarchy.* We often write $\bigwedge x \in y \varphi$ for $\bigwedge x (x \in y \to \varphi)$, and $\bigvee x \in y \varphi$ for $\bigvee x (x \in y \wedge \varphi)$. Azriel Levy defined a hierarchy of formulae as follows:

A formula is Σ_0 (or Π_0) iff it is in the smallest class Σ of formulae such that every primitive formula is in Σ and $\bigwedge v \in u \varphi$, $\bigvee v \in u \varphi$ are in Σ whenever φ is in Σ and v, u are distinct variables.

(Alternatively, we could introduce $\bigwedge v \in u$, $\bigvee v \in u$ as part of the primitive notation. We could then define a formula as being Σ_0 iff it contains no unbounded quantifiers.)

The Σ_{n+1} formulae are then the formulae of the form $\bigvee v\varphi$, where φ is Π_n. The Π_{n+1} formulae are the formulae of the form $\bigwedge v\varphi$ when φ is Σ_n.

If M is a transitive model, we let $\Sigma_n(M)$ denote the set of relations on M which are definable by a Σ_n formula. Similarly for $\Pi_n(M)$. We say that a relation R is $\Sigma_n(M)(\Pi_n(M))$ *in parameters* $p_1, \ldots, p_m$ iff

$$R(x_1, \ldots, x_n) \leftrightarrow R'(x_1, \ldots, x_n, p_1, \ldots, p_m)$$

and R' is $\Sigma_n(M)(\Pi_n(M))$. $\underline{\Sigma}_1(M)$ then denotes the set of relations which are $\Sigma_1(M)$ in some parameters. Similarly for $\underline{\Pi}_1(M)$.

(12) *Kleene's equation sign.* An equation '$L \simeq R$' means: 'The left side is defined if and only if everything on the right side is defined, in which case the sides are equal'. This is of course not a strict definition and must be interpreted from case to case.

$F(\vec{x}) \simeq G(H_1(\vec{x}), \ldots, H_n(\vec{x}))$ obviously means that the function F is defined at $\langle x_1, \ldots, x_n\rangle$ iff each of the H_i is defined at $\langle\vec{x}\rangle$ and G is defined at $\langle H_1(\vec{x}), \ldots, H_n(\vec{x})\rangle$, in which case equality holds.

The recursion schema of set theory says that, given a function G, there is a function F with:

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).$$

This says that F is defined at $\langle y, \vec{x}\rangle$ iff F is defined at $\langle z, \vec{x}\rangle$ for all $z \in y$ and G is defined at $\langle y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle\rangle$, in which case equality holds.

(13) By the recursion theorem we can define:

$$TC(x) = x \cup \bigcup_{z \in x} TC(z)$$

(the transitive closure of x)

$$\mathrm{rn}(x) = \mathrm{lub}\{\mathrm{rn}(z) | z \in x\}$$

(the rank of x).

(14) By a *normal ultrafilter on* κ we mean an ultrafilter U on $\mathbb{P}(\kappa)$ with the property that whenever $f : \kappa \to \kappa$ is regressive modulo U (i.e. $\{\nu | f(\nu) < \nu\} \in U$), then there is $\alpha < \kappa$ such that $\{\nu | f(\nu) < \nu\} \in U$. Each normal ultrafilter determines an elementary embedding π of V into an inner model W. Letting

$$D = \text{ the class of functions } f \text{ with domain } \kappa,$$

we can characterize the pair $\langle W, \pi\rangle$ uniquely by the conditions:

- $\pi : V \prec W$ and $\operatorname{crit}(\pi) = \kappa$
- $W = \{\pi(f)(\kappa) \mid f \in D\}$
- $\pi(f)(\kappa) \in \pi(g)(\kappa) \leftrightarrow \{\nu | f(\nu) \in g(\nu)\} \in U$.

U can then be recovered from π by:

$$U = \{x \subset \kappa | \kappa \in \pi(x)\}.$$

We shall call $\langle W, \pi \rangle$ the *extension of V by U*. W can be defined from U by the well known *ultrapower construction*: We first define a "term model" $\mathbb{D} = \langle D, \cong, \tilde{\in} \rangle$ by:

$$f \cong g \leftrightarrow: \{\nu | f(\nu) = g(\nu)\} \in U$$
$$f \tilde{\in} g \leftrightarrow: \{\nu | f(\nu) = g(\nu)\} \in U.$$

$\mathbb{D}$ is an *equality model* in the sense that $\cong$ is not the identity relation but rather a congruence relation for $\mathbb{D}$. We can then factor $\mathbb{D}$ by $\cong$, getting an identity model $\mathbb{D} \backslash \cong$, whose are the equivalence classes:

$$[x] = \{y | y \cong x\}$$

$\mathbb{D} \backslash \cong$ turns out to be isomorphic to an inner model W. If σ is the isomorphism, we can define π by:

$$\pi(x) = \sigma([\text{const}_x])$$

where const_x is the constant function x defined on κ. W is then called the *ultrapower of V by U*. π is called the *canonical embedding*.

(15) *(Extenders)* The normal ultrafilter is one way of coding an embedding of V into an inner model by a set. However, many embeddings cannot be so coded, since $\pi(\kappa) \leq 2^\kappa$ whenever $\langle W, \pi \rangle$ is the extension by U. If we wish to surmount this restriction, we can use *extenders* in place of ultrafilters. (The extenders we shall deal with are also known as "short extenders".)

An extender F at κ maps $\bigcup\limits_{n<\omega} \mathbb{P}(u^n)$ into $\bigcup\limits_{n<\omega} \mathbb{P}(\lambda^n)$ for a $\lambda > u$.

It engenders an embedding π of V into an inner model W characterized by:

- $\pi : V \prec W, \operatorname{crit}(\pi) = \wr$
- Every element of W has the form $\pi(f)(\vec{\alpha})$ where $\alpha_1, \ldots, \alpha_n < \lambda$ and f is a function with domain κ^n
- $\pi(f)(\vec{\alpha}) \in \pi(g)(\vec{\alpha}) \leftrightarrow \langle \vec{\alpha} \rangle \in \pi(\{\langle \vec{\xi} \rangle | f(\vec{\xi}) \in g(\vec{\xi})\})$

F is then recoverable from $\langle W, \pi \rangle$ by:

$$F(X) = \pi(X) \cap \lambda^n \text{ for } X \subset \kappa^n.$$

The concept "F is an extender" can be defined in ZFC, but we defer that to Chapter 3. If $\langle W, \pi \rangle$ is as above, we call it the *extension of V by F*. We also call W the *ultrapower of V by F* and π the *canonical* embedding. $\langle W, \pi \rangle$ can be obtained from F by a "term model" construction analogous to that described above.

(16) *(Large Cardinals)*

Definition 0.0.1. We call a cardinal κ *strong* iff for all $\beta > \kappa$ there is an extender F such that if $\langle W, \pi \rangle$ is the extension of V by F, then $V_\beta \subset W$.

Definition 0.0.2. Let A be any class. κ is *A–strong* iff for all $\beta > \kappa$ there is F such that letting $\langle W, \pi \rangle$ be the extension of V by F, we have:

$$A \cap V_\beta = \pi(A) \cap V_\beta.$$

These concepts can of course be relativized to V_τ in place of V when τ is strongly inaccessible. We then say that κ is strong (or A–strong) *up to τ*.)

Definition 0.0.3. τ is *Woodin* iff τ is strongly inaccessible and for every $A \subset V_\tau$ there is $\kappa < \tau$ which is A-strong up to τ.

(17) *(Embeddings)*

Definition 0.0.4. Let M, M' be $\in$–structures and let π be a structure preserving embedding of M into M'. We say that π is *Σ_n–preserving* (in symbols: $\pi : M \to_{\Sigma_n} M'$) iff for all Σ_n formulae we have:

$$M \models \varphi[a_1, \ldots, a_n] \leftrightarrow M' \models \varphi[\pi(a_1), \ldots, (a_n)]$$

for $a_1, \ldots, a_n \in M$. It is *elementary* (in symbols: $\pi : M \prec M'$ or $\pi : M \to_{\Sigma_\omega} M'$) iff the above holds for *all* formulae φ of the M-language. It is easily seen that π is elementary iff it is Σ_n–preserving for all $n < \omega$.

We say that π is *cofinal* iff $M' = \bigcup_{u \in M} \pi(u)$.

We note the following facts, which we shall occasionally use:

Fact 1 Let $\pi : M \to_{\Sigma_0} M'$ cofinally. Then π is Σ_1–preserving.

Fact 2 Let $\pi : M \to_{\Sigma_0} M'$ cofinally, where M is a ZFC^- model. Then M' is a ZFC^- model and π is elementary.

Fact 3 Let $\pi : M \to_{\Sigma_0} M'$ cofinally where M' is a ZFC^- model. Then M is a ZFC^- model and π is elementary.

We call an ordinal κ the *critical point* of an embedding $\pi : M \to M'$ (in symbols: $\kappa = \mathrm{crit}(\pi)$) iff $\pi \restriction \kappa = \mathrm{id}$ and $\pi(\kappa) > \kappa$.

Chapter 1

Transfinite Recursion Theory

1.1 Admissibility

Some fifty years ago Kripke and Platek brought out about a wide ranging generalization of recursion theory — which dealt with "effective" functions and relations on ω — to transfinite domains. This, in turn, gave the impetus for the development of fine structure theory, which became a basic tool of inner model theory. We therefore begin with a discussion of Kripke and Platek's work, in which ω is replaced by an arbitrary "admissible" structure.

1.1.1 Introduction

Ordinary recursion theory on ω can be developed in three different ways. We can take the notion of *algorithm* as basic, defining a recursive function on ω to be one given by an algorithm. Since, however, we have no definition for the general notion of algorithm, this approach involves defining a special class of algorithms and then convincing ourselves that "Church's thesis" holds — i.e. that every function generated by an algorithm is, in fact, generated by one which lies in our class. Alternatively we can take the notion of *calculus* as basic, defining an n–ary relation R on ω to be recursively enumerable (r.e.) if for some calculus involving statements of the form "$R(i_1, \ldots, i_n)$" $(i_1, \ldots, i_n < \omega)$, R is the set of tuples $\langle i_1, \ldots, i_n \rangle$ such that "$R(i_1, \ldots, i_n)$" is provable. R is then recursive if both it and its complement are r.e. A function defined on ω is recursive if it is recursive as a relation. But again, since we have no general definition of calculus, this involves specifying a special class of calculi and appealing to the appropriate form of Church's thesis.

A third alternative is to base the theory on *definability*, taking the r.e. relation as those which are definable in elementary number theory by one of a certain class of formulae. This approach has often been applied, but characterizing the class of defining formula tends to be a bit unnatural. The situation changes radically, however, if we replace ω by the set $H = H_\omega$ of heredetarily finite sets. We consider definability over the structure $\langle H, \in\rangle$, employing the familiar Levy hierarchy of set theoretic formulae:

$\Pi_0 = \Sigma_0 =:$ formulae in which all quantifiers are bounded

$\Sigma_{n+1} =:$ formulae $\bigvee x\varphi$ where φ is Π_n

$\Pi_{n+1} =:$ formulae $\bigwedge x\varphi$ where φ is Σ_n.

We then call a relation on H r.e. (or H–r.e.) iff it is definable by a Σ_1 formula. Recalling that $\omega \subset H$ it then turns out that a relation on ω is H–r.e. iff it is r.e. in the classical sense. Moreover, there is an H–recursive map $\pi : H \leftrightarrow \omega$ such that $A \subset H$ is H–r.e. iff $\pi''A$ is r.e. in the classical sense.

This suggests a very natural way of relativizing recursion theory to transfinite domains. Let $N = \langle |N|, \in, A_1, \ldots, A_n\rangle$ be any transitive structure. We first define:

Definition 1.1.1. A relation on N is $\Sigma_n(N)$ (in the parameters $p_1, \ldots, p_n \in N$) iff it is N–definable (in $\vec{p}$) by a Σ_n formula. It is $\Delta_n(N)$ (in $\vec{p}$) if both it and its completement are $\Sigma_n(N)$ (in $\vec{p}$). It is $\underline{\Sigma}_n(N)$ iff it is $\Sigma_n(N)$ in some parameters. Similarly for $\underline{\Delta}_n(N)$.

Following our above example of $N = \langle H, \in\rangle$, it is natural to define a relation on N as being *N–r.e.* iff it is $\underline{\Sigma}_1(N)$, and *N–recursive* iff it is $\underline{\Delta}_1(N)$. A partial function F on N is *N–r.e.* iff it is N–r.e. as a relation. F is *N–recursive* as a function iff it is N–r.e. and $\mathrm{dom}(F)$ is $\underline{\Delta}_1(N)$.

(Note that $\underline{\Sigma}_1(\langle H, \in\rangle) = \Sigma_1(\langle H, \in\rangle)$, which will not hold for arbitrary N.)

However, this will only work for an N satisfying rather strict conditions since, when we move to transfinite structures N, we must relativize not only the concepts “recursive” and “r.e.”, but also the concept “finite”. In the theory of H the finite sets were simply the elements of H.

Correspondingly we define:

u is *N–finite* iff $u \in N$.

But there are certain basic properties which we expect any recursion theory to have. In particular:

- If A is recursive and u is finite, then $A \cap u$ is finite.
- If u is finite and $F : u \to N$ is recursive, then $F''u$ is finite.

Those transitive structures $N = \langle |N|, \in A_1, \ldots, A_n \rangle$ which yield a satisfactory recursion theory are called *admissible*. An ordinal α is then called *admissible* iff L_α is admissible. The admissible structures were characterized by Kripke and Platek as those transfinite structures which satisfy the following axioms:

(1) $\emptyset, \{x, y\}, \bigcup x$ are sets

(2) *The Σ_0 axiom of subsets:*

$$x \cap \{z | \varphi(z)\} \text{ is a set}$$

(where φ is any Σ_0–formula)

(3) *The Σ_0 axiom of collection:*

$$\bigwedge x \in u \bigvee y\ \varphi(x, y) \to \bigvee v \bigwedge x \in u \bigvee y \in v\ \varphi(x, y),$$

(where φ is any Σ_0–formula).

Note. *Kripke–Platek set theory* (KP) consists of the above axioms together with the axiom of extensionality and the full axiom of foundation (i.e. for all formulae, not just the Σ_0 ones). This axiom can be stated as:

$$\bigwedge y (\bigwedge x \in y \varphi(x) \longrightarrow \varphi(y)) \longrightarrow \bigwedge y \varphi(y)$$

and is also known as the axiom of *induction*.

Note. Although the definability approach is the one most often employed in transfinite recursion theory, the approaches via algorithms and calculi have also been used to define the class of admissible ordinals.

1.1.2 Properties of admissible structures

We now show that admissible structures satisfy the two criteria stated above. In the following let $M = \langle |M|, \in A_a, \ldots, A_n \rangle$ be admissible.

Lemma 1.1.1. *Let $u \in M$. Let A be $\underline{\Delta}_1(M)$. Then $A \cap u \in M$.*

Proof: Let $Ax \leftrightarrow \bigvee y A_0 yx; \neg Ax \leftrightarrow \bigvee y A_1 yx$, where A_0, A_1 are $\underline{\Sigma}_0(M)$. Then $\bigwedge x \in u \bigvee y (A_0 yx \vee A_1 yx)$. Hence there is $v \in M$ such that $\bigwedge x \in u \bigvee y \in v (A_0 yx \vee A_1 yx)$. QED

Before verifying the second criterion we prove:

Lemma 1.1.2. *M satisfies:*

$$\bigwedge x \in u \bigvee y_1 \ldots y_n \varphi(x, \vec{y}) \to \bigvee u \bigwedge x \in u \bigvee y_1 \ldots y_n \in v \varphi(x, \vec{y})$$

for Σ_0–formulae φ.

Proof. Assume $\bigwedge x \in u \bigvee y_1 \ldots y_n \varphi(x, \vec{y})$. Then

$$\bigwedge x \in u \bigvee w \underbrace{\bigvee y_1 \ldots y_n \in w \varphi(x, \vec{y})}_{\Sigma_0}.$$

Hence there is $v' \in M$ such that $\bigwedge x \in u \bigvee w \in v' \bigvee y_1 \ldots y_n \in w \varphi(x, \vec{y})$. Take $v = \bigcup v'$. QED (Lemma 1.1.2)

We now verify the second criterion:

Lemma 1.1.3. *Let $u \in M, u \subset \mathrm{dom}(F)$, where F is a $\underline{\Sigma}_1(M)$ function. Then $F''u \in M$.*

Proof. Let $y = F(x) \leftrightarrow \bigvee z F' zyx$, where F' is a $\underline{\Sigma}_0(M)$ relation. Then $\bigwedge x \in u \bigvee z, y F' zyx$. Hence there is $v \in M$ such that $\bigwedge x \in u \bigvee z, y \in v F' zyx$. Hence $F''u = v \cap \{y | \bigvee x \in u \bigvee z \in v F' zxy\}$. QED (Lemma 1.1.3)

Assuming the admissibility of M, we immediately get from Lemma 1.1.2:

Lemma 1.1.4. *Let $\varphi(y, \vec{x})$ be a Σ_1–formula. Then $\bigvee y \varphi(y, \vec{x})$ is uniformly Σ_1 in M.*

Note. "Uniformly" is a word which recursion theorists like to use. Here it means that $M \models \bigvee y \varphi(y, \vec{x}) \leftrightarrow \Psi(\vec{x})$ for a Σ_1 formula Ψ which depends only on φ and not on the choice of M.

Lemma 1.1.5. *Let $\varphi(y, \vec{x})$ be Σ_1. Then $\bigwedge y \in u \varphi(y, \vec{x})$ is uniformly Σ_1 in M.*

Proof. Let $\varphi(y, \vec{x}) = \bigvee z \varphi'(z, y, x)$, where φ' is Σ_0. Then

$$\bigwedge y \in u \varphi(y, \vec{x}) \leftrightarrow \bigvee v \underbrace{\bigwedge y \in u \bigvee z \in v \varphi'(z, y, x)}_{\Sigma_0}$$

in M. QED (Lemma 1.1.5)

Lemma 1.1.6. *Let $\varphi_0(\vec{x}), \varphi_1(\vec{x})$ be Σ_1. Then $(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x})), (\varphi_0(\vec{x}) \vee \varphi_1(\vec{x}))$ are uniformly Σ_1 in M.*

Proof. Let $\varphi_i(\vec{x}) = \bigvee y_i \varphi'_i(y_i, \vec{x})$ where without loss of generality $y_0 \neq y_1$. Then

$$(\varphi_0(\vec{x}) \wedge \varphi_1(\vec{x})) \leftrightarrow \bigvee y_0 \bigvee y_1 (\varphi'_0(y_0, x) \wedge \varphi'_1(y_1, x)).$$

Similarly for $\vee$. QED (Lemma 1.1.6)

Putting this together:

Lemma 1.1.7. *Let $\varphi_1, \ldots, \varphi_n$ be Σ_1–formulae. Let Ψ be formed from $\varphi_1, \ldots, \varphi_n$ using only conjunction, disjunction, existence quantification and bounded universal quantification. Then $\Psi(x_1, \ldots, x_m)$ is uniformly $\Sigma_1(M)$*

An immediate consequence of Lemma 1.1.7 is:

Lemma 1.1.8. *$R \subset M^n$ is $\Sigma_1(M)$ in the parameter $\emptyset$ iff it is $\Sigma_1(M)$ in no parameter.*

Proof. Let $R(\vec{x}) \leftrightarrow R'(\emptyset, \vec{x})$. Then

$$R(\vec{x}) \leftrightarrow \bigvee z (R'(z, \vec{x}) \wedge \bigwedge y \in z y \neq y).$$

QED (Lemma 1.1.8)

Note. R is in fact *uniformly* $\Sigma_1(M)$ in the sense that its Σ_1 definition depends only on the original Σ_1 definition of R from $\emptyset$, and not on M.

Lemma 1.1.9. *Let $R(y_1, \ldots, y_n)$ be a relation which is $\Sigma_1(M)$ in the the parameter p. For $i = 1, \ldots, n$ let $f_i(x_1, \ldots, x_m)$ be a partial function on M which (as a relation) is $\Sigma_1(M)$ in p. Then the following relation is uniformly $\Sigma_1(M)$ in p:*

$$R(f_1(\vec{x}), \ldots, f_n(\vec{x})) \leftrightarrow: \bigvee y_1 \ldots y_n (\bigwedge_{i=1}^{n} y_i = f_i(\vec{x}) \wedge R(\vec{y})).$$

This follows by Lemma 1.1.7. (“Uniformly” again means that the Σ_1 definition depends only on the Σ_1 definition of $R, f_1, \ldots, f_n$.)

Similarly:

Lemma 1.1.10. *Let $f(y_1, \ldots, y_n), g_i(x_1, \ldots, x_m)(i = 1, \ldots, n)$ be partial functions which are $\Sigma_1(M)$ in p, then the function $h(\vec{x}) \simeq f(g(\vec{x}))$ is uniformly $\Sigma_1(M)$ in p.*

Proof.

$$z = h(\vec{x}) \leftrightarrow \bigvee y_1 \ldots y_n (\bigwedge_{i=1}^{n} y_i = g_i(\vec{x}) \wedge z = f(\vec{y})).$$

QED (Lemma 1.1.10)

Lemma 1.1.11. *Let $f_i(\vec{x})$ be a function which is $\Sigma_1(M)$ in $p(i = 1, \dots, n)$. Let $R_i(\vec{x})(i = 1, \dots, n)$ be mutually exclusive relations which are $\Sigma_1(M)$ in p. Then the function*

$$f(\vec{x}) \simeq f_i(\vec{x}) \text{ if } R_i(\vec{x})$$

is uniformly $\Sigma_1(M)$ in p.

Proof.

$$y = f(\vec{x}) \leftrightarrow \bigvee_{i=1}^{n} (y = f_i(\vec{x}) \wedge R_i(\vec{x})).$$

QED (Lemma 1.1.11)

Using these facts, we see that the restrictions of many standard set theoretic functions to M are $\Sigma_1(M)$.

Lemma 1.1.12. *The following functions are uniformly $\Sigma_1(M)$:*

(a) $f(x) = x, f(x) = \cup x, f(x, y) = x \cup y, f(x, y) = x \cap y, f(x, y) = x \setminus y$ (set difference)

(b) $f(x) = C_n(x)$, where $C_0(x) = x, C_{n+1}(x) = C_n(x) \cup \bigcup C_n(x)$

(c) $f(x_1, \dots, x_n) = \{x_1, \dots, x_n\}$

(d) $f(x) = i$ (where $i < \omega$)

(e) $f(x_1, \dots, x_n) = \langle x_1, \dots, x_n \rangle$

(f) $f(x) = \mathrm{dom}(x)$, $f(x) = \mathrm{rng}(x)$, $f(x, y) = x''y, f(x, y) = x \upharpoonright y$, $f(x) = x^{-1}$

(g) $f(x_1, \dots, x_n) = x_1 \times x_2 \times \dots \times x_n$

(h) $f(x) = (x)_i^n$ where $(\langle z_0, \dots, z_{n-1} \rangle)_i^n = z_i$ and $(u)_i^n = \emptyset$ in all other cases

(i) $f(x, z) = x[z] = \begin{cases} x(z) \text{ if } x \text{ is a function} \\ \text{and } z \in \mathrm{dom}(x) \\ \emptyset \text{ otherwise.} \end{cases}$

Proof. We display sample proofs. (a) is straightforward. (b) follows by induction on n. To see (c), $y = \{x_1, \dots, x_n\}$ can be expressed by the Σ_0–statement

$$x_1, \dots, x_n \in y \wedge \bigwedge z \in y(z = x_1 \vee \dots \vee z = x_n).$$

(d) follows by induction on i, since

$$0 = \emptyset, i + 1 = i \cup \{i\}.$$

The proof of (e) depends on the precise definition of $\langle x_1, \ldots x_n \rangle$. If we want each tuple to have a unique length, then the following definition recommends itself: First define a notion of ordered pair by: $(x, y) =: \{\{x\}, \{x, y\}\}$ Then (x, y) is a Σ_1 function. Then if $\langle x_1, \ldots, x_n \rangle =: \{(x_1, 0), \ldots, (x_n, n-1)\}$, the conclusion is immediate.

For (f) we display the proof that $\mathrm{dom}(x)$ is a Σ_1 function. Note that $x, y \in C_n(\langle x, y \rangle)$ for a sufficient n. But since every element of $\mathrm{dom}(x)$ is a component of a pair lying in x, it follows that $\mathrm{dom}(x) \subset C_n(x)$ for a sufficient n. Hence $y = \mathrm{dom}(x)$ can be expressed as:

$$\bigwedge z \in y \bigvee w \langle w, z \rangle \in x \wedge \bigwedge z, w \in C_n(x)(\langle w, z \rangle \in x \to z \in y).$$

To see (g), note that $y = x_1 \times \ldots \times x_n$ can be expressed by:

$$\begin{array}{l} \bigwedge z_1 \in x_1 \ldots \bigwedge z_n \in x_n \langle z_1, \ldots, z_n \rangle \in y \\ \wedge \bigwedge w \in y \bigvee z_1 \in x_1 \ldots \bigvee z_n \in x_n w = \langle z_1, \ldots, z_n \rangle. \end{array}$$

To see (h) note that, for sufficiently large $m, y = (x)_i^n$ can be expressed by:

$$\begin{array}{l} \bigvee z_0 \ldots z_{n-1}(x = \langle z_0, \ldots, z_{n-1} \rangle \wedge y = z_i) \\ \vee(y = \emptyset \wedge \bigwedge z_0 \ldots z_{n-1} \in C_m(x) x \neq \langle z_0, \ldots, z_{n-1} \rangle) \end{array}$$

(i) is similarly straightforward. QED (Lemma 1.1.12)

The *recursion theorem* of classical recursion theory says that if $g(n, m)$ is recursive on ω and $f : \omega \to \omega$ is defined by:

$$f(0) = k, f(n+1) = g(n, f(n)),$$

then f is recursive. The point is that the value of f at any n is determined by its values at smaller numbers. Working with H instead of ω we can express this in the elegant form:

Let $g : \omega \times H \to \omega$ be Σ_1.
Then $f : \omega \to \omega$ is Σ_1, where $f(n) = g(n, f \restriction n)$.

If we take $g : H^2 \to H$, then f will be Σ_1 where $f(x) = g(x, f \restriction x)$ for $x \in H$. We can even take g as being a partial function on H^2. Then f is Σ_1 where:

$$f(x) \simeq g(x, \langle f(z) | z \in x \rangle).$$

(This means that $f(x)$ is defined if and only if $f(z)$ is defined for $z \in x$ and g is defined at $\langle x, f \upharpoonright x\rangle$, in which case the above equality holds.)

We now prove the same thing for an arbitrary admissible M. If f is a partial $\underline{\Sigma}_1$ function and $x \subset \mathrm{dom}(f)$, we know by Lemma 1.1.3 that $f''x \in M$. But then $f \upharpoonright x \in M$, since $f^*(z) \simeq \langle f(z), z\rangle$ is a $\underline{\Sigma}_1$ function with $x \subset \mathrm{dom}(f^*)$, and $f^{*\prime\prime}x = f \upharpoonright x$. The *recursion theorem for admissibles* $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ then reads:

Lemma 1.1.13. *Let $G(y, \vec{x}, u)$ be a $\Sigma_1(M)$ function in the parameter p. Then there is exactly one function $F(y, \vec{x})$ such that*

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).$$

Moreover, F is uniformly $\Sigma_1(M)$ in p (i.e. the Σ_1 definition depends only on the Σ_1 definition of G.)

Proof. We first show existence. Set:

$$\begin{aligned}\Gamma_{\vec{x}} =: \ & \{f \in M | f \text{ is a function } \wedge \mathrm{dom}(f) \text{ is} \\ & \text{transitive } \wedge \bigwedge y \in \mathrm{dom}(f) f(y) = G(y, \vec{x}, f \upharpoonright y)\}\end{aligned}$$

Set $F_{\vec{x}} = \bigcup \Gamma_{\vec{x}}; F = \{\langle y, \vec{x}\rangle | y \in F_{\vec{x}}\}$. Then F is $\Sigma_1(M)$ in p uniformly.

(1) F is a function.

Proof. Suppose not. Then for some $\vec{x}$ there are $f, f' \in \Gamma_{\vec{x}}$, $y \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ such that $f(y) \neq f'(y)$. Let y be $\in$–minimal with this property. Then $f \upharpoonright y = f' \upharpoonright y$. But then $f(y) = G(y, \vec{x}, f \upharpoonright y) = G(y, \vec{x}, f' \upharpoonright y) = f'(y)$. Contradiction! QED (1)

Hence $F(y, \vec{x}) = f(y)$ if $y \in \mathrm{dom}(f)$ and $f \in \Gamma_{\vec{x}}$.

(2) Let $\langle y, \vec{x}\rangle \in \mathrm{dom}(F)$. Then $y \subset \mathrm{dom}(F_{\vec{x}}), \langle y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle\rangle \in \mathrm{dom}(G)$ and

$$F(y, \vec{x}) = G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).$$

Proof. Let $y \in \mathrm{dom}(f), f \in \Gamma_{\vec{x}}$. Then

$$\begin{aligned} F(y, \vec{x}) = f(y) \ & = G(y, \vec{x}, f \upharpoonright x) \\ & = G(y, \vec{x}, \langle F(z, \vec{x}) | z \in y\rangle).\end{aligned}$$

QED (2)

(3) Let $y \subset \mathrm{dom}(F_{\vec{x}}), \langle y, \vec{x}, F_{\vec{x}} \upharpoonright y\rangle \in \mathrm{dom}(G)$. Then $y \in \mathrm{dom}(F_{\vec{x}})$.

Proof. By our assumption: $\bigwedge z \in y \bigvee f (f \in \Gamma_{\vec{x}} \wedge z \in \mathrm{dom}(f))$. Hence there is $u \in M$ such that

$$\bigwedge z \in y \bigvee f \in u (f \in \Gamma_{\vec{x}} \wedge z \in \mathrm{dom}(f)).$$

Set: $f' = \bigcup(u \cap \Gamma_{\vec{x}})$. Then $f' \in \Gamma_{\vec{x}}$ and $y \subset \mathrm{dom}(f')$. Moreover $f' \restriction y = F_{\vec{x}} \restriction y$. Set $f'' = f' \cup \{\langle G(y, \vec{x}, f' \restriction y), y\rangle\}$. Then $f'' \in \Gamma_{\vec{x}}$ and $y \in \mathrm{dom}(f'')$, where $f'' \subset F_{\vec{x}}$. QED (3)

This proves existence. To show uniqueness, we virtually repeat the proof of (1): Let F^* satisfy the same condition. Set $F^*_{\vec{x}}(y) \simeq F^*(y, \vec{x})$. Suppose $F^* \neq F$. Then $F^*_{\vec{x}}(y) \not\simeq F_{\vec{x}}(y)$ for some $\vec{x}, y$. Let y be $\in$–minimal such that $F^*_{\vec{x}}(y) \not\simeq F_{\vec{x}}(y)$. Then $F^*_{\vec{x}} \restriction y = F_{\vec{x}} \restriction y$. Hence

$$\begin{aligned} F^*_{\vec{x}}(y) &\simeq G(y, \vec{x}, \langle F^*_{\vec{x}}(z) | z \in y\rangle) \\ &\simeq G(y, \vec{x}, \langle F_{\vec{x}}(z) | z \in y\rangle) \\ &\simeq F_{\vec{x}}(y). \end{aligned}$$

Contradiction! QED (Lemma 1.1.13)

We recall that the transitive closure $TC(x)$ of a set x is recursively definable by: $TC(x) = x \cup \bigcup_{z \in x} TC(z)$. Similarly, the rank $rn(x)$ of a set is definable by $rn(x) = \mathrm{lub}\{rn(z) | z \in x\}$. Hence:

Corollary 1.1.14. *TC, rn are uniformly $\Sigma_1(M)$.*

The successor function $s\alpha = \alpha + 1$ on the ordinals is defined by:

$$sx = \begin{cases} x \cup \{x\} \text{ if } x \in On \\ \text{undefined if not} \end{cases}$$

which is Σ_1. The function $\alpha + \beta$ is defined by:

$$\begin{aligned} &\alpha + 0 = \alpha \\ &\alpha + s\nu = s(\alpha + \nu) \\ &\alpha + \lambda = \textstyle\bigcup_{\nu < \lambda} \alpha + \nu \text{ for limit } \lambda. \end{aligned}$$

This has the form:

$$x + y \simeq G(y, x, \langle x + z | z \in y\rangle).$$

Similarly for the function $x \cdot y, x^y, \ldots$ etc. Hence:

Corollary 1.1.15. *The ordinal functions $\alpha + 1, \alpha + \beta, \alpha^\beta, \ldots$ etc. are uniformly $\Sigma_1(M)$.*

We note that there is an even more useful form of Lemma 1.1.13:

Lemma 1.1.16. *Let G be as in Lemma 1.1.13. Let $h : M \to M$ be $\Sigma_1(M)$ in p such that $\{\langle x, y\rangle | x \in h(y)\}$ is well founded. There is a unique F such that*

$$F(y, \vec{x}) \simeq G(y, \vec{x}, \langle F(z, \vec{x}) | x \in h(y)\rangle).$$

Moreover, F is uniformly[1] $\Sigma_1(M)$ *in* p.

The proof is exactly like that of Lemma 1.1.13, using minimality in the relation $\{\langle x,y\rangle | x \in h(y)\}$ in place of $\in$–minimality. We now consider the structure of "really finite" sets in an admissible M.

Lemma 1.1.17. *Let $u \in H_\omega$. The class u and the constant function $f(x) = u$ are uniformly $\Sigma_1(M)$.*

Proof. By $\in$–induction on u: Let $u = \{z_1, \dots, z_n\}$.

$$x \in u \leftrightarrow \bigvee_{i=1}^{n} x = z_i$$
$$x = u \leftrightarrow \bigwedge y \in x\, y \in u \wedge \bigwedge_{i=1}^{n} z_i \in x.$$

QED

$x \in \omega$ is clearly a Σ_0 condition. But then:

Lemma 1.1.18. *Let $\omega \in M$. Then the constant function $f(x) = \omega$ is uniformly $\Sigma_1(M)$.*

Proof.

$$x = \omega \leftrightarrow (\bigwedge z \in xz \in \omega \wedge \emptyset \in x \wedge \bigwedge z \in xz \cup \{z\} \in x)$$

(where '$z \in \omega$' is Σ_0) QED

Lemma 1.1.19. *The class* Fin *and the function $f(x) = \mathbb{P}_\omega(x)$ are uniformly $\Sigma_1(M)$, where* $\text{Fin} = \{x \in M | \overline{\overline{x}} < \omega\}, \mathbb{P}_\omega(x) = \mathbb{P}(x) \cap \text{Fin}$.

Proof.

$$\begin{aligned} x \in \text{Fin} &\leftrightarrow \bigvee n \in \omega \bigvee f\, f : n \leftrightarrow x \\ y = \mathbb{P}_\omega(x) &\leftrightarrow \bigwedge u \in y(u \subset x \wedge u \in \text{Fin}) \wedge \emptyset \in y \wedge \\ &\wedge \bigwedge z \in x\{z\} \in y \wedge \bigwedge u, v \in yu \cup v \in y. \end{aligned}$$

We must show that $\mathbb{P}_\omega(x) \in M$. If $\omega \notin M$, then $rn(x) < \omega$ for all $x \in M$, Hence $M = H_\omega$ is closed under $\mathbb{P}_\omega$. If $\omega \in M$, there is $\underline{\Sigma}_1(M)$ f defined by

$$f(0) = \{\{z\} | z \in x\}, f(n+1) = \{u \cup v | \langle u, v\rangle \in f(n)^2\}.$$

Then $\mathbb{P}_\omega(x) = \bigcup f''\omega \in M$. QED (Lemma 1.1.19)

But then:

[1](“uniformly” meaning, of course, that the Σ_1 definition of F depends only on the Σ_1 definition of G, h.)

Lemma 1.1.20. *If $\omega \in M$, then $H_\omega \in M$ and the constant function $f(x) = H_\omega$ is uniformly $\Sigma_1(M)$.*

Proof. $H_\omega \in M$, since there is a $\Sigma_1(M)$ function g defined by $g(0) = \emptyset, g(n+1) = \mathbb{P}_\omega(g(n))$. Then $H_\omega = \bigcup g''\omega \in M$ and $f(x) = H_\omega$ is $\Sigma_1(M)$ since g and the constant function ω are $\Sigma_1(M)$. QED (Lemma 1.1.20)

1.1.3 The constructible hierarchy

We recall Gödel's definition of the *constructible hierarchy* $\langle L_r | r \in \mathrm{On}\rangle$:

$$\begin{aligned} &L_0 = \emptyset \\ &L_{\nu+1} = \mathrm{Def}(L_\nu) \\ &L_\lambda = \bigcup_{\nu<\lambda} L_\nu \text{ for limit } \lambda, \end{aligned}$$

where $\mathrm{Def}(u)$ is the set of all $z \subset u$ which are $\langle u, \in\rangle$–definable in parameters from u (taking $\mathrm{Def}(\emptyset) = \{\emptyset\}$). (Note that if u is transitive, then $u \subset \mathrm{Def}(u)$ and $\mathrm{Def}(u)$ is transitive.) Gödel's *constructible universe* is then $L =: \bigcup_{\nu \in \mathrm{On}} L_\nu$.

By fairly standard methods one can show:

Lemma 1.1.21. *Let $\omega \in M$. Then the function $f(u) = \mathrm{Def}(u)$ is uniformly $\Sigma_1(M)$.*

We omit the proof, which is quite lengthy. It involves "arithmetizing" the language of first order set theory by identifying formulae with elements of ω or H_ω, and then showing that the relevant syntactic and semantic concepts are M–recursive.

By the recursion theorem we can of course conclude:

Corollary 1.1.22. *Let $\omega \in M$. The function $f(\alpha) = L_\alpha$ is uniformly $\Sigma_1(M)$.*

The constructible hierarchy *over* a set u is defined by:

$$\begin{aligned} &L_0(u) = TC(\{u\}) \\ &L_{\nu+1}(u) = \mathrm{Def}(L_\nu(u)) \\ &L_\lambda(u) = \bigcup_{\nu<\lambda} L_\nu(u) \text{ for limit } \lambda. \end{aligned}$$

Obviously:

Corollary 1.1.23. *Let $\omega \in M$. The function $f(u, \alpha) = L_\alpha(u)$ is uniformly $\Sigma_1(M)$.*

The constructible hierarchy *relative to* classes $A_1, \ldots, A_n$ is defined by:

$$
\begin{aligned}
&L_0[\vec{A}] = \emptyset \\
&L_{\nu+1}[\vec{A}] = \mathrm{Def}(L_\nu[\vec{A}], \vec{A}) \\
&L_\lambda[\vec{A}] = \bigcup_{\nu<\lambda} L_\nu[\vec{A}] \text{ for limit } \lambda,
\end{aligned}
$$

where $\mathrm{Def}(U, A_1, \ldots, A_n)$ is the set of all $z \subset u$ which are $\langle u, \in, A_1 \cap u, \ldots, A_n \cap u\rangle$–definable in parameters from u.

Much as before we have:

Lemma 1.1.24. *Let $\omega \in M$. Let $A_1, \ldots, A_n$ be $\Delta_1(M)$ in the parameter p. Then the function $f(u) = \mathrm{Def}(u, A_1, \ldots, A_n)$ is uniformly $\Sigma_1(M)$ in p.*

Corollary 1.1.25. *Let $\omega \in M$. Let $A_1, \ldots, A_n$ be as above. Then the function $f(\alpha) = L_\alpha[\vec{A}]$ is uniformly $\Sigma_1(M)$ in p.*

(In particular, if $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$. Then $f(\alpha) = L_\alpha[\vec{A}]$ is uniformly $\Sigma_1(M)$.)

(One could, of course, also define $L_\alpha(u)[\vec{A}]$ and prove the corresponding results.)

Any well ordering r of a set u induces a well ordering of $\mathrm{Def}(u)$, since each element of $\mathrm{Def}(u)$ is defined over $\langle u, \in\rangle$ by a tuple $\langle \varphi, x_1, \ldots, x_n\rangle$, where φ is a formula and $x_1, \ldots, x_n$ are elements of u which interpret free variables of φ. If u is transitive (hence $u \subset \mathrm{Def}(u)$), we can also arrange that the well ordering, which we shall call $<(u, r)$, is an end extension of r. The function $<(u, r)$ is uniformly Σ_1. If we then set:

$$
\begin{aligned}
&<_0 = \emptyset, <_{\nu+1} = <(L_\nu, <_\nu) \\
&<_\lambda = \bigcup_{\nu<\lambda} <_\nu \ \text{ for limit } \lambda,
\end{aligned}
$$

it follows that $<_\nu$ is a well ordering of L_ν for all ν. Moreover $<_\alpha$ is an end extension of $<_\nu$ for $\nu < \alpha$.

Similarly, if A is $\Sigma_1(M)$ in p, there is a hierarchy $<_\nu^A$ ($\nu \in \mathrm{On} \cap M$) such that $<_\nu^A$ well orders $L_\nu[A]$ and the function $f(\nu) = <_\nu^A$ is $\Sigma_1(M)$ in p (uniformly relative to the Σ_1 definition of A).

By Corollary 1.1.25 we easily get:

Lemma 1.1.26. *Let* $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ *be admissible. Let* $\alpha = \mathrm{On} \cap M$. *Then* $\langle L_\alpha[\vec{A}], \in, \vec{A} \rangle$ *is admissible.*

Proof: Set: $L_\nu^{\vec{A}} = \langle L_\alpha[\vec{A}], \in, \vec{A} \rangle$. Axiom (1) holds trivially in $L_\nu^{\vec{A}}$.

To verify the Σ_0–axiom of subsets, let B be $\underline{\Sigma}_0(L_\alpha^{\vec{A}})$. Let $u \in L_\alpha^{\vec{A}}$.

Claim $u \cap B \in L_\alpha^{\vec{A}}$.

Proof: Pick $\nu < \alpha$ such that $u \in L_\nu^{\vec{A}}$ and B is $\underline{\Sigma}_0$ in parameters from $L_\nu^{\vec{A}}$. By $\underline{\Sigma}_0$–absoluteness we have:

$$u \cap B \in \mathrm{Def}(L_\nu^{\vec{A}}) = L_{\nu+1}^{\vec{A}} \subset L_\alpha^{\vec{A}}.$$

QED (Claim)

We now prove Σ_0–collection. Let Rxy be a $\underline{\Sigma}_0$–relation. Let $u \in L_\alpha^{\vec{A}}$ such that $\bigwedge x \in u \bigvee y Rxy$.

Claim $\bigvee v \in L_\alpha^{\vec{A}} \bigwedge x \in u \bigvee y \in v Rxy$.

For each $x \in u$ let $g(x)$ be the least $\nu < \alpha$ such that $x \in L_\nu^{\vec{A}}$. Then g is in $\underline{\Sigma}_1(M)$ and $u \subset \mathrm{dom}(g)$. Hence $\delta = \sup g''u < \alpha$ and

$$\bigwedge x \in u \bigvee y \in L_\delta^{\vec{A}} Rxy.$$

QED (Lemma 1.1.26)

Definition 1.1.2. Let α be an ordinal.

- α is *admissible* iff L_α is admissible
- α is *admissible in* $A_1, \ldots, A_n \subset$ iff $L_\alpha^{\vec{A}} =: \langle L_\alpha[\vec{A}], \in \vec{A} \rangle$ is admissible
- $f : \alpha^n \to \alpha$ is α*–recursive* (in $\vec{A}$) iff f is $\underline{\Sigma}_1(L_\alpha)(\underline{\Sigma}_1(L_\alpha^{\vec{A}}))$
- $R \subset \alpha^n$ is *r.e.* (in $\vec{A}$) iff R is $\Sigma_1(L_\alpha)(\Sigma_1(L_\alpha^{\vec{A}}))$.

Note. The theory of α–recursive functions and relations on an admissible α has been built up without references to L_α, using a formalized notion of α–bounded calculus (Kripke) or α–bounded algorithm (Platek).

Similarly for α–recursiveness in $A_1, \ldots, A_n$, taking the A_i as "oracles".

A transitive structure $M = \langle |M|, \in \vec{A}\rangle$ is called *strongly admissible* iff, in addition to the Kripke–Platek axioms, it satisfies the Σ_1 *axiom of subsets*:

$$x \cap \{z|\varphi(z)\} \text{ is a set (for } \Sigma_1 \text{ formulae } \varphi).$$

Kripke defines the *projectum* δ_α of an admissible ordinal α to be the least δ such that $A \cap \delta \notin L_\alpha$ for some $\underline{\Sigma}_1(L_\alpha)$ set A. He shows that $\delta_\alpha = \alpha$ iff α is strongly admissible. He calls α *projectible* iff $\delta_\alpha < \alpha$. There are many projectible admissibles — e.g. $\delta_\alpha = \omega$ if α is the least admissible greater than ω. He shows that for every admissible α there is a $\underline{\Sigma}_1(L_\alpha)$ injection f_α of L_α into δ_α.

The definition of projectum of course makes sense for *any* $\alpha \geq \omega$. By refinements of Kripke's methods it can be shown that f_α exists for every $\alpha \geq \omega$ and that $\delta_\alpha < \alpha$ whenever $\alpha \geq \omega$ is not strongly admissible. We shall — essentially — prove these facts in chapter 2 (except that, for technical reasons, we shall employ a modified version of the constructible hierarchy).

1.2 Primitive Recursive Set Functions

1.2.1 PR Functions

The *primitive recursive set functions* comprise a collection of functions

$$f : V^n \to V$$

which form a natural analogue of the primitive recursive number functions in ordinary recursion theory. As with admissibility theory, their discovery arose from the attempt to generalize ordinary recursion theory. These functions are ubiquitous in set theory and have very attractive absoluteness properties. In this section we give an account of these functions and their connection with admissibility theory, though — just as in §1 — we shall suppress some proofs.

Definition 1.2.1. $f : V^n \to V$ is a *primitive recursive (pr) function* iff it is generated by successive application of the following schemata:

(i) $f(\vec{x}) = x_i$ (here $\vec{x}$ is $x_1, \ldots, x_n$)

(ii) $f(\vec{x}) = \{x_i, x_j\}$

(iii) $f(\vec{x}) = x_i \setminus x_j$

(iv) $f(\vec{x}) = g(h_1(\vec{x}), \ldots, h_m(\vec{x}))$

(v) $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$

(vi) $f(y, \vec{x}) = g(y, \vec{x}, \langle f(z, \vec{x}) | z \in y \rangle)$

We also define:

Definition 1.2.2. $R \subset V^n$ is a *primitive recursive relation* iff there is a primitive recursive function r such that $R = \{\langle \vec{x} \rangle | r(\vec{x}) \neq \emptyset\}$.

(**Note** It is possible for a function on V to be primitive recursive as a relation but not as a function!)

We begin by developing some elementary consequences of these definitions:

Lemma 1.2.1. *If $f : V^n \to V$ is primitive recursive and $k : n \to m$, then g is primitive recursive, where*

$$g(x_0, \dots, x_{m-1}) = f(x_{k(0)}, \dots, x_{k(n-1)}).$$

Proof. By (i), (iv).

Lemma 1.2.2. *The following functions are primitive recursive*

(a) $f(\vec{x}) = \bigcup x_j$

(b) $f(\vec{x}) = x_i \cup x_j$

(c) $f(\vec{x}) = \{\vec{x}\}$

(d) $f(\vec{x}) = n$, *where* $n < \omega$

(e) $f(\vec{x}) = \langle \vec{x} \rangle$

Proof.

(a) By (i), (v), Lemma 1.2.1, since $\bigcup x_j = \bigcup_{z \in x_j} z$

(b) $x_i \cup x_j = \bigcup \{x_i, x_j\}$

(c) $\{\vec{x}\} = \{x_1\} \cup \ldots \cup \{x_m\}$

(d) By in induction on n, since $0 = x \setminus x, n + 1 = n \cup \{n\}$

(e) The proof depends on the precise definition of n–tuple. We could for instance define $\langle x, y \rangle = \{\{x\}, \{x, y\}\}$ and $\langle x_1, \dots, x_n \rangle = \langle x_1, \langle x_2, \dots, x_n \rangle \rangle$ for $n > 2$.

If, on the other hand, we wanted each tuple to have a unique length, we could call the above defined ordered pair (x, y) and define:

$$\langle x_1, \ldots, x_n \rangle = \{(x_1, 0), \ldots, (x_n, n-1)\}.$$

QED (Lemma 1.2.2)

Lemma 1.2.3. *(a)* $\notin$ *is pr*

(b) If $f : V^n \to V, R \subset V^n$ *are primitive recursive, then so is*

$$g(\vec{x}) = \begin{cases} f(\vec{x}) \text{ if } R\vec{x} \\ \emptyset \text{ if not} \end{cases}$$

(c) $R \subset V^n$ *is primitive recursive iff its characteristic functions* χ_R *is a primitive recursive function*

(d) If $R \subset V^n$ *is primitive recursive so is* $\neg R =: V^n \setminus R$

(e) Let $f_i : V^n \to V, R_i \subset V^n$ *be* $pr(i = 1, \ldots, m)$ *where* $R_1, \ldots, R_m$ *are mutually disjoint and* $\bigcup_{i=1}^{m} R_i = V^n$*. Then* f *is pr where:*

$$f(\vec{x}) = f_i(x) \text{ when } R_i\vec{x}.$$

(f) If $Rz\vec{x}$ *is primitive recursive, so is the function*

$$f(y, \vec{x}) = y \cap \{z | Rz\vec{x}\}$$

(g) If $Rz\vec{x}$ *is primitive recursive so is* $\bigvee z \in yRz\vec{x}$

(h) If $R_i\vec{x}$ *is primitive recursive* $(i = 1, \ldots, m)$*, then so is* $\bigvee_{i=1}^{m} R_i\vec{x}$

(i) If $R_1, \ldots, R_n$ *are primitive recursive relations and* φ *is a* Σ_0 *formula, then* $\{\langle\vec{x}\rangle | \langle V, R_1, \ldots, R_n\rangle \models \varphi[\vec{x}]\}$ *is primitive recursive.*

(j) If $f(z, \vec{x})$ *is primitive recursive, then so are:*

$$g(y, \vec{x}) = \{f(z, \vec{x}) : z \in y\}$$
$$g'(y, \vec{x}) = \langle f(z, \vec{x}) : z \in y\rangle$$

(k) If $R(z, \vec{x})$ *is primitive recursive, then so is*

$$f(y, \vec{x}) = \begin{cases} \text{That } z \in y \text{ such that } Rz\vec{x} \text{ if exactly} \\ \text{one such } z \in y \text{ exists;} \\ \emptyset \text{ if not.} \end{cases}$$

Proof.

(a) $x \notin y \leftrightarrow \{x\} \setminus y \neq \emptyset$

(b) Let $R\vec{x} \leftrightarrow r(\vec{x}) \neq \emptyset$. Then $g(\vec{x}) = \bigcup_{z \in r(\vec{x})} f(\vec{x})$.

(c) $\chi_r(\vec{x}) = \begin{cases} 1 \text{ if } R\vec{x} \\ 0 \text{ if not} \end{cases}$

(d) $\chi_{\neg R}(\vec{x}) = 1 \setminus \chi_R(\vec{x})$

(e) Let $f_i'(\vec{x}) = \begin{cases} f_i(\vec{x}) \text{ if } R_i\vec{x} \\ \emptyset \text{ if not} \end{cases}$
Then $f(\vec{x}) = f_i'(\vec{x}) \cup \ldots \cup f_m'(\vec{x})$.

(f) $f(y, \vec{x}) = \bigcup_{z \in y} h(z, \vec{x})$, where:

$$h(z, \vec{x}) = \begin{cases} \{z\} \text{ if } Rz\vec{x} \\ \emptyset \text{ if not} \end{cases}$$

(g) Let $Py\vec{x} \leftrightarrow: \bigvee z \in yRz\vec{x}$. Then $\chi_P(\vec{x}) = \bigcup_{z \in y} \chi_R(z, \vec{x})$.

(h) Let $P\vec{x} \leftrightarrow \bigvee_{i=1}^{m} R_i\vec{x}$. Then

$$X_P(\vec{x}) = X_{R_1} \cup \ldots \cup X_{R_n}(\vec{x}).$$

(i) is immediate by (d), (g), (h)

(j) $g(y, \vec{x}) = \bigcup_{z \in y} \{f(z, \vec{x})\}, g'(y, \vec{x}) = \bigcup_{z \in y} \{\langle f(z, \vec{x}), z\rangle\}$

(k) $R'zu\vec{x} \leftrightarrow: (z \in u \wedge Rz\vec{x} \wedge \bigwedge z' \in u(z \neq z' \rightarrow \neg Rz'\vec{x}))$ is primitive recursive by (i). But then:

$$f(y, \vec{x}) = \bigcup (y \cap \{z | R'zy\vec{x}\})$$

QED (Lemma 1.2.3)

Lemma 1.2.4. *Each of the functions listed in §1 Lemma 1.1.12 is primitive recursive.*

Proof. (a) $\bigcup x = \bigcup_{z \in x} z$, $x \cup y = \bigcup \{x, y\}$, $x \cap y$, $x \setminus y$ are primitive recursive by Lemma 1.2.3 (f).

(b)–(e) follow by computation from (a).

(g) $x_1 \times x_2 \times \cdots \times x_n = f_n^n(\vec{x})$

- $f_0^n(\vec{x}) = \{\langle \vec{x} \rangle\}$
- $f_{i+1}^n(\vec{x}) = \bigcup_{z \in x_i} f_i^n(x_0, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n)$

(f) then follows by Lemma 1.2.3 (f), since for sufficient n we have:

- $\mathrm{dom}(x) = C_n(x) \cap \{z \mid \bigvee w \in C_n(x) \langle w, z\rangle x\}$
- $\mathrm{rng}(x) = C_n(x) \cap \{z \mid \bigvee w \in C_n(x) \langle z, w\rangle \in x\}$
- $x"y = C_n(x) \cap \{u \mid \bigvee z, w \in C_n(y)(u = \langle z, w\rangle \in x \wedge w \in y)\}$
- $x^{-1} = C_n(x) \cap \{u \mid \bigvee z, w \in C_n(x)(\langle z, w\rangle \in x \wedge u = \langle w, z\rangle)\}$

(h), (i) then follow by Lemma 1.2.3 (f).

QED (Lemma 1.2.4)

Note Up until now we have only made use of the schemata (i) – (v). This will be important later. The functions and relations obtainable from (i) – (v) alone are called *rudimentary* and will play a significant role in fine structure theory. We shall use the fact that Lemmas 1.2.1 – 1.2.3 hold with "rudimentary" in place of "primitive recursive".

Using the recursion schema (vi) we then get:

Lemma 1.2.5. *The functions $TC(x)$, $\mathrm{rn}(x)$ are primitive recursive.*

The proof is the same as before (§1 Corollary 1.1.14).

Definition 1.2.3. $f : \mathrm{On}^n \times V^m \to V$ is primitive recursive iff f' is primitive recursive, where

$$f'(\vec{y}, \vec{x}) = \begin{cases} f(\vec{y}, \vec{x}) \text{ if } y_1, \ldots, y_n \in \mathrm{On} \\ \emptyset \text{ if not} \end{cases}$$

As before:

Lemma 1.2.6. *The ordinal function $\alpha + 1, \alpha + \beta, \alpha \cdot \beta, \alpha^\beta, \ldots$ are primitive recursive.*

Definition 1.2.4. Let $f : V^{n+1} \to V$.

$f^\alpha(\alpha \in \mathrm{On})$ is defined by:

$$\begin{aligned} &f^0(y, \vec{x}) = y \\ &f^{\alpha+1}(y, \vec{x}) = f(f^\alpha(y, \vec{x}), \vec{x}) \\ &f^\lambda(y, \vec{x}) = \bigcup_{r<\lambda} f^r(y, \vec{x}) \text{ for limit } \lambda. \end{aligned}$$

Then:

Lemma 1.2.7. *If f is primitive recursive, so is $g(\alpha, y, \vec{x}) = f^\alpha(y, \vec{x})$.*

There is a strengthening of the recursion schema (vi) which is analogous to §1 Lemma 1.1.16. We first define:

Definition 1.2.5. Let $h : V \to V$ be primitive recursive. h is *manageable* iff there is a primitive recursive $\sigma : V \to \mathrm{On}$ such that

$$x \in h(y) \to \sigma(x) < \sigma(y).$$

(Hence the relation $x \in h(y)$ is well founded.)

Lemma 1.2.8. *Let h be manageable. Let $g : V^{n+2} \to V$ be primitive recursive. Then $f : V^{n+1} \to V$ is primitive recursive, where:*

$$f(y, \vec{x}) = g(y, \vec{x}, \langle f(z, \vec{x}) | z \in h(y) \rangle).$$

Proof. Let σ be as in the above definition. Let $|x| = \mathrm{lub}\{|y| | y \in h(x)\}$ be the rank of x in the relation $y \in h(x)$. Then $|x| \leq \sigma(x)$. Set:

$$\Theta(z, \vec{x}, u) = \bigcup \{\langle g(y, \vec{x}, z \upharpoonright h(y)), y \rangle | y \in u \wedge h(y) \subset \mathrm{dom}(z)\}.$$

By induction on α, if u is h–closed (i.e. $x \in u \to h(x) \subset u$), then:

$$\Theta^\alpha(\emptyset, \vec{x}, u) = \langle f(y, \vec{x}) | y \in u \wedge |y| < \alpha \rangle$$

Set $\tilde{h}(v) = v \cup \bigcup\limits_{z \in v} h(z)$. Then $\tilde{h}^\alpha(\{y\})$ is h–closed for $\alpha \geq |y|$. Hence:

$$f(y, \vec{x}) = \Theta^{\sigma(y)+1}(\emptyset, \vec{x}, \tilde{h}^{\sigma(y)}(\{y\}))(y).$$

QED (Lemma 1.2.8)

Corresponding to §1 Lemma 1.1.17 we have:

Lemma 1.2.9. *Let $u \in H_\omega$. The constant function $f(x) = u$ is primitive recursive.*

Proof: By $\in$–induction on u. QED

As we shall see, the constant function $f(x) = \omega$ is not primitive recursive, so the analog of §1 Lemma 1.1.18 fails. We say that f is primitive recursive *in the parameters* $p_1, \ldots, p_m H$:

$$f(\vec{x}) = g(\vec{x}, \vec{p}), \text{ where } g \text{ is primitive recursive.}$$

In place of §1 Lemma 1.1.19 we get:

Lemma 1.2.10. *The class* Fin *and the function* $f(x) = \mathbb{P}_\omega(x)$ *are primitive recursive in the parameter* ω.

Proof: Let f be primitive recursive such that $f(0, x) = \{\emptyset\} \cup \{\{z\}|z \in x\}$, $f(n+1, x) = \{u \cup v|\langle u, v\rangle \in f(n, x)^2\}$. Then $\mathbb{P}_\omega(x) = \bigcup_{n\in\omega} f(n, x)$. But then:

$$x \in \mathrm{Fin} \leftrightarrow \bigvee n \in \omega \bigvee g \in \bigcup_{n<\omega} \mathbb{P}^n_\omega(x \times \omega) g : n \leftrightarrow x.$$

QED

Corollary 1.2.11. *The constant function* $f(x) = H_\omega$ *is primitive recursive in the parameter* ω.

Proof: $H_\omega = \bigcup_{n<\omega} \mathbb{P}^n_\omega(\emptyset)$. QED

Corresponding to Lemma 1.1.21 of §1 we have:

Lemma 1.2.12. *The function* Def(u) *is primitive recursive in the parameter* ω.

The proof involves carrying out the proof of §1 Lemma 1.1.21 (which we also omitted) while ensuring that the relevant classes and functions are primitive recursive. We give not further details here (though filling in the details can be an arduous task). A fuller account can be found in [PR] or [AS].

Hence:

Corollary 1.2.13. *The function* $f(\alpha) = L_\alpha$ *is primitive recursive in* ω.

Similarly:

Lemma 1.2.14. *The function* $f(\alpha, x) = L_\alpha(x)$ *is primitive recursive in* ω.

Lemma 1.2.15. *Let* $A \subset V$ *be primitive recursive in the parameter* p. *Then* $f(\alpha) = L^A_\alpha$ *is primitive recursive in* p.

One can generalize the notion primitive recursive to *primitive recursive in the class* $A \subset V$ (or *in the classes* $A_1, \ldots, A_n \subset V$).

We define:

Definition 1.2.6. Let $A_1, \ldots, A_n \subset V$. The function $f : V^n \to V$ is *primitive recursive in* $A_1, \ldots, A_n$ iff it is obtained by successive applications of the schemata (i) – (vi) together with the schemata:

$$f(x) = \chi_{A_i}(x)(i = 1, \ldots, n).$$

A relation R is primitive recursive in $A_1, \ldots, A_n$ iff

$$R = \{\langle \vec{x} \rangle | f(\vec{x}) \neq 0\}$$

for a function f which is primitive recursive in $A_1, \ldots, A_n$.

It is obvious that all of the previous results hold with "primitive recursive in $A_1, \ldots, A_n$" in place of "primitive recursive".

By induction on the defining schemata of f we can show:

Lemma 1.2.16. *Let f be primitive recursive in $A_1, \ldots, A_n$, where each A_i is primitive recursive in $B_1, \ldots, B_m$. Then f is primitive recursive in $B_1, \ldots, B_m$.*

The proof is by induction on the defining schemata leading from $A_1, \ldots, A_n$ to f. The details are left to the reader. It is clear, however, that this proof is *uniform* in the sense that the schemata which give in f from $B_1, \ldots, B_m$ are not dependent on $B_1, \ldots, B_m$ or $A_1, \ldots, A_n$, but only on the schemata which lead from $A_1, \ldots, A_n$ to f and the schemata which led from $B_1, \ldots, B_m$ to $A_i (i = 1, \ldots, n)$.

This will be made more precise in §1.2.2

1.2.2 PR Definitions

Since primitive recursive functions are proper classes, the foregoing discussion must ostensibly be carried out in second order set theory. However, we can translate it into ZF by talking about *primitive recursive definitions*. By a primitive recursive definition we mean a finite sequence of equations of the form (i) – (vi) such that:

- The function variable on the left side does not occur in a previous equation in the sequence
- every function variable on the right side occurs previously on the left side with the same number of argument places.

We assume that the language in which we write these equation has been *arithmetized* — i.e. formulae, terms, variables etc. have been identified in a natural way with elements of ω (or at least H_ω).

Every primitive recursive definition s defines a function F_s. If $s = \langle s_0, \ldots, s_{n-1} \rangle$, then $F_s = F_s^{n-1}$, where F_s^i interprets the leftmost function variable of s_i.

This is defined in a straightforward way. If e.g. s_i is "$f(y,\vec{x}) = \bigcup\limits_{z\in y} g(z,\vec{x})$" and g was leftmost in s_j, then we get

$$F^i(y,\vec{x}) = \bigcup_{z\in y} F^j(z,\vec{x}).$$

Let PD be the class of primitive recursive definitions. In order to define $\{\langle x,s\rangle | s \in PD \wedge x \in F_s\}$ in ZF we proceed as follows:

Let $s = \langle s_0,\ldots,s_{n-1}\rangle \in PD$. Let M be any admissible structure. By induction we can then define $\langle F_s^{i,M} | i < n\rangle$ where F_s^i a function on M^{n_i} (n_i being the number of argument places). By admissibility we know that F_s^i exists and is defined on all of M^{n_i}. We then set: $F_s^M = F_s^{n-1,M}$. This defines the set $\langle F_s^M | s \in PD\rangle$. If $M \subseteq M'$ and M' is also admissible, it follows by any induction on $i < n$ that $F^{i,M} = F^{i,M'} \restriction M$. Hence $F_s^M \subset F_s^{M'}$. We can then set:

$$F_s = \bigcup\{F_s^M | M \text{ is admissible}\}.$$

Note that by §1, each F_s^M has a uniform Σ_1 definition φ_s which defines F_s^M over *every* admissible M. It follows that φ_s defines F_s in V. Thus we have won an important absoluteness result: Every primitive recursive function has a Σ_1 definition which is absolute in all inner models, in all generic extensions of V, and indeed, in all admissible structures $M = \langle |M|, \in\rangle$. This absoluteness phenomenon is perhaps the main reason for using the theory of primitive recursive functions in set theory. Carol Karp was the first to notice the phenomenon — and to plumb its depths. She proved results going well beyond what I have stated here, showing for instance that the canonical Σ_1 definition can be so chosen, that $F_s \restriction M$ is the function defined over M by φ_s whenever M is transitive and closed under primitive recursive function. She also improved the characterization of such M: Call an ordinal α *nice* if it is closed under each of the function:

$$f_0(\alpha,\beta) = \alpha+\beta; f_1(\alpha,\beta) = \alpha\cdot\beta, f_2(\alpha,\beta) = \alpha^\beta \ldots \text{ etc.}$$

(More precisely: $f_{i+1}(\alpha,\beta) = \tilde{f}_i^{\beta}(\alpha)$ for $i \geq 1$, where $\tilde{f}_i(\alpha) = f_i(\alpha,\alpha)$, $g^\beta(\alpha)$ is defined by: $g^0(\alpha) = \alpha$, $g^{\beta+1}(\alpha) = g(g^\beta(\alpha))$, $g^\lambda(\alpha) = \sup\limits_{v<\lambda} g^v(\alpha)$ for limit λ.) She showed that L_α is primitive recursively closed iff α is nice. Moreover, $L_\alpha[A_1,\ldots,A_n]$ is closed under functions primitive recursive in $A_1,\ldots,A_n$ iff α is nice.

Primitive recursiveness in classes $A_1,\ldots,A_n$ can also be discussed in terms of primitive recursive definitions. To this end we appoint new designated function variable $\dot{a}_i (i = 1,\ldots,n)$, which will be interpreted by $\chi_{A_i} (i = 1,\ldots,n)$. By a *primitive recursive definition in* $\dot{a}_1,\ldots,\dot{a}_n$ we mean a sequence of equation having either the form (i) – (vi), in which $\dot{a}_1,\ldots,\dot{a}_n$ do not appear, or the form

(*) $f(x_1, \ldots, x_p) = \dot{a}_i(x_j)(i = 1, \ldots, n, j = 1, \ldots, p)$

We impose our previous two requirements on all equations not of the form (*).

If $s = \langle s_0, \ldots, s_{n-1}\rangle$ is a pr definition in $\dot{a}_1, \ldots, \dot{a}_n$, we successively define $F_s^{i,A_1,\ldots,A_n}(i < n)$ as before, setting $F_s^{i,\vec{A}}(x_1, \ldots, x_p) = X_{A_i}(x_j)$ if s_i has the form (*). We again set $F_s^{\vec{A}} = F_s^{n-1,\vec{A}}$. The fact that $\{\langle x, s\rangle | x \in F_s^{\vec{A}}\}$ is uniformly $\langle V, \in, A_1, \ldots, A_n\rangle$ definable is shown essentially as before:

Given an admissible $M = \langle |M|, \in, a_1, \ldots, a_n\rangle$ we define $F_s^{i,M}, F_s^M = F_s^{n-1,M}$ as before, restricting to M. The existence of the total function $F_s^{i,M}$ follows as before by admissibility. Admissibility also gives a canonical Σ_1 definition φ_s such that

$$y = F_s^M(\vec{x}) \leftrightarrow M \models \varphi_s[y, \vec{x}].$$

(Thus F_s^M is uniformly Σ_1 regardless of M.) If M, M' are admissibles of the same type and $M \subseteq M'$ (i.e. M is structurally included in M'), then $F_s^M = F_s^{M'} \upharpoonright M$. Thus we can let $F_s^{A_1,\ldots,A_n}$ be the union of all F_s^M such that $M = \langle |M|, \in, A_1 \cap |M|, \ldots, A_n \cap |M|\rangle$ is admissible. φ_s then defines $F_s^{\vec{A}}$ over $\langle V, \vec{A}\rangle$. (Here, Karp refined the construction so as to show that $F_s^{\vec{A}} \upharpoonright M = F_s^M$ whenever $M = \langle |M|, \in, A_1 \cap |M|, \ldots, A_n \cap |M|\rangle$ is transitive and closed under function primitive recursive in $A_1, \ldots, A_n$. It can also be shown that $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ is closed under functions primitive recursive in $A_1, \ldots, A_n$ iff $|M|$ is primitive recursive closed and M is amenable, (i.e. $x \cap A_i \in M$ for all $x \in M$, $v = 1, \ldots, n$).

A full account of these results can be found in [PR] or [AS].

We can now state the uniformity involved in Lemma 2.2.19: Let $A_i \subset V$ be primitive recursive in $B_1, \ldots, B_m$ with primitive recursive def s_i in $\dot{b}_1, \ldots, \dot{b}_m$ $(i = 1, \ldots, m)$. Let f be primitive recursive in $A_1, \ldots, A_n$ with primitive recursive definition s in $\dot{a}_1, \ldots, \dot{a}_n$. Then f is primitive recursive in $B_1, \ldots, B_n$ by a primitive recursive definition s' in $\dot{b}_1, \ldots, \dot{b}_m$. s' is *uniform* in the sense that it depends only on $s_1, \ldots, s_n$ and s, not on $B_1, \ldots, B_m$. In fact, the induction on the schemata in s implicitly describes an algorithm for a function

$$s_1, \ldots, s_m, s \mapsto s'$$

with the following property: Let $B_1, \ldots, B_m$ be any classes. Let s_i define g_i from $\vec{B}(i = 1, \ldots, n)$. Set: $A_i = \{x | g_i(x) \neq 0\}$ in $i = 1, \ldots, n$. Let f be the function defined by s from $\vec{A}$. Then s' defines f from $\vec{B}$.

Note $\langle H_\omega, \in\rangle$ is an admissible structure; hence $F_s \upharpoonright H_\omega = f_s^{H_\omega}$. This shows that the constant function ω is not primitive recursive, since $\omega \notin H_\omega$. It can be shown that $f : \omega \to \omega$ is primitive recursive in the sense of ordinary recursion theory iff

$$f^*(x) = \begin{cases} f(x) \text{ if } x \in \omega \\ 0 \text{ if not} \end{cases}$$

is primitive recursive over H_ω. Conversely, there is a primitive recursive map $\sigma : H_\omega \leftrightarrow \omega$ such that $f : H_\omega \to H_\omega$ is primitive recursive over H_ω iff $\sigma f \sigma^{-1}$ is primitive recursive in sense of ordinary recursion theory.

1.3 Ill founded ZF^- models

We now prove a lemma about arbitrary (possibly ill founded) models of ZF^- (where the language of ZF^- may contain predicates other than $\in$). Let $\mathbb{A} = \langle A, \in_{\mathbb{A}}, B_1, \ldots, B_n \rangle$ be such a model. For $X \subset A$ we of course write $\mathbb{A}|X = \langle X, \in_A \cap X^2, \ldots \rangle$. By the *well founded core* of $\mathbb{A}$ we mean the set of all $v \in \mathbb{A}$ such that $\in_{\mathbb{A}} \cap C(x)^2$ is well founded, where $C(x)$ is the closure of $\{x\}$ under $\in_{\mathbb{A}}$. Let wfc($\mathbb{A}$) be the restriction $\mathbb{A}|C$ of $\mathbb{A}$ to its well founded core C. Then wfc($\mathbb{A}$) is a well founded structure satisfying the axiom of extensionality, and is, therefore, isomorphic to a transitive structure. Hence $\mathbb{A}$ is isomorphic to a structure $\mathbb{A}'$ such that wfc($\mathbb{A}'$) is transitive (i.e. wfc($\mathbb{A}'$) $= \langle A', \in, m \rangle$ where A' is transitive). We call such $\mathbb{A}'$ *grounded*, defining:

Definition 1.3.1. $\mathbb{A} = \langle A, \in_{\mathbb{A}}, \ldots \rangle$ is *grounded* iff wfc($\mathbb{A}$) is transitive.

Note. Elsewhere we have called these models "solid" instead of "grounded". We avoid that usage here, however, since *solidity* — in quite another sense — is an important concept in inner model theory.

By the argument just given, every consistent set of sentences in ZF^- has a grounded model. Clearly

(1) $\omega \subset$ wfc($\mathbb{A}$) if $\mathbb{A}$ is grounded.

For any ZF^- model $\mathbb{A}$ we have:

(2) If $x \in \mathbb{A}$ and $\{z | z \in_{\mathbb{A}} x\} \subset$ wfc($\mathbb{A}$), then $x \in$ wfc($\mathbb{A}$).

Proof: $C(x) = \{x\} \cup \bigcup \{C(z) | z \in_{\mathbb{A}} x\}$. QED

By Σ_0–absoluteness we have:

(3) Let $\mathbb{A}$ be grounded. Let φ be Σ_0 and let $x_1, \ldots, x_n \in \mathrm{wfc}(\mathbb{A})$. Then

$$\mathrm{wfc}(\mathbb{A}) \models \varphi[\vec{x}] \leftrightarrow \mathbb{A} \models \varphi[\vec{x}].$$

By $\in$–induction on $x \in \mathrm{wfc}(\mathbb{A})$ it follows that the rank function is absolute:

(4) $\mathrm{rn}(x) = \mathrm{rn}^{\mathbb{A}}(x)$ for $x \in \mathrm{wfc}(\mathbb{A})$ if $\mathbb{A}$ is grounded.

The converse also holds:

(5) Let $\mathrm{rn}^{\mathbb{A}}(x) \in \mathrm{wfc}(\mathbb{A})$. Then $x \in \mathrm{wfc}(\mathbb{A})$.

Proof: Let $r = \mathrm{rn}^{\mathbb{A}}(x)$. Then r is an ordinal by (3). Assume that r is the least counterexample. Then $\mathrm{rn}^{\mathbb{A}}(z) < r$ for $z \in_{\mathbb{A}} x$. Hence $\{z | z \in_{\mathbb{A}} x\} \subset \mathrm{wfc}(\mathbb{A})$ and $x \in \mathrm{wfc}(\mathbb{A})$ by (2).

Contradiction! QED

We now prove:

Lemma 1.3.1. *Let $\mathbb{A}$ be grounded. Then* $\mathrm{wfc}(\mathbb{A})$ *is admissible.*

Proof: Axiom (1) and axiom (2) (Σ_0–subsets) follow trivially from (3). We verify the axiom of Σ_0 collection. Let $R(x, y)$ be $\underline{\Sigma}_0(\mathrm{wfc}(\mathbb{A}))$. Let $u \in \mathrm{wfc}(\mathbb{A})$ such that $\bigwedge x \in u \bigvee y R(x, y)$. It suffices to show:

Claim: $\bigvee v \bigwedge x \in u \bigvee y \in v R(x, y)$.

Let R' be $\underline{\Sigma}_0(\mathbb{A})$ by the same definition in the same parameters as R. Then $R = R' \cap \mathrm{wfc}(\mathbb{A})^2$ by (3). If $\mathbb{A} = \mathrm{wfc}(\mathbb{A})$, there is nothing to prove, so suppose not. Then there is $r \in \mathrm{On}^{\mathbb{A}}$ such that $r \notin \mathrm{wfc}(\mathbb{A})$. Hence

$$\mathbb{A} \models rn(y) < r \text{ for all } y \in \mathrm{wfc}(\mathbb{A})$$

by (4). Hence there is an $r \in \mathrm{On}^{\mathbb{A}}$ such that

(6) $\bigwedge x \in u \bigvee y (R'(x, y) \wedge \mathbb{A} \models rn(y) < r)$

Since $\mathbb{A}$ models ZF^-, there must be a least such r. But then:

(7) $r \in \mathrm{wfc}(\mathbb{A})$.

Since by (2) there would otherwise be an r' such that $\mathbb{A} \models r' < r$ and $r' \notin \text{wfc}(\mathbb{A})$. Hence (6) holds for r', contradicting the minimality of r.

QED (7)

But there is w such that

(8) $\bigwedge x \in u \bigvee y \in w(R'(x,y) \wedge rn(y) < r)$.

Let $\mathbb{A} \models v = \{y \in w | rn(y) < r\}$. Then $rn^{\mathbb{A}}(v) \leq r$. Hence $rn^{\mathbb{A}}(v) \in \text{wfc}(\mathbb{A})$ and $v \in \text{wfc}(\mathbb{A})$ by (5). But:

$$\bigwedge x \in u \bigvee y \in vRxy.$$

QED (Lemma 1.3.1)

As immediate corollaries we have:

Corollary 1.3.2. *Let* $\delta = \text{On} \cap \text{wfc}(\mathbb{A})$. *Then* $L_\delta(u)$ *is admissible whenever* $u \in \text{wfc}(\mathbb{A})$.

Corollary 1.3.3. $L_\delta^A = \langle L_\delta[A], A \cap L_\delta[A]\rangle$ *is admissible whenever* $A \in \underline{\Sigma}_\omega(\mathbb{A})$ *(since* $\langle \mathbb{A}, A\rangle$ *is a* ZF^- *model.*

Note. It is clear from the proof of lemma 1.3.1 that we can replace ZF^- by KP (Kripke–Platek set theory). In this form Lemma 1.3.1 is known as *Ville's Lemma.*

1.4 Barwise Theory

Jon Barwise worked out the syntax and model theory of certain infinitary (but M–finite) languages in countable admissible structures M. In so doing, he created a powerful and flexible tool for set theory, which we shall utilize later in this book. In this chapter we give an introduction to Barwise's work.

1.4.1 Syntax

Let M be admissible. Barwise developed a first order theory in which arbitrary M–finite conjunction and disjunction are allowed. The predicates, however, have only a (genuinely) finite number of argument places and there are no infinite strings of quantifiers. In order that the notion "M–finite" have a meaning for the symbols in our language, we must "arithmetize" the language — i.e. identify its symbols with objects in M. There are many ways of doing this. For the sake of definitness we adopt a specific arithmetization of M–finitary first order logic:

Predicates: For each $x \in M$ and each n such that $1 \leq n < \omega$ we appoint an n–ary predicate $P^n_x =: \langle 0, \langle n, x\rangle\rangle$.

Constants: For each $x \in M$ we appoint a constant $c_x =: \langle 1, x\rangle$.

Variables: For each $x \in M$ we appoint a variable $v_x =: \langle 2, x\rangle$.

Note The set of variables must be M–infinite, since otherwise a single formula might exhaust all the variables.

We let P^2_0 be the identity predicate $\dot{=}$ and also reserve P^2_1 as the $\in$–predicate $(\dot{\in})$.

By a *primitive formula* we mean $Pt_1 \ldots t_n =: \langle 3, \langle P, t_1, \ldots, t_n\rangle\rangle$ where P is an n–ary predicate and $t_1, \ldots, t_n$ are variables or constants.

We then define:

$$\begin{aligned}
&\neg\varphi =: \langle 4, \varphi\rangle, (\varphi \vee \psi) =: \langle 5, \langle \varphi, \psi\rangle\rangle,\\
&(\varphi \wedge \psi) =: \langle 6, \langle \varphi, \psi\rangle\rangle, (\varphi \to \psi) =: \langle 7, \langle \varphi, \psi\rangle\rangle,\\
&(\varphi \leftrightarrow \psi) =: \langle 8, \langle \varphi, \psi\rangle\rangle, \bigwedge v\varphi = \langle 9, \langle v, \varphi\rangle\rangle,\\
&\bigvee v\varphi = \langle 10, \langle v, \varphi\rangle\rangle.
\end{aligned}$$

The infinitary conjunctions and disjunctions are

$$\bigwedge\!\!\!\bigwedge f =: \langle 11, f\rangle, \bigvee\!\!\!\bigvee f =: \langle 12, f\rangle.$$

The set Fml of *first order M–formulae* is then the smallest set X which contains all primitive formulae, is closed under $\neg, \wedge, \vee, \to, \leftrightarrow$, and such that

- If v is a variable and $\varphi \in X$, then $\bigwedge v\varphi \in X$ and $\bigvee v\varphi \in X$.
- If $f = \langle \varphi_i | i \in I\rangle \in M$ and $\varphi_i \in X$ for $i \in I$, then $\bigwedge\!\!\!\bigwedge f \in X$ and $\bigvee\!\!\!\bigvee f \in X$.

(In this case we also write:

$$\bigwedge\!\!\!\bigwedge_{i \in I} \varphi_i =: \bigwedge\!\!\!\bigwedge f, \bigvee\!\!\!\bigvee_{i \in I} \varphi_i =: \bigwedge\!\!\!\bigwedge f.$$

If $B \in M$ is a set of formulae we may also write: $\bigwedge\!\!\!\bigwedge B$ for $\bigwedge\!\!\!\bigwedge_{\varphi \in B} \varphi$.)

It turns out that the usual syntactical notions are $\Delta_1(M)$, including: Fml, $Const$ (set of constants), Vbl (set of variables), $Sent$ (set of all sentences), as are the functions:

$Fr(\varphi) =$ The set of free variables in φ
$\varphi(^v/t) \simeq$ the result of replacing occurences of the variable v by t (where $t \in Vbl \cup Const$), as long as this can be done without a new occurence of t being bound by a quantifier in φ (if t is a variable).

That Vbl, $Const$ are Δ_1 (in fact Σ_0) is immediate. The characteristic function X of Fml is definable by a recursion of the form:

$$X(x) = G(x, \langle X(z) | z \in TC(x))$$

where $G : M^2 \to M$ is Δ_1. (This is an instance of the recursion schema in §1 Lemma 1.1.16. We are of course using the fact that any proper subformula of φ lies in $TC(\varphi)$.)

Now let $h(\varphi)$ be the set of immediate subformulae of φ (e.g. $h(\neg\varphi) = \{\varphi\}$, $h(\bigwedge\!\!\!\bigwedge_{i\in I}\varphi_i) = \{\varphi_i | i \in I\}$, $h(\bigwedge v\varphi) = \{\varphi\}$ etc.) Then h satisfies the condition in §1 Lemma 1.1.16. It is fairly easy to see that

$$Fr(\varphi) = G(\varphi, \langle F(x) | x \in h(\varphi)\rangle)$$

where G is a Σ_1 function defined on Fml. Then $Sent = \{\varphi | Fr(\varphi) = \emptyset\}$.

To define $\varphi(^v/t)$ we first define it on primitive formulae, which is straightforward. We then set:

$(\varphi \wedge \psi)(^v/t) \simeq (\varphi(^v/t) \wedge \psi(^v/t))$ (similarly for $\wedge, \to, \leftrightarrow$)

$\neg\varphi(^v/t) \simeq \neg(\varphi(^v/t))$

$(\bigwedge\!\!\!\bigwedge_{i\in I}\varphi_i)(^v/t) \simeq \bigwedge\!\!\!\bigwedge_{i\in I}(\varphi_i(^v/t))$ similarly for $\bigvee\!\!\!\bigvee$.

$$(\bigwedge u\varphi)(^v/t) \simeq \begin{cases} \bigwedge u\varphi \text{ if } u = v \\ \bigwedge u(\varphi(^v/t)) \text{ if } u \neq v, t \quad \text{(similarly for } \bigvee) \\ \text{otherwise undefined} \end{cases}$$

This has the form:

$$\varphi(^v/t) \simeq G(\varphi, v, t\langle X(^v/t) | X \in h(\varphi)\rangle),$$

where G is $\Sigma_1(M)$. The domain of the function $f(\varphi, v, t) = \varphi(^v/t)$ is $\Delta_1(M)$, however, so f is M–recursive.

(We can then define:

$$\varphi(^{v_1,\ldots,v_n}/t_1,\ldots,t_n) = \varphi(^{v_1}/w_1)\ldots(^{v_n}/w_n)(^{w_1}/t_1)\ldots(^{w_n}/t_n)$$

where $v_1,\ldots,v_n$ is a sequence of distinct variables and $w_1,\ldots,w_n$ is any sequence of distinct variables which are different from $v_1,\ldots,v_n$, $t_1,\ldots,t_n$ and do not occur bound or free in φ. We of cours follow the usual conventions, writing $\varphi(t_1,\ldots,t_n)$ for $\varphi(^{v_1,\ldots,v_n}/t_1,\ldots,t_n)$, taking $v_1,\ldots,v_n$ as known.)

M–finite predicate logic has the axioms:

- all instances of the usual propositional logic axiom schemata (enough to derive all tautologies with the help of modus ponens).
- $\bigwedge\!\!\!\bigwedge_{i\in U}\varphi_i \to \varphi_j,\ \varphi_j \to \bigvee\!\!\!\bigvee_{i\in U}\varphi_i\ (j\in U\in M)$
- $\bigwedge x\varphi \to \varphi(^x/t),\ \varphi(^x/t) \to \bigvee x\varphi$
- $x\dot{=}y \to (\varphi(x) \leftrightarrow \varphi(y))$

The *rules of inference* are:

- $\frac{\varphi,\varphi\to\psi}{\psi}$ (modus ponens)
- $\frac{\varphi\to\psi}{\varphi\to\bigwedge x\psi}$ if $x\notin Fr(\varphi)$
- $\frac{\psi\to\varphi}{\bigvee x\psi\to\varphi}$ if $x\notin Fr(\varphi)$
- $\frac{\varphi\to\psi_i(i\in u)}{\varphi\to\bigwedge\!\!\!\bigwedge\psi_i}$ $(u\in M)$
- $\frac{\psi_i\to\varphi(i\in u)}{\bigvee\!\!\!\bigvee\psi_i\to\varphi}$ $(u\in M)$

We say that φ is *provable* from a set of sentences A iff φ is in the smallest set which contains A and the axioms and is closed under the rules of inference. We write $A\vdash\varphi$ to mean that φ is provable from A. $\vdash\varphi$ means the same as $\emptyset\vdash\varphi$.

However, this definition of provability cannot be stated in the 1st order language of M and rather misses the point which is that a provable formula should have an M–finite proof. This, as it turns out, will be the case whenever A is $\underline{\Sigma}_1(M)$. In order to state and prove this, we must first formalize the notion of proof. Because we have not assumed the axiom of choice to hold in our admissible structure M, we adopt a somewhat unorthodox concept of proof:

Definition 1.4.1. By a *proof from A* we mean a sequence $\langle p_i | i < \alpha\rangle$ such that $\alpha \in \mathrm{On}$ and for each $i < \alpha$, $p_i \subset Fml$ and whenever $\psi \in p_i$, then either $\psi \in A$ or ψ is an axiom or ψ follows from $\bigcup_{h<i} p_h$ by a single application of one of the rules.

Definition 1.4.2. $p = \langle p_i | i < \alpha\rangle$ is a *proof of φ from A* iff p is a proof from A and $\varphi \in \bigcup_{i<\alpha} p_i$.

(**Note** that this definition does *not* require a proof to be M–finite.)

It is straightforward to show that φ is provable iff it has a proof. However, we are more interested in M–finite proofs. If A is $\Sigma_1(M)$ in a parameter q, it follows easily that $\{p \in M | p \text{ is a proof from } A\}$ is $\Sigma_1(M)$ in the same parameter. A more interesting conclusion is:

Lemma 1.4.1. *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is an M–finite proof of φ from A.*

Proof: $(\leftarrow)$ trivial. We prove $(\rightarrow)$

Let $X =$ the set of φ such that there is $p \in M$ which proves φ from A.

Claim: $\{\varphi | A \vdash \varphi\} \subset X$.

Proof: We know that $A \subset X$ and all axioms lie in X. Hence it suffices to show that X is closed under the rules of proof. This must be demonstrated rule by rule. As an example we show:

Claim: Let $\varphi \to \psi_i$ be in X for $i \in u$. Then $\varphi \to \bigwedge\!\!\!\bigwedge_{i\in u} \psi_i \in X$.

Proof: Let $P(p, \varphi)$ mean: p is a proof of φ from A. Then P is $\underline{\Sigma}_1(M)$. We have assumed:

(1) $\bigwedge i \in u \bigvee_P \; P(p, \varphi \to \psi_i)$.
Now let $P(p, x) \leftrightarrow \bigvee z P'(z, p, x)$ where P' is Σ_0. We then have:

(2) $\bigwedge i \in u \bigvee p \bigvee z P'(z, p, \varphi \to \psi_i)$.
Hence there is $v \in M$ with:

(3) $\bigwedge i \in u \bigvee p, z \in v P'(z, p, \varphi \to \psi_i)$.
Set: $w = \{p \in v | \bigvee i \in u \bigvee z \in v P'(z, p, \varphi \to \psi_i)\}$
Set: $\alpha = \bigcup_{p\in w} \mathrm{dom}(p)$. For $i < \alpha$ set:

$$q_i = \bigcup \{p_i | p \in w \wedge i \in \mathrm{dom}(p)\}$$

Then $q = \langle q_i | i < \alpha \rangle \in M$ is a proof.

? But then $q^\frown\{\varphi \longrightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \psi_i\}$ is a proof of $\varphi \longrightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \psi_i$. Hence $\varphi \longrightarrow \bigwedge\!\!\!\bigwedge_{i \in U} \psi_i \in X$.

QED (Lemma 1.4.1)

From this we get the *M–finiteness lemma*:

Lemma 1.4.2. *Let A be $\underline{\Sigma}_1(M)$. Then $A \vdash \varphi$ iff there is $a \subset A$ such that $a \in M$ and $a \vdash \varphi$.*

Proof: $(\leftarrow)$ is trivial. We prove $(\rightarrow)$. Let $p \in M$ be a proof of φ from A. Set:

$a =$ the set of ψ such that for some $i \in \mathrm{dom}(p)$, $\psi \in p_i$ and ψ is neither an axiom nor follows from $\bigcup_{l<i} p_l$ by an application of a single rule.

Then $a \subset A$, $a \in M$, and p is a proof of φ from a. QED (Lemma 1.4.2)

Another consequence of Lemma 1.4.1 is:

Lemma 1.4.3. *Let A be $\Sigma_1(M)$ in q. Then $\{\varphi | A \vdash \varphi\}$ is $\Sigma_1(M)$ in the same parameter (uniformly in the Σ_1 definition of A).*

Proof: $\{\varphi | A \vdash \varphi\} = \{\varphi | \bigvee p \in M \; p$ proves φ from $A\}$.

Corollary 1.4.4. *Let A be $\Sigma_1(M)$ in q. Then "A is consistent" is $\Pi_1(M)$ in the same parameter (uniformly in the Σ_1 definition of A).*

"p proves φ from u" is uniformly $\Sigma_i(M)$. Hence:

Lemma 1.4.5. *$\{\langle u, \varphi \rangle | u \in M \wedge u \vdash \varphi\}$ is uniformly $\Sigma_1(M)$.*

Corollary 1.4.6. *$\{\langle u \in M | u$ is consistent$\}$ is uniformly $\Pi_1(M)$.*

Note. Call a proof p *strict* iff $\overline{\overline{p}}_i = 1$ for $i \in \mathrm{dom}(p)$. This corresponds to the more usual notion of proof. If M satisfies the axiom of choice in the form: Every set is enumerable by an ordinal, then Lemma 1.4.1 holds with "strict proof" in place of "proof". We leave this to the reader.

1.4.2 Models

We will not normally employ all of the predicates and constants in our M–finitary first order logic, but cut down to a smaller set of symbols which we intend to interpret in a model. Thus we define a *language* to be a set $\mathbb{L}$ of predicates and constants. By a *model* of $\mathbb{L}$ we mean a structure:

$$\mathbb{A} = \langle |\mathbb{A}|, \langle t^{\mathbb{A}} | t \in \mathbb{L} \rangle \rangle$$

such that $|\mathbb{A}| \neq \emptyset$, $P^{\mathbb{A}} \subset |\mathbb{A}|^n$ whenever P is an n–ary predicate, and $c^{\mathbb{A}} \in |\mathbb{A}|$ whenever c is a constant. By a *variable assignment* we mean a partial map of f of the variables into $\mathbb{A}$. The *satisfaction relation* $\mathbb{A} \models \varphi[f]$ is defined in the usual way, where $\mathbb{A} \models [f]$ means that the formula φ becomes true in $\mathbb{A}$ if the free variables of φ are interpreted by the assignment f. We leave the definition to the reader, remarking only that:

$$\mathbb{A} \models \bigwedge\!\!\!\bigwedge_{i \in u} \varphi_i[f] \leftrightarrow \bigwedge i \in u \; \mathbb{A} \models \varphi_i[f]$$
$$\mathbb{A} \models \bigvee\!\!\!\bigvee_{i \in u} \varphi_i[f] \leftrightarrow \bigvee i \in u \; \mathbb{A} \models \varphi_i[f]$$

We adopt the usual conventions of model theory, writing $\mathbb{A} = \langle |\mathbb{A}|, t_1^{\mathbb{A}}, \ldots \rangle$ if we think of the predicates and constants of $\mathbb{L}$ as being arranged in a fixed sequence $t_1, t_2, \ldots$. Similarly, if $\varphi = \varphi(v_1, \ldots, v_n)$ is a formula in which at most the variables $v_1, \ldots, v_n$ occur free, we write $\mathbb{A} \models \varphi[a_1, \ldots, a_n]$ for:

$$\mathbb{A} \models \varphi[f] \text{ where } f(v_i) = a_i \text{ for } i = 1, \ldots, n.$$

If φ is a sentence we write: $\mathbb{A} \models \varphi$. If A is a set of sentences, we write $\mathbb{A} \models A$ to mean: $\mathbb{A} \models \varphi$ for all $\varphi \in A$.

Proof: The *correctness theorem* says that if A is a set of $\mathbb{L}$ sentences and $\mathbb{A} \models A$, then A is consistent. (We leave this to the reader.)
Barwise's Completeness Theorem says that the converse holds whenever our admissible structure is countable:

Theorem 1.4.7. *Let M be a countable admissible structure. Let $\mathbb{L}$ be an M–language and let A be a set of statements in $\mathbb{L}$. If A is consistent in M–finite predicate logic, then $\mathbb{L}$ has a model $\mathbb{A}$ such that $\mathbb{A} \models A$.*

Proof: (Sketch)
We make use of the following theorem of Rasiowa and Sikorski: Let $\mathbb{B}$ be a Boolean algebra. Let $X_i \subset \mathbb{B} (i < \omega)$ be such that the Boolean union $\bigcup X_i = b_i$ exists in the sense of $\mathbb{B}$. Then $\mathbb{B}$ has an ultrafilter U such that

$$b_i \in U \leftrightarrow X_i \cap U \neq \emptyset \text{ for } i < \omega.$$

(Proof. Successively choose $c_i (i < \omega)$ by: $c_0 = 1$, $c_{i+1} = c_i \cap b \neq 0$, where $b \in X_i \cup \{\neg b_i\}$. Let $\overline{U} = \{a \in \mathbb{B} | \bigvee i(c_i \subset a)\}$. Then $\overline{U}$ is a filter and extends to an ultrafilter on $\mathbb{B}$.)

Extend the language $\mathbb{L}$ by adding an M–infinite set C of new constants. Call the extended language $\mathbb{L}^*$. Set:

$$[\varphi] =: \{\psi | A \vdash (\psi \leftrightarrow \varphi)\}$$

for $\mathbb{L}^*$–sentences φ. Then

$$\mathbb{B} =: \{[\varphi] | \varphi \in Sent_{\mathbb{L}^*}\}$$

is the Lindenbaum algebra of $\mathbb{L}^*$ with the defining equations:

$$\begin{aligned}
&[\varphi] \cup [\psi] = [\varphi \vee \psi], [\varphi] \cap [\psi] = [\varphi \wedge \psi], \neg[\varphi] = [\neg\varphi] \\
&\bigcup_{i \in U} [\varphi_i] = [\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i](i \in u), \ \bigcap_{i \in U} [\varphi_i] = [\bigwedge\!\!\!\bigwedge_{i \in U} \varphi_i](i \in u) \\
&\bigcup_{c \in C} [\varphi(c)] = [\bigvee v\varphi(v)], \ \bigcap_{c \in C} [\varphi(c)] = [\bigwedge v\varphi(v)].
\end{aligned}$$

The last two equations hold because the constants in C, which do not occur in the axiom A, behave like free variables. By Rasiowa and Sikorski there is then an ultrafilter U on $\mathbb{B}$ which respects the above operations. We define a model $\mathbb{A} = \langle |\mathbb{A}|, \langle t^{\mathbb{A}} | t \in \mathbb{L} \rangle \rangle$ as follows: For $c \in C$ set $[c] =: \{c' \in C | [c = c'] \in U\}$. If $P \in \mathbb{L}$ is an n–place predicate, set:

$$P^{\mathbb{A}}([c_1], \ldots, [c_n]) \leftrightarrow: [Pc_1, \ldots, c_n] \in U.$$

If $t \in \mathbb{L}$ is a constant, set:

$$t^{\mathbb{A}} = [c] \text{ where } c \in C, [t = c] \in U.$$

A straightforward induction then shows:

$$\mathbb{A} \models \varphi[[c_1], \ldots, [c_n] \leftrightarrow [\varphi(c_1, \ldots, c_n)] \in U$$

for formulae $\varphi = \varphi(v_1, \ldots, v_n)$ with at most the free variables $v_1, \ldots, v_n$. In particular, $\mathbb{A} \models \varphi \leftrightarrow [\varphi] \in U$ for $\mathbb{L}^*$–statements φ. Hence $\mathbb{A} \models A$.

QED (Theorem 1.4.7)

Combining the completeness theorem with the M–finiteness lemma, we get the well known *Barwise compactness theorem*:

Corollary 1.4.8. *Let M be countable. Let $\mathbb{L}$ be a language. Let A be a $\underline{\Sigma}_1(M)$ set of sentences in $\mathbb{L}$. If every M–finite subset of $\mathbb{A}$ has a model, then so does A.*

1.4.3 Applications

Definition 1.4.3. By a *theory* or *axiomatized language* we mean a pair $\mathbb{L} = \langle \mathbb{L}_0, A \rangle$ such that $\mathbb{L}_0$ is a language and A is a set of $\mathbb{L}_0$–sentences. We say that $\mathbb{A}$ *models* $\mathbb{L}$ iff $\mathbb{A}$ is a model of $\mathbb{L}_0$ and $\mathbb{A} \models A$. We also write $\mathbb{L} \vdash \varphi$ for: ($\varphi \in Fml_{\mathbb{L}_0}$ and $A \vdash \varphi$). We say that $\mathbb{L} = \langle \mathbb{L}_0, A \rangle$ is $\Sigma_1(M)$ (in p) iff $\mathbb{L}_0$ is $\Delta_1(M)$ (in p) and A is $\Sigma_1(M)$ (in p). Similarly for: $\mathbb{L}$ is $\Delta(M)$ (in p).

We now consider the class of axiomatized languages containing a fixed predicate $\dot\in$, the special constants $\underline{x}(x \in M)$ (we can set e.g. $\underline{x} = \langle 1, \langle 0, x \rangle \rangle$), and the *basic axioms*:

- Extensionality
- $\bigwedge v(v \dot\in \underline{x} \leftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v \dot= \underline{z})$ for $x \in M$.

(Further predicates, constants, and axioms are allowed of course.) We call any such theory an "$\in$–theory". Then:

Lemma 1.4.9. *Let $\mathbb{A}$ be a grounded model of an $\in$–theory $\mathbb{L}$. Then $\underline{x}^{\mathbb{A}} = x \in \mathrm{wfc}(\mathbb{A})$ for $x \in M$.*

In an $\in$–theory $\mathbb{L}$ we often adopt the set of axioms ZFC^- (or more precisely $\mathsf{ZFC}^-_{\mathbb{L}}$). This is the collection of all $\mathbb{L}$–sentences φ such that φ is the universal quantifier closure of an instance of the ZFC^- axiom schemata — but does *not* contain infinite conjunctions or disjunctions. (Hence the collection of all subformulae is finite.) (Similarly for ZF^-, ZFC, ZF.)

(**Note** If we omit the sentences containing constants, we get a subset $B \subset \mathsf{ZFC}^-$ which is equivalent to ZFC^- in $\mathbb{L}$. Since each element of B contain at most finitely many variables, we can restrict further to the subset B' of sentences containing only the variables $v_i (i < \omega)$. If $\omega \in M$ and the set of predicates in $\mathbb{L}$ is M–finite, then B' will be M–finite. Hence ZFC^- is equivalent in $\mathbb{L}$ to the statement $\bigwedge\!\!\!\bigwedge B'$.)

We now bring some typical applications of $\in$–theories. We say that an ordinal α is *admissible in* $a \subset \alpha$ iff $\langle L_\alpha[a], \in, a \rangle$ is admissible.

Lemma 1.4.10. *Let $\alpha > \omega$ be a countable admissible ordinal. Then there is $a \subset \omega$ such that α is the least ordinal admissible in a.*

This follows straightforwardly from:

Lemma 1.4.11. *Let M be a countable admissible structure. Let $\mathbb{L}$ be a consistent $\underline{\Sigma}_1(M)$ $\in$–theory such that $\mathbb{L} \vdash ZF^-$. Then $\mathbb{L}$ has a grounded model $\mathbb{A}$ such that $\mathbb{A} \neq \mathrm{wfc}(\mathbb{A})$ and $\mathrm{On} \cap \mathrm{wfc}(\mathbb{A}) = \mathrm{On} \cap M$.*

We first show that lemma 1.4.11 implies lemma 1.4.10. Take $M = L_\alpha$. Let $\mathbb{L}$ be the M–theory with:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in M), \dot{a}$

Axioms: Basic axioms $+\mathsf{ZFC}^- + \underline{\beta}$ is not admissible in $\dot{a}(\beta \in M)$

Then $\mathbb{L}$ is consistent, since $\langle H_{\omega_1}, \in, a\rangle$ is a model, where a is any $a \subset \omega$ which codes a well ordering of type $\geq \alpha$. Let $\mathbb{L}$ be a grounded model of $\mathbb{L}$ such that $\mathrm{wfc}(\mathbb{A}) \neq \mathbb{A}$ and $\mathrm{On} \cap \mathrm{wfc}(\mathbb{A}) = \alpha$. Then $\mathrm{wfc}(\mathbb{A})$ is admissible by §3. Hence so is $L_\alpha[a]$ where $a = \dot{a}^{\mathbb{A}}$. QED

Note This is a very typical application in that Barwise theory hands us an ill founded model, but our interest is entirely concentrated on its well founded part.

Note Pursuing this method a bit further we can use lemma 1.4.11 to prove: Let $\omega < \alpha_0 < \ldots < \alpha_{n-1}$ be a sequence of countable admissible ordinals. There is $a \subset \omega$ such that $\alpha_i =$ the i–th $\alpha < \omega$ which is admissible in $a(1 = 0, \ldots, n-1)$.

We now prove lemma 1.4.11 by modifying the proof of the completeness theorem. Let $\Gamma(v)$ be the set of formulae: $v \in \mathrm{On}$, $v > \underline{\beta}(\beta \in \mathrm{On} \wedge M)$. Add an M–infinite (but $\underline{\Delta}_1(M)$) set E of new constants to $\mathbb{L}$. Let $\mathbb{L}'$ be $\mathbb{L}$ with the new constants and new axioms: $\Gamma(e)$ $(e \in E)$. Then $\mathbb{L}'$ is consistent, since any M–finite subset of the axioms can be modeled in an arbitrary grounded model $\mathbb{A}$ of $\mathbb{L}$ by interpreting the new constants as sufficiently large elements of α. As in the proof of completeness we then add a new class C of constants which is not M–finite. We assume, however, that C is $\Delta_1(M)$. We add no further axioms, so the elements of C behave like free variables. The so–extended language $\mathbb{L}''$ is clearly $\underline{\Sigma}_1(M)$.

Now set:
$$\Delta(v) =: \{v \notin \mathrm{On}\} \cup \bigcup_{\beta \in M} \{v \leq \underline{\beta}\} \cup \bigcup_{e \in E} \{e < v\}.$$

Claim Let $c \in C$. Then $\bigcup\{[\varphi] | \varphi \in \Delta(c)\} = 1$ in the Lindenbaum algebra of $\mathbb{L}''$.

Proof: Suppose not. Then there is ψ such that $A \vdash \varphi \to \psi$ for all $\varphi \in \Delta(c)$ and $A \cup \{\neg\psi\}$ is consistent, where $\mathbb{L}'' = \langle \mathbb{L}''_0, A \rangle$. Pick an $e \in E$ which does not occur in ψ. Let A^* be the result of omitting the axioms $\Gamma(e)$ from A. Then $A^* \cup \{\neg\psi\} \cup \Gamma(e) \vdash c \leq e$. By the finiteness lemma there is $\beta \in M$ such that $A^* \cup \{\neg\psi\} \cup \{\underline{\beta} \leq e\} \vdash c \leq e$. But e behaves here like a free variable, so $A^* \cup \{\neg\psi\} \vdash c \leq \underline{\beta}$. But $A \supset A^*$ and $A \cup \{\neg\psi\} \vdash \underline{\beta} < c$. Hence $A \cup \{\neg\psi\} \vdash \underline{\beta} < \underline{\beta}$ and $A \cup \{\neg\psi\}$ is inconsistent.
Contradiction! QED (Claim)

Now let U be an ultrafilter on the Lindenbaum algebra of $\mathbb{L}''$ which respects both two operations listed in the proof of the completeness theorem and the unions $\bigcup\{[\varphi] | \varphi \in \Delta(c)\}$ for $c \in C$. Let $X = \{\varphi | [\varphi] \in U\}$. Then as before, $\mathbb{L}''$ has a grounded model $\mathbb{A}$, all of whose elementes have the form $c^{\mathbb{A}}$ for $c \in C$ and such that:

$$\mathbb{A} \models \varphi \text{ iff } \varphi \in X$$

for $\mathbb{L}''$–statements φ. But then for each $x \in A$ we have either $x \notin \mathrm{On}_{\mathbb{A}}$ or $x < \beta$ for a $\beta \in \mathrm{On} \cap M$ or $e^{\mathbb{A}} < v$ for all $e \in E$. In particular, if $x \in \mathrm{On}_{\mathbb{A}}$ and $x > \beta$ for all $\beta \in \mathrm{On} \cap M$, then there is $e^{\mathbb{A}} < x$ in $\mathbb{A}$. But $\beta < e^{\mathbb{A}}$ for all $\beta \in \mathrm{On} \cap M$. Hence $\mathrm{On}_{\mathbb{A}} \setminus \mathrm{On}_M$ has no minimal element in $\mathbb{A}$.

QED (Lemma 1.4.11)

Another typical application is:

Lemma 1.4.12. *Let W be an inner model of* ZFC. *Suppose that, in W, U is a normal measure on κ. Let $\tau > \kappa$ be regular in W. Set: $M = \langle H^W_\tau, U \rangle$. Assume that M is countable in V. Then for any $\alpha \leq \kappa$ there is $\overline{M} = \langle \overline{H}, \overline{U} \rangle$ such that*

- $\overline{M} \models \overline{U}$ *is a normal measure on $\overline{\kappa}$ for a $\overline{\kappa} \in \overline{M}$*
- $\overline{M}$ *iterates to M in α many steps.*

(Hence $\overline{M}$ is iterable, since M is.)

Proof: The case $\alpha = 0$ is trivial, so assume $\alpha > 0$. Let δ be least such that $L_\delta(M)$ is admissible. Let $\mathbb{L}$ be the $\in$–theory on $L_\delta(M)$ with:

Predicate: $\dot{\in}$

Constants: $\underline{x} (x \in L_\delta(M)), \dot{M}$

Axiom:
- Basic axioms $+\mathsf{ZFC}^-$
- $\dot{M} = \langle \dot{H}, \dot{U} \rangle \models (\mathsf{ZFC}^- + \dot{U}$ is a normal measure on a $\kappa < \dot{H})$

- $\dot{M}$ iterates to $\underline{M}$ in $\underline{\alpha}$ many steps.

It will suffice to show:

Claim $\mathbb{L}$ is consistent.

We first show that the claim implies the theorem. Let $\mathbb{A}$ be a grounded model of $\mathbb{L}$. Then $\mathbb{L}_\delta(M) \subset \text{wfc}(\mathbb{A})$. Hence $M, \overline{M} \in \text{wfc}(\mathbb{A})$, where $\overline{M} = \dot{M}^{\mathbb{A}}$. But then in $\mathbb{A}$ there is an iteration $\langle \overline{M}_i | i \leq \alpha \rangle$ of $\overline{M}$ to M. By absoluteness $\langle \overline{M}_i | i \leq \alpha \rangle$ really is such an iteration. QED

We now prove the claim.

Case 1 $\alpha < \kappa$
Iterate $\langle W, U \rangle$ α many times, getting $\langle W_i, U_i \rangle (i \leq \alpha)$ with iteraton maps $\pi_{i,j}$. Then $\pi_{0,\alpha}(\alpha) = \alpha$. Set $M_i = \pi_{0,i}(M)$. Then $\langle M_i | i \leq \alpha \rangle$ is an iteration of M with iteration maps $\pi_{i,j} \upharpoonright M_i$. But $M_\alpha = \pi_{0,\alpha}(M)$. Hence $\langle H_{\kappa^+}, M \rangle$ models $\pi_{0,\alpha}(\mathbb{L})$. But then $\pi_{0,\alpha}(\mathbb{L})$ is consistent. Hence so is $\mathbb{L}$. QED

Case 2 $\alpha = \kappa$
Iterate $\langle W, U \rangle$ β many times, where $\pi_{0,\beta}(\kappa) = \beta$. Then $\langle M_i | i \leq \beta \rangle$ iterates M to M_β in β many steps. Hence $\langle H_{\kappa^+}, M \rangle$ models $\pi_{0,\beta}(\mathbb{L})$. Hence $\pi_{0,\beta}(\mathbb{L})$ is consistent and so is $\mathbb{L}$. QED (Lemma 1.4.12)

Barwise theory is useful in situations where one is given a transitive structure Q and wishes to find a transitive structure $\overline{Q}$ with similar properties inside an inner model. Another tool, which is often used in such situations, is Schoenfield's lemma, which, however, requires coding Q by a real. Unsurprizingly, Schoenfield's lemma can itself be derived from Barwise theory. We first note the well known fact that every Σ^1_2 condition on a real is equivalent to a $\Sigma_1(H_{\omega_1})$ condition, and conversely. Thus it suffices to show:

Lemma 1.4.13. *Let $H_{\omega_1} \models \varphi[a], a \subset \omega$, where φ is Σ_1. Then:*

$$H_{\omega_1} \models \varphi[a] \text{ in } L(a).$$

Proof: Let $\varphi = \bigvee z\psi$, where ψ is Σ_0. Let $H_{\omega_1} \models \psi[z, a]$ where $\text{rn}(z) = \delta < \alpha < \omega_1$ and α is admissible in a. Let $\mathbb{L}$ be the language on $L_\alpha(a)$ with:

Predicate: $\dot{\in}$

Constants: $\underline{x} (x \in L_\alpha(a))$

Axioms: Basic axioms $+\mathsf{ZFC}^- + \bigvee z(\psi(z, \underline{a}) \wedge \text{rn}(z) = \underline{\delta})$.

Then $\mathbb{L}$ is consistent, since $\langle H_{\omega_1}, a\rangle$ is a model. We cannot necessarily chose α such that it is countable in $L(a)$, however. Hence, working in $L(a)$, we apply a Skolem–Löwenheim argument to $L_\alpha(a)$, getting countable $\overline{\alpha}, \overline{\delta}, \pi$ such that $\pi : L_{\overline{\alpha}}(a) \prec L_\alpha(a)$ and $\pi(\overline{\delta}) = \delta$. Let $\overline{\mathbb{L}}$ be defined from $\overline{\delta}$ over $L_{\overline{\alpha}}(a)$ as $\mathbb{L}$ was defined from δ over $L_\alpha(a)$. Then $\overline{\mathbb{L}}$ is consistent by corollary 1.4.4. Since $L_{\overline{\alpha}}(a)$ is countable in $L(a)$, $\overline{\mathbb{L}}$ has a grounded model $\mathbb{A} \in L(a)$. But then there is $z \in \mathbb{A}$ such that $\mathbb{A} \models \psi[z, a]$ and $rn^{\mathbb{A}}(z) = \overline{\delta}$. Thus $rn(z) = \overline{\beta} \in \mathrm{wfc}(\mathbb{A})$ and $z \in \mathrm{wfc}(\mathbb{A})$. Thus $\mathrm{wfc}(\mathbb{A}) \models \psi[z, a]$, where $\mathrm{wfc}(\mathbb{A}) \subset H_{\omega_1}$ in $L(a)$. Hence $H_{\omega_1} \models \varphi[a]$ in $L(a)$. QED

Chapter 2

Basic Fine Structure Theory

2.1 Introduction

Fine structure theory arose from the attempt to describe more precisely the way the constructable hierarchy grows. There are many natural questions. We know for instance by Gödel's condensation lemma that there are countable γ such that L_γ models $\mathsf{ZFC}^- + \omega_1$ exists. This means that some $\beta < \gamma$ is a cardinal in L_γ but not in L. Hence there is a subset $b \subset \beta$ lying in L but not in L_γ. Hence there must be a least $\alpha > \gamma$ such that such a subset lies in $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$. What happens there, and what do such α look like? It turns out that there is then a $\underline{\Sigma}_\omega(L_\alpha)$ injection of L_α into β, and that α can be anything — even a successor ordinal. The body of methods used to solve such questions is called *fine structure theory*.

In chapter 1 we developed an elaborate body of methods for dealing with admissible structures. In order to deal with questions like the above ones, we must try to adapt these methods to an arbitrary L_α. A key concept in this endeavor is that of *amenability*:

Definition 2.1.1. A transitive structure $M = \langle |M|, \in, A_1, \ldots, A_n\rangle$ is *amenable* iff $A_i \cap x \in M$ for all $x \in M$, $i = 1, \ldots, n$.

Omitting almost all proofs, we now sketch the fine structural demonstration that if $\beta < \alpha$ and $b \subset \beta$ is a $\underline{\Sigma}_\omega(L_\alpha)$ set with $b \notin L_\alpha$, then there is a $\underline{\Sigma}_\omega(L_\alpha)$ injection of L_α into β. Given any structure of the form $M = \langle L_\alpha, B_1, \ldots, B_n\rangle$ we define its *projectum* to be the least ρ such that there is $A \subset L_\rho$ such that A is $\underline{\Sigma}_1(M)$ and $A \notin M$. (Thus $\langle L_\rho, A\rangle$ is amenable whenever $A \subset L_\rho$ is $\underline{\Sigma}_1(M)$.) It turns out that, whenever ρ is the projectum of L_α, then there is a $\underline{\Sigma}_1(L_\alpha)$ injection of L_α into ρ. Now suppose that b is $\underline{\Sigma}_1(L_\alpha)$, where α, β, b

are as above. Let ρ^0 be the projectum of L_α and let f^0 be a $\underline{\Sigma}_1(L_\alpha)$ injection of L_α into ρ^0. Clearly $\rho^0 \leq \beta$, so f^0 injects L_α into β. Now suppose that b is $\underline{\Sigma}_2(L_\alpha)$ but not $\underline{\Sigma}_1(L_\alpha)$.

If $p^0 \leq \beta$ the result follows as before, so suppose $\beta < \rho^0$. By the existence of f^0 there is an $A^0 \subset \rho^0$ which completely codes L_α and f^0. The structure $N^0 = \langle L_{\rho^0}, A^0 \rangle$ is then called a *reduct* of L_α. It then follows that any set $a \subset L_{\rho^0}$ is $\Sigma_n(N^0)$ if and only if it is $\underline{\Sigma}_{n+1}(L_\alpha)$. In particular b is $\Sigma_1(N^0)$ and $b \notin N^0$. Hence $\rho^1 \leq \beta$, where ρ^1 is the projectum of N^0. It turns out, however, that in very many respects N^0 behaves exactly like an L_α. In particular there is a $\underline{\Sigma}_1(N^0)$ injection f^1 of N^0 into ρ^1. Thus $f^1 \circ f^0$ is a $\underline{\Sigma}_\omega(L_\alpha)$ injection of L_α into β.

Now suppose that b is $\underline{\Sigma}_3(L_\alpha)$ but not $\underline{\Sigma}_2(L_\alpha)$ and that $\beta < \rho^1$. Then b is $\underline{\Sigma}_2(N^0)$ and we can repeat the above proof, using N^0 in place of L_α. This gives us a reduct N^1 of N^0 and a $\underline{\Sigma}_1(N^1)$ injection f^2 of N^1 into the projectum ρ^2 of N^1. But b is $\underline{\Sigma}_1(N^1)$ and $b \notin N^1$. Hence $\rho^2 \leq \beta$. $f^2 \circ f^1 \circ f^0$ is then a $\underline{\Sigma}_\omega(L_\alpha)$ injection of L_α into β. Proceeding in this way, we see that if b is $\underline{\Sigma}_{n+1}(L_\alpha)$, then there is a $\underline{\Sigma}_\omega(L_\alpha)$ map $f = f^n \circ \ldots \circ f^0$ injecting L_α into β. But b is $\underline{\Sigma}_{n+1}$ for some n.

The first proof of the above result was due to Hilary Putnam and did not use the full fine structure analysis we have just outlined. However, our analysis yielded many new insights; giving for instance the first proof that L_α is $\underline{\Sigma}_n$ uniformizable for all $n \geq 1$. (I.e. every $\underline{\Sigma}_n$ relation is uniformizable by a $\underline{\Sigma}_n$ function.)

Not long afterwards fine structure theory was used to prove some deep global properties of L, such as:

$$L \models \Box_\beta \text{ for all infinite cardinals } \beta.$$

It was also used to prove the covering lemma for L. That, in turn, led to extended versions of fine structure theory which could be used to analyze larger inner models, in which some large cardinals could be realized. (Here, however, the fine structure theory was needed not only to analyze the inner model, but even to define it in the first place.)

Carrying out the above analysis of L requires a very fine study of definability over an arbitrary L_α. In order to achieve this, however, one must overcome some formidable technical obstacles which arise from Gödel's definition of the constructible hierarchy: At successors α, L_α is not even closed under ordered pairs, let alone other basic set functions like unit set, crossproduct etc. One solution is to employ the theory of *rudimentary functions* in an auxiliary role. These functions, which were discovered by Gandy and Jensen, are exactly the functions which are generated by the schemata for primitive recursive functions when the recursion schema is omitted. (Cf. the remark following chapter 1, §2, Lemma 1.2.4). If $\text{rn}(x_i) < \gamma$ for $i = 1, \ldots, n$ and f is rudimentary, then $\text{rn}(f(x_1, \ldots, x_n)) < \gamma + \omega$. All reasonable "elementary" set theoretic functions are rudimentary. If α is a limit ordinal, then L_α is closed under rudimentary functions. If α is a successor, then closing L_α under rudimentary functions yields a transitive structure L^*_α of rank $\alpha+\omega$. It then turns out that every $\underline{\Sigma}_\omega(L^*_\alpha)$ definable subset of L_α is already $\underline{\Sigma}_\omega(L_\alpha)$, and conversely. Hence we can, in effect, replace the rather weak definability theory of L_α by the rather nice definability theory of L^*_α. (This method was used in [JH], except that L^*_α was given a different but equivalent definition, since the rudimentary functions were not yet known.) It turns out that if N is transitive and rudimentarily closed, and $\text{Rud}(N)$ is defined to be the closure of $N \cup \{N\}$ under rudimentary functions, then $\mathbb{P}(N) \cap \text{Rud}(N) = \text{Def}(N)$. This suggests an alternative version of the constructible hierarchy in which every level is rudimentarily closed. We shall index this hierarchy by the class Lm of limit ordinals, setting:

$$J_\omega = H_\omega = \text{Rud}(\emptyset)$$

$$J_{\alpha+\omega} = \text{Rud}(J_\alpha) \text{ for } \alpha \in \text{Lm}$$

$$J_\lambda = \bigcup_{\nu<\lambda} J_\nu \text{ for } \lambda \text{ a limit p.t. of Lm.}$$

Note. Setting $J = \bigcup_\alpha J_\alpha$, we have: $J = L$. In fact $J_\alpha = L_\alpha$ whenever α is pr closed.

Note. This indexing was introduced by Sy Friedman. In [FSC] we indexed by *all* ordinals, so that our $J_{\omega\alpha}$ corresponds to the J_α of [FSC]. The usage in [FSC] has been followed by most authors. Nonetheless, we here adopt Friedman's usage, which seems to us more natural, since we then have: $\alpha = \text{rn}(J_\alpha) = \text{On} \cap J_\alpha$.

In the following section, we develop the theory of rudimentary functions.

2.2 Rudimentary Functions

Definition 2.2.1. $f : V^n \to V$ is a *rudimentary* (rud) *function* iff it is generated by successive applications of schemata (i) – (v) in the definition of *primitive recursive* in chapter 1, §2.

A relation $R \subset V^n$ is rud iff there is a rud function f such that: $R\vec{x} \leftrightarrow f(\vec{x}) = 1$. In chapter 1, §1.2 we established that:

Lemma 2.2.1. *Lemmas 1.2.1 – 1.2.4 of chapter 1, §1.2 hold with* 'rud' *in place of 'pr'.*

Note. Our definition of 'rud function', like the definition of 'pr function' is ostensibly in second order set theory, but just as in chapter 1, §1.2 we can work in ZFC by talking about rud *definitions*. The notion of rud definition is defined like that of pr definition, except that instances of schema (vi) are not allowed. As before, we can assign to each rud definition s a rud function $F_s : V^n \to V$ with the property that $F_s^M = F_s \upharpoonright M$ whenever M is admissible and $F_s^M : M^n \to M$ is the function on M defined by s. But then if M is transitive and closed under rud functions, it follows by induction on the length of s that there is a unique $F_s^M = F_s \upharpoonright M$.

A rudimentary function can raise the rank of its arguments by at most a finite amount:

Lemma 2.2.2. *Let $f : V^n \to V$ be* rud*. Then there is $p < \omega$ such that*

$$f(\vec{x}) \subset \mathbb{P}^p(TC(x_1 \cup \ldots \cup x_n)) \text{ for all } x_1, \ldots, x_n.$$

(Hence $\mathrm{rn}(f\vec{x}) \leq \max\{\mathrm{rn}(x_1), \ldots, \mathrm{rn}(x_n)\} + p$ *and* $\bigcup^p f(\vec{x}) \subset TC(x_1 \cup \ldots \cup x_n)$*.)*

Proof: Call any such p *sufficient* for f. Then if p is sufficient, so is every $q \geq p$. By induction on the defining schemata for f, we prove that f has a sufficient p. If f is given by an initial schema, this is trivial. Now let $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$. Let p be sufficient for h and q be sufficient for $g_i (i = 1, \ldots, m)$. It follows easily that $p + q$ is sufficient for f. Now let $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$, where p is sufficient for g. It follows easily that p is sufficient for f. QED

By lemma 2.2.1 and chapter 1 lemma 1.2.3 (i) we know that every Σ_0 relation is rud. We now prove the converse. In fact we shall prove a stronger result. We first define:

Definition 2.2.2. $f : V^n \to V$ is *simple* iff whenever $R(z, \vec{y})$ is a Σ_0 relation, then so is $R(f(\vec{x}), \vec{y})$.

The simple functions are obviously closed under composition. The simplicity of a function f is equivalent to the conjunction of the two conditions:

(i) $x \in f(\vec{y})$ is Σ_0

(ii) If $A(z, \vec{u})$ is Σ_0, then $\bigwedge z \in f(\vec{x}) A(z, \vec{u})$ is Σ_0,

for given these we can verify by induction on the Σ_0 definition of R that $R(f(\vec{x}), \vec{y})$ is Σ_0.
But then:

Lemma 2.2.3. *All* rud *functions are simple.*

Proof: Using the above facts we verify by induction on the defining schemata of f that f is simple. The proof is left to the reader. QED

In particular:

Corollary 2.2.4. *Every* rud *function f is Σ_0 as a relation. Moreover $f \upharpoonright U$ is uniformly $\Sigma_0(U)$ whenever U is transitive and* rud *closed.*

Corollary 2.2.5. *Every* rud *relation is Σ_0.*

We now list some facts which follow easily from the foregoing lemmas.

Fact 1. Let $f \colon V^n \to V$ such that $z \in f(\vec{x})$ is a Σ_0 relation. If there is a rudimentary function g such that $f(\vec{x}) \subset g(\vec{x})$, then f is a rudimentary function.

Proof. Lemma 2.2.1 and Lemma 1.2.3 we have: $f(\vec{x}) = g(\vec{x}) \cap \{z \mid z \in f(\vec{x})\}$.
QED(Fact 1)

Fact 2. Let $f \colon V^n \to V$ such that $y = f(\vec{x})$ is a Σ_0 relation. If there is a rudimentary function g such that $f(\vec{x}) \in g(\vec{x})$, then f is a rudimentary function.

Proof. $z \in f(\vec{x})$ is Σ_0, since it is expressed by: $\bigvee y \in g(\vec{x}) z \in y$. But then $f(\vec{x}) \subset \bigcup g(\vec{x})$. QED(Fact 2)

Definition 2.2.3.

$$\Gamma(u) =: u \cup \bigcup u \cup \{\{x, y\} \mid x, y \in u\} \cup$$
$$\{x \cup y \mid x, y \in u\} \cup \{x \cap y \mid x, y \in u\} \cup \{x \setminus y \mid x, y \in u\}.$$

Definition 2.2.4. We define rudimentary function C^*_n $(n < \omega)$ by: $C^*_0(u) = u$, $C^*_{n+1}(u) = \Gamma(C^*_n(u))$.

Fact 3. Let $n < \omega$. If $p < \omega$ is sufficiently large, then for all n we have:

- If $x_1, \ldots, x_n \in u$, then $\langle x_1, \ldots, x_n \rangle \in C^*_p(u)$
- If $\langle x_1, \ldots, x_n \rangle \in u$, then $x_1, \ldots, x_n \in C^*_p(u)$.

In Chapter 1, §2 we relativized the concept 'pr' to 'pr in $A_1, \ldots, A_n$'. We can do the same thing with 'rud'.

Definition 2.2.5. Let $A_i \subset V (i = 1, \ldots, m)$. $f : V^n \to V$ is *rudimentary in* $A_1, \ldots, A_n$ (rud *in* $A_1, \ldots, A_n$) if and only if it is obtained by successive applications of the schemata (i) – (v) and:

$$f(x) = \chi_{A_i}(x) \; (i = 1, \ldots, n)$$

where χ_A is the characteristic function of A.

Lemma 2.2.1 and 2.2.2 obviously hold with 'rud in $A_1, \ldots, A_n$' in place of 'rud'. Lemma 2.2.3 and its corollaries do *not* hold, however, since e.g. the relation $\{x\} \in A$ is not Σ_0 in A.

However, we do get:

Lemma 2.2.6. *Every function rud in $A_1, \ldots, A_n$ is obtainable as a composition of rud function, and the functions*

$$f(x) = A_i \cap x (i = 1, \ldots, n).$$

Proof: Let RC be the set of such compositions. More precisely, RC is the set of functions obtainable from rud function by successive application of the schemata:

- $f(\vec{x}) = A_i \cap g(\vec{x}) \; (i = 1, \ldots, n)$
- $f(\vec{x}) = g(\vec{h}(\vec{x}))$

It suffices to show:

Claim. If g is in RC, then so is:

$$f(u, \vec{x}) = \bigcup_{z \in u} g(z, \vec{x}).$$

We define:

Definition 2.2.6. Let $f : V^n \to V$ be in RC. f is *viable* if and only the function:

$$f^*(u) = f \restriction (u \cap \mathrm{TP}_n)$$

is in RC, where TP_n =the class of all n-tuples $\langle x_1, \ldots, x_n \rangle$.

Then:

(1) If f is viable, then f' is in RC, where

$$f'(u, \vec{x}) = \bigcup_{z \in u} f(z, \vec{x}).$$

Proof. Set $k(u, \vec{x}) = \{\langle z\, \vec{x}\rangle \mid z \in u\}$. Then k is rud. But $f^*(u, \vec{x}) = f \restriction k(u, \vec{x})$. Hence $\bigcup \mathrm{rng}(f^*(u, \vec{x})) = f'(u, \vec{x})$. QED(1)

Hence it suffices to show:

Claim. Every f in RC is viable.

We prove this by induction on the defining schemata of f. We show:

(A) Every rud function is viable

(B) If $g(\vec{x})$ is viable, so is $f(\vec{x}) = A_i \cap g(\vec{x})$

(C) If $g(y_1, \ldots, y_n)$ is viable and $h_i(\vec{x})$ is viable for $i = 1, \ldots, n$, then $f(\vec{x}) = g(\vec{h}(\vec{x}))$ is viable.

We first prove (A). Let $f(x_1, \ldots, x_n)$ be rud. Set $f_0^n(u, \vec{x}) = \{\langle f(\vec{x}), \langle \vec{x}\rangle\rangle\}$. We then recursively define:

$$f_{i+1}^n(u, x_{i+1}, \ldots, x_n) = \bigcup_{z \in u} f_i^n(u, z, x_{i+1}, \ldots, x_n)$$

for $i < n$. Then $f_n^n(u) = f \restriction u^n$ and $f^*(u) = f_n^n(u) \restriction u$. QED(A)

We now prove (B). Set $k(a, w) = \{\langle a \cap y, x\rangle \mid \langle y, x\rangle \in w\}$. Then k is rudimentary. To see this, note that $x \in k(a, w)$ is Σ_0, since:

$$z \in k(a, w) \iff \bigvee y, x \in C_n^*(w)(z = \langle a \cap y, x\rangle \wedge \langle y, x\rangle \in w)$$

for sufficient n. ut $k(a, w) \subset C_n^*(\{a, w\})$ for sufficient n. But

$$k(a, f^*(u)) = \{\langle a \cap f(\vec{x}), \langle \vec{x}\rangle\rangle \mid \langle \vec{x}\rangle \in u\}.$$

Set: $\tilde{f}(u)0 : \bigcup \mathrm{rng}(f^*) = \bigcup_{\langle \vec{x}\rangle \in u} f(\vec{x})$. Let $a = A_i \cap \tilde{f}(u)$. Then:

$$\begin{aligned} k(a, f^*(u) &= \{\langle A_i \cap f(\vec{x}), \langle \vec{x}\rangle\rangle \mid \langle \vec{x}\rangle \in u\} \\ &= f_{A_i}^*(u) \text{ where } f_{A_i}(\vec{x}) = A_i \cap f(\vec{x}). \end{aligned}$$

Hence $f_{A_i}^*(u)$ lies in RC and f_{A_i} is viable. QED(B)

We now prove (C). Let $f(\vec{x}) = g(\vec{h}(\vec{x}))$, where g is m-ary anf h_i is n-ary for $i = 1, \ldots, m$. Set:

$$y = k(\vec{w}) \iff \bigvee \vec{z}, x(y = \langle\langle\vec{z}\rangle, x\rangle \wedge \bigwedge_{i=1}^{m} \langle z_i, x\rangle \in w_i),$$

where the existence quantifier can be bounded by $C_p^*(\{\vec{w}\})$ for sufficient p, and: $k(\vec{w}) \in C_p^*(\{\vec{w}\})$ for sufficient p. But:

$$k(h_1^*(u), \ldots, h_m^*(u)) = \{\langle h_1(\vec{x}), \ldots, h_m(\vec{x}), \langle\vec{x}\rangle\rangle \mid \langle\vec{x}\rangle \in u \cap \mathrm{TP}_m\}.$$

Set: $\tilde{k}(u) = \mathrm{rng}(k(h_1^*(u), \ldots, h_m^*(u)))$. Then:

$$\tilde{k}(u) = \{\langle h_1(\vec{x}), \ldots, h_m(\vec{x})\rangle \mid \langle\vec{x}\rangle \in u \cap \mathrm{TP}_m\}.$$

Hence:

$$\mathrm{prod}(g^*(u), \tilde{h}(u)) = f \upharpoonright u \cap \mathrm{TP}_n = f^*(u),$$

where:

$$\mathrm{prod}(w, v) = \{\langle y, z \,\mathrm{rg} \mid \bigvee x(\langle y, x\rangle \in w \wedge \langle x, z\rangle \in v)\}.$$

But $u = \mathrm{prod}(w, v)$ is Σ_0 since it is expressed by:

$$\bigvee y, x \in C_P^*(u) \bigvee z \in C_p^*(v))\langle y, x\rangle \in w \wedge \langle x, z\rangle \in v)$$

for sufficient p. Moreover: $\mathrm{prod}(w, v) \subset C_p^*(\{w, v\})$ for sufficient p. Hence prod is a rud function and f^* lies in RC. Hence f is viable.

QED (Lemma 2.2.6)

Definition 2.2.7. X is *rudimentarily closed* (rud closed) if and only if it is closed under rudimentary functions. $\langle M, A_1, \ldots, A_n\rangle$ is rud closed if and only if M is closed under functions rudimentary in $A_1, \ldots, A_n$.

If $M = \langle |M|, A_1, \ldots, A_n\rangle$ is transitive and rud closed, then it is amenable, since it is closed under $f(x) = x \cap A$. By lemma 2.2.6 we then have:

Corollary 2.2.7. *Let $M = \langle |M| A_1, \ldots, A_n\rangle$ be transitive. M is rud closed iff it is amenable and $|M|$ is rud closed.*

Corresponding to corollary 2.2.4 we have:

Corollary 2.2.8. *Every function f which is* rud *in A is Σ_1 in A as a relation. Moreover $f \upharpoonright U$ is $\Sigma_1(\langle U, A \cap U\rangle)$ by the same Σ_1 definition whenever $\langle U, A \cap U\rangle$ is transitive and rud closed. (Similarly for "*rud *in $A_1, \ldots, A_n$".)*

Proof: f is obtained from rud functions by successive application of the schemata:

- $f(\vec{x}) = A \cap g(\vec{x})$
- $f(\vec{x}) = g(\vec{h}(\vec{x}))$.

The result follows by induction on these schemata. QED (Corollary 2.2.8)

In Chapter 1 §2.2 we extended the notion of "pr definition" so as to deal with functions pr in classes $A_1, \ldots, A_n$. We can do the same for rudimentary functions:

We appoint new designated function variables $\dot{a}_1, \ldots, \dot{a}_n$ and define the set of rud *definitions in* $a_1, \ldots, a_n$ exactly as before, except that we omit the schema (vi). Given $A_1, \ldots, A_n$ we can, exactly as before, assign to each rud definition s in $\dot{a}_1, \ldots, \dot{a}_n$ a function $F_s^{A_1, \ldots, A_n}$ are then exactly the functions rud in $A_1, \ldots, A_n$. Since lemma 2.2.6 (and with it, corollary 2.2.8) is proven by induction on the defining schemata, its proof implicitly defines an algorithm which assigns to each s a Σ_1 formula φ_s which defines $F_s^{\vec{A}}$.

Corresponding to chapter 1 §1 Lemma 1.1.13 we have:

Lemma 2.2.9. *Let f be* rud *in $A_1, \ldots, A_n$, where each A_i is* rud *in $B_1, \ldots, B_m$. Then f is* rud *in $B_1, \ldots, B_m$.*

The proof is again by induction on the defining schemata. It shows, in fact that f is *uniformly* rud in $\vec{B}$ in the sense that its rud definition from $\vec{B}$ depends only on its rud definition from $\vec{A}$ and the rud definition of A_i from $\vec{B}$ $(i = 1, \ldots, n)$.

We also note:

Lemma 2.2.10. *Let $\pi : \overline{M} \to_{\Sigma_0} M$, where $\overline{M}, M$ are* rud *closed. Then π preserves rudimentarity in the following sense: Let $\overline{f}$ be defined from the predicates of $\overline{M}$ by the* rud *definition s. Let f be defined from the predicates of M by s. Then $\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x}))$ for $x_1, \ldots, x_n \in \overline{M}$.*

Proof: Let φ_s be the canonical Σ_1 definition. Then $\overline{M} \models \varphi_s[y, \vec{x}] \to M \models \varphi_s[\pi(y), \pi(\vec{x})]$ by Σ_0–preservation. QED (Lemma 2.2.10)

We now define:

Definition 2.2.8.
rud(U) =: The closure of U under rud functions

rud$_{A_1, \ldots, A_n}(U)$ =: The closure of U under functions rud in $A_1, \ldots, A_n$

(Hence $\text{rud}(U) = \text{rud}_\emptyset(U)$.)

Lemma 2.2.11. *If U is transitive, then so is* $\text{rud}(U)$.

Proof: Let $W = \text{rud}(U)$. Let $Q(x)$ mean: $TC(\{x\}) \subset W$. By induction on the defining schemata of f we show:

$$(Q(x_1) \wedge \ldots \wedge Q(x_n)) \rightarrow Q(f(x_1, \ldots, x_n))$$

for $x_1, \ldots, x_n \in W$. The details are left to the reader. But $x \in U \rightarrow Q(x)$ and each $z \in W$ has the form $f(\vec{x})$ where f is rud and $x_1, \ldots, x_n \in U$. Hence $TC(\{z\}) \subset W$ for $z \in W$. QED

The same proof shows:

Corollary 2.2.12. *If U is transitive, then so is* $\text{rud}_{\vec{A}}(U)$.

Using Corollary 2.2.12 and Lemma 2.2.3 we get:

Lemma 2.2.13. *Let U be transitive and $W = \text{rud}(U)$. Then the restriction of any $\underline{\Sigma}_0(W)$ relation to U is $\underline{\Sigma}_0(U)$.*

Proof: Let R be $\underline{\Sigma}_0(W)$. Let $R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p})$ where R' is $\Sigma_0(W)$ and $p_1, \ldots, p_n \in W$. Let $p_i = f_i(\vec{z})$, where f_i is rud and $z_1, \ldots, z_n \in U$. Then for $x_1, \ldots, x_m \in U$:

$$\begin{aligned} R(\vec{x}) \ &\leftrightarrow R'(\vec{x}, \vec{f}(\vec{z})) \\ &\leftrightarrow R''(\vec{x}, \vec{z}) \end{aligned}$$

where R'' is $\Sigma_0(U)$, by lemma 2.2.3. QED (Lemma 2.2.13)

We now define:

Definition 2.2.9. Let U be transitive.

$$\begin{aligned} &\text{Rud}(U) =: \text{rud}(U \cup \{U\}) \\ &\text{Rud}_{\vec{A}}(U) =: \text{rud}_{\vec{A}}(U \cup \{U\}) \end{aligned}$$

Then $\text{Rud}(U)$ is a proper transitive extension of U. By Lemma 2.2.13:

Corollary 2.2.14. $\text{Def}(U) = \mathbb{P}(U) \cap \text{Rud}(U)$ *if $U \neq \emptyset$ is transitive.*

Proof: If $A \in \text{Def}(U)$, then A is $\underline{\Sigma}_0(U \cup \{U\})$. Hence $A \in \text{Rud}(U)$. Conversely, if $A \in \text{Rud}(U)$, then A is $\underline{\Sigma}_0(U \cup \{U\})$ by lemma 2.2.13. It follows easily that $A \in \text{Def}(U)$. QED (Corollary 2.2.14)

Note. To see that $A \in \mathrm{Def}(U)$, consider the $\in$–language augmented by a new constant $\dot{U}$ which is interpreted by U. We assign to every Σ_0 formula φ in this language a first order formula φ' not containing $\dot{U}$ such that for all $x_1, \ldots, x_n \in U$:

$$U \cup \{U\} \models \varphi[\vec{x}] \leftrightarrow U \models \varphi'[\vec{x}].$$

(Here x_i is taken to interpret v_i where $v_1, \ldots, v_n$ is an arbitrarily chosen sequence of distinct variables, including all variables which occur free in φ.) We define φ' by induction on φ. For primitive formulae we set first:

$$(v \in w)' = v \in w, (v \in \dot{U})' = v = v,$$
$$(\dot{U} \in v)' = v \neq v, (\dot{U} \in \dot{U}) = \bigvee v\; v \neq v.$$

For sentential combinations we do the obvious thing:

$$(\varphi \wedge \psi)' = (\varphi' \wedge \psi'), (\neg\varphi)' = \neg\varphi',$$

etc. Quantifiers are treated as follows:

$$(\bigwedge v \in w\varphi)' = \bigwedge v \in w\varphi'$$
$$(\bigwedge v \in \dot{U}\varphi)' = \bigwedge v\varphi'$$

Given finitely many rud functions $s_1, \ldots, s_p$ we say that they constitute a *basis* for the rud function iff every rud function is obtainable by successive application of the schemata:

- $f(x_1, \ldots, x_n) = x_j$ $(j = 1, \ldots, n)$
- $f(\vec{x}) = s_i(g_1(\vec{x}), \ldots, g_m(\vec{x}))$ $(i = 1 \ldots, p)$

Note that if $s_1, \ldots, s_p$ is a basis, then $\mathrm{rud}(U)$ is simply the closure of U under the finitely many functions $s_1, \ldots, s_p$. We shall now prove the *Basis Theorem*, which says that the rud functions possess a finite basis. We first define:

Definition 2.2.10. $(x, y) =: \{\{x\}, \{x, y\}\}; (x) = x$,
$(x_1, \ldots, x_n) = (x_1, (x_2, \ldots, x_n))$ for $n \geq 2$.

(Note: Our "official" notation for n–tuples is $\langle x_1, \ldots, x_n\rangle$. However, we have refrained from specifying its definition. Thus we do not know whether $(\vec{x}) = \langle\vec{x}\rangle$.)

We also set:

Definition 2.2.11.

$$\begin{aligned} &x \otimes y = \{(z,w) | z \in x \wedge w \in y\} \\ &\mathrm{dom}^*(x) = \{z | \bigvee y (y,z) \in x\} \\ &x^* z = \{y | (y,z) \in x\} \end{aligned}$$

Theorem 2.2.15. *The following functions form a basis for the* rud *function:*

$$\begin{aligned} &F_0(x,y) = \{x,y\} \\ &F_1(x,y) = x \setminus y \\ &F_2(x,y) = x \otimes y \\ &F_3(x,y) = \{(u,z,v) | z \in x \wedge (u,v) \in y\} \\ &F_4(x,y) = \{(u,v,z) | z \in x \wedge (u,v) \in y\} \\ &F_5(x,y) = \bigcup x \\ &F_6(x,y) = \mathrm{dom}^*(x) \\ &F_7(x,y) = \{(z,w) | z,w \in x \wedge z \in w\} \\ &F_8(x,y) = \{x^* z | z \in y\} \end{aligned}$$

Proof: The proof stretches over several subclaims. Call a function f *good* iff it is obtainable from $F_0, \ldots, F_8$ by successive applications of the above schemata. Then every good function is rud. We must prove the converse. We first note:

Claim 1 The good functions are closed under composition — i.e. if $g, h_1, \ldots, h_n$ are good, then so is $f(\vec{x}) = g(\vec{h}(\vec{x}))$.

Proof: Set $G =$ the set of good function $g(y_1, \ldots, y_v)$ such that whenever $h_i(\vec{x})$ is good for $i = 1, \ldots, r$, then so is $f(\vec{x}) = g(\vec{h}(\vec{x}))$. By a straightforward induction on the defining schemata it is easily shown that all good functions are in G. QED (Claim 1)

Claim 2 The following functions are good:

$$\begin{aligned} &\{x,y\}, x \setminus y, x \otimes y, x \cup y = \bigcup \{x,y\}, \\ &x \cap y = x \setminus (x \setminus y), \{x_1, \ldots, x_n\} = \{x_1\} \cup \ldots \cup \{x_n\}, \\ &C_n(u) = u \cup \bigcup u \cup \ldots \cup \overbrace{\bigcup \cdots \bigcup}^{n} u, (x_1, \ldots, x_n) \end{aligned}$$

(since $(x_1, \ldots, x_n)$ is obtained by iteration of F_0.) By an $\in$*–formula* we mean a first order formula containing only $\dot{\in}$ as a non logical predicate. If $\varphi = \varphi(v_1, \ldots, v_n)$ is any $\in$–formula in which at most the distinct variables $(v_1, \ldots, v_n)$ occur free, set:

$$t_\varphi(u) =: \{(x_1, \ldots, x_n) | \vec{x} \in u \wedge \langle u, \in \rangle \models \varphi[\vec{x}]\}.$$

Note. We follow the usual convention of suppressing the list of variables. We should, of course, write: $t_{\varphi, v_1, \ldots, v_n}(u)$.

Note. Recall our convention that $\vec{x} \in u$ means that $x_i \in u$ for $i = 1, \ldots, n$.

Then t_φ is *rud*. We claim:

Claim 3 t_φ is good for every $\in$–formula φ.

Proof:

(1) It holds for $\varphi = v_i \in v_j$ $(1 \leq i < j \leq n)$

Proof: For $i = 2, 3$ set:

$$F_i^0(u, w) = w, \ F_i^{m+1}(u, w) = F_i(u, F_i^m(u, w))$$

then F_i^m is good for all m. For $m \geq 1$ we have:

$$\begin{aligned} F_2^m(u, w) &= \{(x_1, \ldots, x_m, z) | \vec{x} \in u \wedge z \in w\} \\ F_3^m(u, w) &= \{(y, x_1, \ldots, x_m, z) | \vec{x} \in u \wedge (y, z) \in w\} \end{aligned}$$

We also set

$$\begin{aligned} u^{(m)} &= \{(x_1, \ldots, x_m) | \vec{x} \in u\} \\ &= F_2^{m-1}(u, u) \end{aligned}$$

If $j = n$, then

$$\begin{aligned} t_\varphi(u) &= \{(x_1, \ldots, x_n) | \vec{x} \in u \wedge x_i \in x_j\} \\ &= F_2^{i-1}(u, F_3^{n-i-1}(u, F_7(u, u))). \end{aligned}$$

Now let $n > j$. Noting that:

$$F_4(u^{(m)}, w) = \{(y, z, x_1, \ldots, x_m) | \vec{x} \in u \wedge (y, z) \in w\},$$

we have:

$$t_\varphi(u) = F_2^{i-1}(u, F_3^{j-i-1}(u, F_4(u^{(n-j)}, F_7(u, u)))).$$

QED (1)

(2) It holds for $\varphi = v_i \in v_i$.

Proof: $t_\varphi(w) = \emptyset = w \setminus w$.

(3) If it holds for $\varphi = \varphi(v_1, \ldots, v_n)$, then for $\neg\varphi$.

Proof:

$$t_{\neg\varphi}(w) = (w^{(n)} \setminus t_\varphi(w)).$$

QED (3)

(4) If it holds for φ, ψ, then for $\varphi \wedge \psi$, $\varphi \vee \psi$. (Hence for $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ by (3).)

Proof:

$$t_{\varphi \vee \psi}(w) = t_\varphi(w) \cup t_\psi(w) = \bigcup\{t_\varphi(w), t_\psi(w)\}$$
$$t_{\varphi \wedge \psi}(w) = t_\varphi(w) \cap t_\psi(w), \text{ where } x \cap y = (x \setminus (x \setminus y)).$$

QED (4)

(5) If it holds for $\varphi = \varphi(u, v_1, \ldots, v_n)$, then for $\bigwedge u\varphi$, $\bigvee_u \varphi$.

Proof:

$$t_{\bigvee u\varphi}(w) = F_6(t_\varphi(\omega), t_\varphi(\omega)) \text{ hence}$$
$$t_{\bigwedge u\varphi}(w) = t_{\neg \bigvee u \neg\varphi}(w) \text{ by (3)}$$

QED (5)

(6) It holds for $\varphi = v_i = v_j$ $(i, j \leq n)$.

Proof: Let $\psi(v_1, \ldots, v_n) = \bigwedge z(z \in v_i \leftrightarrow z \in v_j)$. Then for $(\vec{x}) \in U^{(n)}$ we have:

$$(\vec{x}) \in t_\psi(u \cup \bigcup u) \leftrightarrow x_i = x_j,$$

since $x_i, x_j \subset (u \cup \bigcup u)$. Hence

$$t_\varphi(u) = u^{(n)} \cap t_\psi(u \cup \bigcup u).$$

QED (6)

(7) It holds for $\varphi = v_j \in v_i$ $(i < j)$

Proof:

$$v_j \in v_i \leftrightarrow \bigvee u(u = v_j \wedge u \in v_i).$$

We apply (6), (5) and (4). QED (7)

But then if $\varphi(v_1, \ldots, v_n) = Qu_1, \ldots Qu_n \psi(\vec{u}, \vec{v})$ is any formula in prenex normal form, we apply (1), (2), (6), (7) and (3), (4) to see that t_ψ is good. But then t_φ is good by iterated applications of (5). QED (Claim 3)

In our application we shall use the function t_φ only for Σ_0 formulae φ. We shall make strong use of the following well known fact, which can be proven by induction on n.

Fact Let $\varphi = \varphi(v_1, \ldots, v_m)$ be a Σ_0 formula in which at most n quantifiers occur. Let u be any set and let $x_1, \ldots, x_m \in u$. Then $V \models \varphi[\vec{x}] \leftrightarrow C_n(u) \models \varphi[\vec{x}]$.

Definition 2.2.12. Let $f : V^n \to V$ be rud. f is *verified* iff there is a good $f^* : V \to V$ such that $f''U^n \subset f^*(U)$ for all sets U. We then say that f^* *verifies* f.

Claim 4 Every verified function is good.

Proof: Let f be verified by f^*. Let φ be the Σ_0 formula: $y = f(x_1, \ldots, x_n)$. For sufficient m we know that for any set u we have:

$$y = f(\vec{x}) \leftrightarrow (y, \vec{x}) \in t_\varphi(C_m(u \cup f^*(u)))$$
$$\text{for } y, \vec{x} \in u \cup f^*(u).$$

Define a good function F by:

$$F(u) =: (f^*(u) \otimes u^{(n)}) \cap t_\varphi(C_m(u \cup f^*(u))).$$

Then $F(u)$ is the set of $(f(\vec{x}), \vec{x})$ such that $\vec{x} \in u$. In particular, if $u = \{x_1, \ldots, x_n\}$, then:

$$F_8(F(\{\vec{x}\}), \{(\vec{x})\}) = \{f(\vec{x})\}$$

and $f(\vec{x}) = \bigcup F_8(F(\{\vec{x}\}), \{(\vec{x})\})$. QED (Claim 4)

Thus it remains only to prove:

Claim 5 Every rud function is verified.

Proof: We proceed by induction on the defining schemata of f.

Case 1 $f(\vec{x}) = x_i$

Take $f^*(u) = u = u \setminus (u \setminus u)$.

Case 2 $f(\vec{x}) = x_i \setminus x_j$

Let φ be the formula $z \in x \setminus y$. Then for $z, x, y \in v$ we have

$$\begin{aligned} z \in x \setminus y &\leftrightarrow v \models \varphi[z, x, y] \\ &\leftrightarrow (z, x, y) \in t_\varphi(v). \end{aligned}$$

But $x, y \in u \to x \setminus y \subset \bigcup u$. Hence for all x, y, u and all z we have:

$$z \in x \setminus y \leftrightarrow (z, x, y) \in t_\varphi(u \cup \bigcup u).$$

Hence:

$$f''u^n \subset \{x \setminus y | x, y \in u\} = F_8(t_\varphi(u \cup \bigcup u), u^{(2)}).$$

QED (Case 2)

Case 3 $f(\vec{x}) = \{x_i, x_j\}$

Then $f''u^n = \{\{x, y\} | x, y \in u\} = \bigcup u^{(2)}$. QED (Case 3)

Case 4 $f(\vec{x}) = g(\vec{h}(\vec{x}))$

Let h_i^* verify h_i and g^* verify g. Then $f^*(u) = g^*(\bigcup_i h_i^*(u))$ verifies f.

QED (Case 4)

Case 5 $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$. Let g^* verify g. Let $\varphi = \varphi(w, y, \vec{x})$ be the Σ_0 formula: $\bigvee z \in y\ w \in g(z, \vec{x})$. For sufficient m we have:

$$\bigvee z \in y\ w \in g(z, \vec{x}) \leftrightarrow (w, y, \vec{x}) \in t_\varphi(C_m(u \cup \bigcup g^*(u)))$$

for all $w, y, \vec{x} \in u \cup \bigcup g^*(u)$.

Set $F(u) = t_\varphi(C_m(u \cup \bigcup g^*(u)))$. Then $g(z, \vec{x}) \subset \bigcup g^*(u)$ whenever $y, \vec{x} \in u$ and $z \in y$. Hence

$$F(u)^*(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$$

for $y, \vec{x} \in u$. Hence

$$f''u^{n+1} \subset F_8(F(u), u^{(n+1)}).$$

QED (Theorem 2.2.15)

Combining Theorem 2.2.15 with Lemma 2.2.6 we get:

Corollary 2.2.16. *Let $A_1, \ldots, A_n \subset V$. Then $F_0, \ldots, F_8$ together with the functions $a_i(x) = x \cap A_i (i = 1, \ldots, n)$ form a basis for the functions which are rudimentary in $A_1, \ldots, A_n$.*

Let $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$. '$\models_M$' denotes the satisfaction relation for M and '$\models_M^{\Sigma_n}$' denotes its restriction to Σ_n formulae. We can make good use of the basis theorem in proving:

Lemma 2.2.17. *$\models_M^{\Sigma_0}$ is uniformly $\Sigma_1(M)$ over transitive* rud *closed $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$.*

Proof: We shall prove it for the case $n = 1$, since the extension of our proof to the general case is then obvious. We are then given: $M = \langle |M|, \in, A \rangle$. By a *variable evaluation* we mean a function e which maps a finite set of variables of the M–language into $|M|$. Let E be the set of such evaluations. If $e \in E$, we can extend it to an evaluation e^* of all variables by setting:

$$e^*(v) = \begin{cases} e(v) \text{ if } v \in \mathrm{dom}(e) \\ \emptyset \text{ if not} \end{cases}$$

$\models_M \varphi[e]$ then means that φ becomes true in M if each free variable v in φ is interpreted by $e^*(v)$.

We assume, of course, that the first order language of M has been "arithmetized" in a reasonable way — i.e. the syntactic objects such as formulae and variables have been identified with elements of H_ω in such a way that the basic syntactic relations and operations become recursive. (Without this the assertion we are proving would not make sense.) In particular the set Vbl of variables, the set Fml of formulae, and the set Fml_0 of Σ_0–formulae are all recursive (i.e. $\Delta_1(H_\omega)$). We first note that every $\Sigma_0(M)$ relation is rud, or equivalently:

(1) Let φ be Σ_0. Let $v_1, \ldots, v_n$ be a sequence of distinct variables containing all variables occuring free in φ. There is a function f uniformly rud in A such that

$$\models_M \varphi[e] \leftrightarrow f(e^*(v_1), \ldots e^*(v_n)) = 1$$

for all $e \in E$.

Proof: By induction on φ. We leave the details to the reader.

QED (1)

The notion A–*good* is defined like "good" except that we now add the function $F_9(x, y) = x \cap A$ to our basis. By Corollary 2.2.16 we know that every function rud in A is A–good. We now define in H_ω an auxiliary term language whose terms represent the A–good function. We first set: $\dot{F}_i(x, y) =: \langle i, \langle x, y\rangle\rangle$ for $i = 0, \ldots, 9$: $\dot{x} = \langle 10, x\rangle$. The set Tm of *Terms* is then the smallest set such that

- $\dot{v}$ is a term whenever $v \in Vbl$
- If t, t' are terms, then so is $\dot{F}_i(t, t')$ for $i = 0, \ldots, 9$.

Applying the methods of Chapter 1 to the admissible set H_ω it follows easily that the set Tm is recursive (i.e. $\Delta_1(H_\omega)$). Set

$C(t) \simeq$: The smallest set C such that the term $t \in C$ and C is closed under subterms (i.e. $\dot{F}_i(s, s') \in C \to s, s' \in C$).

Then $C(t) \in H_\omega$ for $t \in Tm$, and the function $C(t)$ is recursive (hence $\Delta_1(H_\omega)$). Since Vbl is recursive, the function
$Vbl(t) \simeq: \{v \in Vbl | \dot{v} \in C(t)\}$ is recursive.

We note that:

(2) Every recursive relation on H_ω is uniformly $\Sigma_1(M)$.

Proof: It suffices to note that: H_ω is uniformly $\Sigma_1(M)$, since

$$x \in H_\omega \leftrightarrow \bigvee f \bigvee u \bigvee n \varphi(f, u, n, x)$$

where φ is the Σ_0 formula: f is a function $\wedge$ u is transitive $\wedge n \in \omega \wedge f : n \leftrightarrow u \wedge x \in u$.

QED (2)

Given $e \in E$ we recursively define an evaluation $\langle \overline{e}(t) | t \in Tm \rangle$ by:

$$\begin{aligned} &\overline{e}(\dot{v}) = e^*(v) \text{ for } v \in Vbl \\ &\overline{e}(\dot{F}_i(t,s)) = F_i(\overline{e}(t), \overline{e}(s)). \end{aligned}$$

Then:

(3) $\{\langle y, e, t\rangle | e \in E \wedge t \in Tm \wedge y = \overline{e}(t)\}$ is uniformly $\Sigma_1(M)$.

Proof: Let $e \in E$, $t \in Tm$. Then $y = \overline{e}(t)$ can be expressed in M by:

$$\bigvee g \bigvee u \bigvee v (u = C(t) \wedge v = Vbl(t) \wedge \varphi(y, e, u, v, y, t))$$

where φ is the Σ_0 formula:

$(g$ is a function $\wedge \operatorname{dom}(g) = u \wedge \bigwedge x \in v \; x \in u$

$$\begin{aligned} &\wedge \bigwedge x \in v((x \in \operatorname{dom}(e) \wedge g(\dot{x}) = e(x)) \vee \\ &\quad \vee (x \notin \operatorname{dom}(e) \wedge g(\dot{x}) = \emptyset)) \\ \wedge \bigwedge_{i=0}^{9} &\bigwedge t, s, i \in u(t = \dot{F}_i(s, s') \to \\ &\quad \to g(t) = F_i(g(s), y(s'')) \\ \wedge y = g(t)) & \end{aligned}$$

QED (3)

(4) Let $f(x_1, \dots, x_n)$ be A–good. Let $v_1, \dots, v_n'$ be any sequence of distinct variables. There is $t \in Tm$ such that

$$f(e^*(v_1), \dots, e^*(v_n)) = \overline{e}(t)$$

for all $e \in E$.

Proof: By induction on the defining schemata of f. If $f(\vec{x}) = x_i$, we take $t = \dot{v}_i$. If $e^*(\vec{v})) = \overline{e}(s_i)$ for $e \in \mathbb{E}(i = 0, 1)$, and $f(\vec{x}) = F_i(g_0(\vec{x}), g_1(\vec{x}))$, we set $t = \dot{F}_i(s_0, s_1)$. Then

$$\overline{e}(t) = F_i(\overline{e}(s_0), \overline{e}(s_1)) = F_i(g_0(\vec{x}), g_1(\vec{x})) = f(\vec{x}).$$

QED (4)

But then:

(5) Let φ be a Σ_0 formula. There is $t \in Tm$ such that $M \models \varphi[e] \leftrightarrow \overline{e}(t) = 1$ for all $e \in E$.

Proof: Let $v_1, \dots, v_n$ be a sequence of distinct variables containing all variables which occur free in φ. Then

$$M \models \varphi[e] \leftrightarrow M \models \varphi[e^*(v_1), \dots, e^*(v_n)]$$

for all $e \in E$. Set

$$(*)\ f(\vec{x}) = \begin{cases} 1 \text{ if } M \models \varphi[\vec{x}] \\ 0 \text{ if not.} \end{cases}$$

Then f is rudimentary, hence A–good. Let $t \in Tm$ such that

$$(**)\ f(e^*(v_1), \ldots, e^*(v_n)) = \overline{e}(t).$$

Then: $M \models \varphi[e] \leftrightarrow \overline{e}(t) = 1.$ QED (5)

(5) is, however, much more than an existence statement, since our proofs are *effective*: Clearly we can effectively assign to each Σ_0 formula φ a sequence $v(\varphi) = \langle v_1, \ldots, v_n \rangle$ of distinct variables containing all variables which occur free in φ. But the proof that the f defined by $(*)$ is rud in fact implicity defines a rud definition D_φ such that D_φ defines such an $f = f_{D_\varphi}$ over any rud closed $M = \langle M, \in, A \rangle$. The proof that f is A–good is by induction on the defining schemata and implicitly defines a term $t = T_\varphi$ which satisfies $(**)$ over any rud closed M. Thus our proofs implicitly describe an algorithm for the function $\varphi \mapsto T_\varphi$. Hence this function is recursive, hence uniformly $\Sigma_1(M)$. But then Σ_0 satisfaction can be defined over M by:

$$M \models \varphi[e] \leftrightarrow: \overline{e}(T_\varphi) = 1.$$

QED (Lemma 2.2.17)

Corollary 2.2.18. *Let $n \geq 1$. $\models_M^{\Sigma_n}$ is uniformly $\Sigma_n(M)$ for transitive* rud *closed structures $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$.*

(We leave this to the reader.)

2.2.1 Condensation

The *condensation lemma* for rud closed sets $U = \langle U, \in \rangle$ reads:

Lemma 2.2.19. *Let $U = \langle U, \in \rangle$ be transitive and* rud *closed. Let $X \prec_{\Sigma_1} U$. Then there is an isomorphism $\pi : \overline{U} \overset{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive and* rud *closed. Moreover, $\pi(f(\vec{x})) = f(\pi(\vec{x}))$ for all* rud *functions f.*

Proof: X satisfies the extensionality axiom. Hence by Mostowski's isomorphism theorem there is $\pi : \overline{U} \overset{\sim}{\longleftrightarrow} X$, where $\overline{U}$ is transitive. Now let f be rud and $x_1, \ldots, x_n \in \overline{U}$. Then there is $y' \in X$ such that $y' = f(\pi(\vec{x}))$, since $X \prec_{\Sigma_1} U$. Let $\pi(y) = y'$. Then $y = f(\vec{x})$, since the condition '$y = f(\vec{x})$' is Σ_0 and π is Σ_1–preserving. QED (Lemma 2.2.19)

The condensation lemma for rud closed $M = \langle |M|, \in, A_1, \ldots, A_n \rangle$ is much weaker, however. We state it for the case $n = 1$.

Lemma 2.2.20. *Let* $M = \langle |M|, \in, A\rangle$ *be transitive and* rud *closed. Let* $X \prec_{\Sigma_1} M$. *There is an isomorphism* $\pi : \overline{M} \overset{\sim}{\longleftrightarrow} X$, *where* $\overline{M} = \langle |\overline{M}|, \in, \overline{A}\rangle$ *is transitive and* rud *closed. Moreover:*

(a) $\pi(\overline{A} \cap x) = A \cap \pi(x)$ *for* $x \in \bar{M}$.

(b) Let f *be* rud *in* A. *Let* f *be characterized by:* $f(\vec{x}) = f_0(\vec{x}, A \cap f_1(\vec{x}))$, *where* f_0, f_1 *are* rud. *Set:* $\overline{f}(\vec{x}) =: f_0(\vec{x}, \overline{A} \cap f_1(\vec{x}))$. *Then:*

$$\pi(\overline{f}(\vec{x})) = f(\pi(\vec{x})).$$

The proof is left to the reader.

2.3 The J_α hierarchy

We are now ready to introduce the alternative to Gödel's constructible hierarchy which we had promised in §1. We index it by ordinals from the class Lm of limit ordinals.

Definition 2.3.1.

$$\begin{aligned} &J_\omega = \mathrm{Rud}(\emptyset) \\ &J_{\beta+\omega} = \mathrm{Rud}(J_\beta) \text{ for } \beta \in \mathrm{Lm} \\ &J_\lambda = \bigcup_{\gamma<\lambda} J_\gamma \text{ for } \lambda \text{ a limit point of } \mathrm{Lm} \end{aligned}$$

It can be shown that $L = \bigcup\limits_\alpha J_\alpha$ and, indeed, that $L_\alpha = J_\alpha$ for a great many α (for instance closed α). Note that $J_\omega = L_\omega = H_\omega$.

By §2 Corollary 2.2.14 we have:

$$\mathbb{P}(J_\alpha) \cap J_{\alpha+\omega} = \mathrm{Def}(J_\alpha),$$

which pinpoints the resemblance of the two hierarchies. However, we shall not dwell further on the relationship of the two hierarchies, since we intend to consequently employ the J–hierarchy in the rest of this book. As usual, we shall often abuse notation by not distinguishing between J_α and $\langle J_\alpha, \in\rangle$.

Lemma 2.3.1. $\mathrm{rn}(J_\alpha) = \mathrm{On} \cap J_\alpha = \alpha$.

Proof: By induction on $\alpha \in \mathrm{Lm}$. For $\alpha = \omega$ it is trivial. Now let $\alpha = \beta + \omega$, where $\beta \in \mathrm{Lm}$. Then $\beta = \mathrm{On} \cap J_\beta \in \mathrm{Def}(J_\beta) \subset J_\alpha$. Hence $\beta + n \in J_\alpha$ for

$n < \omega$ by rud closure. But $\mathrm{rn}(J_\alpha) \leq \beta + \omega = \alpha$ since J_α is the rud closure of $J_\alpha \cup \{J_\alpha\}$. Hence $\mathrm{On} \cap J_\alpha = \alpha = \mathrm{rn}(J_\alpha)$.

If α is a limit point of Lm the conclusion is trivial. QED (Lemma 2.3.1)

To make our notation simpler, define

Definition 2.3.2. $\mathrm{Lm}^* =$ the limit points of Lm.

It is sometimes useful to break the passage from J_α to $J_{\alpha+\omega}$ into ω many steps. Any way of doing this will be rather arbitrary, but we can at least do it in a uniform way. As a preliminary, we use the basis theorem (§2 Theorem 2.2.15) to prove:

Lemma 2.3.2. *There is a* rud *function* $s : V \to V$ *such that for all* U*:*

(a) $U \subset s(U)$

(b) $\mathrm{rud}(U) = \bigcup\limits_{n<\omega} s^n(U)$

(c) If U *is transitive, so is* $s(U)$.

Proof: Define rud functions $G_i (i = 0, 1, 2, 3)$ by:

$$\begin{aligned}
G_0(x, y, z) &= (x, y)\\
G_1(x, y, z) &= (x, y, z)\\
G_2(x, y, z) &= \{x, (y, z)\}\\
G_3(x, y, z) &= x^* y
\end{aligned}$$

Set:

$$s(U) =: U \cup \bigcup_{i=0}^{9} F_i^U U^2 \cup \bigcup_{i=0}^{3} G_i^U U^3.$$

(a) is then immediate, (b) is immediate by the basis theorem. We prove (c).

Let $a \in s(U)$. We claim: $a \subset s(U)$. There are 14 cases: $a \in U$, $a = F_i(x, y)$ for an $i = 0, \ldots, 8$, where $x, y \in U$, and $a = G_i(x, y, z)$ where $x, y, z \in U$ and $i = 0, \ldots, 3$. Each of the cases is quite straightforward. We give some example cases:

- $a = F(x, y) = x \otimes y$. If $z \in a$, then $z = (x', y')$ where $x' \in x$, $y' \in y$. But then $x', y' \in U$ by transitivity and $z = G_0(x', y', x') \in s(U)$.
- $a = F_3(x, y) = \{(w, z, v) | z \in x \wedge (u, v) \in y\}$. If $a' = (w, z, v) \in a$, then $w, z, v \in U$ by transitivity and $a' = G_1(w, z, v) \in s(U)$.

- $a = F_8(x, y)$. If $a' \in a$, then $a' = x^*z$ where $z \in y$. Hence $z \in U$ by transitivity and $a' = G_3(x, z, z) \in s(U)$.
- $a = G_0(x, y, z) = \{\{x\}, \{x, y\}\}$. Then $a \subset F_0''U^2 \subset s(U)$.
- $a = G_1(x, y, z) = (x, y, z) = \{\{x\}, \{x, (y, z)\}\}$. Then $\{x\} = F_0(x, x) \in s(U)$ and $\{x, (y, z)\} = G_2(x, y, z) \in s(U)$. QED (Lemma 2.3.2)

If we then set:

Definition 2.3.3. $S(U) = s(U \cup \{U\})$ we get:

Corollary 2.3.3. *S is a* rud *function such that*

(a) $U \cup \{U\} \subset S(U)$

(b) $\bigcup_{n<\omega} S^n(U) = \mathrm{Rud}(U)$

(c) If U is transitive, so is $S(U)$.

We can then define:

Definition 2.3.4.

$$S_0 = \emptyset$$
$$S_{\nu+1} = S(S_\nu)$$
$$S_\lambda = \bigcup_{\nu<\lambda} S_\nu \text{ for limit } \lambda.$$

Obviously then: $J_\gamma = S_\gamma$ for $\gamma \in \mathrm{Lm}$. (It would be tempting to simply define $J_\nu = S_\nu$ for all $\nu \in \mathrm{On}$. We avoid this, however, since it could lead to confusion: At successors ν the models S_ν do not have very nice properties. Hence we retain the convention that whenever we write J_α we mean α to be a limit ordinal.)

Each J_α has Σ_1 knowledge of its own genesis:

Lemma 2.3.4. *$\langle S_\nu | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof: $y = S_\nu \leftrightarrow \bigvee f(\varphi(f) \wedge y = f(\nu))$, where $\varphi(f)$ is the Σ_0 formula:

$$f \text{ is a function } \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge f(0) = \emptyset$$
$$\wedge \bigwedge \xi \in \mathrm{dom}(f)(\xi + 1 \in \mathrm{dom}(f) \to f(\xi + 1) = S(f(\xi)))$$
$$\wedge \bigwedge \lambda \in \mathrm{dom}(f|(\lambda \text{ is a limit} \to f(\lambda) = \bigcup f''\lambda).$$

Thus it suffices to show that the existence quantifier can be restricted to J_α — i.e.

Claim $\langle S_\nu | \nu < \tau\rangle \in J_\alpha$ for $\tau < \alpha$.

Case 1 $\alpha = \omega$ is trivial.

Case 2 $\alpha = \beta + \omega,\ \beta \in \mathrm{Lm}$.
Then $\langle S_\nu | \nu < \beta\rangle \in \mathrm{Def}(J_\beta) \subset J_\alpha$. Hence $S_\beta = \bigcup_{\nu<\beta} S_\nu \in J_\alpha$. By rud closure it follows that $S_{\beta+n} \in J_\alpha$ for $n \subset w$. Hence $S \restriction \nu \in J_\alpha$ for $\nu < \alpha$. QED (Case 2)

Case 3 $\alpha \in \mathrm{Lm}^*$.
This case is trivial since if $\nu < \beta \in \alpha \cap \mathrm{Lm}$. Then $S \restriction \nu \in J_\beta \subset J_\alpha$.
QED (Lemma 2.3.4)

We now use our methods to show that each J_α has a uniformly $\Sigma_1(J_\alpha)$ well ordering. We first prove:

Lemma 2.3.5. *There is a* rud *function $w : V \to V$ such that whenever r is a well ordering of u, then $w(u, r)$ is a well ordering of $s(u)$ which end extends r.*

Proof: Let r_2 be the r–lexicographic ordering of u^2:

$$\langle x, y\rangle r_2 \langle z, w\rangle \leftrightarrow (xrz \vee (x = z \wedge yrw)).$$

Let r_3 be the r–lexicographic ordering of u^3. Set:

$$u_0 = u,\ u_{1+i} = F_i'' u^2 \text{ for } i = 0, \ldots, 8,\ u_{10+i} = G_i'' u^3 \text{ for } i = 0, \ldots, 3.$$

Define a well ordering w_i of u_i as follows: $w_0 = r$, For $i = 0, \ldots, 9$ set

$$\begin{aligned} &xw_{1+i}y \leftrightarrow \bigvee a, b \in u^2 (x = F_i(a) \wedge y = F_i(b) \wedge \\ &\wedge ar_2b \wedge \bigwedge a' \in u^2(a'r_2a \to x \neq F_i(a')) \wedge \\ &\wedge \bigwedge b' \in u^2(b'r_2b \to y \neq F_i(b'))) \end{aligned}$$

For $i = 0, \ldots, 3$ let w_{10+i} have the same definitions with G_i in place of F_i and u^3, r_3 in place of u^2, r_2.

We then set:

$$\begin{aligned} w = w(u) = \{\langle x, y\rangle \in s(u)^2 | \bigvee_{i=0}^{13} ((xw_iy \wedge x, y \notin \bigcup_{h<i} u_n) \vee \\ \vee (x \in \textstyle\bigcup_{h<i} u_n \wedge y \notin \bigcup_{n<i} u_n))\} \end{aligned}$$

(where $\bigcup_{h<0} u_n = \emptyset$). QED (Lemma 2.3.5)

If r is a well ordering of u, then

$$r_u = \{\langle x, y\rangle | \langle x, y\rangle \in r \vee (x \in u \wedge y = u)\}$$

is a well ordering of $u \cup \{u\}$ which end extends r. Hence if we set:

Definition 2.3.5. $W(u, r) =: w(u \cup \{u\}, r_u)$.

We have:

Corollary 2.3.6. *W is a* rud *function such that whenever r is a well ordering of u, then $W(u, r)$ is a well ordering of $S(u)$ which end extends r.*

If we then set:

Definition 2.3.6.

$$\begin{aligned} &<_{S_0} = \emptyset \\ &<_{S_{\nu+1}} = W(S_\nu, <_{S_\nu}) \\ &<_{S_\lambda} = \bigcup_{\nu<\lambda} <_{S_\nu} \text{ for limit } \lambda, \end{aligned}$$

it follows that $<_{S_\alpha}$ is a well ordering of S_α which end extends $<_{S_\nu}$ for all $\nu < \alpha$.

Definition 2.3.7. $<_\alpha = <_{J_\alpha} =: <_{S_\alpha}$ for $\alpha \in \mathrm{Lm}$.

Then $<_\alpha$ is a well ordering of J_α for $\alpha \in \mathrm{Lm}$.

By a close imitation of the proof of Lemma 2.3.4 we get:

Lemma 2.3.7. *$\langle <_{S_\nu} | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J_\alpha)$.*

Proof:

$$y = <_{S_\nu} \leftrightarrow \bigvee f \bigvee g(\varphi(f) \wedge \psi(f, g) \wedge y = g(\nu))$$

where φ is as in the proof of Lemma 2.3.4 and ψ is the Σ_0 formula:

$$\begin{aligned} &g \text{ is a function } \wedge \mathrm{dom}(g) = \mathrm{dom}(f) \\ &\wedge g(0 = \emptyset \wedge \bigwedge \xi \in \mathrm{dom}(g) | \xi + 1 \in \mathrm{dom}(g) \rightarrow \\ &\rightarrow g(\xi + 1) = W(f(\xi), g(\xi))) \\ &\wedge \bigwedge \lambda \in \mathrm{dom}(g)\ (\lambda \text{ is a limit} \rightarrow g(\lambda) = \bigcup g''\lambda). \end{aligned}$$

Just as before, we show that the existence quantifiers can be restricted to J_α. QED (Lemma 2.3.7)

But then:

Corollary 2.3.8. *$<_\alpha = \bigcup\limits_{\nu<\alpha} <_{S_\nu}$ is a well ordering of J_α which is uniformly $\Sigma_1(J_\alpha)$. Moreover $<_\alpha$ end extends $<_\nu$ for $\nu \in \mathrm{Lm}$, $\nu < \alpha$.*

Corollary 2.3.9. *u_α is uniformly $\Sigma_1(J_\alpha)$, where $u_\alpha(x) \simeq \{z | z <_\alpha x\}$.*

Proof:

$$y = u_\alpha(x) \leftrightarrow \bigvee \nu (x \in S_\nu \wedge y = \{z \in S_\nu | z <_{S_\nu} x\})$$

QED (Corollary 2.3.9)

Note. We shall often write $<_{J_\alpha}$ for $<_\alpha$. We also write $<_\infty$ or $<_J$ or $<_L$ for $\bigcup\limits_{\alpha\in\mathrm{On}} <_\alpha$. Then $<_L$ well orders L and is an end extension of $<_\alpha$.

We obtain a particularly strong form of Gödel's condensation lemma:

Lemma 2.3.10. *Let $X \prec_{\Sigma_1} J_\alpha$. Then there are $\overline{\alpha}, \pi$ such that $\pi : J_{\overline{\alpha}} \overset{\sim}{\longleftrightarrow} X$.*

Proof: By §2 Lemma 2.2.19 there is rud closed U such that U is transitive and $\pi : U \overset{\sim}{\longleftrightarrow} X$. Note that the condition

$$S(f, \nu) \leftrightarrow: f = \langle S_\xi | \xi < \nu \rangle$$

is Σ_0, since:

$$\begin{aligned} S(f,\nu) \leftrightarrow & (f \text{ is a function } \wedge \\ & \wedge \operatorname{dom}(f) = \nu \wedge f(0) = \emptyset \text{ if } 0 < \nu \wedge \\ & \bigwedge \xi \in \operatorname{dom}(f)(\xi + 1 \in \operatorname{dom}(f) \rightarrow \\ & \rightarrow f(\xi+1) = S(f(\xi)))). \end{aligned}$$

Let $\overline{\alpha} = \mathrm{On} \cap U$ and let $\overline{\nu} < \overline{\alpha}$. Let $\pi(\overline{\nu}) = \nu$. Then $f = \langle S_\xi | \xi < \nu \rangle \in X$ since $X \prec_{\Sigma_1} J_\alpha$. Let $\pi(\overline{f}) = f$. Then $\overline{f} = \langle S_\xi | \xi < \overline{\nu} \rangle$, since $S(\overline{f}, \overline{\nu})$. But then $J_{\overline{\alpha}} = \bigcup\limits_{\xi<\overline{\alpha}} S_\xi \subset U$. But since π is Σ_1 preserving we know that

$$\begin{aligned} x \in U \rightarrow & \bigvee f, \nu \in U (S(f,\nu) \wedge x \in U f''\nu) \\ & \rightarrow x \in J_{\overline{\alpha}}. \end{aligned}$$

QED (Lemma 2.3.10)

Corollary 2.3.11. *Let $\pi \restriction J_{\overline{\alpha}} : J_{\overline{\alpha}} \rightarrow_{\Sigma_1} J_\alpha$. Then:*

(a) $\nu < \tau \leftrightarrow \pi(\nu) < \pi(\tau)$ for $\nu, \tau < \overline{\alpha}$.

(b) $x <_L y \leftrightarrow \pi(x) <_L \pi(y)$ for $x, y \in J_{\overline{\alpha}}$.
Hence:

(c) $\nu \leq \pi(\nu)$ *for* $\nu < \overline{\alpha}$.

(d) $x \leq_L \pi(x)$ *for* $x \in J_{\overline{\alpha}}$.

Proof: (a), (b) follow by the fact that $< \cap J_\alpha^2$ and $<_L \cap J_\alpha^2 = <_\alpha$ are uniformly $\Sigma_1(J_\alpha)$. But if $\pi(\nu) < \nu$, then $\nu, \pi(\nu), \pi^2(\nu), \ldots$ would form an infinite decreasing sequence by (a). Hence (c) holds. Similarly for (d). QED (Corollary 2.3.11)

2.3.1 The J_α^A–hierarchy

Given classes $A_1, \ldots, A_n$ one can generalize the previous construction by forming the *constructible hierarchy* $\langle J_\alpha^{A_1,\ldots,A_n} | \alpha \in \mathrm{Lim}\rangle$ *relativized to* $A_1, \ldots, A_n$. We have this far dealt only with the case $n = 0$. We now develop the case $n = 1$, since the generalization to $n > 1$ is then entirely straightforward. (Moreover the case $n = 1$ is sufficient for most applications.)

Definition 2.3.8. Let $A \subset V$. $\langle J_\alpha^A | \alpha \in \mathrm{Lm}\rangle$ is defined by:

$$\begin{aligned}
J_\alpha^A &= \langle J_\alpha[A], \in, A \cap J_\alpha[A]\rangle \\
J_\omega[A] &= \mathrm{Rud}_A(\emptyset) = H_\omega \\
J_{\beta+\omega}[A] &= \mathrm{Rud}_A(J_\beta) \text{ for } \beta \in \mathrm{Lm} \\
J_\lambda[A] &= \bigcup_{\nu<\lambda} J_\nu[A] \text{ for } \lambda \in \mathrm{Lm}^*
\end{aligned}$$

Note. $A \cap J_\alpha[A]$ is treated as an unary predicate.

Thus every J_α^A is rud closed. We set

Definition 2.3.9.

$$\begin{aligned}
L[A] = J[A] &= \bigcup_{\alpha \in \mathrm{On}} J_\alpha[A]; \\
L^A = J^A &= \langle L[A], \in, A \cap L[A]\rangle.
\end{aligned}$$

Note. that $J_\alpha[\emptyset] = J_\alpha$ for all $\alpha \in \mathrm{Lm}$.

Repeating the proof of Lemma 1.1.1 we get:

Lemma 2.3.12. $\mathrm{rn}(J_\alpha^A) = \mathrm{On} \cap J_\alpha^A = \alpha$.

We wish to break $J_{\alpha+\omega}^A$ into ω smaller steps, as we did with $J_{\alpha+\omega}$. To this end we define:

Definition 2.3.10. $S^A(u) = S(u) \cup \{A \cap u\}$.

Corresponding to Corollary 2.3.3 we get:

Lemma 2.3.13. *S^A is a function* rud *in A such that whenever u is transitive, then:*

(a) $u \cup \{u\} \cup \{A \cap u\} \subset S(u)$

(b) $\bigcup\limits_{n<\omega} (S^A)^n(u) = \mathrm{Rud}_A(u)$

(c) $S(u)$ is transitive.

Proof: (a) is immediate. (c) holds, since $S(u)$ is transitive, $a \subset S(u)$ and $A \cap u \subset u$. (b) holds since $S(u) \supset u$ is transitive and $A \cap u \subset u$. But if we set: $U = \bigcup_{n<\omega}(S^A)^n(u)$, then U is rud closed and $\langle U, A \cap U\rangle$ is amenable. QED (Lemma 2.3.13)

We then set:

Definition 2.3.11.

$$\begin{aligned} &S_0^A = \emptyset \\ &S_{\alpha+1}^A = S^A(S_\alpha^A) \\ &S_\lambda^A = \bigcup_{\nu<\lambda} S_\nu^A \text{ for limit } \lambda. \end{aligned}$$

We again have: $J_\alpha[A] = S_\alpha^A$ for $\alpha \in \mathrm{Lm}$. A close imitation of the proof of Lemma 2.3.4 gives:

Lemma 2.3.14. *$\langle S_\nu^A | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J_\alpha^A)$.*

Proof: This is exactly as before except that in the formula $\varphi(f)$ we replace $S(f(\nu))$ by $S^A(f(\nu))$. But this is $\Sigma_0(J_\alpha^A)$, since:

$$x \in S^A(u) \leftrightarrow (x \in S(u) \vee x = A \cap u),$$

hence:

$$\begin{aligned} y = &S^A(u) \leftrightarrow \bigwedge z \in y\ z \in S^A(u) \\ &\wedge \bigwedge z \in S(u) z \in y \wedge \bigvee z \in y\ z = A \cap u. \end{aligned}$$

QED (Lemma 2.3.14)

We now show that J_α^A has a uniformly $\Sigma_1(J_\alpha^A)$ well ordering, which we call $<_\alpha^A$ or $<_{J_\alpha^A}$.

Set:

Definition 2.3.12.

$$\begin{aligned} W^A(u,r) = &\{\langle x,y\rangle | \langle x,y\rangle \in W(u,r) \vee \\ &(x \in S(u) \wedge y = A \cap u \notin S(u))\} \end{aligned}$$

If u is transitive and r well orders u, then $W^A(u,r)$ is a well ordering of $S^A(u)$ which end extends r.

We set:

Definition 2.3.13.

$$\begin{aligned} &<^A_0 = \emptyset \\ &<^A_{\nu+1} = W^A(S^A_\nu, <^A_\nu) \\ &<^A_\lambda = \bigcup_{\nu<\lambda} <^A_\nu \text{ for limit } < . \end{aligned}$$

Then $<^A_\nu$ is a well ordering of S^A_ν which end extends $<^A_\xi$ for $\xi < \nu$. In particular $<^A_\alpha$ well orders J^A_α for $\alpha \in \Gamma$. We also write: $<_{J^A_\alpha} =: <^A_\alpha$. We set: $<_{L^A} = <_{J^A} = <^A_\infty =: \bigcup_{\nu<\infty} <^A_\nu$.

Just as before we get:

Lemma 2.3.15. *$\langle <^A_\nu | \nu < \alpha\rangle$ is uniformly $\Sigma_1(J^A_\alpha)$.*

The proof is left to the reader. Just as before we get:

Lemma 2.3.16. *$<^A_\alpha$ and $f(u) = \{z | z <^A_\alpha u\}$ are uniformly $\Sigma_1(J^A_\alpha)$.*

Up until now almost everything we proved for the J_α hierarchy could be shown to hold for the J^A_α hierarchy. The condensation lemma, however, is available only in a much weaker form:

Lemma 2.3.17. *Let $X \prec_{\Sigma_1} J^A_\alpha$. Then there are $\overline{\alpha}, \pi, \overline{A}$ such that $\pi : J^{\overline{A}}_{\overline{\alpha}} \overset{\sim}{\longleftrightarrow} X$.*

Proof: By Lemma 2.2.19 there is $\langle \overline{U}, \overline{A}\rangle$ such that $\pi : \langle \overline{U}, \overline{A}\rangle \overset{\sim}{\longleftrightarrow} X$ and $\langle \overline{U}, \overline{A}\rangle$ is rud closed. As before, the condition

$$S^A(f,\nu) \leftrightarrow f = \langle S^A_\xi | \nu < \xi\rangle$$

si Σ_0 in A. Now let $\overline{\nu} < \overline{\alpha}, \pi(\overline{\nu}) = \nu$. As before $f = \langle S_\xi | \xi < \nu\rangle \in X$. Let $\pi(\overline{f}) = f$. Then $\overline{f} = \langle S^A_\xi | \xi < \overline{\nu}\rangle$, since $S^{\overline{A}}(\overline{f}, \overline{\nu})$. Then $J^{\overline{A}}_{\overline{\alpha}} \subset \bigcup_{\xi<\overline{\alpha}} S^{\overline{A}}_\xi \subset \overline{U}$. $U \subset J^{\overline{A}}_{\overline{\alpha}}$ then follows as before. QED (Lemma 2.3.17)

A sometimes useful feature of the J^A_α hierarchy is:

Lemma 2.3.18. $x \in J^A_\alpha \to TC(x) \in J^A_\alpha$.

(Hence $\langle TC(x) | x \in J^A_\alpha \rangle$ *is* $\Pi_1(J^A_\alpha)$ *since* $u = TC(x)$ *is defined by:*

$$u \text{ is transitive } \wedge x \subset u \wedge \bigwedge v((v \text{ is transitive } \wedge x \subset v) \to u \subset v)$$

Proof: By induction on α.

Case 1 $\alpha = \omega$ (trivial)

Case 2 $\alpha = \beta + \omega$, $\beta \in \mathrm{Lim}$.
Then every $x \in J^A_\alpha$ has the form $f(\vec{z})$ where $z_1, \ldots, z_n \in J_\beta[A] \cup \{J_\beta[A]\}$ and f is rud in A. By Lemma 2.2.2 we have

$$\bigcup{}^p x \subset \bigcup_{i=1}^n TC(z_i) \subset J_\beta[A] \text{ for some } p < \omega$$

Hence $TC(x) = C_p(x) \cup TC(\bigcup_{i=1}^n TC(z_i))$, where $\langle TC(z) | z \in J_\beta[A] \rangle$ is J^A_β–definable, hence an element of J^A_α.

Case 3 $\alpha \in \mathrm{Lm}^*$ (trivial). QED (Lemma 2.3.18)

Corollary 2.3.19. *If* $\alpha \in \mathrm{Lm}^*$*, then* $\langle TC(x) | x \in J^A_\alpha \rangle$ *is uniformly* $\Delta_1(J^A_\alpha)$.

Proof: We have seen that it is $\Pi_1(J^A_\alpha)$. But $TC \restriction J^A_\beta \in J^A_\alpha$ for all $\beta \in \mathrm{Lm} \cap \alpha$. Hence $u = TC(x)$ is definable in J^A_α by:

$$\begin{aligned} &\bigvee f (f \text{ is a function } \wedge \mathrm{dom}(f) \text{ is transitive } \wedge u = f(x) \\ &\quad \wedge \bigwedge x \in \mathrm{dom}(f) f(x) = x \cup \bigcup f\text{“}x) \end{aligned}$$

QED (Corollary 2.3.19)

2.4 J–models

We can add further unary predicates to the structure $J^{\vec{A}}_\alpha$. We call the structure:

$$M = \langle J^{A_1, \ldots, A_n}_\alpha, B_1, \ldots, B_m \rangle$$

a *J–model* if it is amenable in the sense that $x \cap B_i \in J^{\vec{A}}_\alpha$ whenever $x \in J^{\vec{A}}_\alpha$ and $i = 1, \ldots, m$. The B_i are again taken as unary predicates. The *type* of M is $\langle n, m \rangle$. (Thus e.g. J_α has type $\langle 0, 0 \rangle$, J^A_α has type $\langle 1, 0 \rangle$, and $\langle J_\alpha, B \rangle$

has type $\langle 0,1\rangle$.) By an abuse of notation we shall often fail to distinguish between M and the associated structure:

$$\hat{M} = \langle J_\alpha[\vec{A}], A'_1, \ldots, A'_n, B_1, \ldots, B_m\rangle$$

where $A'_i = A_i \cap J_\alpha[\vec{A}]$ $(i = 1, \ldots, n)$.
We may for instance write $\Sigma_1(M)$ for $\Sigma_1(\hat{M})$ or $\pi : N \to_{\Sigma_n} M$ for $\pi : \hat{N} \to_{\Sigma_n} \hat{M}$. (However, we cannot unambignously identify M with $\hat{M}$, since e.g. for $M = \langle J^A_\alpha, B\rangle$ we might have: $\hat{M} = J^{A,B}_\alpha$.)

In practice we shall usually deal with J models of type $\langle 1,1\rangle$, $\langle 1,0\rangle$, or $\langle 0,0\rangle$. In any case, following the precedent in earlier section, when we prove general theorem about J–models, we shall often display only the proof for type $\langle 1,1\rangle$ or $\langle 1,0\rangle$, since the general case is then straightforward.

Definition 2.4.1. If $M = \langle J^{\vec{A}}_\alpha, \vec{B}\rangle$ is a J–model and $\beta \leq \alpha$ in Lm, we set:

$$M|\beta =: \langle J^{\vec{A}}_\beta, B_1 \cap J^{\vec{A}}_\beta, \ldots, B_n \cap J^{\vec{A}}_\beta\rangle.$$

In this section we consider $\Sigma_1(M)$ definability over an arbitrary $M = \langle J^{\vec{A}}_\alpha, \vec{B}\rangle$. If the context permits, we write simply Σ_1 instead of $\Sigma_1(M)$. We first list some properties which follow by rud closure alone:

- $\models^{\Sigma_1}_M$ is uniformly Σ_1, by corollary 2.2.18 (**Note** 'Uniformly' here means that the Σ_1 definition is the same for any two M having the same type.)

- If $R(y, x_1, \ldots, x_n)$ is a Σ_1 relation, then so is $\bigvee y R(y, x_1, \ldots, x_n)$ (since $\bigvee y \bigvee z P(y, z, \vec{x}) \leftrightarrow \bigvee u \bigvee y, z \in u P(y, z, \vec{x})$ where $R(y, \vec{x}) \leftrightarrow \bigvee z P(y, z, \vec{x})$ and P is Σ_0).

  By an n–ary $\Sigma_1(M)$ *function* we mean a partial function on M^n which is $\Sigma_1(M)$ as an $n+1$–ary relation.

- If R, R' are n–ary Σ_1 relations, then so are $R \cap R'$, $R \cup R'$. (Since e.g.

$$\begin{aligned}(\bigvee y P(y, \vec{x}) \wedge \bigvee P'(y, \vec{x})) \leftrightarrow \\ \bigvee y y'(P(y, \vec{x}) \wedge P'(y', \vec{x})).)\end{aligned}$$

- If $R(y_1, \ldots, y_m)$ is an n–ary Σ_1 relation and $f_i(\vec{x})$ is an n–ary Σ_1 function for $i = 1, \ldots, m$, then so is the n–ary relation

$$R(\vec{f}(\vec{x})) \leftrightarrow: \bigvee y_1, \ldots, y_m (\bigwedge_{i=1}^{m} y_i = f_i(\vec{x}) \wedge R(\vec{y})).$$

- If $g(y_1, \ldots, y_m)$ is an m–ary Σ_1 function and $f_i(\vec{x})$ is an n–ary Σ_1 function for $i = 1, \ldots, m$ then $h(\vec{x}) \simeq g(\vec{f}(\vec{x}))$ is an n–ary Σ_1 function. (Since $z = h(\vec{x}) \leftrightarrow \bigvee y_1, \ldots, y_m(\bigwedge_{i=1}^{m} y_i = f_i(\vec{x}) \wedge z = g(\vec{y}))$.)

  Since $f(x_1, \ldots, x_n) = x_i$ is Σ_1 function, we have:

- If $R(x_1, \ldots, x_n)$ is Σ_1 and $\sigma : n \to m$, then

$$P(z_1, \ldots, z_m) \leftrightarrow: R(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$$

  is Σ_1.

- If $f(x_1, \ldots, x_n)$ is a Σ_1 function and $\sigma : n \to m$, then the function:

$$g(z_1, \ldots, z_m) \simeq: f(z_{\sigma(1)}, \ldots, z_{\sigma n})$$

  is Σ_1.

J–models have the further property that every binary Σ_1 relation is uniformizable by a Σ_1 function. We define

Definition 2.4.2. A relation $R(y, \vec{x})$ is *uniformized* by the function $F(\vec{x})$ iff the following hold:

- $\bigvee y R(y, \vec{x}) \to F(\vec{x})$ is defined
- If $F(\vec{x})$ is defined, then $R(F(\vec{x}), \vec{x})$

We shall, in fact, prove that M has a uniformly Σ_1 definable *Skolem function*. We define:

Definition 2.4.3. $h(i, x)$ is a Σ_1–*Skolem function* for M iff h is a $\Sigma_1(M)$ partial map from $\omega \times M$ to M and, whenever $R(y, x)$ is a $\Sigma_1(M)$ relation, there is $i < \omega$ such that h_i uniformizes R, where $h_i(x) \simeq h(i, x)$.

Lemma 2.4.1. *M has a Σ_1–Skolem function which is uniformly $\Sigma_1(M)$.*

Proof: $\models_M^{\Sigma_1}$ is uniformly Σ_1. Let $\langle \varphi_i | i < \omega \rangle$ be a recursive enumeration of the Σ_1 formulae in which at most the two variables v_0, v_1 occur free. Then the relation:

$$T(i, y, x) \leftrightarrow: \models_M^{\Sigma_1} \varphi_i[y, x]$$

is uniformly Σ_1. But then for any Σ_1 relation R there is $i < \omega$ such that

$$R(y, x) \leftrightarrow T(i, y, x).$$

Since T is Σ_1, it has the form:

$$\bigvee zT'(z,i,y,x)$$

where T' is Σ_0. Writing $<_M$ for $<_\alpha^{\vec{A}}$, we define:

$$\begin{aligned} y = h(i,x) \leftrightarrow \bigvee z(&\langle z,y\rangle \text{ is the } <_M\text{–least} \\ &\text{pair } \langle z',y'\rangle \text{ such that } T'(z',i,y',x)). \end{aligned}$$

Recalling that the function $f(x) = \{z | z <_M x\}$ is Σ_1, we have:

$$\begin{aligned} y =& h(i,x) \leftrightarrow \bigvee z \bigvee u(T'(z,i,y,x)\wedge \\ &\wedge u = \{w | w <_M \langle z,y\rangle\}\wedge \\ &\wedge \bigwedge \langle z',y'\rangle \in u \neg T'(z,i,y,x)) \end{aligned}$$

QED 2.4.1

We call the function h defined above the *canonical* Σ_1 *Skolem function* for M and denote it by h_M. The existence of h implies that every $\Sigma_1(M)$ relation is uniformizable by a $\Sigma_1(M)$ function:

Corollary 2.4.2. *Let $R(y,x_1,\ldots,x_n)$ be Σ_1. R is uniformizable by a Σ_1 function.*

Proof: Let h_i uniformize the binary relation

$$\{\langle y,z\rangle | \bigvee x_1 \ldots x_n(R(y,\vec{x}) \wedge z = \langle x_1,\ldots,x_n\rangle)\}.$$

Then $f(\vec{x}) \simeq: h_i(\langle\vec{x}\rangle)$ uniformizes R. QED

We say that a $\Sigma_1(M)$ function has a *functionally absolute* definition if it has a Σ_1 definition which defines a function over every J–model of the same type.

Corollary 2.4.3. *Every $\Sigma_1(M)$ function g has functionally absolute definition.*

Proof: Apply the construction in Corollary 2.4.2 to $R(y,\vec{x}) \leftrightarrow y = g(\vec{x})$. Then $f(x) \simeq: h_i(\langle\vec{x}\rangle)$ is functionally absolute since h_i is.

QED (Corollary 2.4.2)

Lemma 2.4.4. *Every $x \in M$ is $\Sigma_1(M)$ in parameters from* $\mathrm{On} \cap M$.

Proof: We must show: $x = f(\xi_1,\ldots,\xi_n)$ where f is $\Sigma_1(M)$. If $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$, it obviously suffices to show it for the model $M' = J_\alpha^{\vec{A}}$. For the sake of simplicity we display the proof for J_α^A. (i.e. M has type $\langle 1,0\rangle$). We proceed by induction on $\alpha \in \mathrm{Lm}$.

Case 1 $\alpha = \omega$.

Then $J^A_\alpha = \text{Rud}(\emptyset)$ and $x = f(\{0\})$ where f is rudimentary.

Case 2 $\alpha = \beta + \omega$, $\beta \in \text{Lm}$.

Then $x = f(z_1, \ldots, z_n, J^A_\beta)$ where $z_1, \ldots, z_n \in J^A_\beta$ and f is rud in A. (This is meant to include the case: $n = 0$ and $x = f(J^A_\beta)$.) By the induction hypothesis there are $\vec{\xi} \in \beta$ such that $z_i = g_i(\vec{\xi})$ $(i = 1, \ldots, n)$ and g_i is $\Sigma_1(J^A_\beta)$. For each i pick a functionally absolute Σ_1 definition for g_i and let g'_i be $\Sigma_1(J^A_\alpha)$ by the same definition. Then $z_i = g'_i(\vec{\xi})$ since the condition is Σ_1. Hence $x = f'(\vec{\xi}, \beta) = f(\vec{g}'(\vec{\xi}), J^A_\beta))$ where f' is Σ_1. QED (Case 2)

Case 3 $\alpha \in \text{Lm}^*$.

Then $x \in J^A_\beta$ for a $\beta < \alpha$. Hence $x = f(\vec{\xi})$ where f is $\Sigma_1(J^A_\beta)$. Pick a functionally absolute Σ_1 definition of f and let f' be $\Sigma_1(J^A_\alpha)$ by the same definition. Then $x = f'(\vec{\xi})$. QED (Lemma 2.4.4)

But being Σ_1 in parameters from $\text{On} \cap M$ is the same as being Σ_1 in a finite subset of $\text{On} \cap M$:

Lemma 2.4.5. *Let $x = f(\vec{\xi})$ where f is $\Sigma_1(M)$. Let $a \subset \text{On} \cap M$ be finite such that $\xi_1, \ldots, \xi_n \in a$. Then $x = g(a)$ for a $\Sigma_1(M)$ function g.*

Proof: Set:

$$k_i(a) = \begin{cases} \text{the } i\text{–th element of } a \text{ in order} \\ \text{of size if } a \subset \text{On is finite} \\ \text{and } \text{card}(a) > i, \\ \text{undefined if not.} \end{cases}$$

Then k_i is $\Sigma_1(M)$ since:

$$y = k_i(a) \leftrightarrow \bigvee f \bigvee n < \omega(f : n \leftrightarrow a \wedge \bigwedge i, j < n(f(i) < f(j) \leftrightarrow i < j) \\ \wedge a \subset \text{On} \wedge y = f(i))$$

Thus $x = f(k_{i_1}(a), \ldots, k_{i_n}(a))$ where $\xi_l = k_{i_l}(a)$ for $l = 1, \ldots, n$.

QED (Lemma 2.4.5)

We now show that for every J–model M there is a $\underline{\Sigma}_1(M)$ partial map of $\text{On} \cap M$ onto M. As a preliminary we prove:

Lemma 2.4.6. *There is a partial $\underline{\Sigma}_1(M)$ map of $\text{On} \cap M$ onto $(\text{On} \cap M)^2$.*

Proof: Order the class of pairs On^2 by setting: $\langle \alpha, \beta \rangle <^* \langle \gamma, \delta \rangle$ iff $\langle \max(\alpha, \beta), \alpha, \beta \rangle$ is lexicographically less than $\langle \max(\gamma, \delta), \gamma, \delta \rangle$. This ordering has the property that the collection of predecessors of any pair form a set. Hence there is a function $p : \mathrm{On} \to \mathrm{On}^2$ which enumerates the pairs in order $<^*$.

Claim 1 $p \restriction \mathrm{On}_M$ is $\Sigma_1(M)$.

Proof: If $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$, it suffices to prove it for $J_\alpha^{\vec{A}}$. To simplify notation, we assume: $M = J_\alpha^A$ for an $A \subset M$ (i.e. M is of type $\langle 1, 0 \rangle$.)

We know:

$$y = p(\nu) \leftrightarrow \bigvee f (\varphi(f) \wedge y = f(\nu))$$

where φ is the Σ_0 formula:

$$\begin{aligned}
&f \text{ is a function } \wedge \mathrm{dom}(f) \in \mathrm{On} \wedge \\
&\wedge \bigwedge u \in \mathrm{rng}(f) \bigvee \beta, \gamma \in C_n(u) u = \langle \beta, \gamma \rangle \wedge \\
&\wedge \bigwedge \nu, \tau \in \mathrm{dom}(f) (\nu < \tau \leftrightarrow f(\nu) <^* f(\tau)) \\
&\wedge \bigwedge u \in \mathrm{rng}(f) \bigwedge \mu, \xi \leq \max(u) (\langle \mu, \xi \rangle <^* u \to \langle \mu, \xi \rangle \in \mathrm{rng}(f)).
\end{aligned}$$

Thus it suffices to show that the existence quantifier can be restricted to J_α^A — i.e. that $p \restriction \xi \in J_\alpha^A$ for $\xi < \alpha$. This follows by induction on α in the usual way (cf. the proof of Lemma 2.3.14). QED (Claim 1)

We now proceed by induction on $\alpha = \mathrm{On}_M$, considering three cases:

Case 1 $p(\alpha) = \langle 0, \alpha \rangle$.

Then $p \restriction \alpha$ maps α onto

$$\{u | u <_* \langle 0, \alpha \rangle\} = \alpha^2$$

and we are done, since $p \restriction \alpha$ is $\Sigma_1(J_\alpha^A)$. (Note that ω satisfies Case 1.)

Case 2 $\alpha = \beta + \omega, \beta \in \mathrm{Lm}$ and Case 1 fails.

There is a $\Sigma_1(J_\alpha^A)$ bijection of β onto α defined by:

$$\begin{aligned}
&f(2n) = \beta + n \text{ for } n < \omega \\
&f(2n+1) = n \text{ for } n < \omega \\
&f(\nu) = \nu \text{ for } \omega \leq \nu < \beta
\end{aligned}$$

Let g be a $\underline{\Sigma}_1(J_\beta^A)$ partial map of β onto β^2. Set $(\langle \gamma_0, \gamma_1 \rangle)_i = \gamma_i$ for $i = 0, 1$.

$$g_i(\nu) \simeq (g(\nu))_i (i = 0, 1).$$

Then $\tilde{f}(\nu) \simeq \langle f g_0(\nu, f g_1(\nu)) \rangle$ maps β onto α^2. QED (Case 2)

Case 3 The above cases fail.

Then $p(\alpha) = \langle \nu, \tau \rangle$, where $\nu, \tau < \alpha$. Let $\gamma \in \mathrm{Lm}$ such that $\max(\nu, \tau) < \gamma < \alpha$. Let g be a partial $\underline{\Sigma}_1(J_\alpha^A)$ map of γ onto γ^2. Then $g \in M, p^{-1}$ is a partial map of γ^2 onto α; hence $f = p^{-1} \circ g$ is a partial map of γ onto α. Set: $\widetilde{f}(\langle \xi, \delta \rangle) \simeq \langle f(\xi), f(\delta) \rangle$ for ξ, δ, γ. Then $\widetilde{f} g$ is a partial map of γ onto α^2. QED (Lemma 2.4.6)

We can now prove:

Lemma 2.4.7. *There is a partial* $\underline{\Sigma}_1(M)$ *map of* On_M *onto* M.

Proof: We again simplify things by taking $M = J^A_\alpha$. Let g be a partial map of α onto α^2 which is $\Sigma_1(J^A_\alpha)$ in the parameters $p \in J^A_\alpha$. Define "ordered pairs" of ordinals $< \alpha$ by:

$$(\nu, \tau) =: g^{-1}(\langle \nu, \tau \rangle).$$

We can then, for each $n \geq 1$, define "ordered n–tuples" by:

$$(\nu) =: \nu, (\nu_1, \ldots, \nu_n) = (\nu_1, (\nu_2, \ldots, \nu_n))(n \geq 2).$$

We know by Lemma 2.4.4 that every $y \in J^A_\alpha$ has the form: $y = f(\nu_1, \ldots, \nu_n)$ where $\nu_1, \ldots, \nu_n < \alpha$ and f is $\Sigma_1(J^A_\alpha)$. Define a function f^* by:

$$y = f^*(\tau) \leftrightarrow \bigvee \nu_1, \ldots, \nu_n(\tau = (\nu_1, \ldots, \nu_n) \wedge \\ \wedge y = f(\nu_1, \ldots, \nu_n)).$$

Then f^* is $\Sigma_1(J^A_\alpha)$ in p and $y \in f^{*\prime\prime}\alpha$. If we set: $h^*(i, x) \simeq h(i, \langle x, p \rangle)$, then each binary relation which is $\Sigma_1(J^A_\alpha)$ in p is uniformized by one of the functions $h^*_i(x) \simeq h^*(i, x)$. Hence $y = h^*(i, \gamma)$ for some $\gamma < \alpha$. Hence $J^A_\alpha = h^{*\prime\prime}(\omega \times \alpha)$. But, setting:

$$y = \hat{h}(\mu) \leftrightarrow \bigvee i, \nu(\mu = (i, \nu) \wedge y = h^*(i, \nu))$$

we see that $\hat{h}$ is $\Sigma_1(J^A_\alpha)$ in p and $y \in \hat{h}''\alpha$. Hence $J^A_\alpha = \hat{h}''\alpha$, where $\hat{h}$ is $\Sigma_1(J^A_\alpha)$ in p. QED (Lemma 2.4.7)

Corollary 2.4.8. *Let* $x \in M$. *There are* $f, \gamma \in J^A_\alpha$ *such that* f *maps* γ *onto* x.

Proof: We again prove it for $M = J^A_\alpha$. If $\alpha = \omega$ it is trivial since $J^A_\alpha = H_\omega$. If $\alpha \in \mathrm{Lm}^*$ then $x \in J^A_\beta$ for a $\beta < \alpha$ and there is $f \in J^A_\alpha$ mapping β onto J^A_β by Lemma 2.4.7. There remains only the case $\alpha = \beta + \omega$ where β is a limit ordinal. By induction on $n < \omega$ we prove:

Claim There is $f \in J^A_\alpha$ mapping β onto $S^A_{\beta+n}$. If $n = 0$ this follows by Lemma 2.4.7.

Now let $n = m + 1$.
Let $f : \beta \overset{\text{onto}}{\longrightarrow} S^A_{\beta+m}$ and define f' by $f'(0) = S^A_{\beta+m}, f'(n+1) = f(n)$ for $n < \omega$, $f'(\xi) = f(\xi)$ for $\xi \geq \omega$. Then f' maps β onto $U = S^A_{\beta+m} \cup \{S^A_{\beta+m}\}$ and $S^A_{\beta+m} = \bigcup\limits_{\delta=\beta}^{8} F''_i U^2 \cup \bigcup\limits_{i=0}^{3} G''_i U^3 \cup \{A \cap S^A_{\beta+m}\}$.

Set:

$$g_i = \{\langle F_i(f'(\xi), f'(\zeta)), \langle i, \langle \xi, \zeta \rangle\rangle\rangle | \xi, \zeta < \beta\}$$
for $i = 0, \ldots, 8$
$$g_{8+i+1} = \{\langle G_i | f'(\xi), f'(\zeta), f'(\mu)), \langle 8+i+1, \langle \xi, \zeta, \mu \rangle\rangle\rangle | \xi, \zeta, \mu < \beta\}$$
for $i = 0, \ldots, 3$
$$g_{13} = \{\langle A \cap S^A_{\beta+m} \langle 13, \emptyset \rangle\rangle\}$$

Then $g = \bigcup_{i=0}^{13} g_i \in J^A_\alpha$ is a partial map of J^A_β onto $S^A_{\beta+n}$ and $gh \in J^A_\alpha$ is a partial map of β onto $S^A_{\beta+m}$ where h is a partial $\Sigma_1(J^A_\beta)$ map of β onto J^A_β where h is a partial $\Sigma_1(J^A_\beta)$ map of β onto J^A_β.

QED (Corollary 2.4.8)

Define the *cardinal* of x in M by:

Definition 2.4.4. $\overline{\overline{x}} = \overline{\overline{x}}^M$ =: the least γ such that some $f \in M$ maps γ onto x.

Note. this is a non standard definition of cardinal numbers. If M is e.g. *pr* closed, we get that there is $f \in M$ bijecting $\overline{\overline{x}}$ onto x.

Definition 2.4.5. Let $X \subset M$. $h(X) = h_M(X)$ =: The set of all $y \in M$ such that $y = f(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n \in X$ and f is a $\Sigma_1(M)$ function

Since $\Sigma_1(M)$ functions are closed under composition, it follows easily that $Y = h(X)$ is closed under $\Sigma_1(M)$ functions.

By Corollary 2.4.2 we then have:

Lemma 2.4.9. *Let $Y = h(X)$. Then $M|Y \prec_{\Sigma_1} M$ where*

$$M|Y =: \langle Y, A_1 \cap Y, \ldots, A_n \cap Y, B_1 \cap Y, \ldots, B_m \cap Y\rangle.$$

Note. We shall often ignore the distinction between Y and $M|Y$, writing simply: $Y \prec_{\Sigma_1} M$.

If f is a $\Sigma_1(M)$ function, there is $i < \omega$ such that $h(i, \langle \vec{x} \rangle) \simeq f(\vec{x})$. Hence:

Corollary 2.4.10. $h(X) = \bigcup_{n<\omega} h''(\omega \times X^n)$.

There are many cases in which $h(X) = h''(\omega \times X)$, for instance:

Corollary 2.4.11. $h(\{x\}) = h''(\omega \times \{x\})$.

Gödel's pair function on ordinals is defined by:

Definition 2.4.6. $\prec \gamma, \delta \succ =: p^{-1}(\prec \gamma, \delta \succ)$, where p is the function defined in the proof of Lemma 2.4.6.

We can then define *Gödel n–tuples* by iterating the pair function:

Definition 2.4.7. $\prec \gamma \succ =: \gamma; \prec \gamma_1, \ldots, \gamma_n \succ =: \prec \gamma_1, \prec \gamma_2, \ldots, \gamma_n \succ \succ$ $(n \geq 2)$.

Hence any X which is closed under Gödel pairs is closed under the tuple–function. Imitating the proof of Lemma 2.4.7 we get:

Corollary 2.4.12. *If $Y \subset \mathrm{On}_M$ is closed under Gödel pairs, then:*

(a) $h(Y) = h''(\omega \times Y)$

(b) $h(Y \cup \{p\}) = h''(\omega \times (Y \times \{p\}))$ *for* $p \in M$.

Proof: We display the proof of (b). Let $y \in h(Y \cup \{p\})$. Then $y = f(\gamma_1, \ldots, \gamma_n, p)$, where $\gamma_1, \ldots, \gamma_n \in Y$ and f is $\Sigma_1(M)$.

Hence $y = f^*(\langle \delta, p \rangle)$ where $\delta = \prec \gamma_1, \ldots, \gamma_n \succ$ and

$$y = f^*(z) \leftrightarrow \bigvee \gamma_1, \ldots, \gamma_n \bigvee p(z = \langle \prec \gamma_1, \ldots, \gamma_n \succ, p \rangle \wedge \\ \wedge y = f(\vec{\gamma}, p)).$$

Hence $y = h(i, \langle \delta, p \rangle)$ for some i. QED (Corollary 2.4.12)

Similarly we of course get:

Corollary 2.4.13. *If $Y \subset M$ is closed under ordered pairs, then:*

(a) $h(Y) = h''(\omega \times Y)$

(b) $h(Y \cup \{p\}) = h''(\omega \times (Y \times \{p\}))$ *for* $p \in M$.

By Lemma 2.4.5 we easily get:

Corollary 2.4.14. *Let $Y \subset \mathrm{On}_M$. Then $h(Y) = h''(\omega \times \mathbb{P}_\omega(Y))$.*

In fact:

Corollary 2.4.15. *Let $A \subset \mathbb{P}_\omega(\mathrm{On}_M)$ be directed (i.e. $a, b \in A \to \bigvee c \in A\ a, b \subset c$). Let $Y = \bigcup A$. Then $h(Y) = h''(\omega \times A)$.*

By the condensation lemma we get:

Lemma 2.4.16. *Let $\pi : \overline{M} \to_{\Sigma_1} M$ where M is a J–model and $\overline{M}$ is transitive. Then $\overline{M}$ is a J–model.*

Proof: $\overline{M}$ is amenable by Σ_1 preservation. But then it is a J–model by the condensation lemma. QED (Lemma 2.4.16)

We can get a theorem in the other direction as well. We first define:

Definition 2.4.8. Let $\overline{M}, M$ be transitive structures. $\sigma : \overline{M} \to M$ *cofinally* iff σ is a structural embedding of $\overline{M}$ into M and $M = \bigcup \sigma''\overline{M}$.

Then:

Lemma 2.4.17. *If $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally. Then σ is Σ_1 preserving.*

Proof: Let $R(y, \vec{x})$ be $\Sigma_0(M)$ and let $\overline{R}(y, \vec{x})$ be $\Sigma_0(\overline{M})$ by the same definition. We claim:

$$\bigvee y R(y, \sigma(\vec{x})) \to \bigvee y \overline{R}(y, \vec{x})$$

for $x_1, \ldots, x_n \in \overline{M}$. To see this, let $R(y, \sigma(\vec{x}))$. Then $y \in \sigma(u)$ for a $u \in \overline{M}$. Hence $\bigvee y \in \sigma(u) R(y, \sigma(\vec{x}))$, which is a Σ_0 statement about $\sigma(u), \sigma(\vec{x})$. Hence $\bigvee y \in u \overline{R}(y, \vec{x})$. QED (Lemma 2.4.17)

Lemma 2.4.18. *Let $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally, where $\overline{M}$ is a J–model. Then M is a J–model.*

Proof: Let e.g. $\overline{M} = \langle J_{\overline{\alpha}}^{\overline{A}} \rangle, M = \langle U, A, \overline{B} \rangle$.

Claim 1 $U = J_\alpha^A$ where $\alpha = \mathrm{On}_M$.

> **Proof:** $y = S^{\overline{A}} \upharpoonright \nu$ is a Σ_0 condition, so $\sigma(S^{\overline{A}} \upharpoonright \nu) = S^A \upharpoonright \sigma(\nu)$. But σ takes $\overline{\alpha}$ cofinally to α, so if $\xi < \alpha, \xi < \sigma(\nu)$, then $S_\xi^A(S^A \upharpoonright \sigma(\nu))(\xi) \in U$. Hence $J_\alpha^A \subset U$. To see $U \subset J_\alpha^A$, let $x \in U$. Then $x \in \sigma(u)$ where $u \in J_{\overline{\alpha}}^{\overline{A}}$. Hence $u \subset S_\nu^{\overline{A}}$ and $x \in \sigma(S_\nu^{\overline{A}}) = S_{\sigma(\nu)}^A \subset J_\alpha^A$. QED (Claim 1)

Claim 2 M is amenable.

> Let $x \in S_{\sigma(\nu)}^A$. Then $\sigma(\overline{B} \cap S_\nu^{\overline{A}}) = B \cap S_{\sigma(\nu)}^A$ and $x \cap B = (B \cap S_\nu^A) \cap x \in U$, since S_ν^A is transitive. QED (Lemma 2.4.18)

Lemma 2.4.19. *Let $\overline{M}, M$ be J–models. Then $\sigma : \overline{M} \to_{\Sigma_0} M$ cofinally iff $\sigma : \overline{M} \to_{\Sigma_0} M$ and σ takes $\mathrm{On}_{\overline{M}}$ to On_M cofinally.*

Proof: $(\rightarrow)$ is obvious. We prove $(\leftarrow)$. The proof of $\sigma(S_{\nu}^{\overline{A}}) = S_{\sigma(\nu)}^{A}$ goes through as before. Thus if $x \in M$, we have $x \in S_{\xi}^{A}$ for some ξ. Let $\xi \leq \sigma(\nu)$. Then $x \in S_{\sigma(\nu)}^{A} = \sigma(S_{\nu}^{\overline{A}})$. QED (Lemma 2.4.19)

2.5 The Σ_1 projectum

2.5.1 Acceptability

We begin by defining a class of J–models which we call *acceptable*. Every J_α is acceptable, and we shall see later that there are many other naturally occurring acceptable structures. Accepability says essentially that if something dramatic happens to β at some later stage ν of the construction, then ν is, in fact, collapsed to β at that stage:

Definition 2.5.1. $J_{\alpha}^{\vec{A}}$ is *acceptable* iff for all $\beta \leq \nu < \alpha$ in Lm we have:

$$\text{If } a \subset \beta \text{ and } a \in J_{\nu+\omega}^{\vec{A}} \setminus J_{\nu}^{\vec{A}}, \text{ then } \overline{\overline{\nu}} \leq \beta \text{ in } J_{\nu+\omega}^{\vec{A}}.$$

In the following we shall always suppose M to be acceptable unless otherwise stated. We recall that by Corollary 2.4.8 every $x \in M$ has a cardinal $\overline{\overline{x}} = \overline{\overline{x}}^{M}$. We call γ a cardinal in M iff $\gamma = \overline{\overline{\gamma}}$ (i.e. no smaller ordinal is mappable onto γ in M).

Lemma 2.5.1. *Let $M = \langle J_{\alpha}^{A}, B\rangle$ be acceptable. Let $\gamma > \omega$ be a cardinal in M. Then:*

(a) $\gamma \in \mathrm{Lm}^*$

(b) $x \in J_{\gamma}^{A} \rightarrow M \cap \mathbb{P}(x) \subset J_{\gamma}^{A}$.

Proof: We first prove (a). Suppose not. Then $\gamma = \beta + \omega$, where $\beta \in \mathrm{Lm}, \beta \geq \omega$. Then $f \in M$ maps β onto γ where: $f(2i) = i, f(2i+1) = \beta + i, f(\xi) = \xi$ for $\xi \geq \omega$.
Contradiction! QED (a)

To prove (b) suppose not. Then x is not finite. Let $\beta = \overline{\overline{x}}$ in J_{γ}^{A}. Then $\beta \geq \omega, \beta \in \mathrm{Lm}$ by (a). Let $f \in J_{\gamma}^{A}$ map β onto x. Let $u \subset x$ such that $u \notin J_{\gamma}^{A}$. Then $v = f^{-1}{}''u \notin J_{\gamma}^{A}$. Let $\nu \geq \gamma$ such that $v \in J_{\nu+\omega}^{A} \setminus J_{\nu}^{A}$. Then $\gamma \leq \overline{\overline{\nu}} \leq \beta$.
Contradiction! QED (Lemma 2.5.1)

Remark We have stated and proven this lemma for M of type $\langle 1, 1\rangle$, since the extension to M of arbitrary type is self evident.

The most general form of GCH says that if $\mathbb{P}(x)$ exists and $\overline{\overline{x}} \geq \omega$, then $\overline{\overline{\mathbb{P}(x)}} = \overline{\overline{x}}^+$ (where α^+ is the least cardinal $> \alpha$).

As a corollary of Lemma 2.5.1 we have:

Corollary 2.5.2. *Let M, γ be as above. Let $a \in M, a \subset J^A_\gamma$. Then:*

(a) $\langle J^A_\gamma, a\rangle$ models the axiom of subsets and GCH.

(b) If γ is a successor cardinal in M, then $\langle J^A_\gamma, a\rangle$ models ZFC^-.

(c) If γ is a limit cardinal in M, then $\langle J^A_\gamma, a\rangle$ models Zermelo set theory.

Proof: (a) follows easily from Lemma 2.5.1 (b). (c) follows from (a) and rud closure of J^A_γ. We prove (b). We know that J^A_γ is rud closed and that the axiom of choice holds in the strong form: $\bigwedge x \bigvee \nu \bigvee f$ f maps ν onto x. We must prove the axiom of collection. Let $R(x, y)$ be $\underline{\Sigma}_\omega(J^A_\gamma)$ and let $u \in J^A_\gamma$ such that $\bigwedge x \in u \bigvee y R(x, y)$.

Claim $\bigvee \nu < \gamma \bigwedge x \in u \bigvee y \in J^A_\nu R(x, y)$. Suppose not.

Let $\gamma = \beta^+$ in M. For each $\nu < \gamma$ there is a partial map $f \in M$ of β onto ν. But then $f \in J^A_\gamma$ since $f \subset \nu \times \beta \in J^A_\gamma$. Set f_ν — the $<_{J^A_\gamma}$ — least such f. For $x \in u$ set:

$$h(x) = \text{ the least } \mu \text{ such that } \bigvee y \in J^A_\mu R(y, x).$$

Then $\sup h''u = \gamma$ by our assumption. Define a partial map k on $u \times \beta$ by: $k(x, \xi) \simeq f_{h(x)}(\xi)$. Then k is onto γ. But $k \in M$, since k is $\underline{\Sigma}_1(J^A_\gamma)$. Clearly $\overline{\overline{u \times \beta}} = \beta$ in M, so $\overline{\overline{\gamma}} \leq \beta < \gamma$ in M.
Contradiction! QED (Corollary 2.5.2)

Corollary 2.5.3. *Let M, γ be as above. Then*

$$|J^A_\gamma| = H^M_\gamma =: \bigcup \{u \in M | u \textit{ is transitive } \wedge \overline{\overline{u}} < \gamma \textit{ in } M\}.$$

Proof: Let $u \in M$ be transitive and $\overline{\overline{u}} < \gamma$ in M. It suffices to show that $u \in J^A_\gamma$. Let $\nu = \overline{\overline{u}} < \gamma$ in M. Let $f \in M$ map ν onto u. Set:

$$r = \{\langle \xi, \delta\rangle \in \nu^2 | f(\xi) \in f(\delta)\}.$$

Then $r \in J^A_\gamma$ by Lemma 2.5.1 (c), since $\nu^2 \in J^A_\gamma$. Let $\beta = \overline{\overline{\nu}}^+$ = the least cardinal $> \nu$ in M. then J^A_β models ZFC^- and $r, \nu \in J^A_\beta$. But then $f \in J^A_\beta \subset J^A_\gamma$, since f is defined by recursion on $r : f(x) = f''r''\{x\}$ for $x \in \nu$. Hence $u = \text{rng}(f) \in J^A_\gamma$. QED (Corollary 2.5.3)

Lemma 2.5.4. *If $\pi : \overline{M} \to_{\Sigma_1} M$ and M is acceptable, then so is $\overline{M}$.*

Proof: $\overline{M}$ is a J–model by §4. Let e.g. $M = J^A_\alpha, \overline{M} = J^{\overline{A}}_{\overline{\alpha}}$. Then $\overline{M}$ has a counterexample — i.e. there are $\overline{\nu} < \overline{\alpha}, \overline{\beta} < \overline{\nu}, \overline{a}$ such that $\text{card}(\overline{\nu}) > \overline{\beta}$ in $J_{\overline{\nu}+\omega}$ and $\overline{a} \subset \overline{\beta}$ and $\overline{a} \in J^{\overline{\mathbb{A}}}_{\overline{\nu}+\omega} \setminus J^A_{\overline{\nu}}$. But then letting $\pi(\overline{\beta}, \overline{\nu}, \overline{a}) = \beta, \nu, a$ it follows easily that β, ν, a is a counterexample in M.
Contradiction! QED (Lemma 2.5.4)

Lemma 2.5.5. *If $\pi : \overline{M} \to_{\Sigma_0} M$ cofinally and $\overline{M}$ is acceptable, then so is M.*

Proof: M is a J–model by §4. Let $M = J^A_\alpha, \overline{M} = J^{\overline{A}}_{\overline{\alpha}}$.

Case 1 $\overline{\alpha} = \omega$.
Then $\overline{M} = M = J^A_\omega, \pi = \text{id}$.

Case 2 $\overline{\alpha} \in \text{Lm}^*$.
Then "$\overline{M}$ is acceptable" is a $\Pi_1(\overline{M})$ condition. But then $\alpha \in \text{Lm}^*$ and M must satisfy the same Π_1 condition.

Case 3 $\overline{a} = \overline{\beta} + \omega, \overline{\beta} \in \text{Lm}$.
Then $\alpha = \beta + \omega, \beta \in \text{Lm}$ and $\beta = \pi(\overline{\beta})$. Then $J^A_\beta = \pi(J^{\overline{A}}_{\overline{\beta}})$ is acceptable, so there can be no counterexample $\langle \delta, \nu, a \rangle \in J^A_\beta$.

We show that there can be no counterexample of the form $\langle \delta, \beta, a \rangle$. Let $\overline{\gamma} = \text{card}(\overline{\beta})$ in $\overline{M}$. The statement $\text{card}(\overline{\beta}) \leq \overline{\gamma}$ is $\Sigma_1(M)$. Hence $\text{card}(\beta) \leq \gamma = \pi(\overline{\gamma})$ in M. Hence there is no counterexample $\langle \delta, \beta, a \rangle$ with $\delta \geq \gamma$. But since $\overline{M}$ is acceptable and $\overline{\gamma} \leq \overline{\beta}$ is a cardinal in $\overline{M}$, the following Π_1 statement holds in $\overline{M}$ by Lemma 2.5.1

$$\bigwedge \delta < \overline{\gamma} \bigwedge a \subset \delta a \in J^{\overline{A}}_{\overline{\gamma}}.$$

But then the corresponding statement holds in M. Hence $\langle \delta, \beta, a \rangle$ cannot be a counterexample for $\delta < \gamma$. QED (Lemma 2.5.5)

2.5.2 The projectum

We now come to a central concept of fine structure theory.

Definition 2.5.2. Let M be acceptable. The Σ_1–*projectum* of M (in symbols ρ_M) is the least $\rho \leq \mathrm{On}_M$, such that there is a $\underline{\Sigma}_1(M)$ set $a \subset \rho$ with $a \notin M$.

Lemma 2.5.6. *Let $M = \langle J^A_\alpha, B\rangle, \rho = \rho_M$. Then*

(a) If $\rho \in M$, then ρ is cardinal in M.

(b) If D is $\underline{\Sigma}_1(M)$ and $D \subset J^A_\rho$, then $\langle J^A_\rho, D\rangle$ is amenable.

(c) If $u \in J^A_\rho$, there is no $\underline{\Sigma}_1(M)$ partial map of u onto J^A_ρ.

(d) $\rho \in \mathrm{Lim}^*$

Proof:

(a) Suppose not. Then there are $f \in M$, $\gamma < \rho$ such that f maps γ onto ρ. Let $a \subset \rho$ be $\underline{\Sigma}_1(M)$ such that $a \notin M$. Set $\tilde{a} = f^{-1}{}''a$. Then $\tilde{a}$ is $\Sigma_1(M)$ and $\tilde{a} \subset \gamma$. Hence $\tilde{a} \in M$. But then $a = f''\tilde{a} \in M$ by rud closure.
Contradiction! QED (a)

(b) Suppose not. Let $u \in J^A_\rho$ such that $D \cap u \notin J^A_\rho$. We first note:

Claim $D \cap u \notin M$.

If $\rho = \alpha$ this is trivial, so let $\rho < \alpha$. Then ρ is a cardinal by (a) and by Lemma 2.5.1 we know that $\mathbb{P}(u) \cap M \subset J^A_\rho$. QED (Claim)

By Corollary 2.5.2 there is $f \in J^A_\rho$ mapping a $\nu < \rho$ onto u. Then $d = f^{-1u}(D\cap u)$ is $\underline{\Sigma}_1(M)$ and $d \subset \nu < \rho$. Hence $d \in M$. Hence $D\cap u = f''d \in M$ by rud closure. QED (b)

(c) Suppose not. Let f ba a counterexample. Set $a = \{x \in u | x \in \mathrm{dom}(f) \wedge x \notin f(x)\}$. Then a is $\underline{\Sigma}_1(M)$, $a \subset u \in M$. Hence $a \in J^A_\rho$ by (b). Let $a = f(x)$. Then $x \in f(x) \leftrightarrow x \notin f(x)$.
Contradiction! QED (c)

(d) If not, then $\rho = \beta + \omega$ where $\beta \in \mathrm{Lim}$. But then there is a $\underline{\Sigma}_1(M)$ partial map of β onto ρ, violating (c). QED (Lemma 2.5.6)

Remark We have again stated and proven the theorem for the special case $M = \langle J^A_\alpha, B\rangle$, since the general case is then obvious. We shall continue this practice for the rest of the book. A *good parameter* is a $p \in M$ which witnesses that $\rho = \rho_M$ is the projectum — i.e. there is $B \subset M$ which is $\Sigma_1(M)$ in p with $B \cap H^M_\rho \notin M$. But by §3 any $p \in M$ has the form $p = f(a)$

where f is a $\Sigma_1(M)$ function and a is a finite set of ordinals. Hence a is good if p is. For technical reasons we shall restrict ourselves to good parameters which are finite sets of ordinals:

Definition 2.5.3. $P = P_M =:$ The set of $p \in [\mathrm{On}_M]^{<\omega}$ which are good parameters.

Lemma 2.5.7. *If $p \in P$, then $p \setminus \rho_M \in P$.*

Proof: It suffices to show that if $\nu = \min(p)$ and $\nu < \rho$, then $p' = p\setminus(\nu+1) \in P$. Let B be $\Sigma_1(M)$ in p such that $B \cap H_\rho^M \notin M$. Let $B(x) \leftrightarrow B'(x,p)$ where B' is $\Sigma_1(M)$.

Set:

$$B^*(x) \leftrightarrow: \bigvee z \bigvee \nu (x = \langle z, \nu\rangle \wedge B'(z, p' \cup \{\nu\})).$$

Then $B^* \cap H_\rho \notin M$, since otherwise

$$B \cap H_\rho = \{x | \langle x, \nu\rangle \in B^* \cap H_\rho\} \in M.$$

Contradiction! QED (Lemma 2.5.7)

For any $p \in [\mathrm{On}_M]^{<\omega}$ we define the *standard code T^p determined by p* as:

Definition 2.5.4.

$$T^p = T^p_M =: \{\langle i, x\rangle | \models_M \varphi_i[x,p]\} \cap H^M_{\rho_M}\}$$

where $\langle \varphi_i | i < \omega\rangle$ is a fixed recursive enumeration of the Σ_1–fomulae.

Lemma 2.5.8. *$p \in P \leftrightarrow T^p \notin M$.*

Proof:

($\leftarrow$) $T^p = T \cap H_p^M$ for a T which is $\Sigma_1(M)$ in p.

($\rightarrow$) Let B be $\Sigma_1(M)$ in p such that $B \cap H_p^M \notin M$. Then for some i:

$$B(x) \leftrightarrow \langle i, x\rangle \in T^p$$

for $x \in H_p^M$. Hence $T^p \notin M$. QED (Lemma 2.5.8)

A parameter p is *very good* if every element of M is Σ_1 definable from parameters in $\rho_M \cup \{p\}$. R is the set of very good parameters lying in $[\mathrm{On}_M]^{<\omega}$.

Definition 2.5.5. $R = R_M =$: the set of $r \in [\mathrm{On}_M]^{<\omega}$ such that $M = h_M(\rho_M \cup \{r\})$.

Note. This is the same as saying $M = h_M(\rho_M \cup r)$, since

$$h(\rho \cup r) = h\text{"}(\omega \times [\rho \cup r]^{<\omega}).$$

But $\rho \cup r = \rho \cup (r \setminus \rho)$. Hence:

Lemma 2.5.9. *If $r \in R$, then $r \setminus \rho \in R$. We also note:*

Lemma 2.5.10. $R \subset P$.

Proof: Let $r \in R$. We must find $B \subset M$ such that B is $\Sigma_1(M)$ in r and $B \cap H^M_\rho \notin M$. Set:

$$B = \{\langle i, x\rangle | \bigvee y\; y = h(i, \langle x, r\rangle) \wedge \langle i, x\rangle \notin y\}.$$

If $b = B \cap H^M_\rho \in M$, then $b = h(i, \langle x, r\rangle)$ for some i. Then $\langle i, x\rangle \in b \leftrightarrow \langle i, x\rangle \notin b$.
Contradiction! QED (Lemma 2.5.10)

However, R can be empty.

Lemma 2.5.11. *There is a function h^r uniformly $\Sigma_1(M)$ in r such that whenever $r \in R_M$, then $M = h^{r\prime\prime}\rho_M$.*

Proof: Let $x \in M$. Since $x \in h(\rho \cup \{r\})$ there is an f which is $\Sigma_1(M)$ in r such that $x = f(\xi_1, \dots, \xi_n)$. But ρ is closed under Gödel pairs, so $x = f'(\prec \xi_1, \dots, \xi_n \succ)$, where

$$x = f'(\xi) \leftrightarrow \bigvee \xi_1, \dots, \xi_n(\xi = \prec \vec{\xi} \succ \wedge x = f(\vec{\xi})).$$

f' is $\Sigma_1(M)$ in r. Hence $x = h(i, \langle \prec \vec{\xi} \succ, r\rangle)$ for some $i < \omega$. Set

$$x = h^r(\delta) \leftrightarrow \bigvee \xi \bigvee i < \omega(\delta = \prec i, \xi \succ \wedge x = h(i, \langle \xi, r\rangle)).$$

Then $x = h^r(\prec i, \prec \vec{\xi} \succ\succ)$. QED (Lemma 2.5.11)

Lemma 2.5.11 explains why we called T^p a *code*: If $r \in R$, then T^r gives complete information about M. Thus the relation $\in' = \{\langle x, \tau\rangle | h^r(\nu) \in h^r(\tau)\}$ is rud in T^r, since $\nu \in' \tau \leftrightarrow \langle i, \langle \nu, \tau\rangle\rangle \in T^r$ for some $i < \omega$. Similarly, if $M = \langle J^{\vec{A}}_\alpha, \vec{B}\rangle$, then $A'_i = \{\nu | h^r(\nu) \in A_i\}$ and $B'_j = \{\nu | h^r(\nu) \in B_i\}$ are rud in T^r (as is, indeed, R' whenever R is a relation which is $\Sigma_1(M)$ in p). Note, too, that if $B \subset H^M_\rho$ is $\underline{\Sigma}_1(M)$, then B is rud in T^r. However, if $p \in P^1 \setminus R^1$, then T^p does not completely code M.

Definition 2.5.6. Let $p \in [\mathrm{On}_M]^{<\omega}$. Let $M = \langle J_\alpha^{\vec{A}}, \vec{B}\rangle$.

The *reduct of M by p* is defined to be

$$M^p =: \langle J_{\rho_M}^{\vec{A}}, T_M^p\rangle.$$

Thus M^p is an acceptable model which — if $p \in R_M$ — incorporates complete information about M.

The *downward extension of embeddings lemma* says:

Lemma 2.5.12. *Let $\pi : N \to_{\Sigma_0} M^p$ where N is a J–model and $p \in [\mathrm{On}_M]^{<\omega}$.*

(a) There are unique $\overline{M}, \overline{p}$ such that $\overline{M}$ is acceptable, $\overline{p} \in R_{\overline{M}}, N = \overline{M}^{\overline{p}}$.

(b) There is a unique $\tilde{\pi} \supset \pi$ such that $\tilde{\pi} : \overline{M} \to_{\Sigma_0} M$ and $\pi(\overline{p}) = p$.

(c) $\tilde{\pi} : \overline{M} \to_{\Sigma_1} M$.

Proof: We first prove the existence claim. We then prove the uniqueness claimed in (a) and (b).

Let e.g. $M = \langle J_\alpha^A, B\rangle, M^p = \langle J_\rho^A, T\rangle, N = \langle J_{\overline{\rho}}^{\overline{A}}, \overline{T}\rangle$. Set: $\tilde{\rho} = \sup \pi''\overline{\rho}$, $\tilde{M} = M^p|\tilde{\rho} = \langle J_{\tilde{\rho}}^A, \tilde{T}\rangle$ where $\tilde{T} = T \cap J_{\tilde{\rho}}^A$. Set $X = \mathrm{rng}(\pi)$, $Y = h_M(X \cup \{p\})$. Then $\tilde{\pi} : N \to_{\Sigma_0} \tilde{M}$ cofinally.

(1) $Y \cap \tilde{M} = X$
Proof: Let $y \in Y \cap \tilde{M}$. Since X is closed under ordered pairs, we have $y = f(x,p)$ where $x \in X$ and f is $\Sigma_1(M)$. Then

$$\begin{aligned} y = f(x,p) \quad &\leftrightarrow \models_M \varphi_i[\langle y,x\rangle, p] \\ &\leftrightarrow \langle i, \langle y,x\rangle\rangle \in \tilde{T}. \end{aligned}$$

Since $X \prec_{\Sigma_1} \tilde{M}$, there is $y \in X$ such that $\langle i, \langle y,x\rangle\rangle \in \tilde{T}$. Hence $y = f(x,p) \in X$. QED (1)

Now let $\tilde{\pi} : \overline{M} \overset{\sim}{\leftrightarrow} Y$, where $\overline{M}$ is transitive. Clearly $p \in Y$, so let $\tilde{\pi}(\overline{p}) = p$. Then:

(2) $\tilde{\pi} : \overline{M} \to_{\Sigma_1} M$, $\tilde{\pi} \restriction N = \pi$, $\tilde{\pi}(\overline{p}) = p$.
But then:

(3) $\overline{M} = h_{\overline{M}}(N \cup \{\overline{p}\})$.
Proof: Let $y \in \overline{M}$. Then $\tilde{\pi}(y) \in Y = h_M''(\omega x(Xx\{p\}))$, since X is closed under ordered pairs. Hence $\tilde{\pi}(y) = h_M(i, \langle \pi(x), p\rangle)$ for an $x \in \overline{M}$. Hence $y = h_{\overline{M}}(i, \langle x, \overline{p})$. QED (3)

(4) $\overline{\rho} \geq \rho_{\overline{M}}$ where $\bar{\rho} = \mathrm{On} \cap N$.
Proof: It suffices to find a $\Sigma_1(\overline{M})$ set b such that $b \subset N$ and $b \notin \overline{M}$. Set

$$b = \{\langle i, x\rangle \in \omega \times N | \bigvee y \quad (y = h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \\ \wedge \langle i, x\rangle \notin y)\}$$

If $b \in \overline{M}$, then $b = h_{\overline{M}}(i, \langle x, \overline{p}\rangle)$ for some $x \in N$. Hence

$$\langle i, x\rangle \in b \leftrightarrow \langle i, x\rangle \notin b.$$

Contradiction! QED (4)

(5) $\overline{T} = \{\langle i, x\rangle \in \omega \times N | \models_{\overline{M}} \varphi_i[i, \langle x, p\rangle]\}$.
Proof: $\overline{T} \subset \omega \times N$, since $\tilde{T} \subset \omega \times \tilde{M}$. But for $\langle i, x\rangle \in \omega \times N$ we have:

$$\begin{aligned} \langle i, x\rangle \in \overline{T} &\leftrightarrow \langle i, \pi(x)\rangle \in \tilde{T} \\ &\leftrightarrow M \models \varphi_i[\langle (x), p\rangle] \\ &\leftrightarrow \overline{M} \models \varphi_i[\langle x, p\rangle] \text{ by } (2) \end{aligned}$$

QED (5)

(6) $\overline{\rho} = \rho_{\overline{M}}$.
Proof: By (4) we need only prove $\overline{\rho} \leq \rho_{\overline{M}}$. It suffices to show that if $b \subset N$ is $\underline{\Sigma}_1(\overline{M})$, then $\langle J^{\overline{A}}_{\overline{\rho}}, b\rangle$ is amenable. By (3) b is $\Sigma_1(\overline{M})$ in $x, \overline{p}$ where $x \in \overline{N}$.
Hence

$$b = \{z | \overline{M} \models \varphi_i[\langle z, x\rangle, \overline{p}]\} = \\ = \{z | \langle i, z, x\rangle \in \overline{T}\}$$

Hence b is rud in $\overline{T}$ where $N = \langle J^{\overline{A}}_{\overline{\rho}}, \overline{T}\rangle$ is amenable. QED (6)

But then $\overline{M} = h_{\overline{M}}(\overline{\rho} \cup \{\overline{p}\})$ by (3) and the fact that $h_{J^{\overline{A}}_{\overline{\rho}}}(\overline{\rho}) = J^{\overline{A}}_{\overline{\rho}}$. Hence

(7) $\overline{p} \in R_{\overline{M}}$.
By (6) we then conclude:

(8) $N = \overline{M}^{\overline{p}}$.
This proves the existence assertions. We now prove the uniqueness assertion of (a). Let $\hat{M}^{\hat{p}} = N$ where $\hat{p} \in R_{\hat{M}}$.
We claim: $\hat{M} = \overline{M}$, $\hat{p} = \overline{p}$.

Since the Skolem function is uniformly Σ_1 there is a $j < \omega$ such that

$$\begin{aligned} & h_{\hat{M}}(i, \langle x, \hat{p}\rangle) \in h_{\hat{M}}(i, \langle y, \hat{p}\rangle \leftrightarrow \\ & \quad \leftrightarrow \hat{M} \models \varphi_j[\langle x, y\rangle, p] \leftrightarrow \langle j, \langle x, y\rangle\rangle \in \overline{T} \\ & \quad \leftrightarrow h_{\overline{M}}(i, \langle x, \overline{p}\rangle) \in h_{\overline{M}}(i, \langle y, \overline{p}\rangle) \end{aligned}$$

Similarly:

$$h_{\hat{M}}(i,\langle x,\hat{p}\rangle)\in\hat{A}\leftrightarrow h_{\overline{M}}(i,\langle x,\overline{p}\rangle)\in\overline{A}$$
$$h_{\hat{M}}(i,\langle x,\hat{p}\rangle)\in\hat{B}\leftrightarrow h_{\overline{M}}(i,\langle x,\overline{p}\rangle)\in\overline{B}$$

where $\hat{M}=\langle J^{\hat{A}}_{\hat{\alpha}},\hat{B}\rangle$, $\overline{M}=\langle J^{\overline{A}}_{\overline{\alpha}},\overline{B}\rangle$. Then there is an isomorphism $\sigma:\hat{M}\overset{\sim}{\leftrightarrow}\overline{M}$ defined by $\sigma(h_{\hat{M}}(i,\langle x,\hat{p}\rangle)\simeq h_{\overline{M}}(i,\langle x,\overline{p}\rangle)$ for $x\in N$. Clearly $\sigma(\hat{p})=\overline{p}$. Hence $\sigma=\mathrm{id},\hat{M},\overline{M},\hat{p}=\overline{p}$, since $\overline{M},\hat{M}$ are transitive.

We now prove (b). Let $\hat{\pi}\supset\pi$ such that $\hat{\pi}:\overline{M}\to_{\Sigma_0}M$ and $\hat{\pi}(\overline{p})=p$. If $x\in N$ and $h_{\overline{M}}(i,\langle x,\overline{p}\rangle)$ is defined, it follows that:

$$\hat{\pi}(h_{\overline{M}}(i,\langle x,\overline{p}\rangle)=h_M(i,\langle\pi(x),p\rangle)=\tilde{\pi}(h_M(i,\langle x,\overline{p}\rangle)).$$

Hence $\hat{\pi}=\pi$. QED (Lemma 2.5.12)

If we make the further assumption that $p\in R_M$ we get a stronger result:

Lemma 2.5.13. *Let $M,N,\overline{M},\pi,\overline{\pi},p,\overline{p}$ be as above where $p\in R_M$ and $\pi:N\to_{\Sigma_l}M^p$ for an $l<\omega$. Then $\tilde{\pi}:\overline{M}\to_{\Sigma_{l+1}}M$.*

Proof: For $l=0$ it is proven, so let $l\geq 1$ and let it hold at l. Let R be $\Sigma_{l+1}(M)$ if l is even and $\Pi_{l+1}(M)$ if l is odd. Let $\overline{R}$ have the same definition over $\overline{M}$. It suffices to show:

$$\overline{R}(\vec{x})\leftrightarrow R(\tilde{\pi}(\vec{x}))\text{ for }x_1,\ldots,x_n\in\overline{M}.$$

But:

$$R(\vec{x})\leftrightarrow Q_1y_1\in M\ldots Q_ly_l\in MR'(\vec{y},\vec{x})$$

and

$$\overline{R}(\vec{x})\leftrightarrow Q_1y_1\in\overline{M}\ldots Q_ly_l\in\overline{MR}'(\vec{y},\vec{x})$$

where $Q_1\ldots Q_l$ is a string of alternating quantifiers, R' is $\Sigma_1(M)$, and $\overline{R}'$ is $\Sigma_1(\overline{M})$ by the same definition. Set

$$D=:\{\langle i,x\rangle\in\omega\times J^A_\rho|h_M(i,\langle x,p\rangle)\text{ is defined}\}$$
$$\overline{D}=:\{\langle i,x\rangle\in\omega\times J^{\overline{A}}_{\overline{\rho}}|h_{\overline{M}}(i,\langle x,\overline{p}\rangle)\text{ is defined}\}.$$

Then D is $\Sigma_1(M)$ in p and $\overline{D}$ is $\Sigma_1(\overline{M})$ in $\overline{p}$ by the same definition. Then D is rud in T^p_M and $\overline{D}$ is rud in $T^{\overline{p}}_{\overline{M}}$ by the same definition, since for some $j<\omega$ we have:

$$\langle i,x\rangle\in D\leftrightarrow\langle j,x\rangle\in T^p_M,\ x\in\overline{D}\leftrightarrow\langle j,x\rangle\in T^{\overline{p}}_{\overline{M}}.$$

Define k on D

$$k(\langle i,x\rangle)=h_M(i,\langle x,p\rangle);\ \overline{k}(\langle i,x\rangle)=h_{\overline{M}}(i,\langle x,\overline{p}\rangle).$$

Set:

$$P(\vec{w},\vec{z}) \leftrightarrow (\vec{w},\vec{z} \in D \wedge R'(k(\vec{w}),k(\vec{z}))$$
$$\overline{P}(\vec{w},\vec{z}) \leftrightarrow (\vec{w},\vec{z} \in \overline{D} \wedge \overline{R}'(\overline{k}(\vec{w}),\overline{k}(\vec{z}))$$

Then: as before, P is rud in T^p_M and $\overline{D}$ is rud in $T^{\overline{p}}_{\overline{M}}$ by the same definition. Now let $x_i = k(z_i)$ for $i = 1,\ldots,n$. Then $\tilde{\pi}(x_i) = k(\pi(z_i))$. But since π is Σ_l–preserving, we have:

$$\begin{aligned}\overline{R}(\vec{x}) &\leftrightarrow Q_1 w_1 \in \overline{D} \ldots Q_l w_l \in \overline{D}\ \overline{P}(\vec{w},\vec{z}) \\ &\leftrightarrow Q_1 w_1 \in D \ldots Q_l w_l \in D P(\vec{w},\pi(\vec{z})) \\ &\leftrightarrow R(\tilde{\pi}(\vec{x}))\end{aligned}$$

QED (Lemma 2.5.13)

2.5.3 Soundness and iterated projecta

The reduct of an acceptable structure is itself acceptable, so we can take its reduct etc., yielding a sequence of reducts and nonincreasing projecta $\langle \rho^n_M | n < \omega\rangle$. this is the classical method of doing fine structure theory, which was used to analyse the constructible hierarchy, yielding such results as the $\square$ principles and the covering lemma. In this section we expound the basic elements of this classical theory. As we shall see, however, it only works well when our acceptable structures have a property called *soundness*. In this book we shall often have to deal with unsound structures, and will, therefore, take recourse to a further elaboration of fine structure theory, which is developed in §2.6.

It is easily seen that:

Lemma 2.5.14. *Let $p \in R_M$. Let B be $\underline{\Sigma}_1(M)$. Then $B \cap J^A_\rho$ is* rud *in parameters over M^p.*

Proof: Let B be Σ_1 in r, where $r = h_M(i,\langle v,p\rangle)$ and $\nu < \rho$. Then B is Σ_1 in ν, p. Let:

$$B(x) \leftrightarrow M \models \varphi_i[\langle x,\nu\rangle, p]$$

where $\langle \varphi_i | i < \omega\rangle$ is our canonical enumeration of Σ_1 formulae. Then:

$$x \in B \leftrightarrow \langle i, \langle x,\nu\rangle\rangle \in T^p$$

QED(Lemma 2.5.14)

It follows easily that:

Corollary 2.5.15. *Let $p, q \in R_M$. Let $D \subset J_\rho^A$. Then D is $\underline{\Sigma}_1(M^p)$ iff it is $\underline{\Sigma}_1(M^q)$.*

Assuming that $R_M \neq \emptyset$, there is then a uniquely defined *second projectum* defined by:

Definition 2.5.7. $\rho_M^2 \simeq: \rho_{M^p}$ for $p \in R_M$.

We can then define:

$$\begin{aligned} R_M^2 =: &\ \text{The set of } a \in [\mathrm{On}_M]^{<w} \text{ such that} \\ &\ a \in R_M \text{ and } a \cap \rho \in R_{M^{(a \setminus \rho)}}. \end{aligned}$$

If $R_M^2 \neq \emptyset$ we can define the *second reduct:*

$$M^{2,a} =: (M^a)^{a \cap \rho} \text{ for } a \in R_M^2.$$

But then we can define the *third projectum:*

$$\rho^3 = \rho_{M^{2,a}} \text{ for } a \in R_M^2.$$

Carrying this on, we get R_M^n, $M^{n,a}$ for $a \in R_M^n$ and ρ^{n+1}, as long as $R_M^n \neq \emptyset$. We shall call M *weakly n–sound* if $R_M^n \neq \emptyset$.

The formal definitions are as follows:

Definition 2.5.8. Let $M = \langle J_\alpha^A, B \rangle$ be acceptable.

By induction on n we define:

- The set R_M^n of *very good n–parameters.*
- If $R_M^n \neq \emptyset$, we define the $n+1$*st projectum* ρ_M^{n+1}.
- For all $a \in R_M^n$ the *n–th reduct* $M^{n,a}$.

We inductively verify:

* If $D \subset J_{\rho^n}^A$ and $a, b \in R^n$, then D is $\underline{\Sigma}_1(M^{n,a})$ iff it is $\underline{\Sigma}_1(M^{n,b})$.

Case 1 $n = 0$. Then $R^0 =: [\mathrm{On}_M]^{<\omega}, \rho^0 = \mathrm{On}_M, M^{0,a} = M$.

Case 2 $n = m + 1$. If $R^m = \emptyset$, then $R^n = \emptyset$ and ρ^n is undefined. Now let $R^m \neq \emptyset$. Since (*) holds at m, we can define

- $\rho^n =: \rho_{M^{m,a}}$ whenever $a \in R^m$.
- $R^n =:$ the set of $a \in [\alpha]^{<\omega}$ such that $a \in R^m$ and $a \cap \rho^m \in R_{M^{m,a}}$.
- $M^{n,a} =: (M^{m,a})^{a \cap \rho^m}$ for $a \in R^n$.

Note. It follows inductively that $a \setminus \rho^n \in R^n$ whenever $a \in R^n$.

We now verify (*). It suffices to prove the direction ($\rightarrow$). We first note that $M^{n,a}$ has the form $\langle J^A_{\rho n}, T\rangle$, where T is the restriction of a $\underline{\Sigma}_1(M^{m,a})$ set T' to $J^A_{\rho n}$. But then T' is $\underline{\Sigma}_1(M^{m,b})$ by the induction hypothesis. Hence T is rudimentary in parameters over $M^{n,b} = (M^{m,b})^{b \cap \rho^n}$ by Lemma 2.5.14.

Hence, if $D \subset J^A_{\rho n}$ is $\underline{\Sigma}_1(M^{n,a})$, it is also $\underline{\Sigma}_1(M^{n,b})$. QED

This concludes the definition and the verification of (*). Note that $R^1_M = R_M$, $\rho^1 = \rho^1_M$, and $M^{1,a} = M^a$ for $a \in R_M$.

We say that M is *weakly n–sound* iff $R^n_M \neq \emptyset$. It is *weakly sound* iff it is weakly n–sound for $n < \omega$. A stronger notion is that of *full soundness*:

Definition 2.5.9. M is *n–sound* (or *fully n– sound*) iff it is weakly n–sound and for all $i < n$ we have: If $a \in R^i$, then $P_{M^{i,a}} = R_{M^{i,a}}$.

Thus $R_M = P_M$, $R_{M^{1,a}} = P_{M^{1,a}}$ for $a \in P_M$ etc. If M is n–sound we write P^i_M for $R^i_M (i \leq n)$, since then: $a \in P^{i+1} \leftrightarrow (a \setminus \rho^i \in P^i \wedge a \cap \rho^i \in R_{M^{i,a \cap \rho^i}}$ for $i < n$).

There is an alternative, but equivalent, definition of soundness in terms of *standard parameters*. in order to formulate this we first define:

Definition 2.5.10. Let $a, b \in [\mathrm{On}]^{<\omega}$.

$$a <_* b \leftrightarrow = \bigvee \mu (a \setminus \mu = b \setminus \mu \wedge \mu \in b \setminus a).$$

Lemma 2.5.16. *$<_*$ is a well ordering of $[\mathrm{On}]^{<\omega}$.*

Proof: It suffices to show that every non empty $A \subset [\mathrm{On}]^{<\omega}$ has a unique $<_*$–minimal element. Suppose not. We derive a contradiction by defining an infinite descending chain of ordinals $\langle \mu_i | i < \omega\rangle$ with the properties:

- $\{\mu_0, \ldots, \mu_n\} \leq_* b$ for all $b \in A$.
- There is $b \in A$ such that $b \setminus \mu_n = \{\mu_0, \ldots, \mu_n\}$.

$\emptyset \notin A$, since otherwise $\emptyset$ would be the unique minimal element, so set: $\mu_0 = \min\{\max(b)|b \in A\}$. Given μ_n we know that $\{\mu_0, \ldots, \mu_n\} \notin A$, since it would otherwise be the $<_*$–minimal element. Set:

$$\mu_{n+1} = \min\{\max(b \cap \mu_n)|b \in A \cap b \setminus \mu_n = \{\mu_0, \ldots, \mu_n\}\}.$$

QED (Lemma 2.5.16)

Definition 2.5.11. The *first standard parameter* p_M is defined by:

$$p_M =: \text{ The } <_*\text{–least element of } P_M.$$

Lemma 2.5.17. *$P_M = R_M$ iff $p_M \in R_M$.*

Proof: ($\rightarrow$) is trivial. We prove ($\leftarrow$). Suppose not. Then there is $r \in P \setminus R$. Hence $p <_* r$, where $p = p_M$. Hence in M the statement:

(1) $\bigvee q <_* r \ r = h(i, \langle \nu, q \rangle)$

holds for some $i < \omega$, $\nu < p_M$. Form M^r and let $\overline{M}, \overline{r}, \pi$ be such that $\overline{M}^{\overline{r}} = M^r$, $\overline{r} \in R_{\overline{M}}$, $\pi : \overline{M} \rightarrow_{\Sigma_1} M$, and $\pi(\overline{r}) = r$. The statement (1) then holds of $\overline{r}$ in $\overline{M}$.

Let $\overline{q} \in \overline{M}$, $\overline{r} = h_{\overline{M}}(i, \overline{q})$ where $\overline{q} <_* \overline{r}$. Set $q = \pi(\overline{q})$. Then $r = h(i, q)$ in M, where $q <_* r$. Hence $q \in P_M$. But then $q \in R_M$ by the minimality of r. This impossible however, since

$$q \in \pi''\overline{M} = h_M(\rho_M \cup r) \neq M.$$

Contradiction! QED (Lemma 2.5.17)

Definition 2.5.12. The n–th *standard parameter* p^n_M is defined by induction on n as follows:

Case 1 $n = 0$. $p^0 = \emptyset$.

Case 2 $n = m + 1$. If $p^m \in R^m$
$p^n = p^m \cup p_{M^{m,p^m}}$

Note. that we always have: $p^n \cap \rho^{n+1} = \emptyset$ by $<_*$–minimality and Lemma 2.5.7.

If $p^m \notin R^m$, then p^n is undefined. By Lemma 2.5.17 it follows easily that:

Corollary 2.5.18. *M is n–sound iff p^n_M is defined and $p^n_M \in R^n_M$.*

This is the definition of soundness usually found in the literature.

Note. That the sequences of projecta ρ^n will stabilize at some n, since it is monotony non increasing. If it stabilizes at n, we have $R^{n+h} = R^n$ and $P^{n+h} = P^n$ for $h < \omega$.

By iterated application of Lemma 2.5.13 we get:

Lemma 2.5.19. *Let $a \in R^n_M$ and let $\overline{\pi} : N \to_{\Sigma_l} M^{na}$. Then there are $\overline{M}, \overline{a}$ and $\pi \supset \overline{\pi}$ such that $\overline{M}^{n\overline{a}} = M^{na}$, $\overline{a} \in R^n_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_{n+l+1}} M$ and $\pi(\overline{a}) = a$.*

We also have:

Lemma 2.5.20. *Let $a \in R^n_M$. There is an M–definable partial map of ρ^n onto M which is M–definable in the parameter a.*

Proof: By induction on n. The case $n = 0$ is trivial. Now let $n = m + 1$. Let f be a partial map of ρ^m onto M which is definable in $a \setminus \rho^m$. Let $N = M^{m,a\setminus\rho^n}$, $b = a \cap \rho^m$. Then $N = h_N(\rho^n \cup \{b\}) = h_N''(w \times (\rho^n \times \{b\}))$. Set:

$$g(\prec i, \nu \succ) \simeq: h_N(i, \langle \nu, b\rangle) \text{ for } \nu < \rho^n.$$

Then $N = g''\rho^n$. Hence $M = fg''\rho^n$, where fg is M–definable in a. QED

We have now developend the "classical" fine structure theory which was used to analyze L. Its applicability to L is given by:

Lemma 2.5.21. *Every J_α is acceptable and sound.*

Unfortunately, in this book we shall sometimes have to deal with acceptable structures which are not sound and can even fail to be weakly 1–sound. This means that the structure is not coded by any of its reducts. How can we deal with it? It can be claimed that the totality of reducts contains full information about the structure, but this totality is a very unwieldy object. In §2.6 we shall develop methods to "tame the wilderness".

We now turn to the proof of Lemma 2.5.21:

We first show:

(A) If J_α is acceptable, then it is sound.

Proof: By induction on n we show that J_α is n–sound. The case $n = 0$ is trivial. Now let $n = m + 1$. Let $p = p^m_M$. Let $q = p_{M^{m,p}} =$ The $<_*$–least $q \in P_{M^{m,p}}$.

Claim $q \in R_{M^{m,p}}$.

Suppose not. Let $X = h_{M^{m,p}}(\rho^n \cup q)$. Let $\overline{\pi} : N \overset{\sim}{\longleftrightarrow} X$, where N is transitive. Then $\overline{\pi} : N \to_{\Sigma_1} M^{np}$ and there are $\overline{M}, \overline{p}, \pi \supset \overline{\pi}$ such that $\overline{M}^{m\overline{p}} = M^{mp}$, $\overline{p} \in R^m_{\overline{M}}$, $\pi : \overline{M} \to_{\Sigma_n} M$, and $\pi(\overline{p}) = p$. Then $\overline{M} = J_{\overline{\alpha}}$ for some $\overline{\alpha} \leq \alpha$ by the condensation lemma for L.

Let A be $\Sigma_1(M^{mp})$ in q such that $A \cap \rho^n_M \notin M^{m,p}$ Then $A \cap \rho^n_M \notin M$.

Let $\overline{A}$ be $\Sigma_1(N)$ in $\overline{q} = \pi^{-1}(q)$ by the same definition. Then $A \cap \rho^n = \overline{A} \cap \rho^n$ is $J_{\overline{\alpha}}$ definable in $\overline{q}$. Hence $\overline{\alpha} = \alpha$, $\overline{M} = M$, since otherwise $A \cap \rho^n \in M$. But then $\pi = id$ and $N = \overline{M}^{m\overline{p}} = M^m$. But by definition: $N = h_{M^{m,p}}(\rho^n \cup q)$. Hence $q \in R_{M^{np}}$. QED

By induction on α we then prove:

(B) J_α is acceptable.

Proof: The case $\alpha = \omega$ is trivial. The case $\alpha \in \text{Lim}^*$ is also trivial. There remains the case $\alpha = \beta + \omega$, where β is a limit ordinal. By the induction hypothesis J_β is acceptable, hence sound.

We know that $\underline{\Sigma}_\omega(J_\alpha) = \underline{\Sigma}^n(J_\alpha)$ by soundeness. But we also know: $\mathbb{P}(J_\alpha) \cap J_{\alpha+\omega} \subset \underline{\Sigma}_\omega(J_\alpha)$. Let $\rho = \rho^\omega_{J_\alpha}$. Clearly, no $\delta > \rho$ is a cardinal in $J_{\alpha+1}$. But if $a \in J_{\alpha+\omega}$ and $a \subset \gamma < \rho$, then $a \in J_\rho$, since this $a \in \Sigma^*(J_\alpha)$ and $\langle J_\rho, A \cap J_\rho \rangle$ is amenable for all $A \in \underline{\Sigma}^*(J_\alpha)$. QED (Lemma 2.5.21)

The fact that $\mathbb{P}(J_\alpha) \cap J_{\alpha+1} \subset \underline{\Sigma}_\omega(J_\alpha)$ was derived from Corollary 2.2.14, which says that if $U \neq \emptyset$ is any traisntive set, then:

$$\underline{\Sigma}_\omega(\langle U, \in \rangle) = \mathbb{P}(U) \cap \text{rud}(U \cup \{U\}),$$

where $\text{rud}(X)$ =:the closure of X under rudimentary functions. However, a slight modification of the proof of Corollary 2.2.14 yields the stronger result:

Lemma 2.5.22. *Let $U \neq \emptyset$ be transitive. Let $A_1, \ldots, A_m \subset U$. Then:*

$$\underline{\Sigma}_\omega(\langle U, \in \vec{A} \rangle) = \mathbb{P}(U) \cap \text{rud}(U \cup \{U, \vec{A}\})$$

(We leave this to the reader.)

This is especially interesting if U is rudimentary closed and $\langle U, A_1, \ldots, A_m \rangle$ is amenable.

Definition 2.5.13. $N = J^A_\beta$ is a *constructible extension* of $M = J^A_\alpha$ if and only if $A \subset J_\alpha[A]$ and $\alpha \leq \beta$.

By Lemma 2.5.22 we get:

Lemma 2.5.23. *Let J^A_β be a constructible extension of J^A_α. Then $\underline{\Sigma}_\omega(J^A_\beta) = \mathbb{P}(J^A_\beta) \cap J^A_{\beta+\omega}$.*

Using this we can repeat the proof of Lemma 2.5.21 to get:

Lemma 2.5.24. *Let J^A_β be a constructible extension of J^A_α such that $\rho^\omega_{J^A_\gamma} \geq \alpha$ for $\alpha \leq \gamma \leq \beta$. Then J^A_β is sound and acceptable.*

Suppose now that $\langle J^A_\alpha, B\rangle$ is a J-model. It is natural to define an extension $A * B$ of the predicate A by: $A * B = A \cup (B \times \{\alpha\})$. Then:

$$(A * B) \cap J^A_\alpha = A, B \in J^{A*B}_{\alpha+\omega}.$$

Clearly $J^{A*B}_{\alpha+\omega} = \text{rud}(J_\alpha[A] \cup \{J_\alpha[A], A, B\})$. Hence by Lemma 2.5.22:

Lemma 2.5.25. $\Sigma_\omega(\langle J^A_\alpha, B\rangle) = \mathbb{P}(J^A_\alpha) \cap J^{A*B}_{\alpha+\omega}$.

We can the repeat the last part of the proof of Lemma 2.5.21 to get:

Lemma 2.5.26. *Let $\langle J^A_\alpha, B\rangle$ be sound and acceptable. Then $J^{A*B}_{\alpha+\omega}$ is acceptable.*

(However, it does not follow that $J^{A*B}_{\alpha+\omega}$ is sound.)

2.6 Σ^*–theory

There is an alternative to the Levy hierarchy of relations on an acceptable structure $M = \langle J^A_\alpha, B\rangle$ which — at first sight — seems more natural. Σ_0, we recall, consists of the relation on M which are Σ_0 definable in the predicates of M. Σ_1 then consists of relations of the form $\bigvee y R(y, \vec{x})$ where R is Σ_0. Call these levels $\Sigma^{(0)}_0$ and $\Sigma^{(0)}_1$. Our next level in the new hierarchy, call it $\Sigma^{(1)}_0$, consists of relations which are "Σ_0 in $\Sigma^{(0)}_1$" — i.e. $\Sigma_0(\langle M, \vec{A}\rangle)$ where $A_1, \ldots, A_n$ are $\Sigma^{(0)}_1$. $\Sigma^{(1)}_1$ then consists of relations of the form $\bigvee y R(y, \vec{x})$ where R is $\Sigma^{(1)}_0$. $\Sigma^{(2)}_0$ then consists of relations which are Σ_0 in $\Sigma^{(1)}_1 \ldots$ etc. By a $\underline{\Sigma}^{(n)}_i$ relation we of course mean a relation of the form

$$R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p}),$$

where $p_1, \ldots, p_m \in M$ and R' is $\Sigma^{(n)}_i(m)$. It is clear that there is natural class of $\Sigma^{(n)}_i$–formulae such that R is a $\Sigma^{(n)}_i$–relation iff it is defined by a $\Sigma^{(n)}_i$–formula. Thus e.g. we can define the $\Sigma^{(1)}_0$ formula to be the smallest set Σ of formulae such that

- All primitive formulae are in Σ.
- All $\Sigma_1^{(0)}$ formulae are in Σ.
- Σ is closed under the sentential operations $\vee, \rightarrow, \leftrightarrow, \neg$.
- If φ is in Σ, then so are $\bigwedge v \in u\, \varphi$, $\bigvee v \in u\, \varphi$ (where $v \neq u$).

By a $\Sigma_1^{(1)}$ formula we then mean a formula of the form $\bigvee v\varphi$, where φ is $\Sigma_0^{(1)}$.

How does this hierarchy compare with the Levy hierarchy? If no projectum drops, it turns out to be a useful refinement of the Levy hierarchy:
If $\rho_M^n = \alpha$, then $\Sigma_0^{(n)} \subset \Delta_{n+1}$ and $\Sigma_1^{(n)} = \Sigma_{n+1}$. If, however, a projectum drops, it trivializes and becomes useless. Suppose e.g. that $M = J_\alpha$ and $\rho = \rho_M^1 < \alpha$. Then *every* M–definable relation becomes $\underline{\Sigma}_0^{(1)}(M)$. To see this let $R(\vec{x})$ be defined by the formula $\varphi(\vec{v})$, which we may suppose to be in prenex normal form:

$$\varphi(\vec{v}) = Q_1 u_1 \dots Q_m u_m \varphi'(\vec{v}, \vec{u}),$$

where φ' is quantifier free (hence Σ_0). Then:

$$R(\vec{x}) \leftrightarrow Q_1 y_1 \in M \dots Q_m y_m \in M R'(\vec{x}, \vec{y})$$

where R' is Σ_0. By soundness we know that there is a $\underline{\Sigma}_1(M)$ partial map f of ρ onto M. But then:

$$R(\vec{x}) \leftrightarrow Q_1 \xi_\xi \in \mathrm{dom}(f) \dots Q_m \xi_m \in \mathrm{dom}(f) R'(\vec{x}, f(\vec{\xi})).$$

Since f is $\underline{\Sigma}_1$, the relation $R'(\vec{x}, f(\vec{\xi}))$ is $\underline{\Sigma}_1$. But $\mathrm{dom}(f)$ is $\underline{\Sigma}_1$ and $\mathrm{dom}(f) \subset \rho$, hence by induction on m:

$$R(\vec{x}) \leftrightarrow Q_1 \xi_1 \in \rho \dots Q_m \xi_m \in \rho R''(\vec{x}, \vec{\xi}),$$

where R'' is a sentential combination of $\underline{\Sigma}_1$ relations. Hence R'' is $\underline{\Sigma}_0^{(1)}(M)$ and so is R.

The problem is that, in passing from $\Sigma_1^{(0)}$ to $\Sigma_0^{(1)}$ our variables continued to range over the whole of M, despite the fact that M had grown "soft" with respect to $\underline{\Sigma}_1$ sets. Thus we were able to reduce unbounded quantification over M to quantification bounded by ρ, which lies in the "soft" part of M. in section 2.5 we acknowledged softness by reducing to the part $H = H_\rho^M$ which remained "hard" wrt $\underline{\Sigma}_1$ sets. We then formed a reduct M^p containing just the sets in H. If M is sound, we can choose p such that M^p contains complete information about M. In the general case, however, this may not be possible. It can happen that *every* reduct entails a loss of information. Thus we want to hold on to the original structure M. In passing to $\Sigma_0^{(1)}$, however, we want to restrict our variables to H. We resolve this conundrum by introducing *new* varibles which range only over H. We call these variables of *Type 1*, the old ones being of *Type 0*. Using $u^h, v^h (h = 0, 1)$ as metavariables for variables of Type h, we can then reformulate the definition of $\Sigma_0^{(1)}$ formula, replacing the last clause by:

- If φ is in Σ, then so are $\bigwedge v^i \in u^1 \varphi$, $\bigvee v^i \in u^1 \varphi$ where $i = 0, 1$ and $v^i \neq u^1$.

A $\Sigma_1^{(1)}$ formula is then a formula of the form $\bigvee v^1 \varphi$, where φ is $\Sigma_0^{(1)}$. We call $A \subset M$ a $\underline{\Sigma}_1^{(1)}$ set if it is definable in parameters by a $\Sigma_1^{(1)}$ formula. The *second projectum* ρ^2 is then the least ρ such that $\rho \cap B \notin M$ for some $\underline{\Sigma}_1^{(1)}$ set B. We then introduce type 2 variables $v^2, u^2, \ldots$ ranging over $|J_{\rho^2}^A|$ ($|J_\gamma^A|$ being the set of elements of the structure J_γ^A, where e.g. $M = \langle J_\alpha^A, B \rangle$.) Proceeding in this way, we arrive at a many sorted language with variables of type n for each $n < \omega$. The resulting hierarchy of $\Sigma_h^{(n)}$ formulae $(h = 0, 1)$ offers a much finer analysis of M–definabilty than was possible with the Levy hierarchy alone. This analysis is known as Σ^* theory. In this section we shall develop Σ^* theory systematically and *ab ovo*.

Before beginning, however, we address a remark to the reader: Most people react negatively on their first encounter with Σ^* theory. The introduction of a many sorted language seems awkward and cumbersome. It is especially annoying that the variable domains diminish as the types increase. The author confesses to having felt these doubts himself. After developing Σ^*–theory and making its first applications, we spent a couple of months trying vainly to redo the proofs without it. The result was messier proofs and a pronounced loss of perspicuity. It has, in fact, been our consistent experience that Σ^* theory facilitates the fine structural analysis which lies at the heart of inner model theory. We therefore urge the reader to bear with us.

Definition 2.6.1. Let $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$ be acceptable.

The Σ^* *M–language* $\mathbb{L}^* = \mathbb{L}_M^*$ has

- a binary predicate $\dot{\in}$
- unary predicates $\dot{A}_1, \ldots, \dot{A}_n, \dot{B}_1, \ldots, \dot{B}_m$
- variables $v_i^j (i, j < \omega)$

Definition 2.6.2. By induction on $n < \omega$ we define sets $\Sigma_h^{(n)} (h = 0, 1)$ of formulae

$\Sigma_0^{(n)}$ = the smallest set of formulae such that

- all primitive formulae are in Σ.
- $\Sigma_0^{(m)} \cup \Sigma_1^{(m)} \subset \Sigma$ for $m < n$.
- Σ is closed under sentential operations $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$.
- If φ is in $\Sigma, j \leq n$, and $v^j \neq u^n$, then $\bigwedge v^j \in u^n \varphi$, $\bigvee v^j \in u^n \varphi$ are in Σ.

We then set:

$$\Sigma_1^{(n)} =: \text{ The set of formulae } \bigvee v^n \varphi, \text{ where } \varphi \in \Sigma_0^{(n)}.$$

We also generalize the last part of this definition by setting:

Definition 2.6.3. Let $n < \omega$, $1 \leq h < \omega$. $\Sigma_h^{(n)}$ is the set of formulae

$$\bigvee v_1^n \bigwedge v_2^n \ldots Q v_h^n \varphi,$$

where φ is $\Sigma_0^{(n)}$ (and Q is $\bigvee$ if h is odd and $\bigwedge$ if h is even).

We now turn to the interpretation of the formualae in M.

Definition 2.6.4. Let Fml^n be the set of formulae in which only variables of type $\leq n$ occur.

By recursion on n we define:

- The n–th projectum $\rho^n = \rho_M^n$.
- The n–th variable domain $H^n = H_M^n$.
- The satisfaction relation $\models^n$ for formulae in Fml^n.

$\models^n$ is defined by interpreting variables of type i as ranging over H^i for $i \leq n$. We set: $\rho^0 = \alpha$, $H^0 = |M| = |J_\alpha^{\vec{A}}|$, when $M = \langle J_\alpha^{\vec{A}}, \vec{B} \rangle$.

Now let ρ^n, H^n be given (hence $\models^n$ is given). Call a set $D \in H^n$ a $\underline{\Sigma}_1^{(n)}$ *set.* if it is definable from parameters by a $\Sigma_1^{(n)}$ formula φ:

$$Dx \leftrightarrow M \models^n \varphi[x, a_1, \ldots, a_p],$$

where $\varphi = \varphi(v^n, u^{i_1}, \ldots, u^{i_m})$ is $\Sigma_1^{(n)}$. ρ^{n+1} is then the least ρ such that there is a $\underline{\Sigma}_1^{(n)}$ set $D \subset \rho$ with $D \notin M$. We then set:

$$H^{n+1} = |J_\rho^{\vec{A}}|.$$

This then defines $\models^{n+1}$.

It is obvious that $\models^i$ is contained in $\models^j$ for $i \leq j$, so we can define the full Σ^* satisfaction relation for M by:

$$\models = \bigcup_{n<\omega} \models^n .$$

Satisfaction is defined in the usual way. We employ v^i, u^i, ω^i etc. as metavariables for variables of type i. We also employ x^i, y^i, z^i etc. as metavariables for elements of H^i. We call $v_1^{i_1}, \ldots, v_n^{i_n}$ a *good sequence* for the formula φ iff it is a sequence of distinct variables containing all the variables which occur free in φ. If $v_1^{i_1}, \ldots, v_n^{i_n}$ is good we write:

$$\models_M \varphi[v_1^{i_1}, \ldots, v_n^{i_n}/x_1^{i_1}, \ldots, x_n^{i_n}]$$

to mean that φ becomes true if $v_h^{i_n}$ is interpreted by $x_h^{i_n}(h = 1, \ldots, n)$. We shall follow normal usage in suppressing the sequence $v_1^{i_1}, \ldots, v_n^{i_n}$ writing only:

$$\models_M \varphi[x_1^{i_1}, \ldots, x_n^{i_n}].$$

(However, it is often important for our understanding to retain the upper indices $i_1, \ldots, i_n$.) We often write $\varphi = \varphi(v_1^{i_1}, \ldots, v_n^{i_n})$ to indicate that these are the suppressed variables. φ (together with $v_1^{i_1}, \ldots, v_n^{i_n}$) defines a relation:

$$R(x_1^{i_1}, \ldots, x_n^{i_n}) \leftrightarrow \models_M \varphi[x_1^{i_1}, \ldots, x_n^{i_n}].$$

Since we are using a many sorted language, however, we must also employ *many sorted relations.*

The number of argument places of an ordinary one sorted relation is often called its "arity". In the case of a many sorted relation, however, we must know not only the number of argument places, but also the type of each argument place. We refer to this information as its "arity". Thus the arity of the above relation is not n but $\langle i_1, \ldots, i_n \rangle$. An ordinary 1–sorted relation is usually identified with its field. We shall identify a many sorted relation with the pair consisting of its field and its arity:

Definition 2.6.5. A *many sorted relation* R on M is a pair $\langle |R|, r \rangle$ such that for some n:

(a) $|R| \subset M^n$

(b) $r = \langle r_1, \ldots, r_n \rangle$ where $r_i < \omega$

(c) $R(x_1, \ldots, x_n) \to x_i \subset H^{r_i}$ for $i = 1, \ldots, n$.

$|R|$ is called the *field* of R and r is called the *arity* of R.

In practice we adopt a rough and ready notation, writing $R(x_1^{i_1}, \ldots, x_n^{i_n})$ to indicate that R is a many sorted relation of arity $\langle i_1, \ldots, i_n \rangle$.

Note. Let $\mathbb{L} = \mathbb{L}_M$ be the ordinary first order language of M (i.e. it has only variables of type 0.

Since $H^n \in M$ or $H^n = M$ for all $n < \omega$, it follows that every $\mathbb{L}^*$–definable many sorted relation has a field which is $\mathbb{L}$–definable in parameters from M.)

Note. If R is a relation of arity $\langle i_1, \ldots, i_n \rangle$, then its *complement* is $\Gamma \setminus R$, where:

$$\Gamma = \{\langle x_1, \ldots, x_n \rangle | x_h \in H^{i_n} \text{ for } h = 1, \ldots, n\},$$

the arity remaining unchanged.

Definition 2.6.6. $R(x_1^{i_1}, \ldots, x_m^{i_m})$ is a $\Sigma_h^{(n)}(M)$ *relation* iff it is defined by a $\Sigma_h^{(n)}$ formula. R is $\Sigma_h^{(n)}(M)$ *in the parameters* $p_1, \ldots, p_r$ iff $R(\vec{x}) \leftrightarrow R'(\vec{x}, \vec{p})$, where R' is $\Sigma_h^{(n)}(M)$. R is a $\underline{\Sigma}_h^{(n)}(M)$ relation iff it is $\Sigma_h^{(n)}(M)$ in some parameters.

It is easily checked that:

Lemma 2.6.1.

- *If $R(y^n, \vec{x})$ is $\Sigma_1^{(n)}$, so is $\bigvee y^n R(y^n, \vec{x})$*
- *If $R(\vec{x}), P(\vec{x})$ are $\Sigma_1^{(n)}$, then so are $R(\vec{x}) \vee P(\vec{x})$, $R(\vec{x}) \wedge P(\vec{x})$.*

Moreover, if $R(x_0^{i_0}, \ldots, x_{m-1}^{i_{m-1}})$ is $\Sigma_1^{(n)}$, so is any relation $R'(y_0^{j_0}, \ldots, y_{r-1}^{j_{r-1}})$ obtained from R by permutation of arguments, insertion of dummy arguments and fusion of arguments having the same type — i.e.

$$R'(y_0^{j_0}, \ldots, y_{r-1}^{j_{r-1}}) \leftrightarrow R(y_{\sigma(0)}^{j_{\sigma(0)}}, \ldots y_{\sigma(m-1)}^{j_{\sigma(m-1)}})$$

where $\sigma : m \to r$ such that $j_{\sigma(l)} = i_l$ for $l < m$.

Using this we get the analogue of Lemma 2.5.6

Lemma 2.6.2. *Let $M = \langle J_\alpha^A, B \rangle$ be acceptable. Let $\rho = \rho^n$, $H = H^n$. Then*

(a) If $\rho \in M$, then ρ is a cardinal in M. (Hence $H = H_\rho^M$)

(b) If D is $\Sigma_1^{(n)}(M)$ and $D \subset H$, then $\langle H, D\rangle$ is amenable.

(c) If $u \in H$, there is no $\Sigma_1^{(n)}(M)$ partial map of u onto H.

(d) $\rho \in \mathrm{Lm}^*$ if $n > 0$.

Proof: By induction on n. The induction step is a virtual repetition of the proof of Lemma 2.5.6. QED (Lemma 2.6.2)

Definition 2.6.7. Let $R(x_1^{i_1}, \ldots, x_m^{i_m})$ be a many sorted relation. By an *n–specialization* of R we mean a relation $R'(x_1^{j_1}, \ldots, x_m^{j_m})$ such that

- $j_l \geq i_l$ for $l = 1, \ldots, m$
- $j_l = i_l$ if $l < n$
- If $z_1, \ldots, z_m$ are such that $z_l \in H^{j_l}$ for $l = 1, \ldots, m$, then: $R(\vec{z}) \leftrightarrow R'(\vec{z})$.

Given a formula φ in which all bound quantifiers are of type $\leq n$, we can easily devise a formula φ' which defines a specialization of the relation defined by φ:

Fact Let $\varphi = \varphi(v_1^{i_1}, \ldots, v_m^{i_m})$ be a formula in which all bound variables are of type $\leq n$. Let $u_1^{j_1}, \ldots, u_m^{j_m}$ be a sequence of distinct variables such that $j_l \geq i_l$ and $j_l = i_l$ if $i_l < n (l = 1, \ldots, m)$. Suppose that $\varphi' = \varphi'(\vec{u})$ is obtained by replacing each free occurence of $v_l^{i_l}$ by a free occurence of $u_l^{j_l}$ for $l = 1, \ldots, m$. Then for all $x_1, \ldots, x_m$ such that $x_l \in H^{j_l}$ for $l = 1, \ldots, m$ we have:

$$\models_M \varphi(\vec{v})[\vec{x}] \leftrightarrow \models_M \varphi'(\vec{u})[\vec{x}].$$

The proof is by induction on φ. We leave it to the reader. Using this, we get:

Lemma 2.6.3. *Let $R(x_1^{i_1}, \ldots, x_m^{i_m})$ be $\Sigma_l^{(n)}$. Then every n–specialization of R is $\Sigma_l^{(n)}$.*

Proof: $R'(x_1^{i_1}, \ldots, x_m^{i_m})$ be an n–spezialization. Let R be defined by $\varphi(v_1^{i_1}, \ldots, v_m^{i_m})$. Suppose $(u_1^{j_1}, \ldots, v_m^{j_m})$ is a sequence of distinct variables which are new — i.e. none of them occur free or bound in φ. Let φ' be obtained by replacing every free occurence of $v_l^{i_l}$ by $u_l^{j_l} (l = 1, \ldots, m)$. Then $\varphi'(u_1^{j_1}, \ldots, v_m^{j_m})$ defines R' by the above fact. QED (Lemma 2.6.3)

Corollary 2.6.4. *Let R be $\Sigma_1^{(n)}$ in the parameter p. Then every n–spezialization of R is $\Sigma_1^{(n)}$ in p.*

Lemma 2.6.5. *Let $R'(x_1^{j_1}, \ldots, x_m^{j_m})$ be $\Sigma_1^{(n)}$. Then R' is an n–specialization of a $\Sigma_1^{(n)}$ relation $R(x_1^{i_1}, \ldots, x_m^{i_m})$ such that $i_l \le n$ for $l = 1, \ldots, m$.*

Proof: Let R' be defined by $\varphi'(u_1^{j_1}, \ldots, v_m^{j_m})$, when φ' is $\Sigma_1^{(n)}$. Let $v_1^{i_n}, \ldots, v_m^{i_m}$ be a sequence of distinct new variables, where $i_l = \min(n, j_l)$ for $l = 1, \ldots, m$. Replace each free occurence of $u_l^{j_l}$ by $v_l^{i_l}$ for $l = 1, \ldots, m$ to get $\varphi(u_1^{i_1}, \ldots, v_m^{i_m})$. Let R be defined by φ. Then R' is a specialization of R by the above fact. QED (Lemma 2.6.5)

Corollary 2.6.6. *Let $R'(x_1^{j_1}, \ldots, x_m^{j_m})$ be $\Sigma_1^{(n)}$ in p. Then R' is a specialization of a relation $R(x_1^{i_1}, \ldots, x_m^{i_m})$ which is $\Sigma_1^{(n)}$ in p with $i_l \le n$ for $l = 1, \ldots, m$.*

Every $\Sigma_1^{(m)}$ formula can appear as a "primitive" component of a $\Sigma_0^{(m+1)}$ formula. We utilize this fact in proving:

Lemma 2.6.7. *Let $n = m+1$. Let $Q_j(z_{j,1}^n, \ldots, z_{j,p_j}^n, x_1^{i_1}, \ldots, x^{i_p})$ be $\Sigma_1^{(m)} (j = 1, \ldots, r)$.*
Set: $Q_{j,\vec{x}} =: \{\langle \vec{z}_j^n \rangle | Q_j(\vec{z}_j^n, \vec{x})\}$.
Set: $H_{\vec{x}} =: \langle H^n, Q_{1,\vec{x}}, \ldots, Q_{r,\vec{x}} \rangle$.
Let $\varphi = \varphi(v_1, \ldots, v_q)$ be Σ_l in the language of $H_{\vec{x}}$. Then

$$\{\langle \vec{x}^n, \vec{x} \rangle | H_{\vec{x}} \models \varphi[\vec{x}^n]\} \text{ is } \Sigma_l^{(n)}.$$

Proof: We first prove it for $l = 0$, showing by induction on φ that the conclusion holds for any sequence $v_1, \ldots, v_l$ of variables which is good for φ.

We describe some typical cases of the induction.

Case 1 φ is primitive.
Let e.g. $\varphi = \dot{Q}_j(v_{h_1}, \ldots, v_{h_{p_i}})$, where $\dot{Q}_j$ is the predicate for $Q_{j\vec{x}}$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to: $Q_j(x_{h_1}^n, \ldots, x_{h_{p_j}}^n, \vec{x})$, which is $\Sigma_1^{(m)}$ (hence $\Sigma_0^{(n)}$). QED (Case 1)

Case 2 φ arises from a sentential operation.
Let e.g. $\varphi = (\varphi_0 \wedge \varphi_1)$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to:

$$H_{\vec{x}} \models \varphi_0[\vec{x}^n] \wedge H_{\vec{x}} \models \varphi_1[\vec{x}^n]$$

which, by the induction hypothesis is $\Sigma_0^{(n)}$. QED (Case 2)

Case 3 φ arises from a quantification.

Let e.g. $\varphi = \bigwedge w \in v_i \Psi$. By bound relettering we can assume *w.l.o.g.* that w is not among $v_1, \dots, v_p$. We apply the induction hypothesis to $\Psi(w, v_1, \dots, v_p)$. Then $H_{\vec{x}} \models \varphi[\vec{x}^n]$ is equivalent to:

$$\bigwedge z \in x_i^n H_{\vec{x}} \models \Psi[w, \vec{x}^n]$$

which is $\Sigma_0^{(n)}$ by the induction hypothesis. QED (Case 3)

This proves the case $l = 0$. We then prove it for $l > 0$ by induction on l, essentially repeating the proof in case 3. QED (Lemma 2.6.7)

Note. It is clear from the proof that the set $\{\langle \vec{x}^n, \vec{x} \rangle | H_{\vec{x}} \models \varphi[\vec{x}^n]\}$ is *uniformly* $\Sigma_l^{(n)}$ — i.e. its defining formula χ depends only on φ and the defining formula Ψ_i for $Q_i (i = 1, \dots, p)$. In fact, the proof implicitly describes an algorithm for the function $\varphi, \Psi_1, \dots, \Psi_p \mapsto \chi$.

We can invert the argument of Lemma 2.6.7 to get a weak converse:

Lemma 2.6.8. *Let $n = m + 1$. Let $R(\vec{x}^n, x_1^{i_1}, \dots, x_g^{i_g})$ be $\Sigma_l^{(n)}$ where $i_l \leq m$ for $l = 1, \dots, g$. Then there are $\Sigma_1^{(n)}$ relations $Q_i(\vec{z}_i^n, \vec{x})(i = 1, \dots, p)$ and a Σ_l formula φ such that*

$$R(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n],$$

where $H_{\vec{x}}$ is defined as above.

Note. This is weaker, since we now require $i_l \leq m$.

Proof: We first prove it for $l = 0$. By induction on χ we prove:

Claim Let χ be $\Sigma_0^{(n)}$. Let $\vec{v}^n, v_1^{i_1}, \dots, v_q^{i_q}$ be good for χ, where $i_1, \dots, i_q \leq m$. Let $\chi(\vec{v}^n, \vec{v})$ define the relation $R(\vec{x}^n, \vec{x})$. Then the conclusion of Lemma 2.6.8 holds for this R (with $l = 0$).

Case 1 χ is $\Sigma_1^{(m)}$.

Let $\chi(\vec{x}^n, \vec{x})$ define $Q(\vec{x}^n, \vec{x})$. Then $R(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \dot{Q}\vec{v}^n[\vec{x}^n]$.

QED (Case 1)

Case 2 χ arises from a sentential operation.

Let e.g. $\chi = (\Psi \wedge \Psi')$. Appliyng the induction hypothesis we get $Q_i(\vec{x}_i^n, \vec{x})(i = 1, \dots, p)$ and φ such that

$$M \models \Psi[\vec{x}^n, \vec{x}] \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n]$$

where $H_{\vec{x}} = \langle H^n, Q_{1\vec{x}}, \dots, Q_{p\vec{x}} \rangle$. Similarly we get $Q'_i(\vec{y}^n_i, \vec{x})(i = 1, \dots, q')$ and φ'

$$M \models \Psi'[\vec{x}^n, \vec{x}] \leftrightarrow H'_{\vec{x}} \models \varphi'[\vec{x}^n].$$

Let $\dot{Q}_i$ be the predicate for $Q_{i\vec{x}}$ in the language of $H_{\vec{x}}$. Let $\dot{Q}'_i$ be the predicate for $Q'_{i\vec{x}}$ in the language of $H'_{\vec{x}}$. Assume *w.l.o.g.* that $\dot{Q}_i \neq \dot{Q}'_j$ for all i, j. Putting the two languages together we get a language for

$$H^*_{\vec{x}} = \langle H^n, \vec{Q}_{\vec{x}}, \vec{Q}'_{\vec{x}'} \rangle.$$

Clearly:

$$M \models (\chi \wedge \chi')[\vec{x}^n, \vec{x}] \leftrightarrow H^*_{\vec{x}} \models (\varphi \wedge \varphi')[\vec{x}^n].$$

QED (Case 2)

Case 3 χ arises from the application of a bounded quantifier.
Let e.g. $\chi = \bigwedge w^n \in v^n_j \chi'$. By bound relettering we can assume *w.l.o.g.* that w^n is not among $\vec{v}^n$. Then $w^n \vec{v}^n, \vec{v}$ is a good sequence for χ' and by the induction hypothesis we have for $\chi' = \chi'(w^n, \vec{v}^n, \vec{v})$:

$$M \models \chi'[z^n, \vec{x}^n, x] \leftrightarrow H_{\vec{x}} \models \varphi[z^n, \vec{x}^n, \vec{x}].$$

But then:

$$\begin{aligned} M \models \chi[\vec{x}^n, \vec{x}] \quad &\leftrightarrow \bigwedge z^n \in x^n_j M \models \chi'[z^n, \vec{x}^n, \vec{x}] \\ &\leftrightarrow \bigwedge z^n \in x^n_j H_{\vec{x}} \models \varphi[z^n, \vec{x}^n] \\ &\leftrightarrow H_{\vec{x}} \models \bigwedge w \in v_j \varphi[\vec{x}^n]. \end{aligned}$$

QED (Lemma 2.6.8)

Note. Our proof again establishes uniformity. In fact, if χ is the $\Sigma^{(n)}_l$–definition of R, the proof implicitely describes an algorithm for the function

$$\chi \mapsto \varphi, \Psi_1, \dots, \Psi_p$$

where Ψ_i is a $\Sigma^{(m)}_1$ definition of Q_i.

Remark. Lemma 2.6.7 and 2.6.8 taken together give an inductive definition of "$\Sigma^{(n)}_l$ relation" which avoids the many sorted language. It would, however, be difficult to work directly from this definition.

By a function of *arity* $\langle i_1, \dots, i_n \rangle$ *to* H^j we mean a relation $F(y^j, x^{i_1}, \dots, x^{i_n})$ such that for all $x^{i_1}, \dots, x^{i_n}$ there is at most one such y^j. If this y exists, we denote it by $F(x^{i_1}, \dots, x^{i_n})$. Of particular interest are the $\Sigma^{(i)}_1$ functions to H^i.

Lemma 2.6.9. *$R(y^n, \vec{x})$ be a $\Sigma^{(n)}_1$ relation. Then R has a $\Sigma^{(n)}_1$ uniformizing function $F(\vec{x})$.*

Proof: We can assume *w.l.o.g* that the arguments of R are all of type $\leq n$. (Otherwise let R be a specialization of R', where the arguments of R' are of type $\leq n$. Let F' uniformize R'. Then the appropriate specialization F of F' uniformizes R.)

Case 1 $n = 0$.

Set:

$$F(\vec{x}) \simeq: y \text{ where } \langle z, y\rangle \text{ is } <_M\text{-least such that } R'(z, y, \vec{x}).$$

By section 2.3 we know that $u_M(x)$ is Σ_1, where $u_M(x) = \{y | y <_M x\}$. Thus for sufficient r we have:

$$\begin{gathered} y = F(\vec{x}) \leftrightarrow \bigvee z(R'(z, y, \vec{x})\wedge \\ \wedge w \in u_M(\langle z, y\rangle) \bigwedge z', y' \in C_r(w) \\ (w = \langle z', y'\rangle \to \neg R(z', y', \vec{x})), \end{gathered}$$

which is uniformly $\Sigma_1(M)$.

Case 2 $n > 0$. Let $n = m + 1$.

Rearranging the arguments of R if necessary, we can assume that R has the form $R(y^n, \vec{x}^n, \vec{x})$, where the $\vec{x}$ are of type $\leq m$. Then there are $Q_i(\vec{z}_i^n, \vec{x}^n, \vec{x})(i = 1, \ldots, p)$ such that Q_i is $\Sigma_1^{(m)}$ and

$$R(y^n, \vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[y^n, \vec{x}^n],$$

where φ is Σ_1 and

$$H_{\vec{x}} = \langle H^n, Q_{1\vec{x}}, \ldots, Q_{n\vec{x}}\rangle.$$

If e.g. $M = \langle J^A, B\rangle$, we can assume *w.l.o.g.* that $Q_1(z^n, \vec{x}) \leftrightarrow A(z^n)$. Then $<_{H\vec{x}}, u_{H\vec{x}}$ are uniformly $\Sigma_1(H_{\vec{x}})$ and by the argument of Case 1 there is a Σ_1 formula φ' such that F uniformies R where

$$y = F(\vec{x}^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi'[\vec{x}^n, \vec{x}].$$

QED (2.6.9)

Note. The proof shows that $F(\vec{x})$ is *uniformly* $\Sigma_1^{(n)}$ — i.e. its $\Sigma_1^{(n)}$ definition depends only on the $\Sigma_1^{(n)}$ definition of $R(y^n, \vec{x})$, regardless of M.

Note. It is clear from the proof that the $\Sigma_1^{(n)}$ definition of F is *functionally absolute* — i.e. it defines a function over every acceptable M of the same type. Thus:

Corollary 2.6.10. *Every $\Sigma_1^{(n)}$ function $F(\vec{x})$ to H^n has a functionally absolute $\Sigma_1^{(n)}$ definition.*

Note. The $\Sigma_1^{(n)}$ functions are closed under permutation of arguments, insertion of dummy arguments, and fusion of arguments of same type. Thus if $F(x_1^{i_1}, \ldots x_n^{i_n})$ is $\Sigma_1^{(n)}$, so is $F'(y_1^{j_1}, \ldots, y_m^{j_m})$ where

$$F'(y_1^{j_1}, \ldots, y_m^{j_m}) \simeq F(y_{\sigma(1)}^{j_{\sigma(1)}}, \ldots, y_{\sigma(n)}^{j_{\sigma(n)}})$$

and $\sigma : n \to m$ such that $j_{\sigma(l)} = i_l$ for $l < n$.

If $R(x_1^{j_1}, \ldots, x_p^{j_p})$ is a relation and $F_i(\vec{z})$ is a function to H^{j_i} for $i = 1, \ldots, n$, we sometimes use the abbreviation:

$$R(\vec{F}(\vec{z})) \leftrightarrow: \bigvee x_1^{j_1}, \ldots x_p^{j_p} (\bigwedge_{i=1}^{p} x_i^{j_i} = F_i(\vec{z}) \wedge R(\vec{x})).$$

Note that $R(\vec{F}(\vec{z}))$ is then false if some $F_i(\vec{z})$ does not exist. $\Sigma_1^{(n)}$ relations are not, in general, closed under substitution of $\Sigma_1^{(n)}$ functions, but we do get:

Lemma 2.6.11. *Let $R(x_1^{j_1}, \ldots, x_p^{j_p})$ be $\Sigma_1^{(n)}$ such that $j_i \leq n$ for $i = 1, \ldots, p$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(j_i)}$ map to H^{j_i} for $i = 1, \ldots, p$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$ (uniformly in the $\Sigma_1^{(n)}$ definitions of $R, F_1, \ldots, F_p$)*

Before proving Lemma 2.6.11 we show that it has the following corollary:

Corollary 2.6.12. *Let $R(\vec{x}, y_1^{j_1}, \ldots, y_p^{j_p})$ be $\Sigma_1^{(n)}$ where $j_i \leq n$ for $i = 1, \ldots, p$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(j_i)}$ map to H^{j_i} for $i = 1, \ldots, p$. Then $R(\vec{x}, \vec{F}(\vec{z}))$ is (uniformly) $\Sigma_1^{(n)}$.*

Proof: We can assume *w.l.o.g.* that each of $\vec{x}$ has type $\leq n$, since otherwise R is a specialization of an R' with this property. But then $R(\vec{x}, \vec{F}(z))$ is a specialization of $R'(\vec{x}, \vec{F}(z))$. Let $\vec{x} = x_1^{h_1}, \ldots, x_q^{h_q}$ with $h_i \leq n$ for $i = 1, \ldots, q$. For $i = 1, \ldots, p$ set:

$$F'(\vec{x}, \vec{z}) \simeq F(\vec{z}).$$

For $i = 1, \ldots, q$ set:

$$G_h(\vec{x}, \vec{z}) \simeq x_i^{h_i}.$$

By Lemma 2.6.11, $R(\vec{G}(\vec{x}, \vec{z}), F'(\vec{x}, \vec{z}))$ is $\Sigma_1^{(n)}$. But

$$R(\vec{G}(\vec{x}, \vec{z}), F'(\vec{x}, \vec{z})) \leftrightarrow R(\vec{x}, \vec{F}(\vec{z})).$$

QED (Corollary 2.6.12)

We now prove Lemma 2.6.11 by induction on n.

Case 1 $n = 0$.

The conclusion is immediate by the definition of $R(\vec{F}(\vec{z}))$:

$$R(\vec{F}(\vec{z})) \leftrightarrow \bigvee x_1^0 \dots x_p^0 (\bigwedge_{i=1}^{p} x_1^0 = F_i(\vec{z}) \wedge R(\vec{x})).$$

Case 2 $n = m + 1$.

Then Lemma 2.6.11 holds at m and it is clear from the above proof that Corollary 2.6.12 does, too.

Rearranging the arguments of R if necessary, we can bring R into the form:

$$R(\vec{x}^n, x_1^{l_1}, \dots, x_q^{l_q}) \text{ where } l_i \leq m \text{ for } i = 1, \dots, q.$$

We first show:

Claim $R(\vec{x}^n, \vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$.

Proof: Let $Q_i(\vec{z}_i^n, \vec{x})$ be $\Sigma_1^{(m)} (i = 1, \dots, r)$ such that

$$R(x^n, \vec{x}) \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n]$$

where φ is Σ_1 and:

$$H_{\vec{x}} = \langle H^n, Q_{1,\vec{x}}, \dots, Q_{r,\vec{x}} \rangle.$$

Set:

$$\begin{aligned} \overline{Q}_i(\vec{z}_i^n, \vec{z}) \quad & \leftrightarrow: Q_i(z_i^n, F(\vec{z})) \\ & \leftrightarrow \bigvee \vec{x} (\textstyle\bigwedge_{i=1}^{q} x_i^{l_i} = F_i(\vec{z}) \wedge R(\vec{x})) \\ & \overline{H}_{\vec{z}} =: \langle H^n, \overline{Q}_{1,\vec{z}}, \dots, \overline{Q}_{r,\vec{z}} \rangle. \end{aligned}$$

If $x_i^{l_i} = F_i(\vec{z})$ for $i = 1, \dots, q$, then $\overline{Q}_i(\vec{z}_i^n, \vec{z}) \leftrightarrow Q_i(\vec{z}^n, \vec{x})$ and $\overline{H}_{\vec{z}} = H_{\vec{x}}$. Hence:

$$\begin{aligned} \overline{H}_{\vec{z}} \models \varphi[\vec{x}^n] \quad & \leftrightarrow H_{\vec{x}} \models \varphi[\vec{x}^n] \\ & \leftrightarrow R(\vec{x}^n, \vec{x}) \\ & \leftrightarrow R(\vec{x}^n, \vec{F}(\vec{z})). \end{aligned}$$

If, on the other hand, $F_i(\vec{z})$ does not exist for some i, then $R(\vec{x}^n, \vec{F}(\vec{z}))$ is false. Hence:

$$\begin{aligned} R(\vec{x}^n, \vec{F}(\vec{z})) \quad & \leftrightarrow (\textstyle\bigwedge_{i=1}^{q} \bigvee x_i^{l_i} (x_i^{l_i} = F_i(\vec{z})) \\ & \wedge \overline{H}_{\vec{z}} \models \varphi[\vec{x}^n]). \end{aligned}$$

But $\bigwedge_{i=1}^{q} \bigvee x_i^{l_i} (x_i^{l_i} = F_i(\vec{z}))$ is $\Sigma_0^{(n)}$, so the result follows by applying Lemma 2.6.7 to φ. QED (Claim)

But then, setting: $R'(\vec{x}^n, \vec{z}) \leftrightarrow R(\vec{x}^n, F(\vec{z}))$, we have:

$$R(\vec{F}(\vec{x})) \leftrightarrow \vee \vec{x}^n (\bigwedge_{i=1}^{q} x_i^n = F_i(\vec{z}) \wedge R'(\vec{x}^n, \vec{z})).$$

QED (Lemma 2.6.11)

Note that if, in the last claim, we took $R(\vec{x}^n, x_1^{l_1}, \ldots, x_q^{l_q})$ as being $\Sigma_0^{(n)}$ instead of $\Sigma_1^{(n)}$, then in the proof of the claim we could take φ as being Σ_0 instead of Σ_1. But then the application of Lemma 2.6.7 to $\overline{H}_{\vec{z}} \models \varphi[\vec{x}^n]$ yields a $\Sigma_0^{(n)}$ formula. Then we have, in effect, also proven:

Corollary 2.6.13. *Let $R(\vec{x}^n, y_1^{l_1}, \ldots, y_q^{l_q})$ be $\Sigma_0^{(n)}$ where $l_1, \ldots, l_r < n$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(l_i)}$ map to H^{l_i} for $i = 1, \ldots, r$. Then $R(x^n, \vec{F}(\vec{z}))$ is (uniformly) $\Sigma_0^{(n)}$.*

As corollaries of Lemma 2.6.11 we then get:

Corollary 2.6.14. *Let $G(x_1^{j_1}, \ldots, x_p^{j_p})$ be a $\Sigma_1^{(n)}$ map to H^n, where $j_1, \ldots, j_p \leq n$. Let $F_i(\vec{z})$ be a $\Sigma_1^{(n)}$ map to H^{j_i} for $i = 1, \ldots, p$. Then $H(\vec{z}) \simeq G(\vec{F}(\vec{z}))$ is uniformly $\Sigma_1^{(n)}$.*

Proof:

$$y = H(\vec{z}) \leftrightarrow \bigvee \vec{x} (\bigwedge_{i=1}^{p} x_i^{j_i} = F_i(\vec{z}) \wedge y = G(\vec{x})).$$

QED (Corollary 2.6.14)

Corollary 2.6.15. *Let $R(x_1^{j_1}, \ldots, x_p^{j_p})$ be $\Sigma_1^{(n)}$ where $j_i \leq n$ for $i = 1, \ldots, p$. There is a $\Sigma_1^{(n)}$ relation $R'(z_1^0, \ldots, z_p^0)$ with the same field*

Proof: Set:

$$R'(\vec{z}) \leftrightarrow: \bigvee \vec{x} (\bigwedge_{i=1}^{p} x_i^{j_i} = z_i^0 \wedge R(\vec{x})).$$

QED (Corollary 2.6.15)

Thus in theory we can always get by with relations that have only arguments of type 0. (Lest one make too much of this, however, we remark that the defining formula of R' will still have bounded many sorted variables.)

Generalizing this, we see that if R is a relation with arguments of type $\leq n$, then the property of being $\Sigma_1^{(n)}$ depends only on the field of R. Let us define:

Definition 2.6.8. $R'(z_1^{j_1}, \ldots, z_r^{j_r})$ is a *reindexing* of the relation $R(x_1^{i_1}, \ldots, x_r^{i_r})$ iff both relations have the same field i.e.

$$R'(\vec{y}) \leftrightarrow R(\vec{y}) \text{ for } y_1, \ldots, y_r \in M.$$

Then:

Corollary 2.6.16. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_1^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $R'(z_1^{j_1}, \ldots, z_r^{j_r})$ be a reindexing of R, where $j_1, \ldots, j_r \leq n$. Then R' is $\Sigma_1^{(n)}$.*

Proof:

$$\begin{aligned} R'(\vec{z}) &\leftrightarrow R(F_1(z_1), \ldots, F_r(z_r)) \\ &\leftrightarrow \vee\vec{x}(\bigvee_{l=1}^{r} x_l^{i_l} = z_l^{j_l} \wedge R(\vec{x})) \end{aligned}$$

where

$$x^{i_l} = F_l(z^{j_l}) \leftrightarrow: x^{i_l} = z^{j_l}.$$

QED (Corollary 2.6.16)

We now consider the relationship between Σ^* theory and the theory developed in §2.5. $\Sigma_1^{(0)}$ is of course the same as Σ_1 and ρ_1 is the same as the Σ_1 projectum ρ which we defined in §2.5.2. In §2.5.2 we also defined the set P of good parameters and the set R of very good parameters. We then defined the reduct M of M_P for any $p \in [\mathrm{On}_M]^{<\omega}$. We now generalize these notions to $\Sigma_1^{(n)}$. We have already defined the $\Sigma_1^{(n)}$ projectum ρ^n. In analogy with the above we now define the sets P^n, R^n of $\Sigma_1^{(n)}$*–good* parameters. We also define the $\Sigma_1^{(n)}$ *reduct* M^{np} of M by $p \in [\mathrm{On}_M]^{<\omega}$.

Under the special assumption of soundness, these will turn out to be the same as the concepts defined in §2.5.3.

Definition 2.6.9. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. We define sets $M^n_{x^{n-1},\ldots,x^0}$ and predicates $T^n(x^n, \ldots, x^0)$ as follows:

$$\begin{aligned} &M^0 =: M, T^0 =: B \text{ (i.e. } M^n_{\vec{x}} = M \text{ for } n = 0) \\ &M^{n+1}_{\vec{x}} =: \langle J^A_{\rho^{n+1}}, T^{n+1}_{\vec{x}}\rangle \text{ for } \vec{x} = x^n, \ldots, x^0 \\ &T^{n+1}(x^{n+1}, \vec{x}) \leftrightarrow \bigvee z^{n+1} \bigvee i < \omega (x^{n+1} = \langle i, z^{n+1}\rangle \\ &\qquad \wedge M^n_{x^{n-1},\ldots,x^0} \models \varphi_i[z^{n+1}, x^n]) \end{aligned}$$

(where $\langle\varphi_i | i < \omega\rangle$ is our fixed canonical enumeration of Σ_1 formulae.)

(Then $T^{n+1}(\langle i, x^{n+1}\rangle, x^n, \ldots, x^0) \leftrightarrow M^n_{x^{n-1},\ldots,x^0} \models \varphi_i[x^{n+1}, x^n]$).

Clearly T^{n+1} is uniformly $\Sigma_1^{(n)}(M)$.

Lemma 2.6.17.

(a) T^{n+1} *is* $\Sigma_1^{(n)}$

(b) Let φ *be* Σ_j. *Then* $\{\langle\vec{x}^{n+1},\vec{x}\rangle | M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}]\}$ *is* $\Sigma_j^{(n+1)}$.

Proof: We first note that $M_{\vec{x}}^{n+1}$ can be written as $H_{\vec{x}} = \langle H^{n+1}, A_{\vec{x}}^{n+1}, T_{\vec{x}}^{n+1}\rangle$, where $A^{n+1}(x^{n+1},\vec{x}) \leftrightarrow: A(x^{n+1})$. Hence by Lemma 2.6.7:

(1) If (a) holds at n, so does (b). But (a) then follows by induction on n:

Case 1 $n = 0$ is trivial since $\Vdash_N^{\Sigma_1}$ is $\Sigma_1(N)$ for all rud closed N.

Case 2 $n = m + 1$. Then $T^{(n+1)}$ is $\Sigma_1^{(n)}$ by (1) applied to m.

QED (Lemma 2.6.17)

We now prove *a* converse to Lemma 2.6.17.

Lemma 2.6.18. *(a) Let* $R(x^{n+1},\ldots,x^0)$ *be* $\Sigma_1^{(n)}$. *Then there is* $i < \omega$ *such that*

$$R(x^{n+1},\vec{x}) \leftrightarrow T^{n+1}(\langle i, x^{n+1}\rangle, \vec{x}).$$

(b) Let $R(\vec{x}^{n+1},\ldots,x^0)$ *be* $\Sigma_1^{(n+1)}$. *Then there is a* Σ_1 *formula* φ *such that*

$$R(\vec{x}^{n+1},\vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}].$$

Proof:

(1) Let (a) hold at n. Then so does (b).

Proof: We know that

$$R(\vec{x}^{n+1},\vec{x}) \leftrightarrow \bigvee z^{n+1} P(z^{n+1}, x^{n+1}, \vec{x})$$

for a $\Sigma_0^{(n+1)}$ formula P. Hence it suffices to show:

Claim Let $P(\vec{x}^{n+1},\vec{x})$ be $\Sigma_0^{(n+1)}$. Then there is a Σ_1 formula φ such that

$$P(\vec{x}^{n+1},\vec{x}) \leftrightarrow M_{\vec{x}}^{n+1} \models \varphi[\vec{x}^{n+1}].$$

Proof: We know that there are $Q_i(\vec{z}_i^{n+1},\vec{x})(i = 1,\ldots,p)$ such that Q_i is $\Sigma_1^{(n)}$ and

(2) $P(\vec{x}^{n+1}, \vec{x}) \leftrightarrow H^{n+1}_{\vec{x}} \models \Psi[\vec{x}^{n+1}]$ where Ψ is Σ_0 and

$$H^n_{\vec{x}} = \langle H^{n+1}, \vec{Q}_{\vec{x}} \rangle.$$

Applying (a) to the relation:

$$\bigvee u^{n+1}(u^{n+1} = \langle \vec{z}^{n+1}_i \rangle \wedge Q_i(\vec{z}^{n+1}_i, \vec{x}))$$

we see that for each i there is $j_i < \omega$ such that

$$Q_i(\vec{z}^{n+1}_i, \vec{x}) \leftrightarrow \langle j_i, \langle \vec{z}^{n+1} \rangle \rangle \in T^{n+1}_{vecx}.$$

Thus $Q_i, \vec{x}$ is uniformly rud in $T^{n+1}_{\vec{x}}$ for $i = 1, \ldots, p$. $P_{\vec{x}}$ is the restriction of a relation rud in $Q_{i,\vec{x}}(i = 1, \ldots, p)$ to H^{n+1}, by (2). By §2 Corollary 2.2.8 it follows that $P_{\vec{x}}$ is the restriction of a relation rud in $T^{n+1}_{\vec{x}}$ to H^{n+1} uniformly. Since $M^{n+1}_{\vec{x}} = \langle J^A_{\rho n+1}, T^{n+1}_{\vec{x}} \rangle$ is rud closed, it follows by §2 Corollary 2.2.8 that:

$$P(\vec{x}^{n+1}, \vec{x}) \leftrightarrow M^{n+1}_{\vec{x}} \models \varphi[\vec{x}^{n+1}]$$

for a Σ_1 formula φ. QED (1)

Given (1) we can now prove (a) by induction on n.

Case 1 $n = 0$.

Since $\Sigma_1 = \Sigma^{(0)}_1$, there is φ_i such that

$$\begin{aligned} R(x^1, x^0) &\leftrightarrow M \models \varphi_i[x^1, x^0] \\ &\leftrightarrow T^1(\langle i, x^1 \rangle, x^0). \end{aligned}$$

Case 2 $n = m + 1$.

Let $R(x^{n+1}, \ldots, x^0)$ be $\Sigma^{(n)}_1$. By the induction hypothesis and (1) we know that (b) holds at n. Hence:

$$\begin{aligned} &R(x^{n+1}, x^{m+1}, x^m, \ldots, x^0) \leftrightarrow \\ &\leftrightarrow M^n_{x^m,\ldots,x^0} \models \varphi_i[x^{n+1}, x^{m+1}] \end{aligned}$$

for some i. But then

$$R(x^{n+1}, \ldots, x^0) \leftrightarrow T^{n+1}(\langle i, x^{n+1} \rangle, x^{m+1}, \ldots, x^0).$$

QED (Lemma 2.6.18)

Note. The reductions in (a) and (b) are both uniform. We have in fact implicitly defined algorithms which in case (a) takes us from the $\Sigma^{(n)}_1$ definition of R to the integer i, and in case (b) takes us from the $\Sigma^{(n+1)}_1$ definition of R to the Σ_1 formula φ.

We now generalize the definition of *reduct* given in §2.5.2 as follows:

Definition 2.6.10. Let $a \in [\mathrm{On}_M]^{<\omega}$. $M^{0,a} =: M$; $M^{n+1,a} =: M^{n+1}_{a^{(0)},\ldots,a^{(n)}}$ where $a^{(i)} = a \cap \rho^i_M$.

Thus $M^{n+1,a} = \langle J^A_{\rho^{n+1}}, T^{n+1,a}\rangle$ where $T^{n+1,a} =: T^{n+1}_{a^{(0)},\ldots,a^{(n)}}$.

Thus by Lemma 2.6.18

Corollary 2.6.19. *Set* $a^{(i)} = a \cap \rho^i$ *for* $a \in [\mathrm{On}_M]^{<\omega}$.

(a) If $D \subset H^{n+1}$ is $\Sigma^{(n)}_1$ in $a^{(0)},\ldots,a^{(n)}$, there is (uniformly) an $i < \omega$ such that
$$D(x^{n+1}) \leftrightarrow \langle i, x^{n+1}\rangle \in T^{n+1,a}$$

(b) If $D(\vec{x}^{n+1})$ is $\Sigma^{(n+1)}_1$ in $a^{(0)},\ldots,a^{(n)}$ there is (uniformly) a Σ_1 formula φ such that $D(\vec{x}^{n+1}) \leftrightarrow M^{n+1,a} \models \varphi[\vec{x}^{n+1}]$.

Note. Being $\Sigma^{(n)}_1$ in a is the same as being $\Sigma^{(n)}_1$ in $a^{(0)},\ldots,a^{(n)}$, but I do not see how this is uniformly so. To see that a $\Sigma^{(n)}_1$ relation R in $a^{(0)},\ldots,a^{(n)}$ is $\Sigma^{(n)}_1$ in a we note that for each n there is k such that $y = a \cap \rho^n \leftrightarrow \bigvee f$ (f is the monotone enumeration of a and $y = f''k$), which is Σ_1 in a. However, k cannot be inferred from the $\Sigma^{(n)}_1$ definition of R, so the reduction is not uniform.

We can generalize the good parameter sets P, R of §2.5.2 as follows:

Definition 2.6.11. $P^0_M =: [\mathrm{On}]^{<\omega}$.

$P^{n+1}_M =:$ the set of $a \in P^n_M$ such that there is D which is $\Sigma^{(n)}_1(M)$ in a with $D \cap H^n_M \notin M$.

(Thus we obviously have $P^1 = P$.)

Similarly:

Definition 2.6.12. $R^0_M =: P^0_M$.

$R^{n+1}_M =:$ The set of $a \in R^n_M$ such that
$$M^{n,a} = h_{M^{n,a}}(\rho^{n+1} \cup (a \cap \rho^n)).$$

Comparing these definitions with those in §2.5.6 it is apparent that R^n_M has the same meaning and that, whenever $a \in R^n_M$, then $M^{n,a}$ is the same structure.

By a virtual repetition of the proof of Lemma 2.5.8 we get:

Lemma 2.6.20. *$a \in P^n \leftrightarrow T^{na} \notin M$.*

We also note the following fact:

Lemma 2.6.21. *Let $a \in R^n$. Let D be $\underline{\Sigma}_1^{(n)}$. Then D is $\Sigma_1^{(n)}$ in parameters from $\rho^{n+1} \cup \{a^{(0)}, \ldots, a^{(n)}\}$, where $a^{(i)} =: a \cap \rho^i$. (Hence D is $\Sigma_1^{(n)}(M)$ in parameters from $\rho^{n+1} \cup \{a\}$.)*

Proof: We use induction on n. Let it hold below n. Then:

$$D(\vec{x}) \leftrightarrow D'(\vec{x}; a^{(0)}, \ldots, a^{(n-1)}, \vec{\xi}),$$

where $\xi_1, \ldots, \xi_r < \rho^n$. (If $n = 0$ the sequence $a^{(0)}, \ldots, a^{(n-1)}$ is vacuous and $\rho^n = \mathrm{On}_M$.)

Let $\xi_i = h_{M^{n+1}}(j_i, \langle \mu_i, a^{(n)} \rangle)$, where $\mu_1, \ldots, \mu_r < \rho^{n+1}$. The functions:

$$F_i(x) \simeq h_{M^{na}}(j_i, \langle x, a^{(n)} \rangle)$$

are $\Sigma_1^{(n)}$ to H^n in the parameters $a^{(0)}, \ldots, a^{(n)}$. But $D(\vec{x})$ then has the form:

$$D'(\vec{x}, a^{(0)}, \ldots, a^{(n-1)}, F_1(\mu_1), \ldots, F_r(\mu_r)),$$

which is $\Sigma_1^{(n)}$ in $a^{(0)}, \ldots, a^{(n)}, \mu_1, \ldots, \mu_k$ by Corollary 2.6.12.

QED (Lemma 2.6.21)

Definition 2.6.13. π is a $\Sigma_h^{(n)}$ *preserving map of* $\overline{M}$ to M (in symbols $\pi : \overline{M} \to_{\Sigma_h^{(n)}} M$) iff the following hold:

- $\overline{M}, M$ are acceptable structures of the same type.
- $\pi'' H^i_{\overline{M}} \subset H^i_M$ for $i \leq n$.
- Let $\varphi = \varphi(v_1^{j_1}, \ldots, v_m^{j_m})$ be a $\Sigma_h^{(n)}$ formula with a good sequence $\vec{v}$ of variables such that $j_1, \ldots, j_m \leq n$. Let $x_i \in H^{j_i}_{\overline{M}}$ for $i = 1, \ldots, m$. Then:
$$\overline{M} \models \varphi[\vec{x}] \leftrightarrow M \models \varphi[\pi(\vec{x})].$$

π is then a structure preserving injection. If it is $\Sigma_h^{(n)}$–preserving, it is $\Sigma_1^{(m)}$–preserving for $m < n$ and $\Sigma_i^{(n)}$–preserving for $i < h$. If $h \geq 1$ then $\pi^{-1}{}'' H^n_M \subset H^n_{\overline{M}}$, as can be seen using:

$$x \in H^n_M \leftrightarrow M \models \bigvee u^n u^n = v^0[x].$$

We say that π is *strictly* $\Sigma_h^{(n)}$ *preserving* (in symbols $\pi : \overline{M} \to_{\Sigma_h^{(n)}} M$ strictly) iff it is $\Sigma_h^{(n)}$ preserving and $\pi^{-1}{}''H_M^n \subset H_{\overline{M}}^n$. (Only if $h = 0$ can the embedding fail to be strict.)

We say that π is Σ^* *preserving* ($\pi : \overline{M} \to_{\Sigma^*} M$) iff it is $\Sigma_1^{(n)}$ preserving for all $n < \omega$. We call π $\Sigma_\omega^{(n)}$ *preserving* iff it is $\Sigma_h^{(n)}$ preserving for all $h < \omega$.

Good functions

Let $n < \omega$. Consider the class $\mathbb{F}$ of all $\Sigma_1^{(n)}$ functions $F(x^{i_1}, \ldots, x^{i_m})$ to H^j, where $j, i_1, \ldots, i_m \leq n$. This class is not necessarily closed under composition. If, however, $\mathbb{G}^0$ is the class of $\Sigma_1^{(j)}$ functions $G(z^{i_1}, \ldots, z^{i_m})$ to H^j where $j, i_1, \ldots, i_m \leq n$, then $\mathbb{G}^0 \subset \mathbb{F}$ and, as we have seen, elements of $\mathbb{G}^0$ can be composed into elements of $\mathbb{F}$ — i.e. if $F(z^{i_1}, \ldots, z^{i_m})$ is in $\mathbb{F}$ and $G_l(\vec{x})$ is in $\mathbb{G}^0$ for $l = 1, \ldots, m$, then $F(\vec{G}(\vec{x}))$ lies in $\mathbb{F}$. The class $\mathbb{G}$ of *good* $\Sigma_1^{(n)}$ functions is the result of closing $\mathbb{G}^0$ under composition. The elements of $\mathbb{G}$ are all $\Sigma_1^{(n)}$ functions and $\mathbb{G}$ is closed under composition. The precise definition is:

Definition 2.6.14. Fix acceptable M. We define sets $\mathbb{G}^k = \mathbb{G}_n^k$ of $\Sigma_1^{(n)}$ functions by:

$\mathbb{G}^0 =$ The set of partial $\Sigma_1^{(i)}$ maps $F(x_1^{j_1}, \ldots, x_m^{j_m})$ to H^i, where $i \leq n$ and $j_1, \ldots, j_m \leq n$.

$\mathbb{G}^{k+1} =$ The set of $H(\vec{x}) \simeq G(\vec{F}(\vec{x}))$, such that $G(y^{j_1}, \ldots, y_m^{j_m})$ is in G^k and $F_l \in \mathbb{G}^0$ is a map to j_l for $l = 1, \ldots, m$.

It follows easily that $\mathbb{G}^k \subset \mathbb{G}_{k+1}^k$ (since $G(\vec{y}) \simeq G(\vec{h}(\vec{y}))$ where $h(y_1^{j_1}, \ldots, y_m^{j_m}) = y_i^{j_i}$ for $i = 1, \ldots, m$). $\mathbb{G} = \mathbb{G}_n =: \bigcup\limits_k \mathbb{G}^k$ is then the set of all *good* $\Sigma_1^{(n)}$ *functions* $\mathbb{G}^* = \bigcup\limits_n \mathbb{G}_n$ is the set of all *good* Σ^* *functions.* All good $\Sigma_1^{(n)}$ functions have a functionally absolute $\Sigma_1^{(n)}$ definition. Moreover, the good $\Sigma_1^{(n)}$ functions are closed under permutation of arguments, insertion of dummy arguments, and fusion of arguments of same type (i.e. if $F(x_0^{i_1}, \ldots, x_{m-1}^{j_p})$ is good, then so is $F'(\vec{y}) \simeq F(y_{\sigma(1)}^{j_{\sigma(1)}}, \ldots, y_{\sigma(m)}^{j_{\sigma(m)}})$ where $\sigma : m \to p$ such that $j_{\sigma(l)} = i_l$ for $l < m$.

To see this, one proves by a simple induction on k that:

Lemma 2.6.22. *Each* $\mathbb{G}_n^k$ *has the above properties.*

The proof is quite straightforward. We then get:

Lemma 2.6.23. *The good $\Sigma_1^{(n)}$ functions are closed under composition: Let $G(y_1^{j_1}, \ldots, y_m^{j_m})$ be good and let $F_l(\vec{x})$ be a good function to H^{j_l} for $l = 1, \ldots, m$. Then the function $G(\vec{F}(\vec{x}))$ is good.*

Proof: By induction in $k < \omega$ we prove:

Claim The above holds for $F_l \in \mathbb{G}^k (l = 1, \ldots, m)$.

Case 1 $k = 0$.

This is trivial by the definition of "good function".

Case 2 $k = h + 1$.

Let:

$$F_l(\vec{x}) \simeq H_l(F_{l,1}(\vec{x}), \ldots, F_{l,p_l}(\vec{x}))$$

for $l = 1, \ldots, m$, where $H_l(z_{l,1}, \ldots, z_{l,p_l})$ is in $\mathbb{G}^h$ and $F_{l,i} \in G^0$ is a map to $H^{j_{l,i}}$ for $l = 1, \ldots, m, i = 1, \ldots, p_l$.

Let $\langle\langle l_\xi, i_\xi\rangle | \xi = 1, \ldots, p\rangle$ enumerate

$$\{\langle l, i\rangle | l = 1, \ldots, m;\ i = 1, \ldots, p_l\}.$$

Define $\sigma_l : \{1, \ldots, p_l\} \to \{1, \ldots, p\}$ by:

$$\sigma_l(i) = \text{ that } \xi \text{ such that } \langle l, i\rangle = \langle l_\xi, i_\xi\rangle.$$

Set:

$$H_l'(z_1, \ldots, z_p) \simeq H_l(z_{\sigma_l(1)}, \ldots, z_{\sigma_l(p_l)})$$

for $l = 1, \ldots, m$. $F_\xi' = F_{l_\xi, i_\xi}$ for $\xi = 1, \ldots, p$.

Clearly we have:

$$F_l(\vec{x}) = H_l'(F_1'(\vec{x}), \ldots, F_p'(\vec{x}))$$

where $H_l' \in \mathbb{G}^h$ for $l = 1, \ldots, m$. Set:

$$G'(z_1, \ldots, z_p| \simeq G(H_1(\vec{z}), \ldots, H_m(\vec{z})).$$

Then G' is a good $\Sigma_1^{(n)}$ function by the induction hypothesis. But:

$$G(\vec{F}(\vec{x})) \simeq G'(F_1'(\vec{x}), \ldots, F_p'(\vec{x})).$$

The conclusion then follows by Case 1, since $F_i' \in \mathbb{G}^0$ for $i = 1, \ldots, p$.

QED (Lemma 2.6.23)

An entirely similar proof yields:

Lemma 2.6.24. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_1^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $F_l(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to $H^{i_l}(L = 1, \ldots, m)$. Then $R(\vec{F}(\vec{z}))$ is $\Sigma_1^{(n)}$.*

Recall that $R(\vec{F}(\vec{z}))$ means:

$$\bigvee y_1, \ldots, y_r (\bigwedge_{l=1}^{r} y_l = F_l(\vec{z}) \wedge R(\vec{y})).)$$

Applying Corollary 2.6.13 we also get:

Lemma 2.6.25. *Let $n = m + 1$. Let $R(\vec{x}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_r \leq m$. Let $F_l(\vec{z})$ be a good $\Sigma_1^{(n)}$ map to H^{i_l} for $l = 1, \ldots, r$. Then $R(\vec{x}^n, \vec{F}(\vec{z}))$ is $\Sigma_0^{(n)}$.*

By a *reindexing* of a function $G(x_1^{i_1}, \ldots, x_r^{i_r})$ we mean any function G' which is a reindexing of G as a relation. (In other words G, G' have the same field, i.e.

$$G(\vec{x}) \simeq G'(\vec{x}) \text{ for all } x_1, \ldots, x_r \in M.)$$

Then:

Corollary 2.6.26. *Let $G(x_1^{i_1}, \ldots, x_r^{i_r})$ be a good $\Sigma_1^{(m)}$ map to H^i. Let $G'(y_1^{j_1}, \ldots, y_r^{j_r})$ be a map to H^j, where $j, j_1, \ldots, j_r \leq n$. If G' is a reindexing of G, then G' is a good $\Sigma_1^{(m)}$ function.*

Proof: $G'(y) \simeq F(G(F_1(y_1^{j_1}), \ldots, F(y_r^{j_r})))$ where F is defined by $x^i = y^i$ and F_l is defined by $x_l^{i_l} = y_l^{j_l}$. (Then e.g.

$$F(y) = \begin{cases} y \text{ if } y \in H_M^{\min\{i,j\}}, \\ \text{undefined if not.} \end{cases}$$

where F is a map to i with arity j.)
But $F, F_1 \ldots, F_r$ are $\Sigma_1^{(n)}$ good. QED (Corollary 2.6.26)

The statement made earlier that every good $\Sigma_1^{(n)}$ function has a functionally absolute $\Sigma_1^{(n)}$ definition can be improved. We define:

Definition 2.6.15. φ is a *good* $\Sigma_1^{(n)}$ *definition* iff φ is a $\Sigma_1^{(n)}$ formula which defines a good $\Sigma_1^{(n)}$ function over any acceptable M of the given type.

Lemma 2.6.27. *Every good $\Sigma_1^{(n)}$ function has a good $\Sigma_1^{(n)}$ definition.*

Proof: By induction on k we show that it is true for all elements of $\mathbb{G}^k$. If $F \in \mathbb{G}^0$, then F is a $\Sigma_1^{(i)}$ map to H^i for an $i \leq n$. Hence any functionally absolute $\Sigma_1^{(i)}$ definition will do. Now let $F \in \mathbb{G}^{k+1}$. Then $F(\vec{x}) \simeq G(H_1(\vec{x}), \ldots, H_p(\vec{x}))$ where $G \in \mathbb{G}^k$ and $H_i \in \mathbb{G}^0$ for $i = 1, \ldots, p$. Then G has a good definition φ and every H_i has a good definition Ψ_i. By the uniformity expressed in Corollary 2.6.14 there is a $\Sigma_1^{(n)}$ formula χ such that, given *any* acceptable M of the given type, if φ defines G' and Ψ_i defines $H_i'(i = 1, \ldots, p)$, then χ defines $F'(\vec{x}) \simeq G'(\vec{H}'(\vec{x}))$. Thus χ is a good $\Sigma_1^{(n)}$ definition of F. QED (Lemma 2.6.27)

Definition 2.6.16. Let $a \in [\mathrm{On}_M]^{<\omega}$. We define partial maps h_a from $\omega \times H^n$ to H^n by:

$$h_a^n(i, x) \simeq: h_{M^{n,a}}(i, \langle x, a^{(n)} \rangle).$$

Then h_a^n is uniformly $\Sigma_1^{(n)}$ in $a^{(n)}, \ldots, a^{(0)}$. We then define maps $\tilde{h}_a^n$ from $\omega \times H^n$ to H^0 by:

$$\begin{aligned} &\tilde{h}_a^0(i, x) \simeq h_a^o(i, x) \\ &\tilde{h}_a^{n+1}(i, x) \simeq \tilde{h}_a^n((i)_0, h_a^{n+1}((i)_1, x)). \end{aligned}$$

Then $\tilde{h}_a^n$ is a good $\Sigma_1^{(n)}$ function uniformly in $a^{(n)}, \ldots, a^{(0)}$.

Clearly, if $a \in R^{n+1}$, then

$$h_a^{n\prime\prime}(\omega \times \rho^{n+1}) = H^n.$$

Hence:

Lemma 2.6.28. *If $a \in R^{n+1}$, then $\tilde{h}_a^{n\prime\prime}(\omega \times \rho^{n+1}) = M$.*

Corollary 2.6.29. *If $R^n \neq \emptyset$, then $\underline{\Sigma}_l \subset \underline{\Sigma}_l^{(n)}$ for $l \geq 1$.*

Proof: Trivial for $n = 0$, since $\Sigma_l^{(0)} = \Sigma_l$. Now let $n = m + 1$. Set: $D = H^n \cap \mathrm{dom}(h_a^n)$, where $a \in R^n$. Then D is $\underline{\Sigma}_1^{(n)}$ by Lemma 2.6.24, since:

$$\begin{aligned} x^n \in D &\leftrightarrow h_a^n(x^n) = h_a^n(x^n) \\ &\leftrightarrow \bigvee z^0 (z^0 = h_a^n(x^n) \wedge z^0 = z^0). \end{aligned}$$

Let $R(\vec{x})$ be $\Sigma_l(M)$. Let

$$R(\vec{x}) \leftrightarrow Q_1 z_1 \ldots Q z_l P(\vec{z}, \vec{x})$$

where P is Σ_0. Set:

$$P'(\vec{u}^n, \vec{x}) \leftrightarrow: P(\vec{h}^n(\vec{u}^n), \vec{x}).$$

Then P' is $\Sigma_1^{(n)}$ in a. But for $u_1^n, \ldots, u_l^n \in D$, $\neg P'(\vec{u}^n, \vec{x})$ can also be written as a $\Sigma_1^{(n)}$ formula. Hence

$$R(\vec{x}) \leftrightarrow Qu_1^n \in D \ldots Qu_l^n \in DP'(\vec{u}^n, \vec{x})$$

is $\Sigma_l^{(n)}$ in a. QED (Corollary 2.6.29)

We have seen that every $\underline{\Sigma}_\omega^{(n)}$ relation is $\underline{\Sigma}_\omega$. Hence:

Corollary 2.6.30. *Let $R^n \neq \emptyset$. Then $\underline{\Sigma}_\omega^{(n)} = \underline{\Sigma}_\omega$.*

An obvious corollary of Lemma 2.6.28 is:

Corollary 2.6.31. *Let $a \in R_M^n$. Then every element of M has the form $F(\xi, a^{(0)}, \ldots, a^{(n)})$ where F is a good $\Sigma_1^{(n)}$ function and $\xi < \rho^n$.*

Using this we now prove a downward extension of embeddings lemma which strengthens and generalizes Lemma 2.5.12

Lemma 2.6.32. *Let $n = m + 1$. Let $a \in [\mathrm{On}_M]^{<\omega}$ and let $N = M^{na}$. Let $\overline{\pi} : \overline{N} \to_{\Sigma_j} N$, where $\overline{N}$ is a J–model. Then:*

(a) There are unique $\overline{M}, \overline{a}$ such that $\overline{a} \in R_{\overline{M}}^n$ and $\overline{M}^{n\overline{a}} = \overline{N}$.

(b) There is a unique $\pi \supset \overline{\pi}$ such that $\pi : \overline{M} \to_{\Sigma_0^{(m)}} M$ strictly and $\pi(\overline{a}) = a$.

(c) $\pi : \overline{M} \to_{\Sigma_j^{(n)}} M$.

Proof: We first prove existence, then uniqueness. The existence assertion in (a) follows by:

Claim 1 There are $\overline{M}, \overline{a}, \hat{\pi} \supset \overline{\pi}$ such that $\overline{M}^{na} = \overline{N}$, $a \in R_{\overline{M}}^n$, $\hat{\pi} : \overline{M} \to_{\Sigma_1} M$, $\hat{\pi}(\overline{a}) = a$.

Proof: We proceed by induction on m. For $m = 0$ this immediate by Lemma 2.5.12. Now let $m = h + 1$. We first apply Lemma 2.5.12 to M^{ma}. It is clear from our definition that $\rho_{M^{m,a}} \geq \rho_M^n$. Set $N' = (M^{m,a})^{a \cap \rho_M^m}$. Then $N' = \langle J_{\rho'}^A, T' \rangle$, where $\rho' = \rho_{M^{ma}}$. But it is clear from our definition that $T^{na} = T' \cap J_{\rho_M^n}^A$. Hence:

(1) $\overline{\pi} : \overline{N} \to_{\Sigma_0} N'$.
By Lemma 2.5.12 there are then $\tilde{M}, \tilde{a}, \tilde{\pi} \supset \overline{\pi}$ such that $\tilde{M}^{\tilde{a}} = N'$, $\tilde{a} \in R_{\tilde{M}}$, $\tilde{\pi} : \tilde{M} \to_{\Sigma_1} M^{m,a}$ and $\tilde{\pi}(\tilde{a}) = a \cap \rho_M^m = a^{(m)}$.

(**Note:** Throughout this proof we use the notation:

$$a^{(i)} =: a \cap \rho^i \text{ for } i = 0, \dots, m.)$$

By the induction hypothesis there are then $\overline{M}, \overline{a}, \hat{\pi} \supset \tilde{\pi}$ such that $\overline{M}^{m\overline{a}} = \tilde{M}$, $\hat{\pi} : \overline{M} \to_{\Sigma_1} M$, and $\hat{\pi}(\overline{a}) = a$.

We observe that:

(2) $\tilde{a} = \overline{a} \cap \rho^m_{\overline{M}}$.

Proof:

($\subset$) Let $\tilde{\rho} =: \rho^m_{\overline{M}} = \mathrm{On} \cap \tilde{M}$. Then $\tilde{a} \subset \tilde{\rho}$. But $\hat{\pi}(\tilde{a}) = \tilde{\pi}(\tilde{a}) = a \cap \rho^m_M \subset a = \hat{\pi}(\overline{a})$. Hence $\tilde{a} \subset a$.

($\supset$) $\hat{\pi}(\overline{a} \cap \tilde{\rho}) = \hat{\pi}''(\overline{a} \cap \tilde{\rho}) \subset \rho^m_M \cap a = \hat{\pi}(\tilde{a})$, since $\hat{\pi}''\tilde{\rho} \subset \rho^m_M$. Hence $\overline{a} \cap \tilde{\rho} = \tilde{a}$. QED (2)

Since $\tilde{a} \in R^{m\overline{a}}_{\overline{M}}$ we conclude that $a \in R^n_{\overline{M}}$ and $\overline{N} = (M^{m\overline{a}})^{a \cap \tilde{\rho}} = \overline{M}^{n,\overline{a}}$. QED (Claim 1)

We now turn to the existence assertion in (b).

Claim 2 Let $\overline{M}^{\overline{a}} = N$ and $\overline{a} \in R^n_{\overline{M}}$. There is $\pi \supset \overline{\pi}$ such that $\pi : \overline{M} \to_{\Sigma_1^{(m)}} M$ and $\pi(\overline{a}) = a$.

Proof: Let $x_1, \dots, x_n \in \overline{M}$ with $x_i = \overline{F}_i(z_i)(i = 1, \dots, r)$, where $\overline{F}_i$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameters $\overline{a}^{(0)}, \dots, \overline{a}^{(n)}$ and $z_i \in \overline{N}$. Let F_i have the same $\Sigma_1^{(m)}(M)$–good definition in $a^{(0)}, \dots, a^{(m)}$. Let $\overline{R}(u_1, \dots, u_r)$ be a $\Sigma_1^{(n)}(\overline{M})$ relation and let R be $\Sigma_1^{(n)}(M)$ by the same definition.

Then $\overline{R}(\overline{F}_1(z_1), \dots, \overline{F}_r(z_r))$ is $\Sigma_1^{(m)}(\overline{M})$ in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$ and $R(F_1(z_1), \dots, F_r(z_r))$ is $\Sigma_1^{(m)}(M)$ in $a^{(0)}, \dots, a^{(m)}$ by the same definition. Hence there is $i < \omega$ such that

$$\overline{R}(\overline{F}(\vec{z}) \leftrightarrow \langle i, \langle \vec{z} \rangle \rangle \in \overline{T}$$

$$R(F(\vec{z})) \leftrightarrow \langle i, \langle \vec{z} \rangle \rangle \in T$$

where $\overline{N} = \langle J^{\overline{A}}_{\overline{\rho}}, \overline{T} \rangle, N = \langle J^A_\rho, T \rangle$. Thus $\overline{R}(\overline{F}(\vec{z}))$ is rud in $\overline{N}$ and $R(F(\vec{z}))$ is rud in N by the same rud definition. But $\overline{\pi} : \overline{N} \to_{\Sigma_0} N$.

Hence:

$$\overline{R}(\overline{F}_1(z_i), \dots, \overline{F}_r(z_r)) \leftrightarrow R(F_1(\overline{\pi}(z_1)), \dots, F_r(\overline{\pi}(z_r))).$$

Thus there is $\pi : \overline{M} \to_{\Sigma_1^{(n)}} M$ defined by $\pi(\overline{F}(\xi)) =: F(\overline{\pi}(\xi))$ whenever $\xi \in \mathrm{On} \cap \overline{N}$, $\overline{F}$ is $\Sigma_1^{(m)}(\overline{M})$– good in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$ and F is $\Sigma_1^{(m)}(M)$–good in $a^{(0)}, \dots, a^{(m)}$ by the same definition. But then

$$\pi(z) = \pi(\mathrm{id}(z)) = \overline{\pi}(z) \text{ for } z \in \overline{N}.$$

Hence $\pi \supset \overline{\pi}$. But clearly

$$\begin{aligned}\pi(\overline{a}) &= \pi(\overline{a}^{(0)} \cup \ldots \cup \overline{a}^{(m)}) \\ &= a^{(0)} \cup \ldots \cup a^{(m)} = a.\end{aligned}$$

QED (Claim 2)

We now verify (c):

Claim 3 Let $\overline{M}, \overline{a}, \pi$ be as in Claim 2. Then $\pi : \overline{M} \to_{\Sigma_j^{(n)}} M$.

Proof: We first note that π, being $\Sigma_1^{(n)}$–preserving, is *strictly* so — i.e. $\rho^i_{\overline{M}} = \pi^{-1}{}''\rho^i_M$ for $i = 0, \ldots, m$. It follows easily that:

$$\pi(\overline{a}^{(i)}) = \pi''\overline{a}^{(i)} = a^{(i)} \text{ for } i = 0, \ldots, m.$$

We now proceed the cases.

Case 1 $j = 0$.
It suffices to show that if φ is $\Sigma_1^{(n)}$ and $x_1, \ldots, x_r \in \overline{N}$, then

$$\overline{M} \models \varphi[x_1, \ldots, x_r] \to M \models \varphi[\pi(x_1), \ldots, \pi(x_r)].$$

Let $x_1, \ldots, x_r \in \overline{M}$. Then $x_i = \overline{F}_i(z_i)(i = 1, \ldots, r)$ where $z_i \in \overline{N}$ and $\overline{F}_i$ is $\Sigma_1^{(m)}(\overline{M})$–good in $\overline{a}^{(0)}, \ldots, \overline{a}^{(m)}$. Let F_i be $\Sigma_1^{(m)}(M)$–good in $a^{(0)}, \ldots, a^{(m)}$ by the same good definition.
By Corollary 2.6.19, we know that $\overline{M} \models \varphi[\overline{F}_1(z_1), \ldots, \overline{F}_r(z_r)]$ is equivalent to

$$\overline{N} \models \Psi[z_1, \ldots, z_r]$$

for a certain Σ_1 formula Ψ. The same reduction on the M side shows that $M \models \varphi[F_1(z_1), \ldots, F_r(z_r)]$ is equivalent to: $N \models \Psi[z_1, \ldots, z_r]$ for $z_1, \ldots, z_r \in N$, where Ψ is the same formula.
Since π is Σ_0–preserving we then get:

$$\begin{aligned}\overline{M} \models \varphi[\vec{x}] &\leftrightarrow \overline{M} \models \varphi[\overline{F}(\vec{z})] \\ &\leftrightarrow \overline{N} \models \Psi[\vec{z}] \\ &\to N \models \Psi[\pi(\vec{z})] \\ &\leftrightarrow M \models \varphi[F(\pi(\vec{z}))] \\ &\leftrightarrow M \models \varphi[\pi(\vec{x})].\end{aligned}$$

QED (Case 1)

Case 2 $j > 0$.
This is entirely similar. Let φ be $\Sigma_j^{(n)}$. By Corollary 2.6.19 it

follows easily that there is a Σ_j formula Ψ such that: $\overline{M} \models \varphi[\overline{F}_1(z_1), \dots, \overline{F}_r(z_r)]$ is equivalent to:

$$\overline{N} \models \Psi[z_1, \dots, z_r].$$

Since the corresponding reduction holds on the M–side, we get

$$\overline{M} \models \varphi[\vec{x}] \leftrightarrow M \models \varphi[\pi(\vec{x})],$$

since $\pi(x_i) = \pi(\overline{F}_i(z_i)) = F_i(\overline{\pi}(z_i))$. QED (Claim 3)

This proves existence. We now prove uniqueness.

Claim 4 The uniqueness assertion of (a) holds.

Proof: Let $\hat{M}, \hat{a}$ be such that $\hat{M}^{n,\hat{a}} = \overline{N}$ and $\hat{a} \in R^N_{\hat{M}}$.
Claim $\hat{M} = \overline{M}, \hat{a} = \overline{a}$.

Proof: By a virtual repetition of the proof in Claim 2 there is a $\pi : \hat{M} \to_{\Sigma_1^{(m)}} \overline{M}$ defined by:

(3) $\pi(\hat{F}(z)) = \overline{F}(z)$ whenever $z \in \overline{N}$, $\hat{F}$ is a good $\Sigma_1^{(m)}(\hat{M})$ function in $\hat{a}^{(0)}, \dots, \hat{a}^{(m)}$ and $\overline{F}$ is the $\Sigma_1^{(m)}(\overline{M})$ function in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$ with the same good definition.

But π is then onto. Hence π is an isomorphism of $\hat{M}$ with $\overline{M}$. Since $\hat{M}, \overline{M}$ are transitive, we conclude that $\overline{M} = \hat{M}, \overline{a} = \hat{a}$.

QED (Claim 4)

Finally we prove the uniqueness assertion of (b):

Claim 5 Let $\pi' : \overline{M} \to_{\Sigma_0^{(m)}} M$ strictly, such that $\pi'(\overline{a}) = a$. Then $\pi' = \pi$.

Proof: By strictness we can again conclude that $\pi'(\overline{a}^{(i)}) = a^{(i)}$ for $i = 0, \dots, m$. Let $x \in \overline{M}$, $x = \overline{F}(z)$, where $z \in \overline{N}$ and $\overline{F}$ is a $\Sigma_1^{(m)}(\overline{M})$ good function in the parameters $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$. Let F be $\Sigma_1^{(m)}(M)$ in $a^{(0)}, \dots, a^{(m)}$ by the same good definition.
The statement: $x = \overline{F}(z)$ is $\Sigma_2^{(m)}(\overline{M})$ in $\overline{a}^{(0)}, \dots, \overline{a}^{(m)}$. Since π' is $\Sigma_0^{(m)}$–preserving, the corresponding statement must hold in M — i.e. $\pi'(x) = F(\overline{\pi}(z)) = \pi(x)$.

QED (Lemma 2.6.32)

2.7 Liftups

2.7.1 The Σ_0 liftup

A concept which, under a variety of names, is frequently used in set theory is the *liftup* (or as we shall call it here, the Σ_0 *liftup*). We can define it as follows:

Definition 2.7.1. Let M be a J–model. Let $\tau > \omega$ be a cardinal in M. Let $H = H_\tau^M \in M$ and let $\pi : H \to_{\Sigma_0} H'$ cofinally. We say that $\langle M', \pi' \rangle$ is a Σ_0 *liftup* of $\langle M, \pi \rangle$ iff M' is transitive and:

(a) $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$

(b) Every element of M' has the form $\pi'(f)(x)$ for an $x \in H'$ and an $f \in \Gamma^0$, where $\Gamma^0 = \Gamma^0(\tau, M)$ is the set of functions $f \in M$ such that $\mathrm{dom}(f) \in H$.

Note. The condition of being a J–model can be relaxed considerably, but that is uninteresting for our purposes.

Until further notice we shall use the word 'liftup' to mean 'Σ_0 liftup'.

If $\langle M', \pi' \rangle$ is a liftup of $\langle M, \pi \rangle$ it follows easily that:

Lemma 2.7.1. *$\pi' : M \to_{\Sigma_0} M'$ cofinally.*

Proof: Let $y \in M'$, $y = \pi'(f)(x)$ where $x \in H'$ and $f \in \Gamma^0$, then $y \in \pi'(\mathrm{rng}(f))$. QED (Lemma 2.7.1)

Lemma 2.7.2. *$\langle M', \pi' \rangle$ is the only liftup of $\langle M, \pi \rangle$.*

Proof: Suppose not. Let $\langle M^*, \pi^* \rangle$ be another liftup. Let $\varphi(v_1, \ldots, v_n)$ be Σ_0. Then

$$\begin{aligned} & M' \models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_n)(x_n)] \leftrightarrow \\ & \langle x_1, \ldots, x_n \rangle \in \pi(\{\langle \vec{z} \rangle | M \models \varphi[\vec{f}(\vec{z})]\}) \leftrightarrow \\ & M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_n)(x_n)]. \end{aligned}$$

Hence there is an isomorphism σ of M' onto M^* defined by:

$$\begin{aligned} & \sigma(\pi'(f)(x)) = \pi^*(f)(x) \\ & \text{for } f \in \Gamma^0,\ x \in \pi(\mathrm{dom}(f)). \end{aligned}$$

But M', M^* are transitive. Hence $\sigma = \mathrm{id}$, $M' = M^*$, $\pi' = \pi^*$. QED (Lemma 2.7.2)

Note. $M \models \varphi[\vec{f}(\vec{z})]$ means the same as

$$\bigvee y_1 \dots y_n (\bigwedge_{i=1}^{n} y_i = f_i(z_i) \wedge M \models \varphi[\vec{y}]).$$

Hence if $e = \{\langle \vec{z} \rangle | M \models \varphi[\vec{f}(\vec{z})]\}$, then $e \subset \overset{n}{\underset{i=1}{\times}} \mathrm{dom}(f_i) \in H$. Hence $e \in M$ by rud closure, since e is $\underline{\Sigma}_0(M)$. But then $e \in H$, since $\mathbb{P}(u) \cap M \subset H$ for $u \in H$.

But when does the liftup exist? In answering this question it is useful to devise a 'term model' for the putative liftup rather like the ultrapower construction:

Definition 2.7.2. Let $M, \tau, \pi : H \to_{\Sigma_0} H'$ be as above. The term model $\mathbb{D} = \mathbb{D}(M, \pi)$ is defined as follows. Let e.g. $M = \langle J_\alpha^A, B \rangle$. $\mathbb{D} =: \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B} \rangle$ where

$D =$ the set of pairs $\langle f, x \rangle$ such that $f \in \Gamma_0$ and $x \in H'$

$$\begin{aligned}
&\langle f, x \rangle \cong \langle g, y \rangle \leftrightarrow: \langle x, y \rangle \in \pi(\{\langle z, w \rangle | f(z) = g(y)\}) \\
&\langle f, x \rangle \tilde{\in} \langle g, y \rangle \leftrightarrow: \langle x, y \rangle \in \pi(\{\langle z, w \rangle | f(z) \in g(y)\}) \\
&\tilde{A}\langle f, x \rangle \leftrightarrow: x \in \pi(\{z | Af(z)\}) \\
&\tilde{B}\langle f, x \rangle \leftrightarrow: x \in \pi(\{z | Bf(z)\})
\end{aligned}$$

Note. $\mathbb{D}$ is an 'equality model', since the identity predicate $=$ is interpreted by $\cong$ rather than the identity.

Łos theorem for $\mathbb{D}$ then reads:

Lemma 2.7.3. *Let $\varphi = \varphi(v_1, \dots, v_n)$ be Σ_0. Then*

$$\mathbb{D} \models \varphi[\langle f_1, x_1 \rangle, \dots, \langle f_n, x_n \rangle] \leftrightarrow \langle x_1, \dots, x_n \rangle \in \pi(\{\langle \vec{z} \rangle | M \models \varphi[\vec{f}(\vec{z})]\}).$$

Proof: (Sketch)
We prove this by induction on the formula φ. We display a typical case of the induction. Let $\varphi = \bigvee u \in v_1 \Psi$. By bound relettering we can assume *w.l.o.g.* that u is not among $v_1, \dots, v_n$. Hence $u, v_1, \dots, v_n$ is a good sequence for Ψ. We first prove $(\to)$. Assume:

$$\mathbb{D} \models \varphi[\langle f_1, x_1 \rangle, \dots, \langle f_n, x_n \rangle].$$

Claim $\langle x_1, \dots, x_n \rangle \in \pi(e)$ where

$$e = \{\langle z_1, \dots, z_n \rangle | M \models \varphi[f_1(z_1) \dots f_n(z_n)]\}.$$

Proof: By our assumption there is $\langle g,y\rangle \in D$ such that $\langle g,y\rangle \tilde{\in} \langle f_1,x_1\rangle$ and:

$$\mathbb{D} \models \Psi[\langle g,y\rangle, \langle f_1,x_1\rangle, \ldots, \langle f_n,x_n\rangle].$$

By the induction hypothesis we conclude that $\langle y,\vec{x}\rangle \in \pi(\tilde{e})$ where:

$$\tilde{e} = \{\langle w,\vec{z}\rangle | g(w) \in f_1(z_1) \wedge M \models \Psi[g(w), \vec{f}(\vec{z})\}.$$

Clearly $e, \tilde{e} \in H$ and

$$H \models \bigwedge w, \vec{z}(\langle w,\vec{z}\rangle \in \tilde{e} \to \langle \vec{z}\rangle \in e).$$

Hence

$$H' \models \bigwedge w, \vec{z}(\langle w,\vec{z}\rangle \in \pi(e) \to \langle \vec{z}\rangle \in \pi(e)).$$

Hence $\langle \vec{x}\rangle \in \pi(e)$. QED $(\to)$

We now prove $(\leftarrow)$
We assume that $\langle x_1,\ldots,x_n\rangle \in \pi(e)$ and must prove:

Claim $\mathbb{D} \models \varphi[\langle f_1,x_1\rangle, \ldots, \langle f_n,x_n\rangle]$.

Proof: Let $r \in M$ be a well ordering of $\mathrm{rng}(f_1)$. For $\langle \vec{z}\rangle \in e$ set:

$$\begin{aligned} g(\langle \vec{z}\rangle) = \quad & \text{the } r\text{–least } w \text{ such that} \\ & M \models \Psi[w, f_1(z_1), \ldots, f_n(z_n)]. \end{aligned}$$

Then $g \in M$ and $\mathrm{dom}(g) = e \in H$. Now let $\tilde{e}$ be defined as above with this g. Then:

$$H \models \bigwedge z_1, \ldots, z_n(\langle \vec{z}\rangle \in e \leftrightarrow \langle\langle \vec{z}\rangle, \vec{z}\rangle \in \tilde{e}).$$

But then the corresponding statement holds of $\pi(e), \pi(\tilde{e})$ in H'. Hence

$$\langle\langle \vec{x}\rangle, \vec{x}\rangle \in \pi(\tilde{e}).$$

By the induction hypothesis we conclude:

$$\mathbb{D} \models \Psi[\langle g, \langle \vec{x}\rangle\rangle, \langle f_1,x_1\rangle, \ldots, \langle f_n,x_n\rangle].$$

The conclusion is immediate. QED (Lemma 2.7.3)

The liftup of $\langle M,\pi\rangle$ can only exist if the relation $\tilde{e}$ is well founded:

Lemma 2.7.4. *Let $\tilde{\in}$ be ill founded. Then there is no $\langle M',\pi'\rangle$ such that $\pi' : M \to_{\Sigma_0} M'$. M' is transitive, and $\pi' \supset \pi$.*

Proof: Suppose not. Let $\langle f_{i+1}, x_{i+1}\rangle \tilde{\in} \langle f_i, x_i\rangle$ for $i < w$. Then

$$\langle x_{i+1}, x_i\rangle \in \pi\{\langle z, w\rangle | f_{i+1}(z) \in f_i(w)\}.$$

Hence $\pi'(f_{i+1})(x_{i+1}) \in \pi'(f_i)(x_i)(i < w)$.
Contradiction! QED (Lemma 2.7.4)

Conversely we have:

Lemma 2.7.5. *Let $\tilde{\in}$ be well founded. Then the liftup of $\langle M, \pi\rangle$ exists.*

Proof: We shall explicitly construct a liftup from the term model $\mathbb{D}$. The proof will stretch over several subclaims.

Definition 2.7.3. $x^* = \pi^*(x) =: \langle \mathrm{const}_x, 0\rangle$, where $\mathrm{const}_x =: \{\langle x, 0\rangle\} =$ the constant function x defined on $\{0\}$.

Then:

(1) $\pi^* : M \to_{\Sigma_0} \mathbb{D}$.
Proof: Let $\varphi(v_1, \ldots, v_n)$ be Σ_0. Set:

$$e = \{\langle z_1, \ldots, z_n\rangle | M \models \varphi[\mathrm{const}_{x_1}(z_1), \ldots, \mathrm{const}_{x_n}(z_n)]\}.$$

Obviously:

$$e = \begin{cases} \{\langle 0, \ldots, 0\rangle\} \text{ if } M \models \varphi[x_1, \ldots, x_n] \\ \emptyset \text{ if not.} \end{cases}$$

Hence by Łoz theorem:

$$\begin{aligned} \mathbb{D} \models \varphi[x_1^*, \ldots, x_n^*] &\leftrightarrow \langle 0, \ldots, 0\rangle \in \pi(e) \\ &\leftrightarrow M \models \varphi[x_1, \ldots, x_n] \end{aligned}$$

(2) $\mathbb{D} \models$ Extensionality.
Proof: Let $\varphi(u, v) =: \bigwedge w \in u\, w \in v \ \wedge \bigwedge w \in v\ w \in u$.
Claim $\mathbb{D} \models \varphi[a, b] \to a \cong b$ for $a, b \in \mathbb{D}$. This reduces to the Claim:
Let $a = \langle f, x\rangle, b = \langle g, y\rangle$. Then

$$\begin{aligned} \mathbb{D} \models \varphi[\langle f, x\rangle, \langle g, y\rangle] &\leftrightarrow \langle x, y\rangle \in \pi(e) \\ &\leftrightarrow \langle f, x\rangle \cong \langle g, y\rangle \end{aligned}$$

where

$$\begin{aligned} e &= \{\langle z, w\rangle | M \models \varphi[z, w]\} \\ &= \{\langle z, w\rangle | f(z) = g(w)\} \end{aligned}$$

QED (2)

Since $\cong$ is a congruence relation for $\mathbb{D}$ we can factor $\mathbb{D}$ by $\cong$, getting:

$$\hat{\mathbb{D}} = (\mathbb{D} \backslash \cong) = \langle \hat{D}, \hat{\in}, \hat{A}, \hat{B} \rangle$$

where:

$$\hat{D} = \{\hat{s} | s \in D\}$$
$$\hat{s} =: \{t | t \cong s\} \text{ for } s \in D$$
$$\hat{s} \hat{\in} \hat{t} \leftrightarrow: s \tilde{\in} t$$
$$\hat{A}\hat{s} \leftrightarrow: \tilde{A}s, \hat{B}\hat{s} \leftrightarrow: \tilde{B}s.$$

Then $\hat{\mathbb{D}}$ is a well founded identity model satisfying extensionality. By Mostowski's isomorphism theorem there is an isomorphism k of $\hat{\mathbb{D}}$ onto M', where $M' = \langle |M'|, \in, A', B' \rangle$ is transitive.

Set:

$$[s] =: k(\hat{s}) \text{ for } s \in D$$
$$\pi'(x) =: [x^*] \text{ for } x \in M.$$

Then by (1):

(3) $\pi' : M \to_{\Sigma_0} M'$.
Lemma 2.7.5 will then follow by:

Lemma 2.7.6. *$\langle M', \pi' \rangle$ is the liftup of $\langle M, \pi \rangle$.*

We shall often write $[f, x]$ for $[\langle f, x \rangle]$. Clearly every $s \in M'$ has the form $[f, x]$ where $f \in M$; $\mathrm{dom}(f) \in H$, $x \in H'$.

Definition 2.7.4. $\tilde{H}$ =: the set of $[f, x]$ such that $\langle f, x \rangle \in D$ and $f \in H$.

We intend to show that $[f, x] = \pi(f)(x)$ for $x \in \tilde{H}$. As a first step we show:

(4) $\tilde{H}$ is transitive.

Proof: Let $s \in [f, x]$ where $f \in H$.

Claim $s = [g, y]$ for a $g \in H$.

Proof: Let $s = [g', y]$. Then $\langle y, x \rangle \in \pi(e)$ where: $e = \{\langle u, v \rangle | g'(u) \in f(v)\}$ set:

$$e' = \{u | g'(u) \in \mathrm{rng}(f)\}, \ g = g' \upharpoonright e'.$$

Then $g \subset \mathrm{rng}(f) \times \mathrm{dom}(g') \in H$. Hence $g \in H$. Then $[g', y] = [g, y]$ since $\pi(g')(y) = \pi(g)(y)$ and hence
$\langle y, y \rangle \in \pi(\{\langle u, v \rangle | g'(u) = g(v)\})$. But $e = \{\langle u, v \rangle | g(u) \in f(v)\}$. Hence $[g, y] \in [f, x]$. QED (4)

But then:

(5) $[f, x] = \pi(f)(x)$ for $f \in H, \langle f, x\rangle \in D$.

Proof: Let $f, g \in H, \langle f, x\rangle, \langle g, y\rangle \in D$. Then:

$$\begin{aligned}[f, x] \in [g, y] &\leftrightarrow \langle x, y\rangle \in \pi(e) \\ &\leftrightarrow \pi(f)(x) \in \pi(g)(y)\end{aligned}$$

where $e = \{\langle u, v\rangle | f(u) \in g(v)\}$. Hence there is an $\in$–isomorphism σ of H onto $\tilde{H}$ defined by:

$$\sigma(\pi(f)(x)) =: [f, x].$$

But then $\sigma = \text{id}$, since $H, \tilde{H}$ are transitive. (5)

But then:

(6) $\pi' \supset \pi$.

Proof: Let $x \in H$. Then $\pi'(x) = [\text{const}_x, 0] = \pi(\text{const}_x)(0) = \pi(x)$ by (5).

(7) $[f, x] = \pi'(f)(x)$ for $\langle f, x\rangle \in D$.

Proof: Let $a = \text{dom}(f)$. Then $[\text{id}_a, x] = \text{id}_{\pi(a)}(x) = x$ by (5). Hence it suffices to show:

$$[f, x] = [\text{const}_f, 0]([\text{id}_a, x]).$$

But this says that $\langle x, 0\rangle \in \pi(e)$ where:

$$\begin{aligned}e &= \{\langle z, u\rangle | f(z) = \text{const}_f(u)(\text{id}_a(z))\} \\ &= \{\langle z, 0\rangle | f(z) = f(z)\} = a \times \{0\}.\end{aligned}$$

QED (7)

Lemma 2.7.6 is then immediate by (3), (6) and (7). QED (Lemma 2.7.6)

Lemma 2.7.7. *Let $\pi^* \supset \pi$ such that $\pi^* : M \to_{\Sigma_0} M^*$. Then the liftup $\langle M', \pi'\rangle$ of $\langle M, \pi\rangle$ exists. Moreover there is a $\sigma : M' \to_{\Sigma_0} M^*$ uniquely defined by the condition:*

$$\sigma \restriction H' = \text{id},\ \sigma\pi' = \pi^*.$$

Proof: $\langle M', \pi'\rangle$ exists, since $\tilde{\in}$ is well founded, since $\langle f, x\rangle \tilde{\in} \langle g, y\rangle \leftrightarrow \pi^*(f)(x) \in \pi^*(g)(y)$. But then:

$$\begin{aligned}M' &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \leftrightarrow \\ &\leftrightarrow \langle x_1, \ldots, x_r\rangle \in \pi(e) \\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)]\end{aligned}$$

where $e = \{\langle z_1, \dots, z_r\rangle | M \models \varphi[\vec{f}(\vec{z})]\}$. Hence there is $\sigma : M' \to_{\Sigma_0} M^*$ defined by:

$$\sigma(\pi'(f)(x)) = \pi^*(f)(x) \text{ for } \langle f, x\rangle \in D.$$

Now let $\tilde{\sigma} : M' \to_{\Sigma_0} M^*$ such that $\tilde{\sigma} \upharpoonright H' = \mathrm{id}$ and $\tilde{\sigma}\pi' = \pi^r$.

Claim $\tilde{\sigma} = \sigma$.
Let $s \in M'$, $s = \pi'(f)(x)$. Then $\tilde{\sigma}(\pi'(f)) = \pi^*(f)$, $\tilde{\sigma}(x) = x$. Hence $\tilde{\sigma}(s) = \pi^*(f)(x) = \sigma(s)$. QED (Lemma 2.7.7)

2.7.2 The $\Sigma_0^{(n)}$ liftup

From now on suppose M to be acceptable. We now attempt to generalize the notion of Σ_0 liftup. We suppose as before that $\tau > w$ is a cardinal in M and $H = H_\tau^M$. As before we suppose that $\pi' : H \to_{\Sigma_0} H'$ cofinally. Now let $\rho^n \geq \tau$. The Σ_0–liftup was the "minimal" $\langle M', \pi'\rangle$ such that $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0} M'$. We shall now consider pairs $\langle M', \pi'\rangle$ such that $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0^n} M'$. Among such pairs $\langle M', \pi'\rangle$ we want to define a "minimal" one and show, if possible, that it exists. The minimality of the Σ_0 liftup was expressed by the condition that every element of M' have the form $\pi'(f)(x)$, where $x \in H'$ and $f \in \Gamma^0(\tau, M)$. As a first step to generalizing this definition we replace $\Gamma^0(\tau, M)$ by a larger class of functions $\Gamma^n(\tau, M)$.

Definition 2.7.5. Let $n > 0$ such that $\tau \leq \rho_M^n$. $\Gamma^n = \Gamma^n(\tau, M)$ is the set of maps f such that

(a) $\mathrm{dom}(f) \in H$

(b) For some $i < n$ there is a good $\Sigma_1^{(i)}(M)$ function G and a parameter $p \in M$ such that $f(x) = G(x, p)$ for all $x \in \mathrm{dom}(f)$.

Note. Good $\Sigma_1^{(i)}$ functions are many sorted, hence any such function can be identified with a pair consisting of its field and its arity. An element of Γ^n, on the other hand, is 1–sorted in the classical sense, and can be identified with its field.

Note. This definition makes sense for the case $n = \omega$, and we will not exclude this case. A $\Sigma_0^{(\omega)}$ formula (or relation) then means any formula (or relation) which is $\Sigma_0^{(i)}$ for an $i < \omega$ — i.e. $\Sigma_0^{(\omega)} = \Sigma^*$.

We note:

Lemma 2.7.8. *Let $f \in \Gamma^n$ such that $\mathrm{rng}(f) \subset H^i$, where $i < n$. Then $f(x) = G(x, p)$ for $x \in \mathrm{dom}(f)$ where G is a good $\Sigma_1^{(h)}$ function to H^i for some $h < n$.*

Proof: Let $f(x) = G'(x,p)$ for $x \in \mathrm{dom}(f)$ where G' is a good $\Sigma_1^{(h)}$ function to H^j where $h, j < n$. Since every good $\Sigma_1^{(h)}$ function is a good Σ_1^k function for $k \geq h$, we can assume *w.l.o.g.* that $i, j \leq h$. Let F be the identity function defined by $v^i = u^j$ (i.e. $y^i = F(x^j) \leftrightarrow y^i = x^j$). Set: $G(x,y) \simeq: F(G'(x,y))$. Then F is a good $\Sigma_1^{(h)}$ function and so is G, where $f(x) = G(x,p)$ for $x \in \mathrm{dom}(f)$.

QED (Lemma 2.7.8)

Lemma 2.7.9. $\Gamma^i(\tau, M) \subset \Gamma^n(\tau, M)$ *for* $i < n$.

Proof: For $0 < i$ this is immediat by the definition. Now let $i = 0$. If $f \in \Gamma^0$, then $f(x) = G(x,f)$ for $x \in \mathrm{dom}(f)$ where G is the $\Sigma_0^{(0)}$ function defined by

$$\begin{aligned} y = G(x,f) \leftrightarrow: \quad & (f \text{ is a function } \wedge \\ & \wedge \langle y, x\rangle \in f). \end{aligned}$$

QED (Lemma 2.7.9)

The "natural" minimality condition for the $\Sigma_0^{(n)}$ liftup would then read: Each element of M has the form $\pi'(f)(x)$ where $x \in H'$ and $f \in \Gamma^n$. But what sense can we make of the expression "$\pi'(f)(x)$" when f is not an element of M? The following lemma rushes to our aid:

Lemma 2.7.10. *Let* $\pi' : M \to_{\Sigma_0^{(n)}} M'$ *where* $n > 0$ *and* $\pi' \supset \pi$. *There is a unique map* π'' *on* $\Gamma^n(\tau, M)$ *with the following property:*

* *Let* $f \in \Gamma^n(\tau, M)$ *such that* $f(x) = G(x,p)$ *for* $x \in \mathrm{dom}(f)$ *where* G *is a good* $\Sigma_1^{(i)}$ *function for an* $i < n$ *and* χ *is a good* $\Sigma_1^{(i)}$ *definition of* G. *Let* G' *be the function defined on* M' *by* χ. *Let* $f' = \pi''(f)$. *Then* $\mathrm{dom}(f') = \pi(\mathrm{dom}(f))$ *and* $f'(x) = G'(x, \pi'(p))$ *for* $x \in \mathrm{dom}(f')$.

Proof: As a first approximation, we simply pick G, χ with the above properties. Let G' then be as above. Let $d = \mathrm{dom}(f)$. The statement $\bigwedge x \in d \bigvee y\ y = G(x,p)$ is $\Sigma_0^{(n)}$ is d, p, so we have:

$$\bigwedge x \in \pi(d) \bigvee y\ y = G'(x, \pi(p)).$$

Define f_0 by $\mathrm{dom}(f_0) = \pi(d)$ and $f_0(x) = G'(x, \pi(p))$ for $x \in \pi(d)$. The problem is, of course, that G, χ were picked arbitrarily. We might also have:

$$f(x) = H(x,q) \text{ for } x \in d,$$

where H is $\Sigma_1^{(j)}(M)$ for a $j < n$ and Ψ is a good $\Sigma_1^{(j)}$ definition of H. Let H' be the good function on M' defined by Ψ. As before we can define f_1

by $\text{dom}(f_1) = \pi(d)$ and $f_1(x) = H'(x, \pi'(q))$ for $x \in \pi(d)$. We must show: $f_0 = f_1$. We note that:

$$\bigwedge x \in dG(x,p) = H(x,q).$$

But this is a $\Sigma_0^{(n)}$ statement. Hence

$$\bigwedge x \in \pi(d) G'(x,p) = H'(x,q).$$

Then $f_0 = f_1$. QED (Lemma 2.7.10)

Moreover, we get:

Lemma 2.7.11. *Let $n, \pi, \tau, \pi', \pi''$ be as above. Then $\pi''(f) = \pi'(f)$ for $f \in \Gamma^0(\tau, M)$.*

Proof: We know $f(x) = G(x, f)$ for $x \in d = \text{dom}(f)$, where:

$$y = G(x,f) \leftrightarrow: (f \text{ is a function } \wedge y = f(x)).$$

Then $\pi''(f)(x) = G'(x, \pi'(f)) = \pi'(f)(x)$ for $x \in \pi(d)$, where G' has the same definition over M'. QED (Lemma 2.7.11)

Thus there is no ambiguity in writing $\pi'(f)$ instead of $\pi''(f)$ for $f \in \Gamma^n$. Doing so, we define:

Definition 2.7.6. Let $\omega < \tau < \rho_M^n$ where $n \leq \omega$ and τ is a cardinal in M. Let $H = H_\tau^M$ and let $\pi : H \to_{\Sigma_0} H'$ cofinally. We call $\langle M', \pi' \rangle$ a $\Sigma_0^{(n)}$ *liftup* of $\langle M, \pi \rangle$ iff the following hold:

(a) $\pi' \supset \pi$ and $\pi' : M \to_{\Sigma_0^{(n)}} M'$.

(b) Each element of M' has the form $\pi'(f)(x)$, where $f \in \Gamma^n(\tau, M)$ and $x \in H'$.

(Thus the old Σ_0 liftup is simply the special case: $n = 0$.)

Definition 2.7.7. $\Gamma_i^n(\tau, M)$ =: the set of $f \in \Gamma^n(\tau, M)$ such that either $i < n$ and $\text{rng}(f) \subset H_M^i$ or $i = n < \omega$ and $f \in H_M^i$.

(Here, as usual, $H^i = J_{\rho_M^i}[A]$ where $M = \langle J_\alpha^A, B \rangle$.)

Lemma 2.7.12. *Let $f \in \Gamma_i^n(\tau, M)$. Let $\pi' : M \to_{\Sigma_0^{(n)}} M'$ where $\pi' \supset \pi$. Then $\pi'(f) \in \Gamma_i^n(\pi'(\tau), M')$.*

Proof:

Case 1 $i = n$. Then $f \in H^M_{\rho^n_M}$. Hence $\pi'(f) \in H^{M'}_{\rho^n_M}$.

Case 2 $i < n$.

By Lemma 2.7.9 for some $h < n$ there is a good $\Sigma_1^{(n)}(M)$ function $G(u, v)$ to H^i and a parameter p such that

$$f(x) = G(x, p) \text{ for } x \in \mathrm{dom}(f).$$

Hence:

$$\pi'(f)(x) = G'(x, \pi'(p)) \text{ for } x \in \mathrm{dom}(\pi(f)),$$

where G' is defined over M' by the same good $\Sigma_1^{(n)}$ definition. Hence $\mathrm{rng}(\pi'(f)) \subset H^i_M$. QED (Lemma 2.7.12)

The following lemma will become our main tool in understanding $\Sigma_0^{(n)}$ liftups.

Lemma 2.7.13. *Let $R(x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_r \leq n$. Let $f_l \in \Gamma^n_{i_l}(l = 1, \ldots, r)$. Then:*

(a) The relation P is $\Sigma_0^{(n)}$ in a parameter p where:

$$P(\vec{z}) \leftrightarrow: R(f_1(z_1), \ldots, f_r(z_r)).$$

(b) Let $\pi' \supset \pi$ such that $\pi' : M \to_{\Sigma_0^{(n)}} M'$. Let R' be $\Sigma_0^{(n)}(M')$ by the same definition as R. Then P' is $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition as P in p, where:

$$P'(\vec{z}) \leftrightarrow: R'(\pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)).$$

Before proving this lemma we note some corollaries:

Corollary 2.7.14. *Let $e = \{\langle\vec{z}\rangle | P(\vec{z})\}$. Then $e \in H$ and $\pi(e) = \{\langle\vec{z}\rangle | P'(\vec{z})\}$.*

Proof: Clearly $e \subset d = \underset{l=1}{\overset{r}{\times}} \mathrm{dom}(f_l) \in H$. But then $d \in H_{\rho^n}$ and $e \in H_{\rho^n}$ since $\langle H_{\rho^n}, P \cap H_{\rho^n}\rangle$ is amenable. Hence $e \in H$, since $H = H^M_\tau$ and therefore $\mathbb{P}(u) \cap M \subset H$ for $u \in H$.

Now set $e' = \{\langle\vec{z}\rangle | P'(\vec{z})\}$. Then $e' \subset \pi(d) = \underset{l=1}{\overset{r}{\times}} \mathrm{dom}(\pi(f_l))$ since $\pi' \supset \pi$ and hence $\pi(\mathrm{dom}(f_l)) = \mathrm{dom}(\pi(f_l))$. But

$$\bigwedge \langle\vec{z}\rangle \in d (\langle\vec{z}\rangle \in e \leftrightarrow P(\vec{z}))$$

which is a $\Sigma_0^{(n)}$ statement about e, p. Hence the same statement holds of $\pi(e), \pi(p)$ in M'. Hence

$$\bigwedge \langle \vec{z} \rangle \in \pi(d)(\langle \vec{z} \rangle \in \pi(e) \leftrightarrow P'(\vec{z})).$$

Hence $\pi(e) = e'$. QED (Corollary 2.7.14)

Corollary 2.7.15. *$\langle M, \pi \rangle$ has at most one $\Sigma_0^{(n)}$ liftup $\langle M', \pi' \rangle$.*

Proof: Let $\langle M^*, \pi^* \rangle$ be a second such. Let $\varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ be a $\Sigma_0^{(n)}$ formula. (In fact, we could take it here as being $\Sigma_0^{(0)}$.) Let $e = \{\langle \vec{z} \rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$ where $f_l \in \Gamma_{i_l}^n (l = 1, \ldots, r)$. Then:

$$\begin{aligned} M' \quad &\models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)(x_r)] \leftrightarrow \\ &\leftrightarrow \langle x_1, \ldots, x_r \rangle \in \pi(e) \\ &\leftrightarrow M^* \models \varphi[\pi^*(f_1)(x_1), \ldots, \pi^*(f_r)(x_r)] \end{aligned}$$

for $x_l \in \pi(\mathrm{dom}(f_l)(l = 1, \ldots, r)$.

Hence there is an isomorphism $\sigma : M' \tilde{\rightarrow} M^*$ defined by:

$$\sigma(\pi'(f)(x)) =: \pi^*(f)(x)$$

for $f \in \Gamma^n,\ x \in \pi(\mathrm{dom}(f))$. But M', M^* are transitive. Hence $\sigma = \mathrm{id}, M' = M^*, \pi' = \pi^*$. QED (Corollary 2.7.15)

We now prove Lemma 2.7.13 by induction on n.

Case 1 $n = 0$.

Then $f_1, \ldots, f_r \in M$ and P is Σ_0 in $p = \langle f_1, \ldots, f_r \rangle$, since f_i is rudimentary in p and for sufficiently large h we have:

$$P(\vec{z}) \leftrightarrow \bigvee{}_{y_1, \ldots, y_r} \in C_h(p)(\bigwedge_{i=1}^{r} y_i = f_i(\vec{z}_i) \wedge R(\vec{y}))$$

where R is Σ_0. If P' has the same Σ_0 definition over M' in $\pi'(p)$, then

$$\begin{aligned} P'(z) \quad &\leftrightarrow \bigvee{}_{y_1, \ldots, y_r} \in C_h(\pi(p))(\bigwedge_{n=1}^{r} y_i = \pi(f_i)(z_i) \wedge R(\vec{y})) \\ &\leftrightarrow R(\pi(\vec{f})(\vec{z})) \end{aligned}$$

QED

Case 2 $n = \omega$.

Then $\Sigma_0^\omega = \bigcup_{h<w} \Sigma_1^{(h)}$. Let $R(x_1^{i_1}, \ldots, x_r^{l_r})$ be $\Sigma_1^{(h)}$. Since every $\Sigma_1^{(h)}$ relation is $\Sigma_1^{(k)}$ for $k \geq h$, we can assume h taken large enough that $i_1, \ldots, i_r \leq h$. We can also choose it large enough that:

$$f_l(z) \simeq G_l(z, p) \text{ for } l = 1, \ldots, v$$

where G_l is a good $\Sigma_1^{(h)}$ map to H^{i_l}. (We assume *w.l.o.g.* that p is the same for $l = 1, \ldots, r$ and that $d_l = \text{dom}(f_l)$ is rudimentary in p.) Set:

$$P(\vec{z}, y) \leftrightarrow: R(G_1 x_1, y), \ldots, G(x_r, y)).$$

By §6 Lemma 2.6.24, P is $\Sigma_1^{(h)}$ (uniformly in the $\Sigma_1^{(h)}$ definition of R and $G_1, \ldots, G_r$). Moreover:

$$P(\vec{z}) \leftrightarrow P(\vec{z}, p).$$

Thus P is uniformly $\Sigma_1^{(h)}$ in p, which proves (a). But letting P' have the same $\Sigma_1^{(h)}$ definition in $\pi'(p)$ over M', we have:

$$\begin{aligned} P'(\vec{z}) \quad & \leftrightarrow P'(\vec{z}, \pi'(p)) \\ & \leftrightarrow R'(\pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)), \end{aligned}$$

which proves (b). QED (Case 2)

Case 3 $0 < n < w$.

Let $n = m + 1$. Rearranging arguments as necessary, we can take R as given in the form:

$$R(y_1^n, \ldots, y_s^n, x_1^{i_1}, \ldots, x_r^{i_r})$$

where $i_1, \ldots, i_r \leq m$. Let $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$ and let $g_1, \ldots, g_1 \in \Gamma_n^n$.

Claim

(a) P is $\Sigma_0^{(n)}$ in a parameter p where

$$P(\vec{w}, \vec{z}) \leftrightarrow: R(\vec{g}(\vec{w}), \vec{f}(\vec{z})).$$

(b) If π', M' are as above and P' is $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition, then

$$P'(w, \vec{z}) \leftrightarrow R'(\pi'(\vec{g})(\vec{w}), \pi'(\vec{f})(\vec{z}))$$

where R' has the same $\Sigma_0^{(n)}$ definition over M'.

We prove this by first substituting $\vec{f}(\vec{z})$ and then $\vec{g}(\vec{w})$, using two different arguments. The claim then follows from the pair of claims:

Claim 1 Let:

$$P_0(\vec{y}^n, \vec{z}) \leftrightarrow= R(y^n, f_1(z_1), \ldots, f_r(z_r)).$$

Then:

(a) P_0 is $\Sigma_0^{(n)}(M)$ in a parameter p_0.

(b) Let π', M', R' be as above. Let P_0' have the same $\Sigma_0^{(n)}(M')$ definition in $\pi'(p_0)$. Then:

$$P_0'(\vec{y}^n, \vec{z}) \leftrightarrow R'(y^n, \pi'(\vec{f})(\vec{z})).$$

Claim 2 Let

$$P(\vec{w}, \vec{z}) \leftrightarrow: P_0(g_1(w_1), \ldots, g_s(w_s), \vec{z}).$$

Then:

(a) P is $\Sigma_0^{(n)}(M)$ in a parameter p.

(b) Let π', M', P_0' be as above. Let P' have the same $\Sigma_1^{(n)}(M')$ definition in $\pi'(p)$. Then

$$P'(\vec{w}, \vec{z}) \leftrightarrow P_0'(\pi'(\vec{g})(\vec{w}), \vec{z}).$$

We prove Claim 1 by imitating the argument in Case 2, taking $h = m$ and using §6 Lemma 2.6.11. The details are left to the reader. We then prove Claim 2 by imitating the argument in Case 1: We know that $g_1, \ldots, g_s \in H^n$. Set: $p = \langle g_1, \ldots, g_n, p\rangle$. Then P is $\Sigma_0^{(n)}(M)$ in p, since:

$$P(\vec{w}, \vec{z}) \leftrightarrow \bigvee y_1 \ldots y_s \in C_h(p)(\bigwedge_{i=1}^{s} y_i = g_i(w_i) \wedge P_0(\vec{y}, \vec{z}))$$

where g_i, p_0 are rud in P, for a sufficiently large h. But if P' is $\Sigma_0^{(n)}(M')$ in $\Pi'(P)$ by the same definition, we obviously have:

$$\begin{aligned} P'(\vec{w}, \vec{z}) &\leftrightarrow \bigvee y_1 \ldots y_r(\bigwedge_{i=1}^{s} y_i = \pi'(g)(w_i) \wedge P_0'(\vec{y}, \vec{z})) \\ &P_0'(\pi'(\vec{g})(\vec{w}), \vec{z}). \end{aligned}$$

QED (Lemma 2.7.13)

We can repeat the proof in Case 3 with "extra" arguments $\vec{u}^n$. Thus, after rearranging arguments we would have $R(\vec{u}^n, \vec{y}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ where $i_1, \ldots, i_r < n$. We would then define

$$P(\vec{u}^n, \vec{w}, \vec{z}) \leftrightarrow: R(\vec{u}^n, \vec{g}(\vec{w}), \vec{f}(\vec{z})).$$

This gives us:

Corollary 2.7.16. *Let $n < w$. Let $R(\vec{u}^n, x_1^{i_1}, \ldots, x_r^{i_r})$ be $\Sigma_0^{(n)}$ where $i_1, \ldots, i_p \leq n$. Let $f_l \in \Gamma_{i_l}^n$ for $l = 1, \ldots, r$. Set:*

$$P(\vec{u}^n, \vec{z}) \leftrightarrow: R(\vec{u}^n, f_1(z_1), \ldots, f_r(z_r)).$$

Then:

(a) $P(\vec{u}^n, \vec{z})$ is $\Sigma_0^{(n)}$ in a parameter p.

(b) Let $\pi' \supset \pi$ such that $\pi' : M \to_{\Sigma_0^{(n)}} M'$. Let R' be $\Sigma_0^{(n)}(M')$ by the same definition. Let P' be $\Sigma_0^{(n)}(M')$ in $\pi'(p)$ by the same definition. Then

$$P'(\vec{u}^n, \vec{z}) \leftrightarrow R'(\vec{u}^n, \pi'(f_1)(z_1), \ldots, \pi'(f_r)(z_r)).$$

By Corollary 2.7.15 $\langle M, \pi\rangle$ can have at most one $\Sigma_0^{(n)}$ liftup. But when does it have a liftup? In order to answer this — as before — define a term model $\mathbb{D} = \mathbb{D}^{(n)}$ for the supposed liftup, which will then exist whenever $\mathbb{D}$ is well founded.

Definition 2.7.8. Let M, τ, H, H', π be as above where $\rho_M^n \geq \tau, n \leq w$. The $\Sigma_0^{(n)}$ *term model* $\mathbb{D} = \mathbb{D}^{(n)}$ is defined as follows: (Let e.g. $M = \langle J_\alpha^A, B\rangle$.) We set: $\mathbb{D} = \langle D, \cong, \tilde{\in}, \tilde{A}, \tilde{B}\rangle$ where:

$$\begin{aligned} D = D^{(n)} =: \quad & \text{the set of pairs } \langle f, x\rangle \\ & \text{such that } f \in \Gamma^n(\tau, M) \text{ and} \\ & x \in \pi(\mathrm{dom}(f)) \end{aligned}$$

$\langle f, x\rangle \cong \langle g, y\rangle \leftrightarrow: \langle x, y\rangle \in \pi(e)$, where

$$e = \{\langle z, w\rangle | f(z) = g(w)\}.$$

$\langle f, x\rangle \tilde{\in} \langle g, y\rangle \leftrightarrow: \langle x, y\rangle \in \pi(e)$, where

$$e = \{\langle z, w\rangle | f(z) \in g(w)\}$$

(similarly for $\tilde{A}, \tilde{B}$).

We shall interpret the model $\mathbb{D}$ in a many sorted language with variables of type $i < \omega$ if $n = \omega$ and otherwise of type $i \leq n$. The variables v^i will range over the domain D_i defined by:

Definition 2.7.9. $D_i = D_i^{(n)} =: \{\langle f, x\rangle \in D | f \in \Gamma_i^n\}.$

Under this interpretation we obtain Łos theorem in the form:

Lemma 2.7.17. *Let $\varphi(v_1^{i_1},\dots,v_r^{i_r})$ be $\Sigma_0^{(n)}$. Then:*

$$\mathbb{D} \models \varphi[\langle f_1, x_1\rangle,\dots,\langle f_r, x_r\rangle] \leftrightarrow \langle x_1,\dots,x_r\rangle \in \pi(e)$$

where $e = \{\langle\vec{z}\rangle | M \models \varphi[f_1(z_1),\dots,f_r(z_r)]\}$ and $\langle f_l, x_l\rangle \in D_{i_l}$ for $l = 1,\dots,r$.

Proof: By induction on i we show:

Claim If $i < n$ or $i = n < \omega$, then the assertion holds for $\Sigma_0^{(i)}$ formulae.

Proof: Let it hold for $j < i$. We proceed by induction on the formula φ.

Case 1 φ is primitive (i.e. φ is $v_i \dot{\in} v_j, v_i \dot{=} v_j, \dot{A}v_i$ or $\dot{B}v_i$ (for $M = \langle J_\alpha^A, B\rangle$). This is immediate by the definition of $\mathbb{D}$.

Case 2 φ is $\Sigma_h^{(j)}$ where $j < i$ and $h = 0$ or 1. If $h = 0$ this is immediate by the induction hypothesis. Let $h = 1$. Then $\varphi = \bigvee u^j \Psi$, where Ψ is $\Sigma_0^{(i)}$. By bound relettering we can assume *w.l.o.g.* that u^i is not in our good sequence $v_1^{i_1},\dots,v_r^{i_r}$. We prove both directions, starting with $(\rightarrow)$:

Let $\mathbb{D} \models \varphi[\langle f_1, x_1\rangle,\dots,\langle f_r, x_r\rangle]$. Then there is $\langle g, y\rangle \in D_j$ such that

$$\mathbb{D} \models \Psi[\langle g, y\rangle, \langle f_1, x_1\rangle,\dots,\langle f_r, x_r\rangle]$$

($u^j, \vec{v}$ being the good sequence for Ψ). Set $e' = \{\langle w, \vec{z}\rangle | M \models \Psi[g(w), \vec{z}(\vec{x})]\}$. Then $\langle y, \vec{x}\rangle \in \pi(e')$ by the induction hypothesis on i. But in M we have:

$$\bigwedge w, \vec{z}(\langle w, \vec{z}\rangle \in e' \rightarrow \langle\vec{z}\rangle \in e).$$

This is a Π_1 statement about e', e. Since $\pi : H \rightarrow_{\Sigma_1} H'$ we can conclude:

$$\bigwedge w, \vec{z}(\langle w, \vec{z}\rangle \in \pi(e') \rightarrow \langle\vec{z} \in \pi(e)).$$

But $\langle y, \vec{x}\rangle \in \pi(e')$ by the induction hypothesis. Hence $\langle\vec{x} \in \pi(e)$. This proves $(\rightarrow)$. We now prove $(\leftarrow)$. Let $\langle\vec{x}\rangle \in \pi(e)$. Let R be the $\Sigma_0^{(j)}$ relation

$$R(w, z_1,\dots,z_r) \leftrightarrow= M \models \varphi[w, z_1,\dots,z_r].$$

Let G be a $\Sigma_0^{(j)}(M)$ map to H^j which uniformizes R. Then G is a spezialization of a function $G'(z_1^{h_1},\dots,z_r^{h_r})$ such that $h_l \le j$ for $l \le j$. Thus G' is a good $\Sigma_0^{(j)}$ function. But

$$f_l(z) = F_l(z, p) \text{ for } z \in \operatorname{dom}(f_l) \text{ for } l = 1,\dots,r$$

where F_l is a good $\Sigma_0^{(k)}$ map to H^{h_l} for $l = 1, \ldots, r$ and $j \leq k < i$. (We assume *w.l.o.g.* that the parameter p is the same for all $l = 1, \ldots, r_n$.) Define $G''(u^k, w)$ by:

$$G''(u, w) \simeq: G'((u)_0^{r-1}, \ldots, (u)_{r-1}^{r-1}, w)$$

then G'' is a good $\Sigma_1^{(k)}$ function. Define g by: $\mathrm{dom}(g) = \mathop{\times}\limits_{i=1}^{r} \mathrm{dom}(f_i)$ and: $g(\langle \vec{z} \rangle) = G''(\langle \vec{z} \rangle, p)$ for $\langle \vec{z} \rangle \in \mathrm{dom}(g)$. Then $g \in \Gamma^n$ and $g(\langle \vec{z} \rangle) = G(f_1(z_1), \ldots, f_r(z_r))$. Hence, letting:

$$e' = \{\langle w, \vec{z} \rangle | M \models \Psi[g(w), \vec{f}(\vec{z})]\},$$

we have:

$$\bigwedge \vec{z}(\langle \vec{z} \rangle \in e \leftrightarrow \langle \langle \vec{z} \rangle, \vec{z} \rangle \in e').$$

This is a Π_1 statement about e, e' in H. Hence in H' we have:

$$\bigwedge \vec{z}(\langle \vec{z} \rangle \in \pi(e) \leftrightarrow \langle \langle \vec{z} \rangle, \vec{z} \rangle \in \pi(e')).$$

But then $\langle \langle \vec{z} \rangle, \vec{z} \rangle \in \pi(e')$. By the induction hypothesis we conclude:

$$\mathbb{D} \models \Psi[\langle g, \langle \vec{z} \rangle \rangle, \langle f_1, x_1 \rangle, \ldots, \langle f_r, x_r \rangle].$$

Hence:

$$\mathbb{D} \models \varphi[\langle f_1, x_1 \rangle, \ldots, \langle f_r, x_r \rangle].$$

QED (Case 2)

Case 3 φ is $\Psi_0 \wedge \Psi_1, \Psi_0 \wedge \Psi_1, \Psi_0 \rightarrow \Psi_1, \Psi_0 \leftrightarrow \Psi_1$, or $\neg\Psi$.

This is straightforward and we leave it to the reader.

Case 4 $\varphi = \bigvee u^i \in v_l \chi$ or $\bigwedge u^i \in v_l \chi$, where v_l has type $\geq i$. We display the proof for the case $\varphi = \bigvee u^i \in v_l \chi$. We again assume *w.l.o.g.* that $u' \neq v_j$ for $j = 1, \ldots, r$. Set: $\Psi = (u^i \in v_l \wedge \chi)$. Then φ is equivalent to $\bigvee u^i \Psi$. Using the induction hypothesis for χ we easily get:

$$(*) \quad \begin{aligned} \mathbb{D} \models \quad & \Psi[\langle g, y \rangle, \langle f_1, x_i \rangle, \ldots, \langle f_r, x_r \rangle] \leftrightarrow \\ & \langle y, x_1, \ldots, x_n \rangle \in \pi(e') \end{aligned}$$

where $e' = \{\langle w, \vec{z} \rangle | M \models \Psi[g(w), \vec{f}(\vec{z})]\}$. Using $(*)$, we consider two subcases:

Case 4.1 $i < n$.
We simply repeat the proof in Case 2, using $(*)$ and with i in place of j.

Case 4.2 $i = n < w$.

(Hence v_l has type n.) For the direction $(\rightarrow)$ we can again repeat the proof in Case 2. For the other direction we essentially revert to the proof used initially for Σ_0 liftups.

We know that $e \in H$ and $\langle \vec{x} \rangle \in \pi(e)$, where $e = \{\langle \vec{z} \rangle | M \models \varphi[f_1(z_1), \ldots, f_r(z_r)]\}$. Set:

$$R(w^n, \vec{z}) \leftrightarrow: M \models \Psi[w^n, f_1(z_1), \ldots, f_r(z_r)].$$

Then R is $\underline{\Sigma}_0^{(n)}$ by Corollary 2.7.16. Moreover $\bigvee w^n R(w^n, \vec{z}) \leftrightarrow \langle \vec{z} \rangle \in e$. Clearly $f_l \in H_M^n$ since $f_l \in \Gamma_n^n$. Let $s \in H_M^n$ be a well odering of $\bigcup \mathrm{rng}(f_l)$. Clearly:

$$\begin{aligned} R(w^n, \vec{z}) &\rightarrow w^n \in f_l(z_l) \\ &\rightarrow w^n \in \bigcup \mathrm{rng}(f_l). \end{aligned}$$

We define a function g with domain e by:

$$g(\langle \vec{z} \rangle) = \text{the } s\text{–least } w \text{ such that } R(w, \vec{z}).$$

Since R is $\underline{\Sigma}_0^{(n)}$, it follows easily that $g \in H_{\rho^n}^M$. Hence $g \in \Gamma_n^n$. But then

$$\bigwedge \vec{z}(\langle \vec{z} \rangle \in e \leftrightarrow \langle \langle \vec{z} \rangle, \vec{z} \rangle \in e'), \text{ where } e' \text{ is defined as above, using this } g.$$

Hence in H' we have:

$$\bigwedge \vec{z}(\langle \vec{z} \rangle \in \pi(e) \leftrightarrow \langle \langle \vec{z} \rangle, \vec{z} \rangle \in \pi(e')).$$

Since $\langle \vec{x} \rangle \in \pi(e)$ we conclude that $\langle \langle \vec{x} \rangle, \vec{x} \rangle \in \pi(e')$. Hence:

$$\mathbb{D} \models \Psi[\langle g, \langle \vec{x} \rangle \rangle, \langle f_1, x_1 \rangle, \ldots, \langle f_r, x_r \rangle].$$

Hence:

$$\mathbb{D} \models \varphi[\langle f_1, x_1 \rangle, \ldots, \langle f_r, x_r \rangle].$$

QED (Lemma 2.7.17)

Exactly as before we get:

Lemma 2.7.18. *If $\tilde{\in}$ is ill founded, then the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi \rangle$ does not exist.*

We leave it to the reader and prove the converse:

Lemma 2.7.19. *If $\tilde{\in}$ is well founded, then the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi \rangle$ exists.*

Proof: We shall again use the term model $\mathbb{D}$ to define an explicit $\Sigma_0^{(n)}$ liftup. We again define:

Definition 2.7.10. $x^* = \pi^*(x) =: \langle \mathrm{const}_x, 0\rangle$, where $\mathrm{const}_x =: \{\langle x, 0\rangle\} =$ the constant function x defined on $\{0\}$.

Using Łos theorem Lemma 2.7.17 we get:

(1) $\pi^* : M \to_{\Sigma_0^{(n)}} \mathbb{D}$
(where the variables v^i range over D_i on the $\mathbb{D}$ side).

The proof is exactly like the corresponding proof for Σ_0–liftups ((1) in Lemma 2.7.5). In particular we have: $\pi^* : M \to_{\Sigma_0} \mathbb{D}$. Repeating the proof of (2) in Lemma 2.7.5 we get:

(2) $\mathbb{D} \models$ Extensionality.
Hence $\cong$ is again a congruence relation and we can factor $\mathbb{D}$, getting:

$$\hat{\mathbb{D}} = (\mathbb{D}\backslash \cong) = \langle \hat{D}, \hat{\in}, \hat{A}, \hat{B}\rangle$$

where

$$\hat{D} =: \{\hat{s} | s \in D\},\ \hat{s} =: \{t | t \cong s\} \text{ for } s \in D$$
$$\hat{s}\hat{\in}\hat{t} \leftrightarrow: s\tilde{\in}t$$
$$\hat{A}\hat{s} \leftrightarrow: \tilde{A}s,\ \hat{B}\hat{s} \leftrightarrow: \tilde{B}s$$

Then $\hat{\mathbb{D}}$ is a well founded identity model satisfying extensionality. By Mostowski's isomorphism theorem there is an isomorphism k of $\hat{\mathbb{D}}$ onto M', where $M' = \langle |M'|, \in, A', B'\rangle$ is transitive. Set:

$$[s] =: k(\hat{s}) \text{ for } s \in D$$
$$\pi'(x) =: [x^*] \text{ for } x \in M$$
$$H_i =: \{\hat{s} | s \in D_i\} (i < n \text{ or } i = n < \omega).$$

We shall *initially* interpret the variables v^i on the M' side as ranging over H_i. We call this the *pseudo interpretation.* Later we shall show that it (almost) coincides with the intended interpretation. By (1) we have

(3) $\pi' : M \to_{\Sigma_0^{(n)}} M'$ in the pseudo interpretation. (Hence $\pi' : M \to_{\Sigma_0^{(n)}} M'$.)

Lemma 2.7.19 then follows from:

Lemma 2.7.20. *$\langle M', \pi'\rangle$ is the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$.*

For $n = 0$ this was proven in Lemma 2.7.6, so assume $n > 0$. We again use the abbreviation:

$$[f, x] =: [\langle f, x \rangle] \text{ for } \langle f, x \rangle \in D.$$

Defining $\tilde{H}$ exactly as in the proof of Lemma 2.7.6, we can literally repeat our previous proofs to get:

(4) $\tilde{H}$ is transitive.

(5) $[f, x] = \pi(f)(x)$ if $f \in H$ and $\langle f, x \rangle \in D$. (Hence $\tilde{H} = H'$.)

(6) $\pi' \supset \pi$.

(However (7) in Lemma 2.7.6 will have to be proven later.)

In order to see that $\pi : M \to_{\Sigma^{(n)}} M'$ in the intended interpretation we must show that $H_i = H^i_M$, for $i < n$ and that $H_n \subset H^n_M$. As a first step we show:

(7) H_i is transitive for $i \leq n$.

Proof: Let $s \in H_i, t \in s$. Let $s = [f, x]$ where $f \in \Gamma^n_i$. We must show that $t = [g, y]$ for $g \in \Gamma^n_i$. Let $t = [g', y]$. Then $\langle y, x \rangle \in \pi(e)$ where

$$e = \{\langle u, v \rangle | g'(u) \in f(v)\}.$$

Set:

$$a =: \{u | g'(u) \in \text{rng}(f)\}, g = g' \restriction a.$$

Claim 1 $g \in \Gamma^n_i$.

Proof: $a \subset \text{dom}(q')$ is $\Sigma^{(n)}_0$. Hence $a \in H$ and $g \in \Gamma^n$. If $i < n$, then $\text{rng}(g) \subset \text{rng}(f) \subset H^i_M$. Hence $g \in \Gamma^n_i$. Now let $i = n$. Then $\text{rng}(f) \in \Gamma^n_n$ and the relation $z = g(y)$ is $\Sigma^{(n)}_0$. Hence $g \in H^n_M$.

QED (Claim 1)

Claim 2 $t = [g, y]$

Proof:

$$\bigwedge u, v(\langle u, v \rangle \in e \to \langle u, u \rangle \in e')$$

where $e' = \{\langle u, w \rangle | g(u) = g'(w)\}$. Hence the same Π_1 statement holds of $\pi(e), \pi(e')$ in H'. Hence $\langle y, y \rangle \in \pi(e')$. Hence $[g, y] = [g', y] = t$.

QED (7)

We can improve (3) to:

(8) Let $\Psi = \bigvee v^{i_1}_{v_1}, \ldots, v^{i_r}_{r} \varphi$, where φ is $\Sigma^{(n)}_0$ and $i_l < n$ or $i_l = n < \omega$ for $l = 1, \ldots, r$. Then π' is "Ψ–elementary" in the sense that:

$$M \models \Psi[\vec{x}] \leftrightarrow M' \models \Psi[\pi'(\vec{x})] \text{ in the pseudo interpretation.}$$

Proof: We first prove $(\to)$. Let $M \models \varphi[\vec{z}, \vec{x}]$. Then $M' \models \varphi[\pi'(\vec{z}), \pi'(\vec{x})]$ by (3).

We now prove $(\leftarrow)$. Let:

$$M' \models \varphi[[f_1, z_1], \ldots, [f_r, z_r], \pi'(\vec{x})]$$

where $f_l \in \Gamma^n_{i_l}$ for $l = 1, \ldots, r$. Since $\pi'(x) = [\text{const}_x, 0]$, we then have: $\langle z_1, \ldots, z_r, 0 \ldots 0\rangle \in \pi(e)$, where:

$$e = \{\langle u_1, \ldots, u_r, 0 \ldots 0\rangle : M \models \varphi[\vec{f}(\vec{u}), \vec{x}]\}.$$

Hence $e \neq \emptyset$. Hence

$$\bigvee v_1 \ldots v_r M \models \varphi[\vec{f}(\vec{v}), \vec{x}]$$

where $\text{rng}(f_l) \subset H^{i_l}$ for $l = 1, \ldots, r$. Hence $M \models \Psi[\vec{x}]$. QED (8)

If $i < n$, then every $\Pi^{(i)}_1$ formula is $\Sigma^{(n)}_0$. Hence by (8):

(9) If $i < n$ then

$$\pi' : M \to_{\Sigma^{(i)}_2} M' \text{ in the pseudo interpretation.}$$

We also get:

(10) Let $n < w$. Then:

$$\pi' \upharpoonright H^n_M : H^n_M \to_{\Sigma_0} H_n \text{ cofinally.}$$

Proof: Let $x \in H_n$. We must show that $x \in \pi'(a)$ for an $a \in H^n_M$. Let $x = [f, y]$, where $f \in \Gamma^n_n$. Let $d = \text{dom}(f), a = \text{rng}(f)$. Then $y \in \pi(d)$ and:

$$\bigwedge z \in d \ \langle z, 0\rangle \in e$$

where

$$\begin{aligned} e &= \{\langle u, v\rangle | f(u) \in \text{const}_a(v)\} \\ &= \{\langle u, 0\rangle | f(u) \in a\}. \end{aligned}$$

This is a Σ_0 statement about d, e. Hence the same statement holds of $\pi(d), \pi(e)$ in H_n. Hence $\langle z, 0\rangle \in \pi(e)$. Hence $[f, y] \in \pi'(a)$. QED (10)

(**Note:** (10) and (3) imply that $\pi' : M \to_{\Sigma^{(n)}_1} M'$ is the pseudo interpretation, but this also follows directly from (8).)

Letting $M = \langle J^A_\alpha, B\rangle$ and $M' = \langle |M'|, A', B'\rangle$ we define:

$$M_i = \langle H^i_M, A \cap H^i_M, B \cap H^i_M\rangle, M'_i = \langle H_i, A' \cap H_i, B' \cap H_i\rangle$$

for $i < n$ or $i = n < w$. Then each M_i is acceptable. It follows that:

(11) M'_i is acceptable.

Proof: If $i = n$, then $\pi' \restriction M_n : M_n \to_{\Sigma_0} M'_n$ cofinally by (3) and (10). Hence M'_n is acceptable by §5 Lemma 2.5.5. If $i < n$, then $\pi' \restriction M_i : M_i \to_{\Sigma_2^{(i)}} M'_i$ by (9). Hence M'_i is acceptable since acceptability is a Π_2 condition. QED (11)

We now examine the "correctness" of the pseudo interpretation. As a first step we show:

(12) Let $i + 1 \leq n$. Let $A \subset H_{i+1}$ be $\underline{\Sigma}_1^{(i)}$ in the pseudo interpretation. Then $\langle H_{i+1}, A\rangle$ is amenable.

Proof: Suppose not. Then there is $A' \subset H_{i+1}$ such that A' is $\underline{\Sigma}_1^{(i)}$ in the pseudo interpretation, but $\langle H_i, A'\rangle$ is not amenable. Let:

$$A'(x) \leftrightarrow B'(x, p)$$

where B' is $\Sigma_1^{(i)}$ in the pseudo interpretation. For $p \in M'$ we set:

$$A'_p =: \{x | B'(x, p)\}.$$

Let B be $\Sigma_1^{(i)}(M)$ by the same definition. For $p \in M$ we set:

$$A_p =: \{x | B(x, p)\}.$$

Case 1 $i + 1 < n$.

Then $\bigvee p \bigvee a^{i+1} \bigwedge b^{i+1} b^{i+1} \neq a^{l+1} \cap A'_p$ holds in the pseudo interpretation. This has the form: $\bigvee p \bigvee a^{i+1} \varphi(p, a^{i+1})$ where φ is $\Pi_1^{(i+1)}$, hence $\Sigma_0^{(n)}$ in the pseudo interpretation. By (8) we conclude that $M \models \varphi(p, a^{i+1})$ for some $p, a^{i+1} \in M$. Hence $\langle H_M^{i+1}, A_p\rangle$ is not amenable, where A_p is $\underline{\Sigma}_1^{(i)}(M)$.
Contradiction! QED (Case 1)

Case 2 Case 1 fails.

Then $i + 1 = n$. Since π' takes H_M^n cofinally to H_n. There must be $a \in H_M^n$ such that $\pi(a) \cap A' \notin H_n$. From this we derive a contradiction. Let $A' = A'_p$ where $p = [f, z]$. Set: $\tilde{B} = \{\langle z, w\rangle | B(w, f(z))\}$. Then $\tilde{B}$ is $\underline{\Sigma}_1^{(i)}(M)$. Set: $b = (d \times a) \cap \tilde{B}$, where $d = \mathrm{dom}(f)$. Then $b \in H_M^n$. Define $g : d \to H_M^n$ by:

$$g(z) =: A_{f(z)} \cap a = \{x \in a | \langle z, x\rangle \in b\}.$$

Then $g \in H_M^n$, since it is rudimentary in $a, b \in H_M^n$. Let $\varphi(u^n, v^n, w)$ be the $\Sigma_0^{(n)}$ statement expressing

$$u = A_w \cap v^n \text{ in } M.$$

Then setting:

$$e = \{\langle v, 0, w\rangle | M \models \varphi[g(v), a, f(z)]\}$$

we have:

$$\bigwedge v \in d \ \langle v, 0, v\rangle \in e.$$

But then the same holds of $\pi(d), \pi(e)$ in H_n. Hence $\langle z, 0, z\rangle \in \pi(e)$. Hence: $[g, z] = A_{[f,z]} \cap \pi(a) \in H_n$.
Contradiction! QED (12)

On the other hand we have:

(13) Let $i+1 < n$. Let $A \subset H_M^{i+1}$ be $\Sigma_1^{(i)}(M)$ in the parameter p such that $A \notin M$. Let A' be $\Sigma_1^{(i)}(M')$ in $\pi'(p)$ by the same $\Sigma_1^{(i)}(M')$ definition in the pseudo interpretation. Then $A' \cap H_{i+1} \notin M'$.

Proof: Suppose not. Then in M' we have:

$$\bigvee a \bigwedge v^{i+1}(v^{i+1} \in a \leftrightarrow A'(v^{i+1})).$$

This has the form $\bigvee a\varphi(a, \pi(p))$ where φ is $\Pi_1^{(i+1)}$ hence $\Sigma_0^{(n)}$. By (8) it then follows that $\bigvee a\varphi(a, p)$ holds in M. Hence $A \in M$.
Contradiction! QED (13)

Recall that for any acceptable $M = \langle J_\alpha^A, B\rangle$ we can define ρ_M^i, H_M^i by:

$$\begin{aligned} \rho^0 = \ & \alpha \\ \rho^{i+1} = \ & \text{the least } \rho \text{ such that there is } A \text{ which is} \\ & \underline{\Sigma}_1^{(i)}(M) \text{ with } A \cap \rho \notin M \\ H^i = \ & J_{\rho_i}[A]. \end{aligned}$$

Hence by (11), (12), (13) we can prove by induction on i that:

(14) Let $i < n$. Then

(a) $\rho_{M'}^i = \rho_i$, $H_{M'}^i = H_i$

(b) The pseudo interpretation is correct for formulae φ, all of whose variables are of type $\leq i$.

By (9) we then have:

(15) $\pi' : M \rightarrow_{\Sigma_2^{(i)}} M'$ for $i < n$.
This means that if $n = \omega$, then π' is automatically Σ^*–preserving. If $n < \omega$, however, it is not necessarily the case that $H_n = H_M^n$, — i.e. the pseudo interpretation is not always correct. By (12), however we do have:

(16) $\rho_n \leq \rho^n_M$, (hence $H_n \subset H^n_{M'}$).
Using this we shall prove that π' is $\Sigma_0^{(n)}$–preserving. As a preliminary we show:

(17) Let $n < w$. Let φ be a $\Sigma_0^{(n)}$ formula containing only variables of type $i \leq n$. Let $v_1^{i_1}, \ldots, v_r^{i_r}$ be a good sequence for φ. Let $x_1, \ldots, x_r \in M'$ such that $x_l \in H_{i_l}$ for $l = 1, \ldots, r$. Then $M \models \varphi[x_1, \ldots, x_r]$ holds in the correct sense iff it holds in the pseudo interpretation.

Proof: (sketch)
Let C_0 be the set of all such φ with: φ is $\Sigma_1^{(i)}$ for an $i < n$. Let C be the closure of C_0 under sentential operation and bounded quantifications of the form $\bigwedge v^n \in w^n \varphi$, $\bigvee v^n \in w^n \varphi$. The claim holds for $\varphi \in C_0$ by (15). We then show by induction on φ that it holds for $\varphi \in C$. In passing from φ to $\bigwedge v^n \in w^n \varphi$ we use the fact that w^n is interpreted by an element of H_n. QED (17)

Since $\pi''' H^i_M \subset H_i$ for $i \leq n$, we then conclude:

(18) $\pi' : M \to_{\Sigma_0^{(n)}} M'$.
It now remains only to show:

(19) $[f, x] = \pi'(f)(x)$.

Proof: Let $f(x) = G(x, p)$ for $x \in \mathrm{dom}(f)$, where G is $\Sigma_1^{(j)}$ good for a $j < n$. Let $a = \mathrm{dom}(f)$. Let $\Psi(u, v, w)$ be a good $\Sigma_1^{(j)}$ definition of G. Set:

$$e = \{\langle z, y, w\rangle | M \models \Psi[f(z), \mathrm{id}_a(y), \mathrm{const}_p(w)]\}.$$

Then $z \in a \to \langle z, z, 0\rangle \in e$. Hence the same holds of $\pi(a), \pi(e)$. But $x \in \pi(a)$. Hence:

$$M' \models \Psi[[f, x], [\mathrm{id}_a, x], [\mathrm{const}_p, x]],$$

where $[\mathrm{id}_a, x] = x, [\mathrm{const}_p, 0] = \pi'(p)$. Hence:

$$[f, x] = G'(x, \pi'(p)) = \pi'(f)(x),$$

where G' has the same $\Sigma_1^{(j)}$ definition. QED (19)

Lemma 2.7.20 is then immediate from (6), (18) and (19).
QED (Lemma 2.7.19)

As a corollary of the proof we have:

Lemma 2.7.21. *Let $\langle M', \pi'\rangle$ be the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$. Let $i < n$. Then* *(a)* $\pi' : M \to_{\Sigma_2^{(i)}} M'$

(b) *If* $\rho_M^i \in M$, *then* $\pi'(\rho_M^i) = \rho_{M'}^i$.

(c) *If* $\rho_M^i = \mathrm{On}_M$, *then* $\rho_{M'}^i = \mathrm{On}_{M'}$.

Proof:

(a) follows by (9) and (14).

(b) In M we have:

$$\bigwedge \xi^0 \bigvee \xi^i (\xi^0 < \rho_M^i \leftrightarrow \xi^0 = \xi^i).$$

This has the form $\bigwedge \xi^0 \Psi(\xi^0, \rho_M^i)$ where Ψ is $\Sigma_0^{(n)}$. But then the same holds of $\pi'(\rho_M^i)$ in M' by (8) and (14) — i.e.

$$\bigwedge \xi^0 \bigvee \xi^i (\xi^0 < \pi(\rho_M^i) \leftrightarrow \xi^0 = \xi^i).$$

(c) In M we have $\bigwedge \xi^0 \bigvee \xi^i \xi^0 = \xi^i$, hence the same holds in M' just as above.

QED (Lemma 2.7.21)

The *interpolation lemma* for $\Sigma_0^{(n)}$ liftups reads:

Lemma 2.7.22. *Let* $\sigma : H' \to_{\Sigma_0} |M^*|$ *and* $\pi^* : M \to_{\Sigma_0^{(n)}} M^*$ *such that* $\pi^* \supset \sigma\pi$. *Then the* $\Sigma_0^{(n)}$ *liftup* $\langle M', \pi' \rangle$ *of* $\langle M, \pi \rangle$ *exists. Moreover there is a unique map* $\sigma' : M' \to_{\Sigma_0^{(n)}} M^*$ *such that* $\sigma' \upharpoonright H' = \sigma$ *and* $\sigma'\pi' = \pi^*$.

Proof: $\tilde{\in}$ is well founded since:

$$\langle f, x \rangle \tilde{\in} \langle g, y \rangle \leftrightarrow \pi^*(f)(\sigma(x)) \in \pi^*(g)(\sigma(y)).$$

Thus $\langle M', \pi' \rangle$ exists. But for $\Sigma_0^{(n)}$ formulae $\varphi = \varphi(v_1^{i_1}, \ldots, v_r^{i_r})$ we have:

$$\begin{aligned}
&M' \models \varphi[\pi'(f_1)(x_1), \ldots, \pi'(f_r)v_r)] \\
&\leftrightarrow \langle x_1, \ldots, x_n \rangle \in \pi(e) \\
&\leftrightarrow \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in \sigma(\pi(e)) = \pi^*(e) \\
&\leftrightarrow M^* \models \varphi[\pi^*(f_1)(\sigma(x_1)), \ldots, \pi^*(f_r)(\sigma(x_r))]
\end{aligned}$$

where:

$$e = \{\langle x_1, \ldots, x_r \rangle | M \models \varphi[f_1(x_1), \ldots, f_r(x_r)]\}$$

and $\langle f_l, x_l\rangle \in \Gamma^n_{i_l}$ for $i = 1, \ldots, r$. Hence there is a $\Sigma_0^{(n)}$–preserving embedding $\sigma : M' \to M^*$ defined by:

$$\sigma'(\pi'(f)(x)) = \pi^*(f)(\sigma(x)) \text{ for } \langle f, x\rangle \in \Gamma^n.$$

Clearly $\sigma' \restriction H' = \sigma$ and $\sigma'\pi' = \pi^*$. But σ' is the unique such embedding, since if $\tilde{\sigma}$ were another one, we have

$$\tilde{\sigma}(\pi'(f)(x)) = \pi^*(f)(\sigma(x)) = \sigma'(\pi'(f)(x)).$$

QED (Lemma 2.7.22)

We can improve this result by making stronger assumptions on the map π, for instance:

Lemma 2.7.23. *Let $\langle M^*, \pi^*\rangle$ be the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi\rangle$. Let $\pi^* \restriction \rho_M^{n+1} = \mathrm{id}$ and $\mathbb{P}(\rho_M^{n+1}) \cap M^* \subset M$. Then $\rho^n_{M^*} = \sup \pi^{*''} \rho^n_M$.*

(Hence the pseudo interpretation is correct and π^* is $\Sigma_1^{(n)}$ preserving.)

Proof: Suppose not. Let $\tilde{\rho} = \sup \pi^{*''} \rho^n_M < \rho^n_{M^*}$. Set:

$$H^n = H^n_M = J^{A_M}_{\rho^n_M};\ \tilde{H} = J^{A_M}_{\tilde{\rho}}.$$

Then $\tilde{H} \in M^*$. Let A be $\Sigma_1^{(n)}(M)$ in p such that $A \cap \rho_M^{n+1} \notin M$. Let:

$$Ax \leftrightarrow \bigvee y^n B(y^n, x),$$

where B is $\Sigma_0^{(n)}$ in p. Let B^* be $\Sigma_0^{(n)}(M^*)$ in $\pi^*(p)$ by the same definition. Then

$$\pi^* \restriction H^n : \langle H^n, B \cap H^n\rangle \to_{\Sigma_1} \langle \tilde{H}, B^* \cap \tilde{H}\rangle.$$

Then $A \cap \rho_M^{n+1} = \tilde{A} \cap \rho_M^{n+1}$, where:

$$\tilde{A} = \{x | \bigvee y^n \in \tilde{H}\ B^*(y, x)\}.$$

But $\tilde{A}$ is $\Sigma_1^{(n)}(M^*)$ in $\pi^*(p)$ and $\tilde{H}$. Hence

$$A \cap \rho_M^{n+1} = \tilde{A} \cap \rho_M^{n+1} \in \mathbb{P}(\rho_M^{n+1}) \cap M^* \subset M.$$

Contradiction! QED (Lemma 2.7.23)

Chapter 3

Mice

3.1 Introduction

In this chapter we develop some of the tools needed to construct fine structural inner models which go beyond L. The concept of "mouse" is central to this endeavor. We begin with a historical introduction which traces the genesis of that notion. This history, and the concepts which it involves, are familiar to many students of set theory, but the thread may grow fainter as the history proceeds. If you, the present reader, find the introduction confusing, we advise you to skim over it lightly and proceed to the formal development in §3.2. The introduction should then make more sense later on.

Fine structure theory was originally developed as a tool for understanding the constructible hierarchy. It was used for instance in showing that $V = L$ implies $\square_\beta$ for all infinite cardinals β, and that every non weakly compact regular cardinal carries a Souslin tree. It was then used to prove the covering lemma for L, a result which pointed in a different direction. It says that, if there is no non trivial elementary embedding of L into itself, then every uncountable set of ordinals is contained in a constructible set having the same cardinality. This implies that if any $\alpha \geq \omega_2$ is regular in L, then its cofinality is the same as its cardinality. In particular, successors of singular cardinals are absolute in L. Any cardinal $\alpha \geq \omega_2$ which is regular in L remains regular in V. In general, the covering lemma says that despite possible local irregularities and cofinalities in L is retained in V.

If, however, L can be imbedded non trivially into itself, then the structure of cardinalities and cofinalities in L is virtually wiped out in V. There is then a countable object known as $0^{\#}$ which encodes complete information about the class L and a non trivial embedding of L. $0^{\#}$ has many concrete representations, one of the most common being a structure $L_\nu^U = \langle L_\nu[U], \in, U\rangle$, where ν is the successor of an inaccessible cardinal κ in L and U is a normal ultrafilter on $\mathbb{P}(\kappa) \cap L$. (Later, however, we shall find it more convenient to work with extenders than with ultrafilters.) This structure, call it M_0, is *iterable* , giving rise to iterates $M_i (i < \infty)$ and embedding $\pi_{ij} : M_i \to_{\Sigma_0} M_j$ $(i \leq j < \infty)$. The iteration points κ_i $(i < \infty)$ are called the *indiscernables* for L and form a closed proper class of ordinals. Each κ_c is inaccessible in L. Thus there are unboundedly many inaccessibles of L which become ω–cofinal cardinals in V. It can also be shown that all infinite successor cardinals in L are collapsed and become ω–cofinal in V. If we chose κ_0 minimally, then $M_0 = 0^{\#}$ is unique. We briefly sketch the argument for this, since it involves a principle which will be of great importance later on. By the minimal choice of κ_0 it can be shown that $h_{M_0}(\emptyset) = M_0$ (i.e. $\rho^1_{M_0} = \omega$ and $\emptyset \in P^1_{M_0}$). Now let $M_0' = L^{U_0'}_{\nu_0'}$ be another such structure. Iterate M_0, M_0' out to ω_1, getting iteration $\langle M_i | i \leq \omega_1\rangle$, $\langle M_i' | i \leq \omega_1\rangle$ with iteration points κ_i, κ_i'. Then $\kappa_{\omega_1} = \kappa'_{\omega_1} = \omega_1$. Moreover the sets:

$$C = \{\kappa_i | i < \omega_1\}, C' = \{\kappa_i' | i < \omega_1\}$$

are club in ω_1. Hence $C \cap C'$ is club in ω_1. But the ultrafilters $U_{\omega_1}, U'_{\omega_1}$ are uniquely determined by $C \cap C'$. Hence $M_{\omega_1} = M'_{\omega_1}$. But then:

$$M_0 \simeq h_{M_{\omega_1}}(\emptyset) = h_{M'_{\omega_1}}(\emptyset) \simeq M_0'.$$

Hence $M_0 = M_0'$. This *comparision iteration* of two iterable structures will play a huge role in later chapters of this book.

The first application of fine structure theory to an inner model which significantly differed from L was made by Solovay in the early 1970's. Solovay developed this fine structure of L^U (where U is a normal measure on $\mathbb{P}(\kappa) \cap L^U$). He showed that each level $M = J^U_\alpha$ had a viable fine structure, with $\rho^n_M, P^n_M, R^n_M (n < \omega)$ defined in the usual way, although M might be neither acceptable nor sound. If e.g. $\alpha > \kappa$ and $\rho^1_M < \kappa$ (a case which certainly occurs), the we clearly have $R^1_M = \emptyset$. However, M has a standard parameter $p = p_M \in P^1_M$ and if we transitivize $h_M(P)$, we get a structure $\overline{M} = J^{\overline{U}}_{\overline{\alpha}}$ which iterates up to M in κ many steps. $\overline{M}$ is then called the *core* of M. ($\overline{M}$ itself might still not be acceptable, since a proper initial segment of $\overline{M}$ might not be sound.) (If $n < 1$ and $\rho^n_M < \kappa$, we can do essentially the same analysis, but when iterating $\overline{M}$ to M we must use $\Sigma_0^{(n)}$–preserving ultrapowers, as defined in the next section.)

Dodd and Jensen then turned Solovay's analysis on its head by defining a *mouse (or Solovay mouse)* to be (roughly) any J_α or iterable structure of the form $M = J_\alpha^U$ where U is a normal measure at some κ on M and $\rho_M^\omega \leq \kappa$. They then defined the *core model* K to be the union of all Solvay mice. They showed that, if there is no non trivial elementary embedding of K into K, then the covering lemma for K holds. If, on the other hand, there is such an embedding π with critical point κ, then U is a normal measure on κ in $L^U = \langle L[u], \in, u\rangle$, where:

$$U = \{x \in \mathbb{P}(\kappa) \cap K | \kappa \in \pi(X)\},$$

(This showed, in contrast to the prevailing ideology, that an inner model with a measurable cardinal can indeed be "reached from below".) The simplest Solovay mouse is $0^\#$ as described above. What K is depends on what there is. If $0^\#$ does not exist, then $K = L$. If $0^\#$ exists but $0^{\#\#}$ does not, then $K = L(0^\#)$ etc. In order to define the general notion of Solovay mouse, one must employ the full paraphanalia of fine stucture theory.

Thus we have reached the situation that fine structure theory is needed not only to analyze a previously defined inner model, but to define the model itself.

If we have reached L^U with U a normal ultrafilter on κ and $\tau = \kappa^+$ in L^U, then we can regard L_τ^U as the "next mouse" and continue the process. If $(L_\tau^\kappa)^\#$ does not exist, however, this will mean that L^U is the core model. The full covering lemma will then not necessarily hold, since V could contain a Prikry sequence for κ.

However, we still get the *weak covering lemma*:

$$cf(\beta) = \text{card}(\beta) \text{ if } \beta \geq \omega_2 \text{ is a cardinal in } K.$$

We also have *generic absoluteness*:

The definition of K is absolute
in every set generic extension of V.

In the ensuring period a host of "core model constructions" were discovered. For instance the "core model below two measurables" defined a unique model with the above properties under the assumption that there is no inner model with two measurable cardinals. Similarly with the "core model up to a measurable limit of measurables" etc. Initially this work was pursued by Dodd and Jensen, on the one hand, and by Bill Mitchell on the other. Mitchell got further, introducing several important innovations. He divided the construction of K into two stages: In the first he constructed an inner model K^C, which may lack the two properties stated above. He then "extracted" K from K^C, in the process defining an elementary embedding of K into K^C. This approach has been basic to everything done since. Mitchell also introduced the concept of *extenders*, having realized that the normal ultrafilters alone could not code the embeddings involved in constructing K.

There are many possible concrete representations of mice, but in general a mouse is regarded as a structure $M = J_\nu^E$ where E describes an indexed sequence of ultrafilters or extenders. A major requirement is that M be *iterable* , which entails that any of the indexed extenders or ultrafilters can be employed in the iteration. But this would seem to imply that eny F lying on the indeved sequence must be *total* — i.e. an ultrafilter or extender on the whole of $\mathbb{P}(\kappa) \cap M$ (κ being the critical point). Unfortunately the most natural representations of mice involve "allowing extenders (or ultrafilters) to die". Letting $M = J_\nu^U$ be the representation of $0^\#$ described above, it is known that $\rho_M^1 = \omega$. Hence $J_{\nu+1}^U$ contains new subsets of κ which are not "measured" by the ultrafilter U. The natural representation of $0^{\#\#}$ would be $M' = J_{\nu'}^{U,U'}$ where:

$$U' = \{X | \kappa' \in \pi(x)\},$$

and π is an embedding of L^U into itself with critical point $\kappa' > \kappa$. But U is not total. How can one iterate such a structure? Because of this conundrum, researchers for many years followed Solovay's lead in allowing only total ultrafilters and extenders to be indexed in a mouse. Thus Solovay's representation of $0^{\#\#}$ was $J_{\nu'}^{U'}$ This structre is not acceptable, however, since there is a $\gamma < \nu'$ set. $\kappa' < \gamma$ and $\rho_{J_\gamma^U}^1 = \omega < \kappa'$. Such representation of mice were unnatural and unwieldy. The conundrum was finally resolved by Mitchell and Stewart Baldwin, who observed that the structures in which extenders are "allowed to die" are in fact, iterable in a very good sense. We shall deal with this in §3.4. All of the innovations mentioned here were then incorporated into [MS] and [CMI]. They where also employed in [MS] and [NFS].

It was originally hoped that one could define the core model below virtually any large cardinal — i.e. on the assumption that no inner model with the cardinal exists one could define a unique inner model K satisfying weak covering and generic absoluteness. It was then noticed, however, that if we assume the existence of a Woodin cardinal, then the existence of a definable K with the above properties is provably false. (This is because Woodin's "stationary tower" forcing would enable us to change the successor of ω_ω while retaining ω_ω as a singular cardinal. Hence, by the covering lemma, K would have to change.) This precludes e.g. the existence of a core model below "an inaccessible above a Woodin", but it does not preclude constructing a core model below one Woodin cardinal. That is, in fact, the main theorem of this book: Assuming that no inner model with a Woodin cardinal exists, we define K with the above two properties.

In 1990 John Steel made an enormous stride toward achieving this goal by proving the following theorem: Let κ be a measurable cardinal. Assume that V_κ has no inner model with a Woodin cardinal. Then there is V–definable inner model K of V_κ which, relativized to V_κ, has he above two properties. This result, which was exposited in [CMI] was an enormous breakthrough, which laid the foundation for all that has been done in inner model theory since then. There remained, however, the pesky problem of doing without the measurable — i.e. constructing K and proving its properties assuming only "ZFC+ there is no inner model with a Woodin". The first step was to construct the model K^C from this assumption. This was almost achieved by Mitchell and Schindler in 2001, except that they needed the additional hypothesis: GCH. Steel then showed that this hypothesis was superfluous. These results were obtained by directly weakening the "background condition" originally used by Steel in constructing K^C. The result of Mitchell and Schindler were published in [UEM]. Independently, Jensen found a construction of K^C using a different background condition called "robustness". This is exposited in [RE]. There reamained the problem of extracting a core model K from K^C. Jensen and Steel finally achieved this result in 2007. It was exposited in [JS].

In the next section we deal with the notion of *extenders*, which is essential to the rest of the book. (We shall, however, deal only with so called "short extenders", whose length is less than or equal to the image of the critical point.)

3.2 Extenders

The *extender* is a generalization of the normal ultrafilter. A normal ultrafilter at κ can be described by a two valued function on $\mathbb{P}(\kappa)$. An extender, on the other hand, is characterized by a map of $\mathbb{P}(\kappa)$ to $\mathbb{P}(\lambda)$, where $\lambda > \kappa$. λ is then called the *length* of the extender. Like a normal ultrafilter an extender F induces a canonical elementary embedding of the universe V into an inner model W. We express this in symbols by: $\pi : V \to_F W$. W is then called the *ultrapower* of V by F and π is called the *canonical embedding* induced by F. The pair $\langle W, \pi \rangle$ is called the *extension* of V by F. We will always have: $\lambda \leq \pi(\kappa)$. However, just as with ultrafilters, we shall also want to apply extenders to transitive models M which may be smaller than V. F might then not be an element of M. Moreover $\mathbb{P}(\kappa)$ might not be a subset of M, in which case F is defined on the smaller set $U = \mathbb{P}(\kappa) \cap M$. Thus we must generalize the notion of extenders, countenancing "suitable" subsets of $\mathbb{P}(\kappa)$ as extender domains. (However, the ultrapower of M by F may not exist.)

We first define:

Definition 3.2.1. S is a *base for* κ iff S is transitive and $\langle S, \in\rangle$ models:

$$\mathsf{ZFC}^- + \ \kappa \text{ is the largest cardinal.}$$

By a *suitable* subset of $\mathbb{P}(\kappa)$ we mean $\mathbb{P}(\kappa) \cap S$, where S is a base for κ.

We note:

Lemma 3.2.1. *Let S be a base for κ. Then S is uniquely determined by $\mathbb{P}(\kappa) \cap S$.*

Proof: For $a, e \in \mathbb{P}(\kappa) \cap S$ set:

$u(a,e) \simeq$: that transitive u such that
$\langle u, \in\rangle$ is isomorphic to $\langle a, \tilde{e}\rangle$,
where $\tilde{e} = \{\langle \nu, \tau\rangle | \prec \nu, \tau \succ \in e\}$.

Claim $S =$ the union of all $u(a,e)$ such that $a, e \in \mathbb{P}(\kappa) \cap S$ and $u(a,e)$ is defined.

Proof: To prove $(\subset)$, note that if $u \in S$ is transitive, then there exist $\alpha \leq \kappa, f \in S$ such that $f : \alpha \leftrightarrow u$. Hence $u = u(\alpha, e)$ where $e = \{\prec \nu, \tau \succ | f(\nu) \in f(\tau)\}$. Conversely, if $u = u(a,e)$ and $a, e \in \mathbb{P}(\kappa) \cap S$, then $u \in S$, since the isomorphism can be constructed in S. QED (Lemma 3.2.1)

Definition 3.2.2. An ordinal λ is called *Gödel closed* iff it is closed under Gödel's pair function $\prec, \succ$ as defined in §2.4. (It follows that λ is closed under Gödel n–tuples $\prec x_1, \ldots, x_n \succ$.)

We now define

Definition 3.2.3. Let S be a base for κ. Let λ be Gödel closed. F is an *extender at κ with length λ, base S and domain* $\mathbb{P}(\kappa) \cap S$ iff the following hold:

- F is a function defined on $\mathbb{P}(\kappa) \cap S$
- There exists a pair $\langle S', \pi\rangle$ such that

  (a) $\pi : S \prec S'$ where S' is transitive

  (b) $\kappa = \mathrm{crit}(\pi), \pi(\kappa) \geq \lambda > \kappa$

(c) Every element of S' has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f \in S$ is a function defined on κ.

(d) $F(X) = \pi(X) \cap \lambda$ for $X \in \mathbb{P}(\kappa) \cap S$.

Note. If F is an extender at κ, then κ is its critical point in the sense that $F \upharpoonright \kappa = \mathrm{id}$, $F(\kappa)$ is defined, and $\kappa < F(\kappa)$. Thus we set: $\mathrm{crit}(F) =: \kappa$.

Note. (c) can be equivalenly replaced by:

$$\pi : S \prec S' \text{ cofinally.}$$

We leave this to the reader.

Note. $\mathbb{P}(\kappa) \cap S \subset S'$ since $X = \pi(X) \cap \kappa \in S'$. But the proof of Lemma 3.2.1 then shows that $S \subset S'$. (We leave this to the reader.)

Note. As an immediate consequence of this definition we get a form of *Łos Theorem* for the base:

$$\begin{aligned} S' \models &\varphi[\pi(f_1)(\alpha_1), \ldots, (f_n)(\alpha_n)] \leftrightarrow \\ &\prec \vec{\alpha} \succ \in F(\{\langle \vec{\xi} \rangle | S \models \varphi[f_1(\xi_1), \ldots, f_n(\xi_n)]\}) \end{aligned}$$

where $\alpha_1, \ldots, \alpha_n < \lambda$ and $f_i \in S$ is a function defined on κ for $i = 1, \ldots, n$.

Note. $\langle S', \pi \rangle$ is uniquely determined by F since if $\langle \tilde{S}, \tilde{\pi} \rangle$ were a second such pair, we would have:

$$\begin{aligned} \pi(f)(\alpha) \in \pi(g)(\beta) &\leftrightarrow \prec \alpha, \beta \succ \in F(\{\prec \xi, \delta \succ | f(\xi) \in g(\xi)\}) \\ &\leftrightarrow \tilde{\pi}(f)(\alpha) \in \tilde{\pi}(g)(\beta). \end{aligned}$$

Thus there is an isomorphism $i : S' \tilde{\leftrightarrow} \tilde{S}$ defined by $i(\pi(f)(\alpha)) = \tilde{\pi}(f)(\alpha)$. Since $S', \tilde{S}$ are transitive, we conclude that $i = id, S' = \tilde{S}$.

But then we can define:

Definition 3.2.4. Let S, F, S', π be as above. We call $\langle S', \pi \rangle$ the *extension of S by F* (in symbols: $\pi : S \to_F S'$).

Note. It is easily seen that:

- S' is a base for $\pi(\kappa)$
- The embedding $\pi : S \to S'$ is cofinal (since $\pi(f)(\alpha) \in \pi(\mathrm{rng}(f))$).

Note. The concept of extender was first introduced by Bill Mitchell. He regarded it as a sequence of ultrafilters (or *measures*) $\langle F_\alpha | \alpha < \lambda \rangle$, where $F_\alpha = \{X | \alpha \in F(X)\}$. For this reason he called it a *hypermeasure*. We shall retain this name and call $\langle F_\alpha | \alpha < \lambda \rangle$ the *hypermeasure representation of F*. We can recover F by: $F(X) = \{\alpha | X \in F_\alpha\}$.

Definition 3.2.5. We call an extender F on κ with base S and extension $\langle S', \pi\rangle$ *full* iff $\pi(\kappa)$ is the length of F.

In later sections we shall work almost entirely with full extenders. We leave it to the reader to show that if S is a ZFC^- model with largest cardinal κ and $\pi : S \prec S'$ cofinally. Then $\pi \restriction \mathbb{P}(\kappa)$ is a full extender with base S and extension $\langle S', \pi\rangle$.

Lemma 3.2.2. *Let F be an extender with base S and extension $\langle S', \pi\rangle$. Then:*

- *(a) $\langle S', \pi\rangle$ is amenable*
- *(b) If F is full, then $\langle S', F\rangle$ is amenable.*
- *(c) If φ is Σ_0, then $\{\langle\vec{x}\rangle : \langle S, \pi\rangle \models \varphi[\vec{x}]\}$ is uniformly $\Sigma_1(\langle S, F\rangle)$ in $x_1, \ldots, x_n$.*

Proof: (b) follows from (a), since then:

$$F \cap u = \{\langle Y, X\rangle \in \pi \cap u | X \subset \kappa \wedge Y \subset \lambda\}.$$

We prove (a). Since π takes S to S' cofinally, it suffices to show: $\pi \cap \pi(u) \in S'$ for $u \in S$. We can assume w.l.o.g. that u is transitive and non empty. If $\langle \pi(X), X\rangle \in \pi \cap \pi(u)$, then $\pi(X) \in \pi(u)$ by transitivity, hence $X \in u$. Thus $\pi \cap \pi(u) = (\pi \restriction u) \cap \pi(u)$ and it suffices to show:

Claim $\pi \restriction u \in S'$.
Let $f = \langle f(i) | i < \kappa\rangle$ enumerate u. Then $\pi \restriction u = \{\langle \pi(f)(i), f(i)\rangle | i < \kappa\}$.

This proves (a). We now prove (c). It suffices to show:
Claim. ($\nu \neq \varnothing$ is transitive and $y = \pi \restriction \nu$) is uniformly $\Sigma_1(\langle S, F\rangle)$ in ν, y, since then $\langle S, \pi\rangle \models \varphi[\vec{x}]$ is expressed by:

$$\bigvee w \bigvee u (u, w \text{ are transitive } \wedge \vec{x} \in u \wedge \pi \restriction u \subset w \wedge \langle w, \pi \restriction u\rangle) \models \varphi[\vec{x}]$$

We prove the Claim. Let $u \neq \varnothing$ be transitive. Then:

$$y = \pi \restriction u \iff \bigvee f (f : k \longrightarrow u \wedge y = \{\langle \pi(f)(i), f(i)\rangle : i < \kappa\}.)$$

$\{\kappa\}, \{\pi(\kappa)\}$ are uniformly $\Sigma_1(\langle S, F\rangle)$, since

$$\langle \pi(\kappa), \kappa\rangle = \text{ the unique } \prec \beta, \alpha \succ \in F.$$

Hence it suffices to show that $\{\pi(f)\}$ is uniformly $\Sigma_1(\langle S, F\rangle)$ in f. Let:

$$X = \{\prec j, i \succ \in \kappa : f(i) \in f(j)\}.$$

Then f is the unique function g such that

$$\mathrm{dom}(g) = \kappa \ \wedge \ g(j) = \{g(i) : \prec j, i \succ \in X\} \text{ for } i < \kappa.$$

Since $F(X) = \pi(X)$ we conclude that $\pi(f)$ is the unique function g such that

$$\mathrm{dom}(g) = \pi(\kappa) \wedge g(j) = \{g(i) : \prec j, i \succ \in F(X)\} \text{ for } i < \pi(\kappa).$$

The conclusion is immediate. QED (Lemma 3.2.2)

Definition 3.2.6. Let F be an extender at κ with base S, length λ, and extension $\langle S', \pi\rangle$. The *expansion* of F is the function F^* on $\bigcup\limits_{n<\omega} \mathbb{P}(\kappa^n) \cap S$ defined by:

$$F^*(X) = \pi(X) \cap \lambda^n \text{ for } X \in \mathbb{P}(\kappa^n) \cap S.$$

We also expand the hypermeasure by setting:

$$F^*_{\alpha_1, \ldots, \alpha_n} = \{X | \langle \vec{\alpha}\rangle \in F^*(X)\}$$

for $\alpha_1, \ldots, \alpha_n < \lambda$. By an abuse of notation we shall usually not distinguish between F and F^*, writing $F(X)$ for $F^*(X)$ and $F_{\vec{\alpha}}$ for $F^*_{\vec{\alpha}}$.

Using this notation we get another version of Łos Lemma:

$$\begin{aligned} S' \models \ & \varphi[\pi(f_1)(\vec{\alpha}), \ldots, \pi(f_n)(\vec{\alpha})] \leftrightarrow \\ & \{\langle \vec{\xi}\rangle | S \models \varphi[f_1(\vec{\xi}), \ldots, f_n(\vec{\xi})]\} \in F_{\vec{\alpha}} \end{aligned}$$

for $\alpha_1, \ldots, \alpha_m < \lambda$ and $f_i \in M$ a function with domain k^m for $i = 1, \ldots, n$.

Note. Most authors permit extenders to have length which are not Gödel closed. We chose not to for a very technical reason: If λ is not Gödel closed, the expanded extender F^* is not necessarily determined by $F = F^* \restriction \mathbb{P}(\kappa)$.

Hence if we drop the requirement of Gödel completeness, we must work with expanded extenders from the beginning. We shall, in fact, have little reason to consider extenders whose length is not Gödel closed, but for the sake of completeness we give the general definition:

Definition 3.2.7. Let S be a base for κ. Let $\lambda > \kappa$. F is an *expanded extender at* κ *with base* S*, length* λ, and *extension* $\langle S', \pi\rangle$ iff the following hold:

- F is a function defined on $\bigcup_{n<\omega}\mathbb{P}(\kappa^n)\cap S$
- $\pi : S \prec S'$ where S' is transitive
- $\kappa = \text{crit}(\pi), \pi(\kappa) \geq \lambda$
- Every element of S' has the form $\pi(f)(\alpha_1,\ldots,\alpha_n)$ where $\alpha_1,\ldots,\alpha_n < \lambda$ and $f \in S$ is a function defined on κ^n
- $F(X) = \pi(X)\cap\kappa^n$ for $X \in \mathbb{P}(\kappa^n)\cap S$.

This makes sense for any $\lambda > \kappa$. If, indeed, λ is Gödel closed and F is an extender of length λ as defined previously, then F^* is the unique expanded extender with $F = F^*\upharpoonright\mathbb{P}(\kappa)$.

Definition 3.2.8. Let F be an extender at κ of length λ with base S and extension $\langle S',\pi\rangle$. $X \subset \lambda$ is a set of *generators* for F iff every $\beta < \lambda$ has the form $\beta = \pi(f)(\vec{\alpha})$ where $\alpha_1,\ldots,\alpha_n \in X$ and $f \in S$.

If X is a set of generators, then every $x \in S'$ will have the form $\pi(f)(\vec{\alpha})$ where $\alpha_1,\ldots,\alpha_n \in X$ and $f \in S$. Thus only the generators are relevant. In some cases $\{\kappa\}$ will be a set of generators. (This will happen for instance if λ is the first admissible above κ or if $\lambda = \kappa+1$ and F is the expanded extender.) This means that every element of S' has the form $\pi(f)(\kappa)$ and that:

$$S' \models \varphi[\pi(\vec{f})(\kappa)] \leftrightarrow \{\xi | S \models \varphi[\vec{f}(\xi)]\} \in F_\kappa.$$

Thus, in this case, S' is the ultrapower of S by the normal ultrafilter F_κ.

In §2.7 we used a "term model" construction to analyze the conditions under which the liftup of a given embedding exists. This construction emulated the well known construction of the ultrapower by a normal ultrafilter. We could use a similar construction to determine wheter a given F is, in fact, an extender with base S — i.e. whether the extension $\langle S',\pi\rangle$ by F exists. However, the only existence theorem for extenders which we shall actually need is:

Lemma 3.2.3. *Let S be a base for κ. Let $\pi^* : S \prec S^*$ such that $\kappa = \text{crit}(\pi^*)$ and $\kappa < \lambda \leq \pi^*(\kappa)$ where λ is Gödel closed. Set*

$$F(X) =: \pi^*(X)\cap\lambda \textit{ for } X \in \mathbb{P}(\kappa)\cap S.$$

Then

(a) F is an extender of length λ.

(b) *Let $\langle S', \pi \rangle$ be the extension by F. Then there is a unique $\sigma : S' \prec S^*$ such that $\sigma\pi = \pi^*$ and $\pi \restriction \lambda = \mathrm{id}$.*

Proof: We first prove (a). Let Z be the set of $\pi^*(f)(\alpha)$ such that $\alpha < \lambda$ and $f \in S$ is a function on κ.

(1) $Z \prec S^*$

Proof: Let $S^* \models \bigvee v\varphi[\vec{x}]$ where $x_1, \ldots, x_n \in Z$. We must show:

Claim $Vy \in ZS^* \models \varphi[y, \vec{x}]$.
We know that there are functions $f_i \in S$ and $\alpha_i < X$ such that $x_i = \pi^*(f_i)(\alpha_i)$ for $i = 1, \ldots, n$. By replacement there is a $g \in S$ such that $\mathrm{dom}(g) = \kappa$ and in S:

$$\bigwedge_{\xi_1 \ldots \xi_n} < \kappa \quad (\bigvee y\varphi(y, f_1(\xi_1), \ldots, f_n(\xi_n)) \rightarrow \\ \varphi(g(\prec \xi_1, \ldots, \xi_n \succ, f_1(\xi_1), \ldots, f_n(\xi_n)))).$$

But then the corresponding statement holds of $\pi^*(\kappa), \pi^*(g), \pi^*(f_1), \ldots, \pi^*(f_n)$ in S^*. Hence, setting $\beta = \prec \alpha_1, \ldots, \alpha_n \succ$ we have:

$$S^* \models \varphi[\pi^*(g)(\beta), \pi^*(f_1)(\alpha_1), \ldots, \pi^*(f_n)(\alpha_n)].$$

QED (1)

Now let $\sigma : S' \overset{\sim}{\leftrightarrow} Z$ where S' is transitive. Set: $\pi = \sigma^{-1}\pi^*$. Then $S \prec S'$. $\sigma : S' \prec S^*$, and $\sigma(\pi(f)(\alpha)) = \pi^*(f)(\alpha)$ for $\alpha < \lambda$. It follows easily that F is an extender and $\langle S', \pi \rangle$ is the extension by F.

This proves (a). It also proves the existence part of (b), since $\sigma \restriction \lambda = \mathrm{id}$ and $\sigma\pi = \pi^*$. But if σ' also has the properties, then $\sigma'(\pi(f)(\alpha)) = \pi^*(f)(\alpha) = \sigma(\pi(f)(\alpha))$. Then $\sigma' = \sigma$ and σ is unique. QED (Lemma 3.2.3)

Definition 3.2.9. Let F be an extender at κ with extension $\langle S', \pi \rangle$. Let $\kappa < \lambda \leq \pi(\kappa)$ where λ is Gödel closed. $F|\lambda$ is the function F' defined by: $\mathrm{dom}(F') = \mathrm{dom}(F)$ and

$$F'(X) = \pi(X) \cap \lambda \text{ for } X \in \mathrm{dom}(F).$$

It follows immediately from Lemma 3.2.3 that $F|\lambda$ is an extender at κ with length λ.

The main use of an extender F with base S is to embed a larger model M with $\mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap S \in M$ into another transitive model M', which we then call the *ultrapower of M by F*. Ther is a wide class of models to which F can be so applied, but we shall confine ourselves to J–models.

Definition 3.2.10. Let M be a J–model. F is an *extender at κ on M* iff F is an extender with base S and $\mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap S \in M$, where κ is the largest cardinal in S. (In other words $S = H_\tau^M \in M$ where $\tau = \kappa^+$.)

Making use of the notion of liftups developed in §2.7.1 we define:

Definition 3.2.11. Let F be an extender at κ on M. Let $H = H_\tau^M$ be the base of F and let $\langle H', \pi' \rangle$ be the extension of H by F. We call $\langle N, \pi \rangle$ the *extension of M by F* (in symbols $\pi : M \to_F N$) iff $\langle N, \pi \rangle$ is the liftup of $\langle M, \pi' \rangle$.

We then call N the *ultrapower of M by F*. We call π the *canonical embedding* given by F.

Note. that π is Σ_0 preserving but not necessarily elementary.

Lemma 3.2.4. *Let F be an extender at κ on M of length λ. Let $\langle N, \pi \rangle$ be the extension of M by F. Then every element of N has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f \in M$ is a function with domain κ.*

Proof: Let $H = H_\tau^M$ and let $\langle H', \pi' \rangle$ be the extension of H by F, where $\tau = \kappa^{+M}$. Each $x \in N$ has the form $x = \pi(f)(z)$, where $f \in M$ is a function, $\mathrm{dom}(f) \in H$ and $z \in \pi(\mathrm{dom}(f))$. But then $z = \pi(g)(\alpha)$ where $\alpha < \lambda, g \in H$ and $\mathrm{dom}(g) = \kappa$. We may assume w.l.o.g. that $\mathrm{rng}(g) \subset \mathrm{dom}(f)$. (Otherwise redefine g slightly.) Thus $x = \pi(f \circ g)(\alpha)$. QED (Lemma 3.2.4)

Using the expanded extenders we then get Łos Theorem in the form:

Lemma 3.2.5. *Let M, F, λ, N, π be as above. Let $\alpha_1, \dots, \alpha_n < \lambda$ and let $f_i \in M$ be such that $f_i : \kappa^m \to M$ for $i = 1, \dots, n$. Let φ be Σ_0. Then*

$$N \models \varphi[\pi(\vec{f}(\vec{\alpha})] \leftrightarrow \{\langle \vec{\xi} \rangle | M \models \varphi[\vec{f}(\vec{\xi})]\} \in F_{\vec{\alpha}}.$$

Proof: As in §2.7.1 we set:

$$\begin{aligned} \Gamma^0 = \quad & \Gamma^0(\tau, M) = \text{ the set of } f \in M \text{ such that} \\ & f \text{ is a function and } \mathrm{dom}(f) \in H_\tau^M. \end{aligned}$$

Then $f_i \in \Gamma^0, \mathrm{dom}(f_i) = \kappa^m$. By Łos Theorem for liftups we get:

$$N \models \varphi[\pi(\vec{f})(\vec{\alpha})] \leftrightarrow \langle \vec{\alpha} \rangle \in \pi(e) \cap \lambda^m = F(e)$$

where

$$e = \{\langle \vec{\xi} \rangle | M \models \varphi[\vec{f}(\vec{\xi})]\}.$$

QED (Lemma 3.2.5)

The following lemma is often useful:

Lemma 3.2.6. *Let F, κ, M, π be as above. Let τ be regular in M such that $\tau \neq \kappa$. Then $\pi(\tau) = \sup \pi''\tau$.*

Proof: If $\tau < \kappa$ this is trivial. Now let $\tau > \kappa$. Let $\xi = \pi(f)(\alpha) < \pi(\tau)$, where $\alpha < \lambda$. Set $\beta = \sup f''\kappa$. Then $\beta < \tau$ by regularity. Hence:

$$\xi = \pi(f)(\alpha) \leq \sup \pi(f)''\pi(\kappa) = \pi(\beta) < \pi(\tau).$$

QED (Lemma 3.2.6)

3.2.1 Extendability

Definition 3.2.12. Let F be an extender at κ on M. M is *extendible* by F iff the extension $\langle N, \pi \rangle$ of M by F exists.

Note. This requires that N be a transitive model.

$\langle N, \pi \rangle$, if it exists, is the liftup of $\langle M, \pi' \rangle$ where $H = H_\tau^M, \tau = \kappa^{+M}$ and $\langle H', \pi' \rangle$ is the extension of its base H by F. In §2.7.1 we formed a term model $\mathbb{D}$ in order to investigate when this liftup exists. The points of $\mathbb{D}$ consisted of pairs $\langle f, z \rangle$ where

$$f \in \Gamma^0(\tau, M) := \text{ the set of functions } f \in M \text{ such that } \operatorname{dom}(f) \in H.$$

The equality and set membership relation were defined by

$$\langle f, z \rangle \simeq \langle g, w \rangle \leftrightarrow: \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$
$$\langle f, z \rangle \tilde{\in} \langle g, w \rangle \leftrightarrow: \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$

Now set:

Definition 3.2.13. $\Gamma_*^0 = \Gamma_*^0(\kappa, M) =: \{f \in \Gamma^0 | \operatorname{dom}(f) = \kappa\}$.

Set $\mathbb{D}^* = \mathbb{D}^*(\kappa, M) =:$ the restriction of $\mathbb{D}$ to terms $\langle t, \alpha \rangle$ such that $t \in \Gamma_*^0$ and $\alpha < \lambda$. The proof of Lemma 3.2.4 implicitly contains a barely disguised proof that:

$$\bigwedge x \in \mathbb{D} \bigvee y \in \mathbb{D}^* x \simeq y.$$

The set membership relation of $\mathbb{D}^*$ is:

$$\langle f, \alpha \rangle \in^* \langle g, \beta \rangle \leftrightarrow \prec \alpha, \beta \succ \in \pi'(\{\xi, \zeta\} | f(\xi) \in g(\zeta)\}).$$

In §2.7.1. we used the term model to show that the liftup $\langle N, \pi \rangle$ exists if and only if $\tilde{\in}$ is well founded. In this case $\mathbb{D}^*$ contains all the points of interest, so we may conclude:

Lemma 3.2.7. *M is extendible iff $\in^*$ is well founded.*

Note. In the future, when dealing with extenders, we shall often fail to distinguish notationally between $\Gamma^0_*, \mathbb{D}^*, \in^*$ and $\Gamma^0, \mathbb{D}, \tilde{\in}$.

Using this principle we develop a further criterion of extendability. We define:

Definition 3.2.14. Let $\overline{F}$ be an extender on $\overline{M}$ at $\overline{\kappa}$ of length $\overline{\lambda}$. Let F be an extender on M at κ of length λ.

$$\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$$

means:

(a) $\pi : \overline{M} \to_{\Sigma_0} M$ and $\pi(\overline{\kappa}) = \kappa$

(b) $g : \overline{\lambda} \to \lambda$

(c) Let $\overline{X} \subset \overline{\kappa}$, $\pi(\overline{X}) = X$, $\alpha_1, \ldots, \alpha_n < \overline{\lambda}$. Let $\beta_i = g(\alpha_i)$ for $i = 1, \ldots, n$. Then

$$\prec \vec{\alpha} \succ \in \overline{F}(\overline{X}) \leftrightarrow \prec \vec{\beta} \succ \in F(X).$$

Lemma 3.2.8. *Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$, where M is extendible by F. Then $\overline{M}$ is extendible by $\overline{F}$. Moreover, if $\langle N, \sigma\rangle$, $\langle \overline{N}, \overline{\sigma}\rangle$ are the extensions of M, N respectively, then there is a unique π' such that*

$$\pi' : \overline{N} \to_{\Sigma_0} N,\ \pi'\overline{\sigma} = \sigma\pi,\ \text{and}\ \pi' \restriction \overline{\lambda} = g.$$

π' is defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$$

for $f \in \Gamma^0$ and $\alpha < \overline{\lambda}$.

Proof: We first show that $\overline{M}$ is extendible by $\overline{F}$. Let $\sigma : M \to_F N$. The relation $\tilde{\in}$ on the term model $\overline{\mathbb{D}} = \mathbb{D}(\overline{\kappa}, \overline{M})$ is well founded, since:

$$\begin{aligned} \langle f, \alpha\rangle \tilde{\in} \langle h, \beta\rangle &\leftrightarrow \prec \alpha, \beta \succ \in \overline{F}(\{\prec \xi, \zeta \succ | f(\xi) \in h(\zeta)\}) \\ &\leftrightarrow \prec g(\alpha), g(\beta) \succ \in F(\{\prec \xi, \zeta \succ | \pi(f)(\xi) \in \pi(h)(\zeta)\}) \\ &\leftrightarrow \sigma\pi(f)(g(\alpha)) \in \sigma\pi(h)(g(\beta)) \end{aligned}$$

Now let $\overline{\sigma} : \overline{M} \to \overline{N}$. Let φ be a Σ_0 formula.

Then:

$$\begin{aligned} \overline{N} &\models \varphi[\overline{\sigma}(f_1)(\alpha_1), \ldots, \overline{\sigma}(f_n)(\alpha_n)] \\ &\leftrightarrow \langle \vec{\alpha}\rangle \in \overline{F}(\{\langle \vec{\xi}\rangle | \overline{M} \models \varphi[\vec{f}(\vec{\xi})]\}) \\ &\leftrightarrow \langle g(\vec{\alpha})\rangle \in F(\{\vec{\xi} | M \models \varphi[\pi(\vec{f})(\vec{\xi})]\}) \\ &\leftrightarrow N \models \varphi[\sigma\pi(f_1)(g(\alpha_1)), \ldots, \sigma\pi(f_n)(g(\alpha_n))]. \end{aligned}$$

Hence there is $\pi' : \overline{N} \to_{\Sigma_0} N$ defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha)).$$

But any π' fulfilling the above conditions will satisfy this definition.

QED (Lemma 3.2.8)

3.2.2 Fine Structural Extensions

These lemmas show that N is the ultrapower of M in the usual sense. However, the canonical embedding can only be shown to be Σ_0–preserving. If, however, M is acceptable and $\kappa < \rho_M^n$, the methods of §2.7.8 suggest another type of ultrapower with a $\Sigma_0^{(n)}$–preserving map. We define:

Definition 3.2.15. Let M be acceptable. Let F be an extender at κ on M. Let $H = H_\tau^M$ be the base of F and let $\langle H', \pi' \rangle$ be the extension of H by F. Let $\rho_M^n > \kappa$ (hence $\rho_M^n \geq \tau$). We call $\langle N, \pi \rangle$ the $\Sigma_0^{(n)}$–*extension of M by F* (in symbols: $\pi : M \to_F^{(n)} N$) iff $\langle N, \pi \rangle$ is the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi' \rangle$.

The extension we originally defined is then the Σ_0 ultrapower (or $\Sigma_0^{(0)}$ ultrapower). The $\Sigma_0^{(n)}$ analogues of Lemma 3.2.4 and Lemma 3.2.5 are obtained by a virtual repetition of our proofs, which we leave to the reader.

Letting $\Gamma^n = \Gamma^n(\tau, M)$ be defined as in §2.7.2 we get the analogue of Lemma 3.2.4.

Lemma 3.2.9. *Let F be an extender at κ on M of length λ. Let $\rho_M^n > \kappa$ and let $\langle N, \pi \rangle$ be the $\Sigma_0^{(n)}$ extension of M by F. Then every element of N has the form $\pi(f)(\alpha)$ where $\alpha < \lambda$ and $f \in \Gamma^n$ such that* $\mathrm{dom}(f) = \kappa$.

Lemma 3.2.10. *Let M, F, λ, N, π be as above. Let $\alpha_1, \ldots, \alpha_m < \lambda$ and let $f_i \in \Gamma^n$ such that* $\mathrm{dom}(f_i) = \kappa^m$ *for $i = 1, \ldots, p$. Let φ be a $\Sigma_0^{(n)}$ formula. Then:*

$$N \models \varphi[\pi(\vec{f})(\vec{\alpha})] \leftrightarrow \{\langle \vec{\xi} \rangle | M \models \varphi[\vec{f}(\vec{\xi})]\} \in F_{\vec{\alpha}}.$$

Note. We remind the reader that an element f of Γ^n is not, in general, an element of M. The meaning of $\pi(f)$ is explained in §2.7.2.

Using Lemma 2.7.22 we get:

Lemma 3.2.11. *Let $\pi^* : M \to_{\Sigma_0^{(n)}} M^*$ where* $\kappa = \mathrm{crit}(\pi^*)$ *and $\pi^*(\kappa) \geq \lambda$, where λ is Gödel closed. Assume: $\mathbb{P}(\kappa) \cap M \in M$. Set:*

$$F(X) =: \pi^*(X) \cap \lambda \textit{ for } X \in \mathbb{P}(\kappa) \cap M.$$

Then:

(a) *F is an extender at κ of length λ on M.*

(b) *The $\Sigma_0^{(n)}$ extension $\langle M', \pi\rangle$ of M by F exists.*

(c) *There is a unique $\sigma : M' \to_{\Sigma_0^{(n)}} M^*$ such that $\sigma' \restriction \lambda = \mathrm{id}$ and $\sigma\pi = \pi^*$.*

Proof: Let $H = H_\tau^M$, $H^* = \pi^*(H)$. Then H is a base for κ and $\pi^* \restriction H : H \prec H^*$. Hence by Lemma 3.2.3 F is an extender at κ with base H and extension $\langle H', \pi'\rangle$. Moreover, there is a unique $\sigma' : H' \prec H^*$ such that $\sigma' \restriction \lambda = \mathrm{id}$ and $\sigma'\pi' = \pi^* \restriction H$. But by Lemma 2.7.22 the $\Sigma_0^{(n)}$ liftup $\langle M', \pi\rangle$ of $\langle M, \pi'\rangle$ exists. Moreover, there is a unique $\sigma : M' \to_{\Sigma_0^{(n)}} M^*$ such that $\sigma \restriction H' = \sigma'$ and $\sigma\pi' = \pi^*$. In particular, $\sigma \restriction \lambda = \mathrm{id}$. But σ is then unique with these properties, since if $\tilde{\sigma}$ had them, we would have:

$$\tilde{\sigma}(\pi(f)(\alpha)) = \pi^*(f)(\alpha) = \sigma(\pi(f)(\alpha))$$

for $f \in \Gamma^n, \mathrm{dom}(f) = \kappa, \alpha < \lambda$. QED (Lemma 3.2.11)

By Lemma 2.7.21 we get:

Lemma 3.2.12. *Let $\pi : M \longrightarrow_F^{(n)} N$. Let $i < n$. Then:*

(a) *π is $\Sigma_2^{(i)}$ preserving.*

(b) *$\pi(\rho_M^i) = \rho_{M'}^i$ if $\rho_M^i \in M$.*

(c) *$\rho_{M'}^i = \mathrm{On} \cap M'$ if $\rho_M^i = \mathrm{On} \cap M$.*

The following definition expresses an important property of extenders:

Definition 3.2.16. Let F be an extender at κ of length λ with base S. F is *weakly amenable* iff whenever $X \in \mathbb{P}(\kappa^2) \cap S$, then $\{\nu < \kappa | \langle \nu, \alpha\rangle \in F(X)\} \in S$ for $\alpha < \lambda$.

Lemma 3.2.13. *Let F be an extender at κ with base S and extension $\langle S', \pi\rangle$. Then F is weakly amenable iff $\mathbb{P}(\kappa) \cap S' \subset S$.*

Proof:

($\rightarrow$) Let $Y \in \mathbb{P}(\kappa) \cap S'$, $Y = \pi(f)(\alpha), \alpha < \lambda$. Set $X = \{\langle \nu, \xi\rangle \in \kappa^2 | \nu \in f(\xi)\}$. Then $\pi(f)(\alpha) = \{\nu < \kappa | \langle \nu, \alpha\rangle \in F(X)\} \in S$, since $F(X) = \pi(X) \cap \lambda$.

($\leftarrow$) Let $X \in \mathbb{P}(\kappa^2) \cap S$, $\alpha < \lambda$. Then $\{\nu < \kappa | \langle \nu, \alpha\rangle \in \pi(X)\} \in \mathbb{P}(\kappa) \cap S' \subset S$.

QED (Lemma 3.2.13)

Corollary 3.2.14. *Let M be acceptable. Let F be a weakly amenable extender at κ on M. Let $\langle N, \pi\rangle$ be the $\Sigma_0^{(n)}$ extension of M by F. Then $\mathbb{P}(\kappa) \cap N \subset M$.*

Proof: Let $H = H_\tau^M$, $\tilde{H} = \bigcup_{u \in H} \pi(u)$, $\tilde{\pi} = \pi \upharpoonright H$. Then H is the base for F and $\langle \tilde{H}, \tilde{\pi}\rangle$ is the extension of H by F. Hence $\mathbb{P}(\kappa) \cap \tilde{H} \subset H \subset M$. Hence it suffices to show:

Claim $\mathbb{P}(\kappa) \cap N \subset \tilde{H}$.

Proof: Since $\pi(\kappa) > \kappa$ is a cardinal in N and N is acceptable, we have:

$$\mathbb{P}(\kappa) \cap N \subset H^N_{\pi(\kappa)} = \pi(H^M_\kappa) \in \tilde{H}.$$

QED (Corollary 3.2.14)

Corollary 3.2.15. *Let M, F, N, π be as above. Then κ is inaccessible in M (hence in N by Corollary 3.2.14).*

Proof:

(1) κ is regular in M.
Proof: If not there is $f \in M$ mapping a $\gamma < \kappa$ cofinally to κ. But then $\pi(f)$ maps γ cofinally to $\pi(\kappa)$. But $\pi(f)(\xi) = \pi(f(\xi)) = f(\xi) < \kappa$ for $\xi < \gamma$. Hence $\sup\{\pi(f)(\xi)|\xi < \gamma\} \subset \kappa$. Contradiction!

(2) $\kappa \neq \gamma^+$ in M for $\gamma < \kappa$.
Proof: Suppose not. Then $\pi(\kappa) = \gamma^+$ in N where $\pi(\kappa) > \kappa$. Hence $\overline{\overline{\kappa}} = \gamma$ in N and N has a new subset of κ. Contradiction!

QED (Corollary 3.2.15)

By Corollary 3.2.14 and Lemma 2.7.23 we get:

Lemma 3.2.16. *Let $\pi : M \to_F^{(n)} N$ where F is weakly amenable. Let n be maximal such that $\rho^n_M > \kappa$. Then $\rho^n_N = \sup \pi\text{"}\rho^n_M$. (Hence π is $\Sigma_1^{(n)}$ preserving.)*

With further conditions on F and n we can considerably improve this result. We define:

Definition 3.2.17. Let F be an extender at κ on M of length λ. F is *close to M* if F is weakly amenable and F_α is $\underline{\Sigma}_1(M)$ for all $\alpha < \lambda$.

This very important notion is due to John Steel. Using it we get the following remarkable result:

Theorem 3.2.17. *Let M be acceptable. Let F be an extender at κ on M which is close to M. Let $n \leq \omega$ be maximal such that $\rho^n > \kappa$ in M. Let $\langle N, \pi\rangle$ be the $\Sigma_0^{(n)}$ extension of M by F. Then π is Σ^* preserving.*

Proof: If $n = \omega$ this is immediate, so let $n < \omega$. Then $\rho^{n+1} \subseteq \kappa < \rho^n$ in M. By the previous lemma π is Σ_1–preserving. Hence $\pi(\kappa)$ is regular in N. Set: $H = H_\kappa^M$. Then $H = H_\kappa^N$ by Corollary 3.2.14.

(1) Let $D \subset H$ be $\underline{\Sigma}_1^{(n)}(N)$. Then D is $\underline{\Sigma}_1^{(n)}(M)$.
Proof: Let:

$$D(z) \leftrightarrow \bigvee x^n D'(x^n, z, \pi(f)(\alpha))$$

where $\alpha < \lambda$, $f \in \Gamma^n$ such that $\mathrm{dom}(f) = \kappa$, and D' is $\Sigma_0^{(n)}$. Then by Lemma 3.2.16:

$$\begin{aligned} D(z) &\leftrightarrow \bigvee u \in H_M^n \bigvee x \in \pi(u) D'(x, z, \pi(f)(\alpha)) \\ &\leftrightarrow \bigvee u \in H_M^n \alpha \in \pi(e) \\ &\leftrightarrow \bigvee u \in H_M^n e \in F_\alpha \end{aligned}$$

where $e = \{\xi | \bigvee x \in u \overline{D}(x, z, f(\xi))\}$ where $\overline{D}$ is $\Sigma_0^{(n)}(M)$ by the same definition as D' over N. QED (1)

By induction on $m > n$ we then prove:

(2) (a) $H_M^m = H_N^m$
(b) $\underline{\Sigma}_1^{(m)}(M) \cap \mathbb{P}(H) = \underline{\Sigma}_1^{(m)}(N) \cap \mathbb{P}(H)$
(c) π is $\Sigma_1^{(m)}$–preserving.

Proof:

Case 1 $m = n + 1$

(a) Let $M = \langle J_\alpha^A, B\rangle$, $N = \langle J_{\alpha'}^{A'}, B'\rangle$. Then: $H = J_\kappa^A = J_\kappa^{A'}$. But

$$\mathbb{P}(\rho) \cap M = \mathbb{P}(\rho) \cap N = \mathbb{P}(\rho) \cap H \text{ for } \rho \leq \kappa.$$

But then in M and N we have:

$$\rho^m = \text{ the least } \rho < \kappa \text{ such that } D \cap J^A_\rho \notin H \text{ for } D \in \underline{\Sigma}^{(n)}_1$$
$$\text{and } H^m = J^A_{\rho^m}.$$

Hence $\rho^m_M = \rho^m_N, H^m_M = H^m_N$. QED (a)

(c) Let $\overline{A}(\vec{x}^m, x_{i_1}, \ldots, x_{i_p})$ be $\Sigma^{(m)}_1(M)$, where $i_1, \ldots, i_p \leq n$. Let A be $\Sigma^{(m)}_1(N)$ by the same definition. Then there are $\Sigma^{(m)}_1(M)$ relations $\overline{B}^j(\vec{x}^m, \vec{x})(j = 1, \ldots, q)$ and a Σ_1 formula φ such that

$$\overline{A}(\vec{x}^m, \vec{x}) \leftrightarrow \overline{H}^m_{\vec{x}} \models \varphi[\vec{x}^m]$$

where $\overline{H}^m_{\vec{x}} = \langle H^m, \overline{B}^1_{\vec{x}}, \ldots, \overline{B}^q_{\vec{x}} \rangle$ and

$$\overline{B}^j_{\vec{x}} = \{\langle \vec{x}^{\,m} \rangle | \overline{B}^j(\vec{z}^{\,m}, \vec{x})\}(j = 1, \ldots, q).$$

Let $B^j(z^m, \vec{x})$ have the same $\Sigma^{(n)}_1$ definition over N. Define $H^m_{\vec{x}}$ the same way, using $B^1, \ldots, B^q$ in place of $\overline{B}^1, \ldots, \overline{B}^q$. Then

$$A(\vec{x}^{\,m}, \vec{x}) \leftrightarrow H^m_{\vec{x}} \models \varphi[\vec{x}^m].$$

But $H^m_M = H^m_N$. Hence, since π is $\Sigma^{(n)}_1$ preserving, we have: $\overline{B}^j_{\vec{x}} = B^j_{\pi(\vec{x})}$. Hence $\overline{H}^m_{\vec{x}} = H^m_{\pi(\vec{x})}$. But then:

$$\begin{aligned} \overline{A}(\vec{x}^m, \vec{x}) \ &\leftrightarrow \overline{H}^m_{\vec{x}} \models \varphi[\vec{x}^m] \\ &\leftrightarrow H^m_{\pi(\vec{x})} \models \varphi[\vec{x}^m] \\ &\leftrightarrow A(\vec{x}^m, \pi(\vec{x})) \\ &\leftrightarrow A(\pi(\vec{x}^m), \pi(\vec{x})) \end{aligned}$$

since $\pi(\vec{x}^m) = \vec{x}^m$. QED (c)

(b) The direction $\subset$ follows straightforwardly from (c). We prove the direction $\supset$. Let $A(\vec{x}^m, x_{i_1}, \cdots, x_{i_r})$ be $\underline{\Sigma}^{(m)}_1(N)$ such that $A \subset H$. Then there are B^j $(j = 1, \ldots, q)$ such that B^j is $\underline{\Sigma}^{(n)}_1(N)$ and

$$A_{\vec{x}}(x^n) \leftrightarrow H^n_{\vec{x}} \models \varphi[\vec{x}, s]$$

where $s \in H^m$ and φ is a Σ_1 formula and $H^m_{\vec{x}} = \langle H^m, B^1_{\vec{x}}, \ldots, B^q_{\vec{x}} \rangle$. By (1) there are $\overline{B}^j$ $(j = 1, \ldots, q)$ such that $\overline{B}^j$ is $\underline{\Sigma}^{(n)}_1(M)$ and $\overline{B}^j_{\vec{x}} = B^j_{\vec{x}}$ whenever $x_{i_1}, \ldots, x_{i_r} \in H$. The conclusion is immediate.

QED (Case 1)

Case 2 $m = h + 1$ where $h > n$.
This is virtually identical to Case 1 except that we use:

$$\underset{\sim}{\Sigma}_1^{(h)} \cap \mathbb{P}(H_M^h) = \underset{\sim}{\Sigma}_1^{(h)} \cap \mathbb{P}(H_N^h)$$

in place of (1). QED (Theorem 3.2.17)

Theorem 3.2.17 justifies us in defining:

Definition 3.2.18. Let F be an extender at κ on M. Let $n \leq \omega$ be maximal such that $\rho_M^m > \kappa$. We call $\langle N, \pi \rangle$ the Σ^**–extension of* M *by* F (in symbols $\pi : M \to_F^* N$) iff F is close to M and $\langle N, \pi \rangle$ is the $\Sigma_0^{(n)}$ extension by F.

As a corollary of the proof of Lemma 3.2.16 we have:

Corollary 3.2.18. *Let* $\pi : M \longrightarrow_F^* N$*. Let* $H = H_\kappa^M$ *and* $\rho_M^{n+1} \leq \kappa$*. Then:*

- $H = H_\kappa^N$
- $M \cap \mathbb{P}(H) = N \cap \mathbb{P}(H)$.
- $\Sigma_1^{(n)}(M) \cap \mathbb{P}(H) = \Sigma_1^{(n)}(N) \cap \mathbb{P}(H)$.
- $H_M^{n+1} = H_N^{n+1}$.

3.2.3 n–extendibility

Definition 3.2.19. Let F be an extender of length λ at κ on M. M is *n–extendible* by F iff $\kappa < \rho_M^n$ and the $\Sigma_0^{(n)}$ extension $\langle N, \pi \rangle$ of M by F exists.

$\langle N, \pi \rangle$, if it exists, is the $\Sigma_0^{(n)}$ liftup of $\langle M, \pi' \rangle$ where $H = H_\tau^M$ is the base of F, $\tau = \kappa^{+M}$, and $\langle M', \pi' \rangle$ is the extension of H by F. To analyse this situation we use the term model $\mathbb{D} = \mathbb{D}^{(n)}(\pi', M)$ defined in §2.7.2. The points of $\mathbb{D}$ are pairs $\langle f, z \rangle$ such that $f \in \Gamma^n = \Gamma^n(\tau, M)$ as defined in §2.7.2. and $z \in \pi'(\mathrm{dom}(f))$. The equality and set membership relation of $\mathbb{D}$ are again defined by:

$$\langle f, z \rangle \simeq \langle g, w \rangle \leftrightarrow \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$
$$\langle f, z \rangle \tilde{\in} \langle g, w \rangle \leftrightarrow \langle z, w \rangle \in \pi'(\{\langle x, y \rangle | f(x) = g(y)\})$$

Set: $\Gamma_*^n = \Gamma_*^n(\kappa, M) =$: the set of $f \in \Gamma^n$ such that $\mathrm{dom}(f) = \kappa$. Let $\mathbb{D}_* = \mathbb{D}_*^{(n)}(F, M)$ be the restriction of $\mathbb{D}$ to points $\langle f, d \rangle$ such that $f \in \Gamma_*^n$ and $\alpha < \lambda$. The proof of Lemma 3.2.7 tells us that

$$\bigwedge x \in \mathbb{D} \bigvee y \in \mathbb{D}_* x \simeq y.$$

Hence M is $\Sigma_0^{(n)}$ extendable iff the restriction $\in^*$ of the relation $\tilde{\in}$ to $\mathbb{D}_*$ is well founded.

We have:

$$\langle f, \alpha \rangle \in^* \langle g, \beta \rangle \leftrightarrow \langle \alpha, \beta \rangle \in F(\{\langle \xi, \zeta \rangle | f(\xi) \in g(\zeta)\}).$$

Note. When dealing with extenders, we shall again sometimes fail to distinguish notationally between $\Gamma_*^n, \mathbb{D}_*^{(n)}, \in^*$ and $\Gamma^n, \mathbb{D}^{(n)}, \tilde{\in}$.

We now prove:

Lemma 3.2.19. *Let $\langle \pi, g \rangle : \langle \overline{M}, \overline{F} \rangle \to \langle M, F \rangle$, where M is m–extendible by F. Let $n \leq m$ and let π be $\Sigma_0^{(n)}$ preserving with $\overline{\kappa} < \rho^m$ in $\overline{M}$, where $\overline{\kappa} = \text{crit}(\overline{F})$. Then $\overline{M}$ is n–extendible by $\overline{F}$. Moreover, if $\langle N, \sigma \rangle$ is the $\Sigma_0^{(m)}$ extension of M by F and $\langle \overline{N}, \overline{\sigma} \rangle$ is the $\Sigma_0^{(n)}$ extension of $\overline{M}$ by F, then there is a unique π' such that*

$$\pi' : \overline{N} \to_{\Sigma_0^{(n)}} N, \pi'\overline{\sigma} = \sigma\overline{N}, \pi' \upharpoonright \overline{\lambda} = g.$$

π' is defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha))$$

for $f \in \Gamma_*^n(\overline{\kappa}, \overline{M}), \alpha < \overline{\beta}$.

Proof: Let $\in^*$ be the set membership relation of $\overline{\mathbb{D}}_* = \overline{\mathbb{D}}_*(\overline{F}, \overline{M})$.

Then:

$$\begin{aligned} \langle f, \alpha \rangle \in^* \langle h, \beta \rangle \quad & \leftrightarrow \langle \alpha, \beta \rangle \in \overline{F}(\{\langle \xi, \zeta \rangle | f(\xi) \in g(\zeta)\}) \\ & \leftrightarrow \langle g(\alpha), g(\beta) \rangle \in F(\{\langle \xi, \zeta \rangle | \pi(f)(\xi) \in \pi(h(\zeta)\}) \\ & \leftrightarrow \sigma\pi(f)(\alpha) \in \sigma\pi(f)(\beta). \end{aligned}$$

Hence there is $\pi' : \overline{N} \to_{\Sigma_0^{(n)}} N$ defined by:

$$\pi'(\overline{\sigma}(f)(\alpha)) = \sigma\pi(f)(g(\alpha)).$$

But any π' fulfilling the above conditions satisfies this definition.

QED (Lemma 3.2.19)

Taking π, g as id, we get:

Corollary 3.2.20. *Let M be $\Sigma_0^{(m)}$ extendible by F. Let $n \leq m$. Then M is $\Sigma_0^{(n)}$ extendible by F. Moreover, if $\sigma : M \to_F^{(m)} N$ and $\overline{\sigma} : M \to_F^{(m)} \overline{N}$, there is $\pi : \overline{N} \to_{\Sigma_0^{(n)}} N$ defined by:*

$$\pi(\overline{\sigma}(f)(\alpha) = \sigma(f)(\alpha) \text{ for } f \in \Gamma^n, \alpha < \lambda.$$

Lemma 3.2.19 is normally applied to the case $n = m$. The condition $\overline{\kappa} < \rho^n_{\overline{M}}$ will be satisfied if the map π is *strictly* $\Sigma^{(n)}_0$-preserving. However, it does not follows that π' is strictly $\Sigma^{(n)}_0$-preserving. Similarly, even if we assume that π is fully $\Sigma^{(n)}_1$-preserving, we get no corresponding strengthening of π'. We can remedy this situation by strengthening our basic premiss:

$$\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \longrightarrow \langle M, F\rangle$$

We define:

Definition 3.2.20. $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to^* \langle M, F\rangle$ iff the following hold:

- $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$
- $\overline{F}, F$ are weakly amenable
- Let $\alpha < \overline{\lambda} =$ length $(\overline{F})$. Then $\overline{F}_\alpha$ is $\Sigma_1(\overline{M})$ in a parameter $\overline{p}$ and $F_{g(\alpha)}$ is $\Sigma_1(M)$ in $p = \pi(\overline{p})$ by the same definition.

(Hence $\overline{F}$ is close to $\overline{M}$.) Taking $n = m$ in Lemma 3.2.19 we prove:

Lemma 3.2.21. *Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to^* \langle M, F\rangle$. Let $\sigma : M \to^{(n)}_F N$ where π is $\Sigma^{(n)}_1$ preserving. Let $\overline{\sigma} : \overline{M} \to^{(n)}_{\overline{F}} \overline{N}$, $\pi' : \overline{N} \to N$ be given by Lemma 3.2.19. Then π' is $\Sigma^{(n)}_1$ preserving.*

We derive this from a stronger lemma:

Lemma 3.2.22. *Let $\langle \pi, g\rangle : \langle \overline{M}, \overline{F}\rangle \to^* \langle M, F\rangle$. Let $n, \overline{N}, N, \pi'$ be as above, where π is $\Sigma^{(n)}_1$ preserving. Let $\overline{D}(y, x_1, \ldots, x_r)$ be $\Sigma^{(n)}_1(\overline{N})$ and $D(\vec{y}, x_1, \ldots, x_r)$ be $\Sigma^{(n)}_1(N)$ by the same definition. Let $\pi'(\overline{x}_i) = x_i (i = 1, \ldots, r)$. Then*

$$\{\langle \vec{y}\rangle \in H^{\overline{M}}_{\overline{\kappa}} | D(\vec{y}, \overline{x}_1, \ldots, \overline{x}_r)\}$$

is $\Sigma^{(n)}_1(\overline{M})$ in a parameter $\overline{p}$

and:

$$\{\langle \vec{y}\rangle \in H^M_\kappa | D(\vec{y}, x_1, \ldots, x_r)\}$$

is $\Sigma^{(n)}_1(M)$ in $p = \pi(\overline{p})$ by the same definition.

Before proving Lemma 3.2.22 we show that it implies Lemma 3.2.21. Let $\overline{D}(x_1, \ldots, x_r)$ be $\Sigma^{(n)}_1(\overline{N})$ and let $D(x_1, \ldots, x_r)$ be $\Sigma^{(n)}_1(N)$ by the same definition. Set:

$$D'(y, \vec{x}) \leftrightarrow: y = \varnothing \wedge D(\vec{x}); \ \overline{D}'(y, \vec{x}) \leftrightarrow: y = \varnothing \wedge \overline{D}(\vec{x}).$$

Let $\pi'(\overline{x}_i) = x_i$ $(i = 1, \ldots, r)$. Applying Lemma 3.2.22 and the $\Sigma_1^{(n)}$ preservation of π we have:

$$\begin{aligned} \overline{D}(\overline{x}_1, \ldots, \overline{x}_r) &\leftrightarrow \varnothing \in \{y \in H_{\overline{\kappa}}^{\overline{M}} | \overline{D}'(y, \overline{x}_1, \ldots, \overline{x}_r)\} \\ &\leftrightarrow \varnothing \in \{y \in H_{\kappa}^{M} | D'(y, x_1, \ldots, x_r)\} \\ &\leftrightarrow D(x_1, \ldots, x_r). \end{aligned}$$

QED

We now prove Lemma 3.2.22. For the sake of simplicity we display the proof for the case $r = 1$. Let $\overline{D}(\vec{y}, x)$ be $\Sigma_1^{(n)}(\overline{N})$ and $D(\vec{y}, x)$ be $\Sigma_1^{(n)}(N)$ by the same definition. We may assume:

$$\overline{D}(\vec{y}, x) \leftrightarrow \bigvee z^n \overline{B}(z^n, y, x), \;\; D(\vec{y}, x) \leftrightarrow \bigvee z^n B(z^n, y, x)$$

where $\overline{B}$ is $\Sigma_0^{(n)}(\overline{N})$ and B is $\Sigma_0^{(n)}(N)$ by the same definition. Let $\overline{A}$ have the same definition over $\overline{M}$ and A the same definition over M. Let $x = \pi'(\overline{x})$. Then $\overline{x} = \overline{\sigma}(f)(\alpha)$ for an $f \in \Gamma^n$ and $\alpha < \overline{\lambda}$. Hence $x = \sigma\pi(f)(g(\alpha))$. Then for $\vec{y} \in H_{\overline{\kappa}}^{\overline{M}}$:

$$\begin{aligned} \overline{D}(\vec{y}, \overline{x}) &\leftrightarrow \bigvee z^n \overline{B}(z^n, \vec{y}, \overline{x}) \\ &\leftrightarrow \bigvee u \in H_{\overline{M}}^{n} \bigvee z \in \overline{\sigma}(u) \overline{B}(z^n, \vec{y}, \overline{\sigma}(f)(\alpha)) \\ &\leftrightarrow \bigvee u \in H_{\overline{M}}^{n} \bigvee \{\xi < \overline{\kappa} | \bigvee z \in u\, \overline{A}(z, \vec{y}, f(\xi))\} \in \overline{F}_{\alpha}. \end{aligned}$$

Similarly for $\vec{y} \in H$ we get:

$$\overline{D}(\vec{y}, \overline{x}) \leftrightarrow \bigvee u \in H_M^n \{\xi < \kappa | \bigvee z \in u A(z, \vec{y}, \pi(f)(\xi))\} \in F_{g(\alpha)}.$$

$\overline{F}_{\alpha}$ is $\Sigma_1(\overline{M})$ in a parameter $\overline{p}$ and $F_{g(\alpha)}$ is $\Sigma_1(M)$ in a parameter $p = \pi(\overline{p})$. But by the definition of Γ^n we know that there are $\overline{q}, q$ such that either:

$$f = \overline{q} \in H_{\overline{M}}^{n} \text{ and } q = \pi(f)$$

or:

$$f(\xi) \simeq \overline{G}(\xi, \overline{q}) \text{ where } \overline{G} \text{ is a good } \Sigma_1^{(i)}(\overline{M}) \text{ map}$$

and:

$$\pi(f)(\xi) \simeq G(\xi q) \text{ where } G \text{ has the same good definition over } M.$$

Hence:

$$\{\langle \vec{y} \rangle \in H_{\overline{\kappa}}^{\overline{M}} | \overline{D}(\vec{y}, \overline{x})\}$$

is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{\kappa}, \overline{q}, \overline{p}$ and:

$$\{\langle \vec{y} \in \rangle H_{\kappa}^{M} | D(\vec{y}, x)\}$$

is $\Sigma_1^{(m)}(M)$ in κ, q, p by the same definition. QED (Lemma 3.2.22)

3.2.4 $*$–extendability

Definition 3.2.21. Let F be an extender of length λ at κ on M. M is $*$*–extendible* by F iff F is close to M and M is n–extendible by F, where $n \leq w$ is maximal such that $\kappa < \rho_M^n$.

(Hence $\pi : M \to_F^* N$ where $\langle N, \pi \rangle$ is the $\Sigma_0^{(n)}$–extension.)

Lemma 3.2.23. *Assume $\langle \pi, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^* \langle M, F \rangle$ where M is $*$–extendible by F. Assume that π is Σ^* preserving. Then $\overline{M}$ is $*$–extendible by E. Moreover, if $\overline{\sigma} : \overline{M} \to_{\overline{F}}^* \overline{N}$ and $\sigma : M \to_F^* N$, there is a unique $\pi' : \overline{N} \to_{\Sigma^*} N$ such that $\pi'\overline{\sigma} = \sigma\pi$ and $\pi' \restriction \overline{\lambda} = g$.*

Proof: Let n be maximal such that $\kappa < \rho_M^n$. Let $\sigma : M \to_F^{(n)} N$. By Lemma 3.2.21 we have $\overline{\kappa} < \rho_{\overline{M}}^n$ and there is $\overline{\sigma} : \overline{M} \to_{\overline{F}}^{(n)} M$. Moreover there is $\pi' : \overline{N} \to_{\Sigma_1^{(n)}} N$ such that $\pi'\overline{\sigma} = \sigma\pi$ and $\pi' \restriction \overline{\lambda} = g$.

Claim 1 n is maximal such that $\overline{\kappa} < \rho_{\overline{M}}^n$.

Proof: If not, then $n < w$ and $\rho_M^{n+1} \leq \kappa < \rho_M^n$. Hence

$$\bigwedge z^{n+1} z^{n+1} \neq \kappa \text{ holds in } M.$$

Thus $\bigwedge z^{n+1} z^{n+1} \neq \overline{\kappa}$ in $\overline{M}$, since π is $\Sigma_0^{(n+1)}$ preserving. Hence $\rho_{\overline{M}}^{n+1} \leq \overline{\kappa} < \rho_{\overline{M}}^n$. (QED Claim 1)

Note. In the case $n < w$ we needed only the $\Sigma_0^{(n+1)}$ preservation of π to establish Claim 1.

By Claim 1 we then have:

(1) $\pi : \overline{M} \to_{\overline{F}}^* \overline{N}$.

Hence $\overline{M}$ is $*$–extendible by $\overline{F}$. It remains only to show:

Claim 2 π' is Σ^* preserving.

Proof: If $n = w$, there is nothing to prove, so assume $n < w$. We must show that π' is $\Sigma_0^{(m)}$ preserving for $n < m < w$. Let $n < m < w$. Since $\sigma : M \to_F^* N$, we know that:

(2) $\rho_M^m = \rho_N^m$ and $\sigma \restriction \rho_M^m = \text{id}$.

By Claim 1 an (1) we similarly conclude:

(3) $\rho^m_{\overline{M}} = \rho^m_{\overline{N}}$ and $\overline{\sigma} \upharpoonright \rho^m_{\overline{M}} = \mathrm{id}$.

Using (2), (3) and Lemma 3.2.22 we can then show:

(4) Let $\overline{D}(\vec{y}^m, \vec{x})$ be $\Sigma_j^{(m)}(\overline{N})$. Let $D(\vec{y}^m, \vec{x})$ be $\Sigma_j^{(m)}(N)$ by the same definition. Let

$$\pi'(\overline{x}_i) = x_i (i = 1, \ldots, r).$$

Then:

$$\overline{D}_{\overline{x}_1,\ldots,\overline{x}_r} =: \{\langle \overline{y}_m \rangle \upharpoonright \overline{D}(\vec{y}^m, \overline{x}_1, \ldots, \overline{x}_r)\}$$

is $\Sigma_j^{(m)}(\overline{M})$ in a parameter $\overline{p}$ and:

$$D_{x_1,\ldots,x_r} =: \{\langle \vec{y}_m \rangle | D(\vec{y}_m, x_1, \ldots, x_r)\}$$

is $\Sigma_j^{(m)}(M)$ in $p = \pi(\overline{p})$ by the same definition.

Proof: By induction on m.

Case 1 $m = n + 1$

We know:

$$\overline{D}(\vec{y}_m, \vec{x}) \leftrightarrow \overline{H}^m_{\vec{x}} \models \varphi[\vec{y}^m]$$

where φ is Σ_j and

$$\overline{H}^m_{\vec{x}} = \langle H^m_{\overline{M}}, \overline{B}^1_{\vec{x}}, \ldots, \overline{B}^q_{\vec{x}} \rangle$$

where $\overline{B}^i_{\vec{x}} = \{\langle \vec{z}^m \rangle | \overline{B}^i(\vec{z}^m, x)\}$ and $\overline{B}^i$ is $\Sigma^m_1(\overline{N})$ for $i = 1, \ldots, q$. Since $D(y^m, \vec{x})$ has the same $\Sigma_j^{(m)}$ definition, we can assume

$$D(\vec{y}m, \vec{x}) \leftrightarrow H^m_{\vec{x}} \models \varphi[\vec{y}m]$$

where:

$$H^m_{\vec{x}} = \langle H^m_M, B^1_{\vec{x}}, \ldots, B^q_{\vec{x}} \rangle$$

where $B^i_{\vec{x}} = \{\langle z^m \rangle | B^i(\vec{z}^m, x)\}$ and B^i is $\Sigma_1^{(n)}(N)$ by the same definition as $\overline{B}^i$ over $\overline{N}$. Letting $\pi'(\overline{x}_i) = x_i$ $(i = q, \ldots, r)$, we know by Lemma 3.2.22 that each of $\overline{B}^i_{\overline{x}_1,\ldots,\overline{x}_r}$ is $\Sigma_1^{(n)}(\overline{M})$ in a parameter $\overline{p}$ and $B^i_{x_1,\ldots,x_r}$ is $\Sigma_1^{(n)}(M)$ in $p = \pi(\overline{p})$ by the same definition. (We can without loss of generality assume that $\overline{p}$ is the same for $i = 1, \ldots, r$.) But then $\overline{D}_{\overline{x},\ldots,\overline{x}_r}$ is $\Sigma_j^{(m)}(\overline{M})$ in $\overline{p}$ and $D_{x_1,\ldots,x_r}$ is $\Sigma_j^{(m)}(M)$ in $p = \pi(p)$ by the same definition. QED (Case 1)

Case 2 $m = h + 1$ where $h > n$.

We repeat the same argument using the induction hypothesis in place of Lemma 3.2.22. QED (4)

But Claim 2 follows easily from Claim 4 and the fact that π is Σ^* preserving. Let $\overline{D}(\vec{x})$ be $\Sigma_0^{(m)}(\overline{N})$ and $D(\vec{x})$ be $\Sigma_0^{(m)}(N)$ by the same definition. Set:

$$\overline{D}'(y, \vec{x}) \leftrightarrow: y = 0 \wedge \overline{D}(\vec{x})$$
$$D'(y, \vec{x}) \leftrightarrow: y = 0 \wedge D(\vec{x})$$

By (4) we have:

$$\overline{D}(\vec{x}) \leftrightarrow 0 \in \overline{D}_{\vec{x}} \leftrightarrow 0 \in D_{\pi'(\vec{x})} \leftrightarrow D(\pi'(\vec{x}))$$

for $x_1, \ldots, x_r \in \overline{M}$, using the $\Sigma_0^{(m)}$ preservation of π and $\pi(0) = 0$.
QED (Lemma 3.2.23)

Note. The last part of the proof also shows that π' is $\Sigma_j^{(m)}$ preserving if π is.

As a corollary of the proof we also get:

Lemma 3.2.24. *Let $\langle \pi, g \rangle : \langle \overline{M}, \overline{F} \rangle \longrightarrow \langle M, F \rangle$. Let M be $*$-extendible by F. Let n be the maximal n such that $\kappa = \mathrm{crit}(F) < \rho_M^n$. Let $n < r < \omega$ and suppose that π is $\Sigma_j^{(r)}$ preserving, where $j < \omega$. Then:*

(a) *n is maximal such that $\overline{\kappa} = \mathrm{crit}(F) < \rho_{\overline{M}}^n$.*

(b) *$\overline{M}$ is $*$-extendible by $\overline{F}$.*

(c) *Let π' be the unique $\pi' : \overline{N} \longrightarrow_{\Sigma_0} N$ such that $\pi'\overline{\sigma} = \sigma\pi$ and $\pi' \upharpoonright \overline{\lambda} = g$. Then π' is $\Sigma_j^{(r)}$ preserving.*

Proof. (a) follows by the proof of Claim 1 in Lemma 3.2.23, since that only need that π is Σ_0^{n+1}-preserving. (1) then follows as before. Hence $\overline{M}$ is $*$-extendible by $\overline{F}$. (2) and (3) follows for $r \geq m > n$, using the $\Sigma_0^{(r)}$ preservation of π. Hence (4) follows as before and we can conclude that π' is $\Sigma_j^{(n)}$ preserving as before.

QED(Lemma 3.2.24)

Notation. $\Gamma_*^n(\kappa, M) = \{f \in \Gamma^n(\tau, M) : \mathrm{dom}(f) = \kappa\}$ and $\Gamma^*(\kappa, M) = \Gamma_*^n(\kappa, M)$ where $n \leq \omega$ is maximal such that $\kappa < \rho_M^n$.

3.2.5 Good Parameters

We now recall some concepts which were developed in §2.5. Let $M = \langle J_\alpha^E, B\rangle$ be acceptable. The set P_M^{n+1} of $n+1$-*good parameters* can be defined by:

> $a \in P_M^{n+1}$ iff $a \in [\mathrm{On}_M]^{<\omega}$ and there is an $A \subset H_M^n$ which is $\Sigma_1^{(n)}(M)$ in parameters from $\rho^{n+1} \cup a$ such that $A \cap H^{n+1} \notin M$.

We then say that A *confirms* $a \in P^{n+1}$. We also set: $P_M^0 = [\mathrm{On}_M]^{<\omega}$. It is not hard to prove:

Fact 1. Let $a \in P^n$. Then:

- $a \subset b \in [\mathrm{On}_M]^{<\omega} \longrightarrow b \in P^M$.
- $a \setminus \rho^n \in P^n$.

The definition of P_M^{n+1} is equivalent to that given in §2.5. However, we thus required $a \in P_M^n$ in place of $a \in [\mathrm{On}_M]^{<\omega}$. To show the equivalence of these definitions, we must prove: $P_M^{n+1} \subset P_M^n$ $(n < \omega)$. With a view to proving this we recall the following definition, which was stated in an equivalent form in §2.5.

With a view to proving this we recall the following definition, which was stated in an equivalent form in §2.5.

Definition 3.2.22. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. Let $a \in [\alpha]^{<\omega}$. For $n < \omega$ we define the n-th reduct $M^{n,a}$ and the n-th standard predicate $T_M^{n,a}$ with respect to a:

$$T^0 = B, M^n = \langle J_{\rho^M}^A,\, T^n\rangle,$$

$$T^{n+1} = \{\langle i, x\rangle : i < \omega \wedge M^n \models \varphi_i[x, a^{(n)}]\}$$

where $a^(n) = a \cap \rho^n$ and $\langle \varphi_i : i < \omega\rangle$ enumerates recursively all Σ_1 formulae $\psi = \varphi(v_0, v_1)$ with at most the free variables v_0, v_1 in the language of M.

By induction on n we get:

Fact 2. Let $a \in [\mathrm{On}_M]^{<\omega}$. Then:

- $T^{n,a}$ is $\Sigma_1^{(n)}(M)$ in a.

- Let $A \subset H^n$ be $\Sigma_1^{(n)}(M)$ in a. There is an $i < \omega$ such that

$$Ax \longrightarrow \langle i, x \rangle \in T^{n,a}$$

From this it follows that:

Fact 3. $a \in P^{n+1} \leftrightarrow T^{n,a}$ confirms $a \in P^{n+1}$. But then:

Fact 4. $P^{n+1} \subset P^n$.

Proof. For $n = 0$ this is trivial. Now let $n = m + 1$. Let $a \in P^{n+1}$. Then $T^{n,a} \cap H^{n+1} \notin M$.

Claim. $T^{m,a} \cap H^n \notin M$.

Suppose not. If $\rho^n \in M$, then:

$$\langle H^n, T^{m,a} \cap H^n \rangle \in M$$

Hence $T^{n,a} \in M$ and $H^{n+1} \cap T^{n,a} \in M$. Contradiction! Now let $\rho^n = \rho^0$. Then for each $x \in M$, there is $i \leq \omega$ such that $\langle i, x \rangle \in T^{m,a}$. If $T^{m,a} \cap H^n = T^{m,a} \in M$, then $\langle i, T^{m,a} \rangle \in T^{m,a}$. Contradiction!

QED(Fact 4.)

We also mention:

Fact 5. $a \in P^{n+1}$ iff there is A which is $\Sigma_1^{(n)}(M)$ in a such that $A \cap \rho^{n+1} \notin M$.

Proof. (Sketch) If $\rho^{n+1} = \rho^0$, take $A = \rho^0$. Now let $\rho^{n+1} < \rho^0$. Then $H^{n+1} = |J^E_{\rho^{n+1}}|$ is a ZFC^- model. Note that for any $N = J^E_\alpha$, the function f_N is uniformly $\Sigma_1(N)$, where

$$f_N(\alpha) = \text{ the } \alpha\text{-th element of } N \text{ in the ordering } <_E .$$

Let A be $\Sigma_1^{(n)}(M)$ such that $A \subset H^n$ and $A \cap H^{n+1} \notin M$. Set:

$$A' = \{\alpha < \rho^n : f(\alpha) \in A\}$$

where $f = f_{J^E_{\rho^n}}$. Then $f \upharpoonright \rho^{n+1} = f_{J^E_{\rho^n}}$ maps ρ^{n+1} onto H^{n+1}. Hence, if $A' \cap \rho^{n+1} \in M$, we have $f``(A' \cap \rho^{n+1}) = A \cap H^{n+1} \in M$. Contradiction!

QED(Fact 5)

Thus $A \cap H^{n+1}$ could have been replaced by $A \cap \rho^n$ in the original definition of P^n.

We now define:

Definition 3.2.23. π is a *strongly Σ^*-preserving map* of M to N (in symbols: $\pi : M \longrightarrow_{\Sigma^*} N$ strongly) iff the following hold:

- $\pi : M \longrightarrow_{\Sigma^*} N$
- If $\rho^{n+1} = \rho^\omega$ in M, then $\rho^{n+1} = \rho^\omega$ in N.
- If $\rho^{n+1} = \rho^\omega$ in M, A confirms $a \in P^{n+1}$ in M, and A' is $\Sigma_1^{(n)}(N)$ in $\pi(a)$ by the same definition, then A' confirms $\pi(a) \in P^{n+1}$ in N.

By Fact 3 and Fact 4 we conclude:

Lemma 3.2.25. *Let $\pi : M \longrightarrow_{\Sigma^*} N$ strongly. Let $\rho^{n+1} = \rho^\omega$ in M. Let $a \in P^{n+1}$ in M. Then $T^{i,a}$ confirms $a \in P^{i+1}$ in M and $T^{i,\pi(a)}$ confirms $\pi(a) \in P^{i+1}$ in N for $i \leq n$.*

We now prove:

Lemma 3.2.26. *Let $\pi : M \longrightarrow_F^* N$. Then $\pi : M \longrightarrow_{\Sigma^*} N$ strongly.*

Proof. Let $\kappa = \mathrm{crit}(F)$. We consider two cases.

Case 1. $\rho_M^\omega \leq \kappa$.

The conclusion is immediate by Corollary 3.2.18.

Case 2. $\kappa < \rho_M^\omega$.

We show that for any $n < \omega$, if A confirms $a \in P^{n+1}$ in M, then A' confirms $\pi(a) \in P^{n+1}$ in N. Suppose not. Let $A' \cap H_M^{n+1} \in N$. Let $y = A' \cap H_N^{n+1}$. Then $y \in H_N^n$ and in N we have:

$$\bigwedge z^{n+1}(z^{n+1} \in y \longleftrightarrow z^{n+1} \in A'),$$

which is a $\Pi_1^{(n+1)}$ statement in $\pi(a), y$. Let $y = \pi(f)(\alpha)$, where $\alpha < \lambda = \lambda_F$ and $f \in \Gamma^*(\kappa, M)$. Thus $\mathrm{dom}(f) = \kappa$ and:

$$f(\xi) = G(\xi, q)$$

where $q \in H_{\kappa^+}^M$ and G is a good $\Sigma_1^{(m)}$ function to H^n for an $m < \omega$. Assume without lose of generality $m > n + 1$.

The statement:

$$\wedge z^{n+1}(z^{n+1} \in f(\xi) \longleftrightarrow z^{n+1} \in A)$$

is then $\Sigma_1^{(m)}(M)$ in q, a, ξ. Hence it is $\Sigma_0^{m+1}(M)$ in q, a, ξ. Set:

$$X = \{\xi < \kappa : \bigwedge z^{n+1}(z^{n+1} \in f(\xi) \longleftrightarrow z^{n+1} \in A)\}.$$

Then $X \in M$. But $\alpha \in \pi(X)$. This is a contradiction, since $X = \pi(X) = \varnothing$ by the fact that $A \cap H_M^{n+1} \notin M$.

Finally we note that for all $n < \omega$ we have $\kappa < \rho_M^{n+1}$. Hence: $\rho_M^n = \pi(\rho_M^n)$ if $\rho_M^n \in M$ and otherwise $\rho_N^n = \mathrm{On}_N$. Thus:

$$\rho_M^{n+1} = \rho_M^\omega \longrightarrow \rho_N^{n+1} = \rho_N^\omega.$$

QED(Lemma 3.2.26)

Obviously we have:

Lemma 3.2.27. *If $\pi_0 : M_0 \longrightarrow_{\Sigma^*} M_1$ strongly and $\pi_1 : M_1 \longrightarrow_{\Sigma^*} M_2$ strongly, then $\pi_1\pi_0$ is a strong Σ^*-preserving map from M_0 to M_2.*

We now prove:

Lemma 3.2.28. *Let $\pi_{ij} : M_i \longrightarrow_{\Sigma^*} M_j$ strongly ($i \leq j < \lambda$) where the π_{ij} commute. Suppose that:*

$$\langle M_i : i < \lambda \rangle, \langle \pi_{ij} : i \leq j < \lambda \rangle$$

has a transitivized direct limit:

$$M, \langle \pi_i : i < \lambda \rangle.$$

Then $\pi_i : M_i \longrightarrow_{\Sigma^*} M$ strongly for $i < \lambda$.

Proof. π_i is Σ_1-preserving, since each π_{ij} is. Hence $M = \langle J_\alpha^E, B \rangle$ is acceptable. If we set:

$$\rho_n = \bigcup_{i<\lambda} \pi_i\text{''}\rho_{M_i}^n, \; H_n = \bigcup_{i<\lambda} \pi_i\text{''}H_n,$$

it follows that $H_n = H_{\rho_n}^M = |J_{\rho_n}^E|$. By induction on n we prove:

Claim. $\rho_n = \rho_M^n$ and $\pi_i : M_i \longrightarrow_{\Sigma_1^{(n)}} M$.

Proof.

Case 1. $n = 0$ is trivial.

Case 2. $n = m + 1$.

Let $r \geq n$ such that $\rho^r_{M_0} = \rho^\omega_{M_0}$. Let $a \in P^r_{M_0}$. Then $T^{m,a_i}_{M_i}$ verifies $a_i \in P_{M_i}$ for $i < \lambda$ where $\pi_{0i}(a_0) = a_i$. Let $a = \pi_i(a_i)$ $(i < \lambda)$. By the induction hypothesis π_i is $\Sigma^{(m)}_1$-preserving. Hence

$$x \in T^{m,a_i}_{M_i} \longleftrightarrow \pi_i(x) \in T^{m,a}_M.$$

Claim. $T^{m,a}_M \cap H_n \notin M$.

Proof. Suppose not. Let $y = T^{m,a}_M \cap H_n$. Let $i < \lambda$ such that $\pi(y_i) = y$. For $x \in H^n_{M_i}$ we have:

$$\begin{aligned} x \in T^{m,a_i}_{M_i} &\longleftrightarrow \pi_i(x) \in T^{m,a}_M \cap H_n \\ &\longleftrightarrow \pi(x) \in \pi(y) \\ &\longleftrightarrow x \in y_i. \end{aligned}$$

Hence $T^{m,a_i}_{M_i} \cap H^n_{M_i} = y_i \cap H^n_{M_i} \in M_i$. Contradiction!

QED(Claim 1)

Claim 2. Let $A \subset H_n$ be $\underline{\Sigma}^{(m)}_1(M)$. Then $\langle H_n, A\rangle$ is amenable.

Proof. Let A be $\Sigma^{(m)}_1(M)$ in q. For i such that $q \in \text{rng}(\pi_i)$, let $q_i = \pi_i^{-1}(q)$ and let A_i be $\Sigma^{(m)}_1(M)$ in q_i by the same definition. Now let $x \in H_n$. We claim that $x \cap A \in H_n$. Let i be large enough that $q \in \text{rng}(\pi_i)$. Set $x_i = \pi_i^{-1}(x)$. Let $z_i = A_i \cap x_i$. Then $x_i \in H^n_{M_i}$ where $\langle H^n_{M_i}, A_i\rangle$ is amenable. Hence $z_i \in H^n_{M_i}$ where $z = \pi_i(z_i) = A \cap x$. Hence $z \in H^n_{M_i}$.

QED(Claim 2)

Hence $\rho^n_M = \rho_n$ and $H^n_M = H_M$. It follows straightforwardly that $\pi_i : M_i \longrightarrow_{\Sigma^{(n)}_1} M$ for $i < \lambda$.

QED(Case 2)

It remains to show:

Claim 3. The embedding π_i is strong.

Proof. Let $\rho^{n+1} = \rho^\omega$ in M_i. Let $A \subset H^n$ confirm $a \in P^{n+1}$ in M_i. Let A_j be $\Sigma^{(n)}_1(M_j)$ in $a_j =: \pi_{ij}(a)$ for $i \leq j < \lambda$. Then $\rho^{n+1} = \rho^\omega$ in M_j and A_j confirms $a_j \in \rho^{n+1}$ in M_j. Let $a' = \pi_i(a)$, and let A' be $\Sigma^{(n)}_1(M)$ in a' by the same definition. We repeat the proof of Claim 1 to show that A' confirms $a' \in P^{n+1}$ in M (i.e. $A' \cap H_{n+1} \notin M$).

QED(Lemma 3.2.28)

3.3 Premice

A major focus of modern set theory is the subject of "strong axioms of infinity". These are principles which posit the existence of a large set or class, not provable in ZFC. Among these principles are the *embedding axioms*, which posit the existence of a non trivial elementary embedding of one inner model into another. The best known example of this is the *measurability axiom*, which posits the existence of a non trivial elementary embedding π of V into an inner model. ("Non trivial" here means simply that $\pi \neq \text{id}$. Hence there is a unique *critical point* $\kappa = \text{crit}(\pi)$ such that $\pi \restriction \kappa = \text{id}$ and $\pi(\kappa) > \kappa$.) The critical point κ of π is then called a *measurable cardinal*, since the existence of such an embedding is equivalent to the existence of an ultrafilter (or *two valued measure*) on κ.

This is a typical example of the recursing case that an axiom positing the existence of a proper class (hence not formulable in ZFC) reduces to a statement about set existence. The weakest embedding axiom posits the existence of a non trivial embedding of L into itself. This is equivalent to the existence of a countable transitive set called $0^{\#}$, which can be coded by a real number. (There are many representations of $0^{\#}$, but all have the same degree of constructability.) The "small" object $0^{\#}$ in fact contains complete information about both the proper class L and an embedding of L into itself. We can then form $L(0^{\#})$, the smallest universe containing the set $0^{\#}$. If $L(0^{\#})$ is embeddable into itself we get $0^{\#\#}$, which gives complete information about $L(0^{\#})$ and its embedding ... etc. This process can be continued very far. Each stage in this progression of embeddings, leading to larger and larger universes, is coded by a specific set, called a *mouse*. $0^{\#}$ and $0^{\#\#}$ are the first two examples of mice. It is not yet known how far this process goes, but it is conjectured that all stages can be represented by mice, as long as the embeddings are representable by extenders. (Extenders in our sense are also called *short extenders*, since one must modify the notion in order to go still further.) The concept of mouse, however hard it is to explicate, will play a central role in this book.

We begin, therefore, with an informal discussion of the *sharp operation* which takes a set a to $a^{\#}$, since applications of this operation give us the smallest mice $0^{\#}, 0^{\#\#}$, etc.

Let a be a set such that $a \in L[a]$. Suppose moreover that there is an elementary embedding π of $L^a = \langle L[a], \in, a \rangle$ into itself such that $a \in L^a_\kappa$, where $\kappa = \mathrm{crit}(\pi)$. We also assume without loss of generality, that κ is minimal for π with this property. Let $\tau = \kappa^{+L^a}$ and $\nu = \sup \pi''\tau$. Then $\tilde{\pi} : L^a_\tau \prec L^a_\nu$ cofinally, where $\tilde{\pi} = \pi \upharpoonright L^a_\tau$. Set $F = \pi \upharpoonright \mathbb{P}(\kappa)$. F is then an extender at κ with base $L_\tau[a]$ and extension $\langle L_\nu[a], \tilde{\pi}\rangle$.

$\langle L^a_\nu, F\rangle = \langle L_\nu[a], a, F\rangle$ is then amenable by Lemma 3.2.2. It can be shown, moreover, that F is uniquely defined by the above condition. We then define:

Definition 3.3.1. $a^\#$ is the structure $\langle L_\nu[a], a, F\rangle$.

Note. In the literature $a^\#$ has many different representations, all of which have the same constructibility degree as $\langle L_\nu[a], a, F\rangle$.

$a^\#$ has a number of interesting properties, which we state here without proof. F is clearly an extender at κ on $\langle L^a_\nu, F\rangle$. Moreover, we can form the extension:

$$\pi_0 : \langle L^a_\nu, F\rangle \to_F \langle L^a_{\nu_1}, F_1\rangle.$$

We then have $\pi_0 \supset \tilde{\pi}$, $\pi_0(\kappa) = \nu$. (In fact $\pi_0 = \pi' \upharpoonright L^a_\nu$.) But we can then apply F_1 to $\langle L^a_{\nu_1}, F_1\rangle \ldots$ etc. This can be repeated indefinitely, showing that $a^\#$ is *iterable* in the following sense:

There are sequences $\kappa_i, \tau_i, \nu_i, F_i (i < \infty)$ and $\pi_{ij} (i \leq j < \infty)$ such that

- $\kappa_0 = \kappa, \tau_0 = \tau, \nu_0 = \nu, F_0 = F$.
- $\kappa_{i+1} = \pi'_{i,i+1}(\kappa_i), \nu_i = \pi'_{i,i+1}(\pi_i), \tau_i = \kappa_i^{+L^a_{\nu_i}}$.
- F_i is a full extender at κ_i with base $L_{\tau_i}[a]$ and extension $\langle L_{\nu_i}[a], \pi'_{i,i+1} \upharpoonright L^{[a]}_{\tau_i}\rangle$.
- $\pi'_{i,i+1} : \langle L^a_{\nu_i}, F_i\rangle \to_{F_i} \langle L^a_{\nu_{i+1}}, F_{i+1}\rangle$.
- The maps π'_{ij} commute — i.e.
$$\pi'_{ii} = \mathrm{id};\ \pi'_{ij}\pi'_{hi} = \pi'_{hj}.$$
- For limit λ, $\langle L^a_{\nu_\lambda}, F_\lambda\rangle$, $\langle \pi'_{i\lambda} | i < \lambda\rangle$ is the transitivized direct limit of
$$\langle\langle L^a_{\nu_0}, F_i\rangle | i < \lambda\rangle, \langle \pi'_{ij} | i \leq j < \lambda\rangle.$$

It turns out that $a^\# = \langle L^a_\nu, F\rangle$ is uniquely defined by the conditions:

- $\langle L^a_\nu, F\rangle$ is iterable in the above sense
- ν is minimal for such $\langle L^a_\nu, F\rangle$.

If $a = \emptyset$ we write: $0^\#$. $0^\# = \langle L_\nu, F\rangle$ is then acceptable. By a Löwenheim–Skolem type argument it follows that $0^\#$ is sound and $\rho^1_{0^\#} = \omega$. (To see this let $M = 0^\#, X = h_M(\omega)$. Let $\sigma : \overline{M} \stackrel{\sim}{\leftrightarrow} X$ be the transitivization of X, where $\overline{M} = \langle L_{\overline{\nu}}, \overline{F}\rangle$. Using the fact that $\sigma : \overline{M} \to M$ is Σ_1–preserving and M is iterable, it can be shown that $\overline{M}$ is iterable. Hence $\overline{M} = M$, since $\overline{\nu} \leq \nu$ and ν is minimal.) But then $0^\#$ is countable and can be coded by a real number. But this is real giving complete information about the proper class L, since we can recover the satisfaction relation for L by:

$$L \models \varphi[\vec{x}] \leftrightarrow L_{\kappa_i} \models \varphi[\vec{x}]$$

where i is chosen large enough that $x_1, \ldots, x_n \in L_{\kappa_i}$. But from $0^\#$ we also recover a nontrivial elementary embedding of L into itself, namely:

$$\pi : L \to_F L \text{ where } 0^\# = \langle L_\nu, F\rangle.$$

$0^\#$ is our first example of a mouse. All of its iterates, however, are not sound, since if $i > 0$, then $\mathrm{rng}(\pi_{0i}) = h_{M_i}(\omega)$, where $\rho^1_{M_i} = \rho^1_{M_0} = \omega$. But $\kappa_0 \notin \mathrm{rng}(\pi_{0i})$.

We can iterate the operation $\#$, getting $0, 0^\#, (0^\#)^\#, \ldots$ etc. This notation is not literally correct, however, since $a^\#$ is defined only when $a \in L[a]$. Thus, setting:

$$0^{\#(n)} = 0^{\overbrace{\# \cdots \#}^{n}},$$

we need to set: $0^{\#(n+1)} = (e^n)^\#$, where e^n codes $0, \ldots, 0^{\#(n)}$. If we do this in a uniform way, we can in fact define $0^{\#(\xi)}$ for all $\xi < \infty$.

Definition 3.3.2. Define $e^i, \nu_i, 0^{\#(i)} = \langle L^{e^i}_{\nu_i}, E_{\nu_i}\rangle (i < \infty)$ as follows:

$$e^i =: \{\langle x, \nu_i\rangle | j < i \wedge x \in E_{\nu_j}\} \text{ (hence } e^0 = \emptyset)$$
$$0^{\#(0)} =: \langle\emptyset, \emptyset\rangle \text{ (hence } \nu_0 = 0)$$
$$0^{\#(i+1)} =: (e^i)^\# \text{ (hence } \nu_{i+1} > \nu_i)$$

For limit λ we set:

$$\nu =: \sup_{i<\lambda} \nu_i,\ 0^{\#(\lambda)} =: \langle L^{e^\lambda}_{\nu_\lambda}, \emptyset\rangle,\ \text{(hence } \emptyset = E_{\nu_\lambda}).$$

By induction on $i < \infty$ it can be shown that each $0^{\#(i)}$ is acceptable and sound, although we skip the details here. Each $0^{\#(i)}$ is also iterable in a sense which we have yet to explicate. As before, it will turn out that the iterates are acceptable but not necessarily sound. Set:

$$E =: \bigcup_{i<\infty} e^i.$$

Then $L[E]$ is the smallest inner model which is closed under the # operation. (For this reason it is also called $L^{\#}$.) We of course set: $L^E =: \langle L[E], \in, E\rangle$.

L^E is a very L–like model, so much so in fact, that we can obtain the next mouse after all the $0^{\#(i)} (i < \infty)$ by repeating the construction of $0^{\#}$ with L^E in place of L: Suppose that $\pi : L^E \prec L^E$ is a nontrivial elementary embedding. Without loss of generality assume the critical point κ of π to be minimal for all such π. Let $\tau = \kappa^{+L^E}$ and $\nu = \sup \pi''\tau$. Then $\tilde{\pi} = \pi \upharpoonright L^E_\tau$. Set: $F = \pi \upharpoonright \mathbb{P}(\kappa)$. Then F is an extender with base $L_\tau[E]$ and extension $\langle L_\nu[E], \tilde{\pi}\rangle$. The new mouse is then $\langle L^E_\nu, F\rangle$.

As before, we can recover full information about L^E from $\langle L^E_\nu, F\rangle$ and we can recover a nontrivial embedding of L^E by: $\pi : L^E \to_F L^E$. $e = E \cup \{\langle x, \nu\rangle | x \in F\}$ then codes all the mice up to and including $\langle L^E_\nu, F\rangle$, so the next mouse is $e^{\#} \ldots$ etc.

Note. that $L^E || \nu = \langle L^E_\nu, \emptyset\rangle$ since, if $\kappa_i = \text{crit}(E_{\nu_{i+1}})$, then the sequence $\langle \kappa_i | i < \infty\rangle$ of all critical points of previous mice is discrete, whereas $\kappa = \text{crit}(F)$ is a fixed point of this sequence.

This process can be continued indefinitely. At each stage it yields a set which encodes full information about an inner model. We call these sets *mice*. Each mouse will be an acceptable structure of the form $M = \langle J^E_\alpha, E_\alpha\rangle$ where $E = \{\langle x, \nu\rangle | \nu < \alpha \wedge x \in E_\nu\}$ codes the set of 'previous' mice. For $\nu = \alpha$ we have: Either $E_\nu = \emptyset$ or ν is a limit ordinal and E_ν is a full extender at a $\kappa < \nu$ with extension $\langle J_\nu[E], \pi\rangle$ and base $J_\tau[E]$, where $\tau = \kappa^{+M}$.

For limit $\xi \leq \alpha$ we set: $M||\xi =: \langle J^E_\xi, E_\xi\rangle$. A class model L^E is called a *weasel* iff $E = \{\langle x, \nu\rangle | \nu < \infty \wedge x \in E_\nu\}$ and $L^E||\alpha =: \langle J^E_\alpha, E_\alpha\rangle$ is a mouse of all limit α.

When dealing with such structures M satisfying, we shall often use the following notation: If $E_\nu \neq \emptyset$, then $\kappa_\nu =$ the critical point of $E_\nu, \tau_\nu = \kappa^+ J^E_\nu$, and $\lambda_\nu =$ the length of $E_\nu = \pi(\kappa_\nu)$, where $\langle J^E_\nu, \pi\rangle$ is the extension of $J^E_{\tau_\nu}$ by E_ν.

In the above examples, the extenders E_ν were so small that τ_ν eventually got collapsed in $L[E_\nu]$. Thus E_ν was no longer an extender in $L[E_\nu]$, since it was not defined on all subsets of κ. However, if we push the construction far enough, we will eventually reach an E_ν which does not have this defect. $L[E_\nu]$ will then be the smallest inner model with a measurable cardinal.

In the above examples the extender E_ν is always generated by $\{\kappa_\nu\}$ Hence we could just as wel have worked with ultrafilters as with extenders. Eventually, however, we shall reach a point where genuine extenders are needed. In the examples we also chose $\lambda_\nu = \pi(\kappa_\nu)$ minimally — i.e. we imposed an *initial segment condition* which says that $E_\nu|\lambda$ is not a full extender for any $\lambda < \lambda_\nu$. This condition can become unduly restrictive, however: It might happen that we wish to add a new extender E_ν and that $E_\nu|\lambda$ is an extender which we added at an earlier stage. In that case we will have: $E_\nu|\lambda \in J^E_\nu$. In order to allow for this situation we modify the initial segment condition to read:

Definition 3.3.3. Let F be a full extender at κ with base S and extension $\langle S', \pi\rangle$. F satisfies the *initial segment condition* iff whenever $\lambda < \pi(\kappa)$ such that $F|\lambda$ is a full extender, then $F|\lambda \in S'$.

As indicated above, we expect our mice to be *iterable*. The example of an iteration given above is quite straightforward, but the general notion of iterability which we shall use is quite complex. We shall, therefore, defer it until later. We mention, however, that, since mice are fine structural etities, we shall iterate by Σ^*–extensions rather than the usual Σ_0–extensions. In the above examples, the minimal choice we made in our construction guaranteed that the mice we constructed were sound. However, in general we want the iterates of mice to themselves be mice. Thus we cannot require all mice to be sound: Suppose e.g. that $M = \langle J^E_\nu, F\rangle$ is a mouse and we form: $\pi : M \to^*_F M'$. Then M' is no longer sound. (To see this, let $p \in P^1_M$. It follows easily that $\pi(p) \in P^1_{M'}$. But $\kappa \notin \mathrm{rng}(\pi)$; hence κ is not $\Sigma_1(M')$ in $\pi(p)$.)

As we said, however, our initial construction is designed to produce sound structures. Hence we *can* require that if $M = \langle J^E_\nu, F\rangle$ is a mouse and $\lambda < \nu$, then $M||\lambda$ is sound, since this property will not be changed by iteration.

By a *premouse* we mean a structure which has the salient properties of a mouse, but is not necessarily iterable. Putting our above remarks together, we arrive at the following definition:

Definition 3.3.4. $M = \langle J^E_\nu, F\rangle$ is a *premouse* iff it is acceptable and:

(a) Either $F = \emptyset$ or F is a full extender at a $\kappa < \nu$ with base $J_\tau[E]$, where $\tau = \kappa^{+M}$, and extension $\langle J_\nu[E], \pi\rangle$. Moreover F is weakly amenable and satisfies the initial segment condition. (Recall that $J = \langle J_\nu[E], E \cap J_\nu[E]\rangle$).

(b) Set $E_\gamma = E''\{\gamma\}$ for $\gamma < \nu$. If $\gamma < \nu$ is a limit ordinal, then $M||\gamma =: \langle J^E_\gamma, E_\gamma\rangle$ is sound and satisfies (a).

(c) $E = \{\langle x, \eta\rangle | x \in E_\eta \cap \eta < \nu \text{ is a limit ordinal}\}$.

By Lemma 2.5.26 we then have:

Lemma 3.3.1. *Let $\langle J^E_\alpha, E_\alpha\rangle$ be a sound premouse. $\langle J^{E'}_{\alpha+\omega}, \emptyset\rangle$ is a premouse, where $E' = E \cup \langle E_\alpha \times \{\alpha\}\rangle$.*

However, it does not follow that $\langle J^{E'}_{\alpha+\omega}, \emptyset\rangle$ is sound.

We call a premouse $M = \langle J^E_\nu, F\rangle$ *active* iff $F \notin \emptyset$. If F is inactive we often write J^E_ν for $\langle J^E_\nu, \emptyset\rangle$. We classify active premice into three *types*:

Definition 3.3.5. Let F be an extender on κ with base S and extension $\langle S', \pi\rangle$. We set:

- $C = C_F =: \{\lambda | \kappa < \lambda < \pi(\kappa) \wedge F|\lambda \text{ is full}\}$
- F is of *type* 1 iff $C = \emptyset$
- F is of *type* 2 iff $C \neq \emptyset$ but is bounded in $\pi(\kappa)$
- F is of *type* 3 iff C is unbounded in $\pi(\kappa)$
- Let $M = \langle J^E_\nu, F\rangle$ be a premouse. The *type of* M is the type of F. We also set: $C_M =: C_F$.

It is evident that F satisfies the initial segment condition iff $F|\lambda \in S'$ whenever $\lambda \in C_F$.

Premice of differing type will very often require different treatment in our proofs. In much of this book we will assume that there is no inner model with a Woodin cardinal, which implies that all mice are of type 1. For now, however, we continue to work in greater generality.

Lemma 3.3.2. *Let F be an extender at κ with base S and extension $\langle S', \pi\rangle$. Let $\kappa < \lambda < \pi(\kappa)$. Then $\lambda \in C_F$ iff $\pi(f)(\alpha_1, \ldots, \alpha_n) < \lambda$ for all $f \in M$ such that $f : \kappa^n \to \kappa$ and all $\alpha_1, \ldots, \alpha_n < \lambda$.*

Proof: We first prove the direction $(\rightarrow)$. Let $F^* = F|\lambda$ be full with extension $\langle S^*, \pi^*\rangle$. Let $f, \alpha_1, \ldots, \alpha_n$ be as above. Let $\beta = \pi^*(f)(\vec{\alpha})$. Set $e = \{\langle \xi_1, \ldots, \xi_n, \delta\rangle | f(\vec{\xi}) = \delta\}$. Then $\beta < \lambda$ and:

$$\langle \vec{\alpha}, \beta\rangle \in F^*(e) = \lambda^{n+1} \cap F(e).$$

Hence $\pi(f)(\vec{\alpha}) = \beta < \lambda$. QED $(\rightarrow)$

We now prove $(\leftarrow)$. Let $f, \alpha_1, \ldots, \alpha_n$ be as above. Then $\pi(f)(\vec{\alpha}) = \beta < \lambda$. Hence

$$\langle \vec{\alpha}, \beta\rangle \in F(e) \cap \lambda^{n+1} = F^*(e).$$

Hence $\pi^*(f)(\vec{\alpha}) = \beta < \lambda$. But each $\gamma < \pi^*(\kappa)$ has the form $\pi^*(f)(\vec{\alpha})$ for some such $f, \alpha_1, \ldots, \alpha_n < \lambda$. Hence $\pi^*(\kappa) = \lambda = \text{length } (F^*)$.

QED (Lemma 3.3.2)

Corollary 3.3.3. *C_F is closed in $\pi(\kappa)$.*

Corollary 3.3.4. *Let F, S, S', π be as above and let F be weakly amenable. Then C_F is uniformly $\Pi_1(\langle S', F\rangle)$ in κ.*

Proof: S' is admissible and the Gödel function $\prec, \succ$ is uniformly Σ_1 over admissible structures. By weak amenability we know that $\mathbb{P}(\kappa^2) \cap S = \mathbb{P}(\kappa^2) \cap S'$. S' is admissible and Gödel's pair function $\prec, \succ$ is $\Sigma_1(S')$ and defined on $(\text{On}_{S'})^2$. Then "λ is Gödel–closed" is $\Delta_1(S')$, since it is expressed by $\bigwedge \xi, \delta < \lambda \prec \xi, \delta \succ < \lambda$. By Lemma 3.3.2, "$\lambda \in C_F$" is equivalent in S' to:

$$\kappa < \lambda \subset \pi(\kappa) \wedge \lambda \text{ is Gödel–closed}$$
$$\wedge \bigwedge f : n \to \kappa \bigwedge \alpha < \lambda \bigvee \beta < \lambda \prec \alpha, \beta \succ \in F(e_f)$$

where $e_f = \{\prec \delta, \xi \succ < \kappa | f(\xi) = \delta\}$. The function $f \mapsto e_f$ is $\Sigma_1(S')$ in κ and defined on $\{f \in S | f : \kappa \to \kappa\}$. Note that $\mu = \pi(\kappa)$ is expressible over $\langle S', F\rangle$ by $\langle \mu, \kappa\rangle \in F$ and $e' = F(e)$ is expressible by $\langle e', e\rangle \in F$. Thus $\lambda \in C_F$ is equivalent to the conjunction of 'λ is Gödel–closed' and:

$$\bigwedge e, e', \mu, f((\langle e', e\rangle \in F \wedge \langle \mu, \kappa\rangle \in F \wedge f : \kappa \to \kappa \wedge e = e_f)$$
$$\to (\kappa < \lambda < \mu \wedge \bigwedge \alpha < \lambda \bigvee \beta < \lambda \prec \alpha, \beta \succ \in e'))$$

QED (Lemma 3.3.4)

We now turn to the task of analyzing the complexity of the property of being a premouse and the circumstances under which this property is preserved by an embedding $\sigma : M \to M'$. If $M = \langle J_\nu^E, F\rangle$ is an active premouse, the answer to these question can vary with the type of F.

We shall be particularly interested in the case that, for some weakly amenable extender G on M at a $\tilde{\kappa} < \rho_M^n$, M' is the $\Sigma_0^{(n)}$ extension $\langle M', \sigma\rangle$ of M by G (i.e. $\sigma : M \to_G^{(n)} M'$). In this case we shall prove:

- M' is a premouse
- If M is active, then M' is active and of the same type
- If M is of type 2, then $\sigma(\max C_M) = \max C_{M'}$.

This will be the content of Theorem 3.3.24 below. Note that if G is close to M in the sense of §3.2, and n is maximal with $\tilde{\kappa} < \rho_M^n$, then M' is a fully Σ^*–preserving ultrapower of M (i.e. $\sigma : M \to_G^* M'$). In later sections we shall consider mainly iterations of premice by Σ^*–ultrapowers.

Note. In later sections we shall mainly restrict ourselves to premice of type 1. For the sake of completeness, however, we here prove the above result in full generality. The proof will be arduous.

We first define:

Definition 3.3.6. $M = \langle J^E_\nu, F\rangle$ is a *mouse precursor (or precursor for short) at* κ iff the following hold:

- M is acceptable
- $\kappa \in M$ and $\tau = \kappa^{+M} \in M$
- F is a full extender at κ on J^E_τ with extension $\langle J^E_\nu, \pi\rangle$.

Note. F then has base $J_\tau[E]$ and extension $\langle J_\nu[E], \pi\rangle$.

Note. F is weakly amenable, since $\mathbb{P}(\kappa) \cap M \subset J_\tau[E]$ by acceptability.

Lemma 3.3.5. $M = \langle J^E_\nu, F\rangle$ *is a precursor at* κ *iff the following hold:*

(a) M *is acceptable*

(b) F *is a function defined on* $\mathbb{P}(\kappa) \cap M$

(c) $F \upharpoonright \kappa = \mathrm{id}$, $\kappa < F(\kappa) = \lambda$, *where* λ *is the largest cardinal in* M.

(d) Let $a_1, \ldots, a_n \in \mathbb{P}(\kappa) \cap M$. *Let* φ *be a* Σ_1 *forumla. Then:*

$$J^E_\tau \models \varphi[\vec{a}] \leftrightarrow J^E_\nu \models \varphi[F(\vec{a})]$$

(e) Let $\xi < \nu$. *There is* $X \in \mathbb{P}(\kappa) \cap M$ *such that*

$$F(X) \notin J^E_\xi.$$

Proof: The direction $(\rightarrow)$ then follows easily. We prove $(\leftarrow)$.

We first note that F injects $\mathbb{P}(\kappa) \cap M$ into $\mathbb{P}(\lambda) \cap M$. F is injective by (d). But if $X \subset \kappa$, then $F(X) \subset F(\kappa) = \lambda$ by (d).

(1) $J^E_\kappa \prec J^E_\lambda$.

Proof: We first recall that by §2.4 each $x \in J^E_\kappa$ has the form $f(a)$ for some first $a \subset \kappa$, where f is $\Sigma_1(J^E_\kappa)$. By §2.4 we can choose the Σ_1 definition of f as being functionally absolute in J–models. Now let $x_1, \ldots, x_n \in J^E_\kappa$.

Let φ be a first order formula. We claim:

$$J_\kappa^E \models \varphi[\vec{x}] \to J_\lambda^E \models \varphi[\vec{x}].$$

Let $x_i = f_i(a_i)$, where $a_i \subset \kappa$ is finite and f_i has a functionally absolute definition '$x = f_i(a)$'. Then $J_\lambda^E \models$ '$x_i = f_i(a_i)$' for $i = 1, \ldots, n$. Let Ψ be the formula:

$$\bigvee x_1 \ldots x_n (\bigwedge_{i=1}^{n} x_i = f_i(a_i) \wedge \varphi(\vec{x})).$$

Then:

$$J_\kappa^E \models \varphi[\vec{x}] \leftrightarrow J_\kappa^E \models \Psi[\vec{a}]$$

and:

$$J_\lambda^E \models \varphi[\vec{x}] \leftrightarrow J_\lambda^E \models \Psi[\vec{a}].$$

But $J_\kappa^E \models \Psi[\vec{a}]$ is $\Sigma_1(M)$ in $\kappa, \vec{a}$ and $J_\lambda^E \models \Psi[\vec{a}]$ is $\Sigma_1(M)$ in $\lambda, \vec{a}$ by the same definition. Moreover $F(a_i) = a_i$ $(i = 1, \ldots, n)$ and $F(\kappa) = \lambda$.

Hence by (d):

$$\begin{aligned} J_\kappa^E \models \varphi[\vec{x}] &\leftrightarrow J_\kappa^E \models \Psi[\vec{a}] \\ &\leftrightarrow J_\lambda^E \models \Psi[\vec{a}] \\ &\leftrightarrow J_\lambda^E \models \varphi[\vec{x}]. \end{aligned}$$

QED (1)

It follows easily, using acceptability, that J_κ^E and J_λ^E are ZFC^- models. Gödel's pair function $\prec, \succ$ then has a uniform definition on J_κ^E and J_λ^E. Hence $\langle \prec \alpha, \beta \succ | \alpha, \beta \in J_\kappa^E \rangle$ is $\Sigma_1(M)$ in κ and $\langle \prec \alpha, \beta \succ | \alpha, \beta \in J_\lambda^E \rangle$ is $\Sigma_1(M)$ in λ by the same definition.

For any $X \subset \kappa$ there is at most one function $\Gamma = \Gamma_X$ defined on κ such that $\Gamma(\alpha) = \{\Gamma(\beta) | \langle \beta, \alpha \rangle \in X\}$ for $\alpha < \kappa$. For $X \in \mathbb{P}(\kappa) \cap M$ the statement $f = \Gamma_X$ is uniformly $\Sigma_1(M)$ in X, f, κ. Moreover the statement $\bigvee f\ f = \Gamma_X$ ('Γ_X is defined') is uniformly $\Sigma_1(M)$ in X, κ. The same is true at λ: For $Y \subset \lambda$ the statement $f = \Gamma_Y$ is uniformly $\Sigma_1(M)$ in Y, f, λ and the statement $\bigvee f\ f = \Gamma_Y$ is uniformly $\Sigma_1(M)$ in Y, λ by the same definition.

We must define a π such that $\langle J_\nu[E], \pi \rangle$ is the extension of F. The above remarks suggest a way of doing so:

Definition 3.3.7. Let $x \in J_\tau^E$, $x \in u$, where $u \in J_\tau^E$ is transitive. Let $f \in J_\tau^E$ map κ onto u. Set:

$$X =: \{\prec \alpha, \beta \succ | f(\alpha) \in f(\beta)\},$$

then $f = \Gamma_X$. Let $f' =: \Gamma_{F(X)}$. Let $x = f(\xi)$ where $\xi < \kappa$. Set:

$$\pi(x) = \pi_{f,\xi}(x) =: f'(\xi).$$

We must first show that π is independent of the choice of f, ξ. Suppose that $x \in v$, where $v \in J^E_\tau$ is transitive, and $g \in J^E_\tau$ maps κ onto v. Then, letting $Y = \{\prec \alpha, \beta \succ | g(\alpha) \in g(\beta)\}$, we have: Let $x = g(\zeta)$. Then by (d):

$$f(\xi) = \Gamma_X(\xi) = \Gamma_Y(\zeta) \to \pi_{f,\xi}(x) = \Gamma_{F(X)}(\xi) = \Gamma_{F(Y)}(\zeta) = \pi_{g,\zeta}(x).$$

Similarly we get:

(2) $\pi : J^E_\tau \to_{\Sigma_0} j^E_\nu$.

Proof: Let $x_1, \ldots, x_n \in J^E_\tau$. Let $x_1, \ldots, x_n \in u$, where $u \in J^E_\tau$ is transitive. Let $f_i \in J^E_\tau$ map κ onto $u(i = 1, \ldots, n)$. Set: $X_i = \{\prec \alpha, \beta \succ | f_i(\alpha) \in f_i(\beta)\}$. Let $x_i = f_i(\xi_i)$. Let φ be Σ_0. By (d) we conclude:

$$\begin{aligned} J^E_\tau \models \varphi[\vec{x}] \quad &\leftrightarrow J^E_\tau \models \varphi(\Gamma_{\vec{X}}(\vec{\xi})) \\ &\leftrightarrow J^E_\tau \models \varphi(\Gamma_{F(\vec{X})}(\vec{\xi})) \end{aligned}$$

where $F(X_i)(\xi_i) = \pi(\xi_i)$. QED (2)

(3) $F(X) = \pi(X)$ for $X \in \mathbb{P}(\kappa) \cap M$.

Proof: Let $X = f(\mu)$ where $\mu < \kappa$, $f \in J^E_\tau$, and $f : \kappa \to u$, where u is transitive. Set: $Y =: \{\prec \alpha, \beta \succ | f(\alpha) \in f(\beta)\}$. Then $f = \Gamma_Y$ and $X = \Gamma_Y(\mu)$. By (d) we conclude:

$$F(X) = \Gamma_{F(Y)}(\mu) = \pi(X).$$

QED (3)

It remains only to show:

(4) $\pi : J^E_\tau \to J^E_\nu$ cofinally.

Proof: Let $y \in J^E_\nu$. If $y \in J^E_\xi$, $\xi < \nu$, there is an $X \in \mathbb{P}(\kappa) \cap M$ such that $F(X) \notin J^E_\xi$. Let $X \in J^E_\mu$, $\mu < \tau$. Then:

$$F(X) = \pi(X) \in J^E_{\pi(\mu)}.$$

Hence $\pi(\mu) > \xi$ and:

$$y \in J^E_{\pi(\mu)} = \pi(J^E_\mu).$$

QED (Lemma 3.3.5)

Corollary 3.3.6. *Let $M = \langle J^E_\nu, F\rangle$. The statement 'M is a precursor' is uniformly* $\Pi_2(M)$.

Proof: The conjunction of (a) – (e) is uniformly $\Pi_2(M)$ in the parameters κ, λ. Let it have the form $R(\kappa, \lambda)$, where R is Π_2. It is evident that if $R(\kappa, \lambda)$ holds, then $\langle\kappa, \lambda\rangle$ is the unique pair of ordinals which is an element of F. Hence the conjunction (a) – (e) is expressible by:

$$\bigvee \kappa, \lambda\langle\kappa, \lambda\rangle \in F \wedge \bigwedge \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \rightarrow R(\kappa, \lambda)).$$

QED (Corollary 3.3.6)

Definition 3.3.8. $M = \langle J^E_\nu, F\rangle$ is a *good precursor* iff M is a precursor and F satisfies the initial segment condition.

Corollary 3.3.7. *Let $M = \langle J^E_\nu, F\rangle$. The statement 'M is a good precursor at κ' is uniformly* $\Pi_3(M)$.

Proof: Let M be a precursor. Then F satisfies the initial segment condition iff in M we have, letting $C =: C_F$:

$$\begin{aligned}&\bigwedge \eta \in C \bigvee F'(F' \text{ is a function } \wedge \operatorname{dom}(F) = \mathbb{P}(\kappa))\\ &\wedge \bigwedge Y, X(\langle Y, X\rangle \in F \rightarrow \langle Y \cap \eta, X\rangle \in F')\end{aligned}$$

This is Π_3 since C is Π_2. QED (Lemma 3.3.7)

Lemma 3.3.8. *Let $M = \langle J_\nu, F\rangle$ be a precursor at κ. Let $\tau = \kappa^{+M}$ and let $\langle J^E_\nu, \pi\rangle$ be the extension of J^E_τ by F. Then π and $\operatorname{dom}(\pi)$ are uniformly* $\Delta_1(M)$.

Proof: π is uniformly $\Sigma_1(M)$ in κ, λ since by the definition of π in the proof of Lemma 3.3.5 we have:

$$\begin{aligned}y &= \pi(x) \leftrightarrow \bigvee f \bigvee u \bigvee X \bigvee \xi \bigvee Y(u \text{ is transitive } \wedge\\ &f : \kappa \xrightarrow{\text{onto}} u \wedge x = f(\xi) \wedge X = \{\prec \alpha, \beta \succ | f(\alpha) \in f(\beta)\}\\ &\wedge Y = F(X) \wedge y = \Gamma_Y(\xi)).\end{aligned}$$

Let $\varphi(\kappa, \lambda, y, x)$ be the uniform Σ_1 definition of π from κ, λ. Then $\langle\kappa, \lambda\rangle$ is the unique pair of ordinals such that $\langle\kappa, \lambda\rangle \in F$. Hence:

$$y = \pi(x) \leftrightarrow \bigvee \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \wedge M \models \varphi[\kappa, \lambda, y, x]).$$

Then π is uniformly $\Sigma_1(M)$. But $\operatorname{dom}(\pi) = J^E_\tau$; hence:

$$\begin{aligned}y \in \operatorname{dom}\pi \quad &\leftrightarrow \bigvee \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \wedge y \in (J^E_{\kappa^+})^{J^E_\lambda})\\ &\bigwedge \kappa, \lambda(\langle\kappa, \lambda\rangle \in F \rightarrow y \in (J^E_{\kappa^+})^{J^E_\lambda}).\end{aligned}$$

Thus $\mathrm{dom}(\pi)$ is uniformly $\Delta_1(M)$. But then

$$\begin{aligned} y = \ & \pi(x) \leftrightarrow (y \in \mathrm{dom}(\pi) \wedge \\ & \bigwedge y' \in M(y \neq y' \rightarrow y' \neq \pi(x))). \end{aligned}$$

Thus π is $\Delta_1(M)$. QED (Lemma 3.3.8)

But then:

Corollary 3.3.9. *Let $\sigma : M \rightarrow_{\Sigma_1} M'$ where $M = \langle J_\nu^E, F\rangle$ and $M' = \langle J_{\nu'}^{E'} F'\rangle$ are precursors. Let $\langle J_\nu^E, \pi\rangle$ be the extension of J_τ^E by F and $\langle J_{\nu'}^{E'}, \pi'\rangle$ be the extension of $J_{\tau'}^{E'}$ by F. Then:*

$$\sigma\pi(x) \simeq \pi'\sigma(x) \text{ for } x \in M.$$

The satisfaction relation for an amenable structure $\langle J_\nu^E, B\rangle$ is uniformly $\Delta_1(M)$ in the parameter $\langle J_\nu^E, B\rangle$ whenever $M \ni \langle J_\nu^E, B\rangle$ is transitive and rudimentarily closed.

(To see this note that, letting $E = E \cap J_\nu^E$, the structure $\langle M, E, B\rangle$ is rud closed. Hence its Σ_0–satisfaction is $\Delta_1(\langle M, E, B\rangle)$ or in other words $\Delta_1(M)$ in E, B. But if φ is any formula in the language of $\langle J_\nu^E, B\rangle$, we can convert it to a Σ_0 formula $\overline{\varphi}$ in the language of $\langle M, E, B\rangle$ simply by bounding all quantifiers by a new variable v. Then:

$$\langle J_\nu^E, B\rangle \models \varphi[\vec{x}] \leftrightarrow \langle M, E, B\rangle \models \overline{\varphi}[J_\nu[E], \vec{x}]$$

for all $x_1, \ldots, x_n \in J_\nu^E$.)

It is apparent from §2.5 that for each n there is a statement φ_n such that

$$\langle J_\nu^E, B\rangle \text{ is } n\text{–sound } \leftrightarrow \langle J_\nu^E, B\rangle \models \varphi_n.$$

Moreover the sequence $\langle \varphi_n | n < \omega\rangle$ is recursive. Thus

Lemma 3.3.10. *"$\langle J_\nu^E, B\rangle$ is sound" is uniformly $\Pi_1(M)$ in $\langle J_\nu^E, B\rangle$ for all transitive rud closed $M \ni \langle J_\nu, B\rangle$.*

Using this we get:

Lemma 3.3.11. *Let J_ν^E be acceptable. The statement '$\langle J_\nu^E, \emptyset\rangle$ is a premouse' is uniformly $\Pi_1(J_\nu^E)$.*

Proof: $\langle J_\nu^E, \emptyset\rangle$ is a premouse iff the following hold in J_ν^E:

- $\bigwedge x \in E \bigvee \nu, z \in TC(x)(x = \langle z, \nu\rangle \wedge \nu \in \mathrm{Lm} \wedge z \in J_\nu^E)$

- $\bigwedge \nu(\nu \in \mathrm{Lm} \to \langle J^E_\nu, E''\{\nu\}\rangle$ is sound)
- $\bigwedge \nu(E''\{\nu\} \neq \emptyset \to \langle J^E_\nu, E''\{v\}\rangle$ is a good precursor).

QED (Lemma 3.3.11)

An immediate corollary is:

Corollary 3.3.12. *Let $\overline{M}, M$ be acceptable. Then:*

- *If $\pi : \overline{M} \to_{\Sigma_1} M$ and $\overline{M}$ is a passive premouse, then so is M.*
- *If $\pi : \overline{M} \to_{\Sigma_0} M$ and M is a passive premouse, then so is $\overline{M}$.*

The property of being an active premouse will be harder to preserve. $\langle J^E_\nu, F\rangle$ is an active premouse iff $\langle J^E_\nu, \emptyset$ is a passive premouse and $\langle J^E_\nu, F\rangle$ is a good precursor. Hence:

Lemma 3.3.13. *'$\langle J^E_\nu, F\rangle$ is an active premouse' is uniformly $\Pi_3(\langle J^E_\nu, F\rangle)$.*

Note. This uses that being acceptable is uniformly $\Pi_1(\langle J^E_\nu, F\rangle)$ when $\nu \in \mathrm{Lm}^*$.

An immediate, but not overly useful, corollary is:

Corollary 3.3.14. *Let $\overline{M}, M$, be J–models.*

- *If $\pi : \overline{M} \to_{\Sigma_3} M$ and $\overline{M}$ is an active premouse, then so is M.*
- *If $\pi : \overline{M} \to_{\Sigma_2} M$ and M is an active premouse, then so is $\overline{M}$.*

In order to get better preservation lemmas, we must think about the *type* of F in $\langle J^E_\nu, F\rangle$. F is of type 1 iff $C_F = \emptyset$. By Corollary 3.3.4 the condition $C_F = \emptyset$ is $\Pi_2(\langle J_\nu, F\rangle)$ uniformly. Hence

Lemma 3.3.15. *The statement 'M is an active premouse of type 1' is uniformly $\Pi_2(M)$ for $M = \langle J^E_\nu, F\rangle$.*

Hence

Corollary 3.3.16. *Let $\overline{M}, M$ be J–models.*

- *If $\pi : \overline{M} \to_{\Sigma_2} M$ and $\overline{M}$ is an active premouse of type 1, then so is M.*
- *If $\pi : \overline{M} \to_{\Sigma_1} M$ and M is an active premouse of type 1, then so is $\overline{M}$.*

A more important theorem is this:

Lemma 3.3.17. *Let M be an active premouse of type 1. Let $M = \langle J_\nu^E, F\rangle$ where $\kappa = \mathrm{crit}(F)$. Let G be a weakly amenable extender on M at $\tilde{\kappa}$, where $\tilde{\kappa} < \rho_M^n$. Let $\langle M', \sigma\rangle$ be the $\Sigma_0^{(n)}$ extension of M by G. Then M' is an active premouse of type 1.*

Proof: We consider two cases:

Case 1 $n = 0$.

Claim 1 $M' = \langle J_{\nu'}^{E'}, F'\rangle$ is a precursor.

(1) F' is a function and $\mathrm{dom}(F') \subset \mathbb{P}(\kappa)$, since these statements are Π_1 and σ is Σ_1 preserving
For $\xi < \tau = \kappa^{+M}$ set: $\pi[\xi] = \pi \restriction J_\xi^E, \pi'[\xi] = \sigma(\pi[\xi])$, then

(2) $\pi'[\xi] : J_{\sigma(\xi)}^E \prec J_{\sigma\pi(\xi)}^E$,
since $\pi[\xi] : J_\xi^E \prec J_{\pi(\xi)}^E$.
Set: $\pi' = \bigcup\limits_\xi \pi'[\xi]$. Since $\sup \pi''\tau = \nu$ and $\sup \sigma''\nu = \nu'$, we have

(3) $\sigma : \langle M, \pi\rangle \to_{\Sigma_0} \langle M', \pi'\rangle$ cofinally.

(4) $\mathrm{dom}(\pi') = \bigcup\limits_{\xi<\tau} \tau(J_\xi^E) = J_{\tau'}^{E'}$,
where $\tau' = \sigma(\tau) = \kappa'^{+M'}$ and $\kappa' = \sigma(\kappa)$. Hence

(5) $\pi' : J_{\tau'}^{E'} \to_{\Sigma_0} J_{\nu'}^{E'}$ cofinally.

(6) $F' = \pi' \restriction \mathbb{P}(\kappa')$
by (3) and:

$$\bigwedge X(X \in J_{\sigma(\xi)}^E \cap \mathbb{P}(\kappa') \to \langle \pi'(X), X\rangle \in F'),$$

since the corresponding Π_1 statement holds of ξ in M.

It follows easily that $\langle J_{\nu'}[E'], \pi'\rangle$ is the extension of $J_{\tau'}^E$ by F'.

QED (Claim 1)

Claim 2 F' is of type 1 (hence F' satisfies the initial segment condition).

Proof: Let $\xi < \lambda' = \pi'(\kappa')$. Using Lemma 3.3.2 we show:

Claim $\xi \notin C_{F'}$.

Let $\zeta \in M$ be least such that $\sigma(\zeta) \geq \zeta$. Since $\zeta \notin C_F$, there is $f : \kappa^n \to \kappa$ in M such that $\pi(f)(\vec{\alpha}) > \zeta$ for some $\alpha_1, \ldots, \alpha_n < \zeta$. But then $\sigma(\alpha_1), \ldots, \sigma(\alpha_n) < \xi$ and

$$\pi'(\sigma(f))(\sigma(\vec{\alpha})) = \sigma(\pi(f))(\vec{\alpha})) > \sigma(\zeta) \geq \xi.$$

Hence $\xi \notin C_{F'}$. QED (Claim 2)

Thus $J^{E'}_{\nu'}$ is a premouse by Corollary 3.3.12 and M' is a good precursor of type 1. Hence M' is a premouse of type 1. QED (Case 1)

Case 2 $n > 1$.

Then σ is Σ_2–preserving by Lemma 3.2.12. Hence M' is a premouse of type 1 by Corollary 3.3.16 QED (Corollary 3.3.17)

We now consider premice of type 2. $M = \langle J^E_\nu, F\rangle$ is a premouse of type 2 iff J^E_ν is a premouse, M is a precursor and $F|\eta \in J^E_\nu$ where $\eta = \max C_F$. (It then follows that $F|\mu = (F|\eta)|\mu \in J^E_\nu$ whenever $\mu \in C_F$.) The statement $e = F|\mu$ is uniformly $\Pi_1(M)$ in e, u, μ, since it says:

$$e \text{ is a function } \wedge \bigwedge x \in \mathbb{P}(\kappa) \cap M e(X) = F(X) \cap \mu.$$

But then the statement:

$$e = F|\eta \wedge \eta = \max C_F$$

is $\Pi_2(M)$ in e, η, κ uniformly, since it says: $e = F|\eta \wedge C_F \setminus \eta = \emptyset$, where C_F is uniformly $\Pi_2(M)$. It then follows easily that:

Lemma 3.3.18. *Let $M = \langle J^E_\nu, F\rangle$, $M = \langle J^{E'}_{\nu'}, \overline{F}\rangle$.*

- *If $\pi : \overline{M} \to_{\Sigma_2} M$ and $\overline{M}$ is a premouse of type 2, then so is M. Moreover, $\pi(\max C_E) = \max C_F$.*
- *If $\pi : \overline{M} \to_{\Sigma_1} M$, M is a premouse of type 2 and $e = F|\max(C_F) \in \operatorname{rng}(\pi)$, then $\overline{M}$ is a premouse of type 2 and $\pi(\max C_{\overline{F}}) = \max C_F$.*

We also get:

Lemma 3.3.19. *Let M be a premouse of type 2. Let G be a weakly amenable extender on M at $\tilde{\kappa}$, where $\tilde{\kappa} < \rho^n_M$. Let $\langle M', \sigma\rangle$ be the $\Sigma^{(n)}_0$ extension of M by G. Then M' is a premouse of type 2. Moreover, $\sigma(\max C_M) = \max C_{M'}$.*

Proof: If $n > 0$, then σ is Σ_2–preserving and the result follows by Lemma 3.3.18. Now let $n = 0$. Let $M = \langle J^E_\nu, F\rangle$ where F is an extender at κ on J^E_τ (where $\tau = \kappa^{+M}$. Let $M' = \langle J^{E'}_{\nu'}, F'\rangle$). It follows exactly as in Lemma 3.3.17 that $J^{E'}_{\nu'}$ is a premouse and M' is a precursor. We must prove:

Claim F' is of type 2. Moreover, $\tau(\max C_F) = \max C_{F'}$.

Proof: Let $\eta = \max C_F$, $e = F|\eta$. Then $\sigma(e) = F'|\eta'$, since this is a Π_1 condition. But then $C_{F'} \setminus \eta' = \emptyset$ follows exactly as in Lemma 3.3.17, since $C_F \setminus \eta = \emptyset$ and σ takes $\lambda = F(\kappa)$ cofinally to $\lambda' = F'(\kappa')$. QED (Lemma 3.3.19)

We now turn to premice of type 3. One very important property of these structures is:

Lemma 3.3.20. *Let $M = \langle J_\nu^E, F\rangle$ be a premouse of type 3. Let $\lambda = F(\kappa)$ where F is at κ. Then $\rho_M^1 = \lambda$.*

Proof:

(1) $h_M(\lambda) = M$. Hence $\rho_M^1 \le \lambda$.

Proof: Note that if $X \in \mathbb{P}(\kappa) \cap M$, then $X \in J_\tau^E \subset h_M(\tau)$. Hence $F(X) \in h_M(\tau)$, since F is $\Sigma_1(M)$. Hence $\xi \in h_M(\tau)$ for a ξ such that $F(X) \in J_\xi^E$. Hence $\mathrm{On} \cap h_M(\tau)$ is cofinal in ν. Let $x \in M$ such that $x \in J_\xi^E$ for a $\xi \in h_M(\tau)$. Then there is $f \in h_M(\tau)$ such that $f\colon \lambda \xrightarrow{\text{onto}} J_\xi^E$. But them $x = f(\alpha)$ for an $\alpha < \lambda$. Hence $x \in f''\lambda \subset h_M(\lambda)$. QED (1)

(2) Let $D \subset \lambda$ be $\underline{\Sigma}_1(M)$. Then $\langle J_\lambda^E, D\rangle$ is amenable. (Hence $\rho_M^1 \ge \lambda$.)

Proof: By (1) D is $\Sigma_1(M)$ in a parameter $\alpha < \lambda$. Let $\eta \in C_F$ such that $\eta > \alpha$. Then $E = F|\eta \in M$. Since J_λ^E is a ZFC^- model, we have:

$$\langle J_{\overline{\nu}}^{\overline{E}}, \overline{F}\rangle \in J_\lambda^E,\text{ where } \pi : J_\tau^E \to_{\overline{F}} J_{\overline{\nu}}^{\overline{E}}.$$

We then observe that there is a unique $\sigma : J_{\overline{\nu}}^{\overline{E}} \prec J_\nu^E$ defined by

$$\sigma(\overline{\pi}(f)(\beta)) = \pi(f)(\beta) \text{ for}$$
$$f \in J_\tau^E, f : \kappa \to J_\tau^E, \beta < \eta.$$

Moreover, $\sigma \restriction \eta = \mathrm{id}$ and σ is cofinal.

(To see that this definition works, let $\beta_1, \dots, \beta_n < \eta$, $f_1, \dots, f_n \in \tau$ such that $f_i : \kappa \to J_\tau^E$ for $i = 1, \dots, n$. Set:

$$X = \{\prec \xi_1, \dots, \xi_n \succ | J_\tau^E \models \varphi[f_1(\xi_1), \dots, f_n(\xi_n)]\}.$$

Then:

$$\begin{aligned} J_{\overline{\nu}}^{\overline{E}} \models \varphi[\overline{\pi}(\vec{f}(\vec{\beta})] \quad &\leftrightarrow \prec \vec{\beta} \succ \in \overline{F}(X) = \eta \cap F(X) \\ &\leftrightarrow J_\nu^E \models \varphi[\pi(\vec{f})(\vec{\beta})].) \end{aligned}$$

But $\sigma(\langle \overline{F}(Z), Z\rangle) = \langle F(Z), Z\rangle$ for $Z \in \mathbb{P}(\kappa) \cap M$. Hence:

$$\sigma(\overline{F} \cap U) = \sigma''(\overline{F} \cap U) = F \cap U.$$

By this we get:

$$\sigma : \langle J_{\overline{\nu}}^{\overline{E}}, \overline{F}\rangle \to_{\Sigma_0} \langle J_\nu^E, F\rangle \text{ cofinally.}$$

Thus $\overline{D} = D \cap \eta$ is $\Sigma_1(\langle J_{\overline{\nu}}^{\overline{E}}, \overline{F}\rangle)$ in α by the same definition as D over $\langle J_\nu^E, F\rangle$. Hence $\overline{D} \in J_\lambda^E$, since $\langle J_{\overline{\nu}}^{\overline{E}}, \overline{F}\rangle \in J_\nu^E$. QED (Lemma 3.3.20)

Note that the argument of (1) holds for arbitrary premice. Hence:

Lemma 3.3.21. *Let $M\langle J^E_\gamma, F\rangle$ be an active premouse. Then $h_M(\lambda) = M$ (hence $\rho^1_M \leq \lambda$).*

If $M = \langle J^E_\nu, F\rangle$ is a precursor, then "F is of type 3" is uniformly $\Pi_3(M)$ in κ, since it is the conjunction:

$$\bigwedge \xi < \lambda \bigvee \eta < \lambda \cdot \eta \in C_F \wedge \bigwedge \xi < \eta \in C_F \bigvee e \in J^E_\lambda e = F|\eta.$$

Hence:

Lemma 3.3.22. *(a) Let $\pi : \overline{M} \longrightarrow_{\Sigma_3} M$ where $\overline{M}$ is a premouse of type 3. Then so is M.*

(b) Let $\pi : \overline{M} \longrightarrow_{\Sigma_2} M$ where M is a premouse of type 3. Then so is $\overline{M}$.

We also get:

Lemma 3.3.23. *Let $M = \langle J^E_\nu, F\rangle$ be a premouse of type 3. Let G be a weakly amenable extender at $\tilde{\kappa}$ on M. Let $\tilde{\kappa} < \rho^n_M$ and let $\langle M', \sigma\rangle$ be the $\Sigma^{(n)}_0$ extension of M by G. Then M' is a premouse of type 3.*

Proof: Let $M' = \langle J^{E'}_{\nu'}, F'\rangle$. We consider three cases:

Case 1 $n = 0$.

Exactly as in the previous lemmas we get: $J^{E'}_{\nu'}$ is a premouse and M' is a precursor. We must show:

Claim F is of type 3.

We know that σ takes λ cofinally to λ'. Let $\eta < \lambda, \eta \in C_F$. Let $e = F|\eta \in M$. Then $\sigma(\eta) \in C_{F'}$ and $\sigma(e) = F'|\sigma(\eta)$, since these statements are Π_1. Hence if $\mu < \lambda'$ there is $\eta \in C_F$ such that $\mu \leq \sigma(\eta)$ and

$$F'|\mu = (F'|\sigma(\eta))|\mu \in J^{E'}_{\lambda'}.$$

QED (Case 1)

Case 2 $n = 1$.

Then σ is Σ_2–preserving. Hence $J^{E'}_{\nu'}$ is a premouse and M' is a precursor. Let $\langle M, \pi\rangle$ be the extension of J^E_τ by F and $\langle M', \pi'\rangle$ the extension of $J^{E'}_{\tau'}$ by F', where $\tau = \kappa^{+M}, \tau' = \sigma(\tau) = \kappa'^{+M'}$.

We know that:

$$\sigma \upharpoonright J^E_\lambda : J^E_\lambda \to_G J^E_{\rho'},$$

where $\lambda = \pi(\kappa) = \rho^1_M$ and $\rho' = \sup \sigma"\lambda = \rho^1_{M'}$. Since τ is a successor cardinal in J^E_λ, we have $\tau \neq \mathrm{crit}(G)$. But then $\tau' = \sup \sigma"\tau$ by Lemma 3.2.6 of §3.2. π takes τ cofinally to ν and π' takes τ' cofinally to ν'. Using this we see:

(1) $\nu' = \sup \sigma"\nu$.

Proof: Let $\xi < \nu'$. Let $\zeta < \tau'$ such that $\pi'(\zeta) > \xi$. Let $\eta < \tau$ such that $\sigma(\eta) > \zeta$. By Corollary 3.3.9 we have:

$$\sigma\pi(\eta) = \pi'\sigma(\eta) > \xi.$$

QED (1)

But then it suffices to show:

Claim $\sigma : M \to_G M'$,

since then we can argue as in Case 1.

Let $x \in M'$. Let $\tilde{\kappa} = \mathrm{crit}(\pi)$. We must show that $x = \sigma(f)(\xi)$ for an $f \in M$ such that $f : \kappa \to M$. Since M' is the $\Sigma^{(1)}_0$–ultrapower, we know:

$$x = \sigma(f)(\xi), \text{ where } f : \kappa \to M \text{ is } \underset{\sim}{\Sigma}_1(M).$$

Choosing a functionally absolute definition for f we have:

$$v = f(w) \leftrightarrow \bigvee y A(y, v, w, p)$$

where A is $\Sigma_0(M)$ and $p \in M$. By functional absoluteness we have:

$$v = \sigma(f)(w) \leftrightarrow \bigvee y A'(\eta, v, w, \sigma(p))$$

where A' is $\Sigma_0(M')$ by the same definition. Let $A'(y, x, \xi, \sigma(p))$. Since σ takes M cofinally to M' there is $a \in M$ such that $y, x \in \sigma(a)$ and $\tilde{\kappa} \subset a$. Set:

$$g(\mu) = \begin{cases} x \text{ if } x \in a \wedge \bigvee y \in a A(y, x, \mu, p) \\ 0 \text{ if no such } x \text{ exists.} \end{cases}$$

Then $g \in M$, $g : \tilde{\kappa} \to M$ and $x = \sigma(g)(\xi)$. QED (Case 2)

Case 3 $n > 1$.

Then $\rho^1_{M'} = \tau(\rho^1_M) = \lambda'$ and σ is $\Sigma^{(1)}_2$–preserving by Lemma 3.2.12. But C_F is now $\Sigma^{(1)}_0(M)$ and $e = F|\eta$ is $\Sigma^{(1)}_0(M)$ for $e, \eta \in J^E_\lambda$. The statements:

$$\bigwedge \xi < \lambda \bigvee \eta < \lambda(\xi < \eta \in C_F, \bigwedge \eta \in C_F(\bigvee e \in J^E_\lambda \; e = F|\eta)$$

are now $\Pi^{(1)}_2(M)$. Hence the corresponding statements hold in M'. Hence $C_{F'}$ is unbounded in λ' and $F'|\eta \in J^{E'}_{\lambda'}$ for $\eta \in C_{F'}$. Then M' is of type 3. QED (Lemma 3.3.23)

Combining lemmas 3.3.12, 3.3.14, 3.3.19 and 3.3.23 we have:

Theorem 3.3.24. *Let M be a premouse. Let G be an extender at $\tilde{\kappa}$ on M where $\rho_M^n > \tilde{\kappa}$. Let $\langle M', \sigma \rangle$ be the $\Sigma_0^{(n)}$ extension of M by G. Then:*

- *M' is a premouse*
- *If M is active then M' is active and of the same type*
- *If M is of type 2, then*

$$\sigma(\max C_M) = \max C_{M'}.$$

In order to show that premousehood is preserved under iteration we shall also need:

Theorem 3.3.25. *Let M_0 be a premouse. Let $\pi_{ij} : M_i \to_{\Sigma_1} M_j$ for $i \leq j \leq \eta$, where:*

- *$\pi_{i,i+1} : M_i \to_{G_i}^{(n_i)} M_{i+1}$, where G_i is an extender at $\tilde{\kappa}_i$ on $G_i (i < \eta)$*
- *M_i is transitive and the π_{ij} commate*
- *If $\lambda \leq \eta$ is a limit ordinal, then $M_\lambda, \langle \pi_i | i < \lambda \rangle$ is the transitivized direct limit of $\langle M_i | i < \lambda \rangle, \langle \pi_{ij} | i \leq j < \lambda \rangle$.*

Then:

- *M_η is a premouse*
- *If M_0 is active, then M_η is active and of the same type as M_0*
- *If M_0 is of type 2, then $\pi_{0\eta}(C_{M_0}) = C_{M'_\eta}$.*

Proof: We proceed by induction on η. Thus the assertion holds at every $i < \eta$. The case $\eta = 0$ is trivial, as is $\eta = \mu + 1$ by Theorem 3.3.24. Hence we assume that η is a limit ordinal. We make the following observation:

(1) Let φ be a Π_3 formula. Let $i < \eta, x_1, \ldots, x_n \in M_i$ such that $M_j \models \varphi[\pi_{ij}(\vec{x})]$ for $i \leq j < \eta$. Then $M_\eta \models \varphi[\pi_{i\eta}(\vec{x})]$.

Proof: Let $y \in M_\eta$. Pick j such that $i \leq j < \eta$ and $y = \pi_{i\eta}(\overline{y})$. Then $M_j \models \Psi[\overline{y}, \pi_{ij}(\vec{x})]$, where $\varphi = \bigwedge v \Psi$. Hence $M_j \models \chi[\overline{z}, \overline{x}, \pi_{ij}(\vec{x})]$ for some $\overline{z}$, where $\Psi = \bigvee u\chi$. Hence $M_\eta \models \chi[z, y, \pi_{i\eta}(\vec{x})]$ where $z = \pi_{i\eta}(\overline{z})$, since $\pi_{j\eta}$ is Σ_1–preserving. QED (1)

Each M_i is a premouse for $i < \eta$. But this condition is uniformly $\Pi_3(M_i)$ by Lemma 3.3.13. Hence M_η is a premouse. If M_0 is of type 1, then $C_{M_i} = \emptyset$ for $i < \eta$. But this condition is uniformly $\Pi_2(M_i)$; Hence M_η is of type 1.

Now let M_0 be of type 2 and let $\mu_0 = \max C_{M_0}$. Then M_i is of type 2 and $\mu_i = \max C_{M_i}$ for $i < \eta$, where $\mu_i = \Pi_{0i}(\mu_0)$. Let $e_0 = F_0|\mu_0$ where $M_0 = \langle J^{E_0}_{\nu_0}, F_0\rangle$. Then $e_i = F_i|\mu_i$ for $i < \eta$, since $e = F|\mu$ is a Π_1 condition. Thus for $i < \rho$ each M_i satisfies the Π_2 condition in e_i, μ_i:

$$e_0 = F_i|\mu_i \wedge C_{F_i} \setminus \mu_i = \emptyset.$$

Hence M_η satisfies the corresponding condition. Hence M_η is of type 2 and $\mu_\eta = \max(C_\eta)$. Clearly $C_{M_i} = C_{F_i} \cup \{\max C_{M_i}\}$ for $i \leq \eta$. Hence $\pi_{ij}(C_{M_i}) = C_{M_i}$.

Now assume that M_0 is of type 3. Then each $M_i (i < \eta)$ satisfies the Π_3 condition:

$$\bigwedge \xi < \lambda_i \bigvee \zeta < \lambda_i (\xi < \zeta \in C_{M_i}),$$
$$\bigwedge \zeta \in C_{M_i} \bigvee e \in J^{E_i}_{\lambda_i} e = F_i|\zeta.$$

But then M_η satisfies the corresponding conditions. Hence M_η is of type 3. QED (Theorem 3.3.25)

We also note:

Lemma 3.3.26. *Let N be a premouse. Let $\nu \in N$ such that $F = E^N_\nu \neq \emptyset$. Let $\kappa = \mathrm{crit}(F)$, $\lambda = \lambda(F)$. If λ is regular in N, then $N||\nu$ is of type 3.*

Proof. Let $\pi : J^E_\tau \longrightarrow_F J^E_\nu$, where $\tau = \tau(F)$. Define $\langle \alpha_i \mid i < \lambda\rangle$ by:

$\alpha_0 = \kappa$,

$\alpha_{i+1} =$ the set of $\pi(f)(\prec\vec{\xi}\succ) < \lambda$ where $\xi_1, \ldots, \xi_n \leq \alpha_i$, and $f \in H^N_\kappa$.

It is easily verified that $F|\alpha_\eta$ is a full extender for all limit $\eta < \lambda$.

QED (Theorem 3.3.26)

If, however, we merely assume λ to be a cardinal in N, the situation is quite different. To see this, let $N' = \langle J^{E'}_{\nu'}, F'\rangle$ be the shortest possible premouse of type 2. Let $\kappa = \mathrm{crit}\, F'$. Then $F = F'|\nu$ is full for a $\nu \in N'$. It is easily seen that $\lambda = \lambda(F)$ is a cardinal in N' although $N = N'||\nu$ is of type 1. Moreover, κ is superstrong in $J^{E'}_\lambda$, although no proper segment of $J^{E'}_\lambda$ is of type > 1.

3.4 Iterating premice

3.4.1 Introduction

We have stated that a mouse will be an iterable premouse, but left the meaning of the term *"iterable"* and *"iteration"* vague. Iteration turns out, indeed, to be a rather complex notion. Let us begin with the simplest example. most logicians are familiar with the iteration of a structure $\langle M, U\rangle$, where M is, say, a transitive ZFC^- model and $U \in M$ is a normal ultrafilter on $\mathbb{P}(U) \cap M$. Set: $M_0 = M, U_0 = U$. Applying U_0 to M_0 gives the ultraproduct $\langle M_1, U_1\rangle$ and the extension $\Pi_{0,1} : \langle M_0, U_0\rangle \to \langle M_1, U_1\rangle$ by U_0. We then repeat the process at $\langle M_1, U_1\rangle$ to get $\langle M_2, U_2\rangle$ etc. After $1 + \mu$ repetitions we get an *iteration of length* μ, consisting of a sequence $\langle\langle M_i, U_i\rangle | i < \mu\rangle$ of models and a commutative sequence $\langle\pi_{ij} | i \leq j < \mu\rangle$ of iteration maps $\pi_{ij} : M_i \to M_j$. These sequences are characterized by the conditions:

- $\pi_{i,i+1} : \langle M_i, U_i\rangle \to \langle M_{i+1}, U_i\rangle$ is the extension by U_i.
- The π_{ij} commute — i.e. $\pi_{ij} = \mathrm{id}$ and $\pi_{ij}\pi_{hi} = \pi_{hj}$ for $h \leq i \leq j < \mu$.
- If $\lambda < \mu$ is a limit ordinal, then $M, \langle\pi_{i\lambda} | i < \lambda\rangle$ is the direct limit of:

$$\langle M_i | i < \lambda\rangle, \langle M_{ij} | i \leq j < \lambda\rangle.$$

Now suppose we are given a structure $\langle M, S\rangle$ where $S = \{\langle X, \kappa\rangle | X \in U_\kappa\}$ and for each $\kappa \in M$, eiter $U_\kappa = \emptyset$ or else κ is a measurable cardinal in M and $U_\kappa \in M$ is a normal ultrafilter on $\mathbb{P}(\kappa) \wedge M$. An *iteration* of $\langle M, S\rangle$ then consists of sequences $\langle\langle M_i, S_i\rangle | i < \mu\rangle$, $\langle M_{ij} | i \leq j < \mu\rangle$ and $\langle\kappa_i | i + 1 < \mu\rangle$.

The first condition above is then replaced by:

> $\pi_{i,i+1} : \langle M_i, S_i\rangle \to \langle M_{i+1}, S_{i+1}\rangle$ is the extension by the ultrafilter
> $U_i = \{X | \langle X, \kappa_i\rangle \in S_i\}$

The other conditions remain unchanged. $\kappa_i | i + 1 \leq \mu\rangle$ is called the sequence of *indices*. κ_i must always be so chosen that U_i is an ultrafilter.

Note. Since we are allowed considerable leeway in the choice of the index κ_i, the purist may question whether the word "iteration" is still appropriate. In fact, the mathematical meaning of this word has rapidly changed as the structures to which it is applied have grown more complex.

An iteration is called *normal* iff the indices are increasing — i.e. $\kappa_i < \kappa_j$ for $i < j < \mu$.

We now attempt to apply these ideas to premice. Let M be a premouse. An iteration of length μ will yield a sequence $\langle M_i | i < \mu \rangle$ of premice. In passing from M_i to M_{i+1} we apply any of the extenders E_ν^M such that $M_i || \nu = \langle J_\nu^E, E_\nu \rangle$ is active. $v = \nu_i$ is then the i–th index. (It would be ambiguous to regard $\kappa_i = \text{crit}(E_{\nu_i})$ as the index, since M_i might have many extenders with this critical point.) In a normal iteration we have that, whenever $i < j$, then:

$$J_{\nu_i}^{E^{M_i}} = J_{\nu_i}^{E^{M_j}} \text{ and } \nu_i \text{ is a cardinal in } M_j.$$

(In fact, $\nu_i = \lambda_i^{+M_j}$, where $\lambda_i = E_{\nu_i}(\kappa_i)$ is inaccessible in M_j.) This follows easily by induction on j. It was originally envisaged that E_{ν_0} would be applied directly to M_i to get M_{i+1}. It turns out, however, that such iterations are unsuitable for may purposes. (In particular, they are unsuited to use in *comparison iteration*, which we shall describe below.) The problem is that $\kappa_i = \text{crit}(E_{\nu_i})$ could be much smaller than λ_i, where $\lambda_i = E_{\nu_i}(\kappa_i)$ is the largest cardinal in the model $J_{\nu_i}^{E^{M_i}}$. In particular, we might have $\kappa_i < \lambda_h$ for an $h < i$. Since λ_h is an inaccessible cardinal in M_i, it follows by acceptability that:

$$\mathbb{P}(\kappa) \cap M_i = \mathbb{P}(\kappa) \cap J_{\lambda_h}^{E^{M_h}} \subset M_h.$$

Hence it should be possible to apply E_{ν_i} to M_h rather than M_i. It turns out that it is most effective to apply E_{ν_i} to the *smalles place possible*: we apply it to $M_{T(i+1)}$, where

$$\begin{aligned} T(i+1) =: \ & \text{the least } h \text{ such that either } h = i \\ & \text{or } h < i \text{ and } \kappa_i < \lambda_h. \end{aligned}$$

This should give us

$$\pi_{h,i+1} : M_h \to M_{i+1}.$$

Here, however, we must deal with a second problem, which can arise even when $T(i+1) = i$. We know that E_{ν_i} is an extender at κ_i on $J_{\nu_i}^E$. Then $\mathbb{P}(\kappa_i) \cap J_{\nu_i}^{E^{M_i}} = \mathbb{P}(\kappa_i) \cap J_{\tau_i}^{E^{M_i}} = \mathbb{P}(\kappa_i) \cap J_{\tau_i}^{E^{M_h}}$, where $\tau_i = \kappa_i^{+J_{\nu_i}^E}$. But M_h might contain subsets of κ_i which do not lie in $J_{\tau_i}^E$ (hence τ_i is not a cardinal in M_h, by acceptability). E_{ν_i} is then only a partial function on M_h and cannot be applied to M_h. The resolution of this difficulty is to apply E_{ν_i} to the *largest possible segment* of M_h. We set:

$$\begin{aligned} M_i^* =: \ & M_h || \eta_h, \text{ where } \eta_i \leq \text{On}_{M_h} \text{ is maximal such that} \\ & \tau_h \text{ is a cardinal in } M_h || \eta. \end{aligned}$$

By acceptability, $\mathbb{P}(\kappa_i) \cap M_i^* = \mathbb{P}(\kappa_i) \cap J_{\tau_i}^E$ and $\rho_{M_i^*}^\omega \leq \kappa_i$ if $\eta_i < \text{On}_{M_h}$.

We then say that M_h *drops* (or *truncates*) to M_i^*, if $M_h \neq M_i^*$. $i+1$ is then called a *drop point* (or *truncation point*). $\pi_{h,i+1} : M_i^* \to M_{i+1}$ is then a *partial* map of M_h to M_{i+1}

This means that iteration is no longer a linear process. Previously π_{ij} was defined whenever $i \leq j < \mu$, μ being the length of the iteration. Now it is defined only when i is less than or equal to j in a tree T on μ. (We write $i \leq_T j$ for $i = j \vee iT_j$.) 0 is the unique minimal point of T. $T(i+1)$ is the unique T–predecessor of $i+1$. The π_{ij} are partial maps and we again have:

$$\pi_{ij} \cdot \pi_{hi} = \pi_{hj} \text{ for } h \leq_T i \leq_T j.$$

We will always have: $iT_j \to i < j$, but the converse may not hold. If $\mu = \omega$, these conditions completely define $T \subset \omega^2$. But how do we then extend the iteration to an iteration of length $\omega + 1$? Previously we simply took a transitivized direct limit of $\langle M_i | i < \omega \rangle$, $\langle \pi_{ij} | i \leq j < \omega \rangle$. Now we must first find a *branch* b in T which is cofinal in ω (i.e. $\sup b = \omega$). We also require that b have at most finitely may drop points. Pick any $i \in b$ such that $b \setminus i$ has no drop point. Then $\pi_{hj} : M_h \to M_j$ is a total map on M_h for $i {\leq_T} h \underset{T\,i}{\leq} \in b$. Form the direct limit:

$$M_b, \langle \pi_{h_i} | i \leq h \in b \rangle$$

of:

$$\langle M_h | i \leq h \in b \rangle,\ \langle \pi_{hj} | i \leq_T h \leq j \in b \rangle.$$

If M_b is well founded, we call b a *well founded branch* and take M_b are being transitive. We can then continue the iteration by setting:

$$M_\omega =: M_b; hT_\omega \leftrightarrow: h \in b \text{ for } h < \omega.$$

$\pi_{j\omega}$ is then defined for $i \leq_T j <_T \omega$. If hTi, we set $\pi_{h\omega} =: \pi_{j\omega} \cdot \pi_{hi}$.

The same procedure is applied at all limit points λ. We then have:

- λ is a limit point of T
- $T''\{\lambda\}$ is cofinal in λ
- $T''\{\lambda\}$ contains at most finitely many truncation points.

By now we have almost given a virtual definition of what is meant by a "normal iteration of a premouse". The only point left vague is what we mean by "applying" the extender E_{ν_i} to M_i^*. We shall, in fact, take the $\Sigma_0^{(n)}$–ultrapower:

$$\pi : M_i^* \to_{E_{\nu_i}}^{(n)} M_{i+1},$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{M_i^*}$.

3.4.2 Normal iteration

We are now ready to write out the formal definition of "normal iteration". We shall employ the following notational devices:

Definition 3.4.1. Let T be a tree. We set:

- $i <_T j \leftrightarrow: \circ T_j$
- $i \leq_T j \leftrightarrow: i = j \vee iT_j$
- $[i, j]_T =: \{h | i \leq_T h \leq_T j\}$ (similarly for $[i, j]_T$, $[i, j]_T$, $[i, j]_T$)
- $T(i) =:$ The immediate T–predecessor of i (if it exists).

We can now define:

Definition 3.4.2. Let M be a premouse. By a *normal iteration* of M of *length* μ we mean:

$$\langle \langle M_i | i < \mu \rangle, \langle \nu_i | i + 1 < \mu \rangle, \langle \pi_{ij} | i \leq_T j \rangle, T \rangle$$

where.

(a) T is a tree on μ such that $iT_j \to j < j$

(b) M_i is a premouse for $i < \mu$

(c) $\nu_i < \nu_j$ if $i < j$. Moreover $M_i || \nu_i = \langle J^E_{\nu_i}, E_{\nu_i} \rangle$ with $E_{\nu_i} \neq \emptyset$. (We set: $\kappa_i =: \mathrm{crit}(E_{\nu_i}), \tau_i =: \kappa_i^+ J^E_{\nu_i}$, $\lambda_i =: E_{\nu_i}(\kappa_i) =$ the largest cardinal in $J^E_{\nu_i}$.)

(d) Let h be least such that $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then $h = T(i+1)$ and $J^{E^{M_h}}_{\tau_i+1} = J^{E^{M_i}}_{\tau_i+1}$.

(e) π_{ij} is a partial map of M_i to M_j. Moreover $\pi_{ij} \circ \pi_{hi} = \pi_{hj}$ for $h \leq_T i \leq_T j$.

(f) Let $h = T(i+1)$. Set: $M_i^* = M_h || \eta_i$, where η_i is maximal such that τ_i is a cardinal in $M_h || \eta_i$. Then $\pi_{h,i+1} : M_i^* \to^{(n)}_{E^{M_i}_{\nu_i}} M_{i+1}$, where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{M_i^*}$. (We call $i+1$ a *drop point* or *truncation point* iff $M_i^* \neq M_h$)

(g) If $k \leq_j$ and $(i, j]_T$ has no drop point, then $\pi_{ij} : M_i \to M_j$ is a total function on M_i.

(h) Let λ be a limit ordinal. Then $T''\{\lambda\}$ is club in λ and contains at most finitely many drop points. Moreover, if $iT\lambda$ and $(i,\lambda)_T$ is free of drops, then:

$$M_\lambda, \langle \pi_{j\lambda} | i \leq_T j <_T \lambda \rangle$$

is the transitivized direct limit of:

$$\langle M_j | i \leq_T j <_T \lambda \rangle, \langle \pi_{hj} | i \leq_T h \leq_T j <_T \lambda \rangle.$$

This completes the definition.

Lemma 3.4.1. *Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$ be a normal iteration. Then*

(a) $J^{E^{M_i}}_{\nu_i} = J^{E^{M_{i+1}}}_{\nu_i}$

(b) *In M_{i+1}, λ_i is inaccessible and $\nu_i = \lambda_i^+$.*

Proof: τ_i is a cardinal in M_i^*. Since κ_i is inaccessible in $J^{E^{M_i}}_{\tau_i}$ and is the largest cardinal in $J^{E^{M_i}}_{\tau_i}$, it follows by acceptability that:

$$\tau_i = \kappa_i^+ \text{ and } \kappa_i \text{ is inaccessible in } M_i^*$$

$F = E^{M_i}{}_{\nu_i}$ is a full extender of length λ_i with base $H = |J^{E^{M_i}}_{\tau_i}|$ and extension $\langle \pi, H' \rangle$, where $H' = |J^{E^{M_i}}_{\nu_i}|$. By acceptability we have:

$$\mathbb{P}(\kappa_i) \cap M_i^* = \mathbb{P}(\kappa_i) \cap J^{E^{M_i}}_{\tau_i}$$

Hence F is an extender on M_i^* (and the condition (f) makes sense). But then $\langle M_{i+1}, \pi_{i,i+1} \rangle$ is the $\Sigma_i^{(n)}$-liftup of $\langle M_i^*, \pi \rangle$, where n is maximal such that $\kappa_i < \rho^n_{M_i^*}$. Hence:

$$\pi_{i,i+1}(\tau_i) = \sup \pi\text{"}\tau_i = \nu_i \text{ and } \pi_{i,i+1}(\kappa_i) = \lambda_i$$

Hence (b) holds, since the corresponding statement is function of κ_i, τ_i in M_i^*.

To see that (a) holds, note that each element of H' has the form $\pi(f)(\alpha)$, where $\alpha < \lambda_0$ and $f \in H$ is a function on κ. But then:

$$\pi(f)(\alpha) \in E^{M_i} \longleftrightarrow \pi(f)(\alpha) \in E^{M_{i+1}} \longleftrightarrow \alpha \in \pi(X)$$

where $X = \{\xi < \kappa_i : f(\xi) \in E^{M_i}\}$. Hence

$$E^{M_i} \cap H' = E^{M_{i+1}} \cap H^i \text{ and } J^{E^{M_i}}_{\nu_i} = J^{E^{M_{i+1}}}_{\nu_i}$$

QED(Lemma 3.4.1)

Using these facts we prove:

Lemma 3.4.2. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration. Let $h < i$. Then*

(a) $J^{E^{M_h}}_{\nu_h} = J^{E^{M_i}}_{\nu_h}$

(b) *λ_h is inaccessible in M_i and $\nu_h = \lambda_h^+$ in M_i*

(c) *Let $h < j <_T i$. Then $\lambda_h \leq \mathrm{crit}(\pi_{j,i}) < \lambda_i$.*

(d) *Let $h <_T i$. $\pi_{h,i}$ is a total function on M_h iff $[H, i]_T$ is drop free.*

The proof is by induction on i. We leave the details to the reader.

Note. $h < i$ implies $\nu_h < \lambda_i$, since $\nu_h < \nu_i$ is a successor cardinal in M_i; hence $\nu_h \notin [\lambda_i, \nu_i)$.

Definition 3.4.3. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration.

- $lh(I)$ denotes the length of I
- If $\eta \leq lh(I)$ we set:

$$I|\eta =: \langle\langle M_i|i < \eta\rangle, \langle\nu_i|i + 1 < \eta\rangle, \langle\pi_{ij}|i \leq_T i < \eta\rangle, T \cap \eta^2\rangle.$$

Definition 3.4.4. Let $I = \langle\langle M_i\rangle, \ldots, T\rangle$ be a normal iteration of limit length η. By a *well founded cofinal branch* in I we mean a branch b in T such that

- $\sup b = \eta$
- b has at most finitely many truncation points
- Let $i \in b$ such that $b \setminus i$ is truncation free. Then

  $$\langle M_j|j \in b\rangle, \langle\pi_{hi}|i \leq h \leq j \text{ in } b\rangle$$

  has a well founded direct limit.

We leave it to the reader to prove:

Lemma 3.4.3. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration of limit length η. Let b be a well founded cofinal branch in I. I has a unique extension I' of length $\eta + 1$ such that $I'|\eta = I$ and $T'''\{\lambda\} = b$. (Moreover, if $i \in b$ and $b \setminus i$ is drop free then:*

$$M'_\eta, \langle\pi'_{h,\eta}|h \in b \setminus i\rangle$$

is the transitivized direct limit of

$$\langle M_h|h \in b \setminus i\rangle, \langle\pi_{h,j}|h \in b \setminus i\rangle.$$

Note. We use Theorem 3.3.25 to show that M'_η is a premouse.

Note. It will be easier to talk about such limits if we have a notion of direct limit which can be applied to directed systems of *partial maps*. This could be defined quite generally, but the following version suffices for our purposes: Let $S = \langle S, <\rangle$ be a linear ordering. Let $\mathbb{A}_i$ be a model and let π_{ij} be a partial injection of $\mathbb{A}_i$ to $\mathbb{A}_j$ for $i \leq j$ in S. Assume that the maps commute (i.e. $\pi_{ij}\pi_{\kappa i} = \pi_{\kappa j}$) and that for sufficiently large $i \in S$ we have:

$$\pi_{ij} \text{ is a total map on } \mathbb{A}_8 \text{ for all } j \geq i \text{ in } I.$$

Let S' be the set of such i. We call:

$$\mathbb{A}, \langle \pi_i | i \in S\rangle$$

a direct limit of:

$$\langle \mathbb{A}_i | i \in S\rangle, \langle \pi_{ij} | i \leq j \text{ in } S\rangle$$

iff:

$$\mathbb{A}, \langle \pi_i | i \in S'\rangle$$

in a direct limit of:

$$\langle \mathbb{A}_i | i \in S'\rangle, \langle \pi_{ij} | i \leq j \text{ in } S'\rangle$$

and π_h is defined by: $\pi_h = \pi_i \pi_{hi}$ for $h \notin S'_1 i \in S$.

In §3.2 we defined $\mathbb{N}$ to be a Σ^**–ultrapower of* M *by* F with Σ^*–extension π (in symbols $\pi : M \to^*_F N$) iff F is close to M and $\pi : M \to^{(n)}_F N$ where $n \leq \omega$ is maximal such that $\text{crit}(F) < \rho^n_M$. Theorem 3.2.17 said that in this case π is Σ^*–preserving. We shall now show that in a normal iteration $E^{M_i}_{\nu_i}$ is always close to M^*_i. In order to utilize the full strength of this fact, we shall formulate it not only for normal iteration, but also for *potential* normal iteration in the following sense:

Let $I = \langle\langle M_i\rangle, \langle \nu_i\rangle, \langle \pi_{ij}\rangle, T\rangle$ be a normal iteration of length $i+1$. If we attempt to extend I to an I' of length $i+2$ by appointing the next ν_i, we call this attempt a *potential normal iteration*. The formal definition is:

Definition 3.4.5. A *potential normal iteration* of length $i+2$ is a structure

$$\mathfrak{T}' = \langle\langle M_j | j \leq i\rangle, \langle \nu_j | j \leq i\rangle, \langle \pi_{ij} | i \leq j \leq i\rangle, T'\rangle$$

where:

- $I = \langle\langle M_j\rangle, \langle \nu_j | j < i\rangle, \langle \pi_{ij}\rangle, T\rangle$ is a normal iteration of length $i+1$, where $T = T' \cap (i+1)^2$

- $E^{M_i}_{\nu_i} \neq \emptyset$ and $\nu_i > \nu_j$ for $j < i$
- $hT'j \leftrightarrow (hTj \vee (h \leq_T \xi \wedge j = i))$ where:

$$\xi = T'(i+1) =: \text{ the least } \xi \text{ such that } \kappa_i < \lambda_\xi.$$

If I' is a potential iteration and $\xi = T'(i+1)$, we define $M^*_i = M_\xi U$ is in the usual way, (but we do not yet know whether M^*_i is extendable by $E^{M_i}_{\nu_i}$).

Note. (a)-(d) in the definition of *normal iteration* continue to hold. ((d) is trivial if $\xi = i$. If $\xi < i$, then $\tau_i < \lambda_\xi$ and $J^{E^{M_\xi}}_{\lambda_\xi} = J^{E^{M_i}}_{\lambda_\xi}$). But then M^*_i is defined and $\tau_i \in M^*_i$ is a cardinal in M^*_i. Let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n_{M^*_i}$. It is easily seen that, if the $\Sigma^{(n)}_0$ extension:

$$\pi' : M^*_i \longrightarrow^{(n)}_{E^{M_i}{}_{\nu_i}} M'$$

exists, we can turn I' into a normal iteration of length $i+2$ by setting:

$$M_{i+1} = M', \ \pi_{\xi,i+1} = \pi'$$

We now prove a basic fact about normal iteration:

Theorem 3.4.4. *Let I be a potential normal iteration of length $i+2$. Let $\xi = T(i+1)$. Then $E^{M_i}_{\nu_i}$ is close to M^*_i.*

Before proving this we note the obvious corollary:

Corollary 3.4.5. *Let I be a normal iteration. If $h = T(i+1)$ in I, then:*

$$\pi_{h,i+1} : M^*_i \to^*_{E_{\nu_i}} M_i.$$

Lemma 3.4.6. *Let I be a normal iteration. Let $h = T(i+1), i+1 \leq_T j$, where $(i+1, j]_T$ has no truncation point. Then:*

$$\pi_{h,j} : M^*_i \longrightarrow_{\Sigma^*} M_j \textit{ strongly}.$$

In particular $\pi_{h,j}$" $P^n_{M^*_i} \subset P^n_{M_j}$ for $\rho^{n+1} = \rho^\omega$ in M^*_i.

Proof. By induction on j using Lemma 3.2.26, Lemma 3.2.27 and Lemma 3.2.28.

QED(Lemma 3.4.6)

We shall derive Theorem 3.4.4 from an even stronger statement:

Lemma 3.4.7. *Let I be a potential normal iteration of length $i+2$. Then*

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i||\nu_i) \subset \underline{\Sigma}_1(M_i^*).$$

We first show that Lemma 3.4.7 implies theorem 3.4.4. Since $F = E_{\nu_i}$ is weakly amenable, we need only show that $F_\alpha \in \underline{\Sigma}_1(M_i^*)$ for $\alpha < \lambda_i$, where:

$$F_\alpha = \{x \subset \kappa_i | x \in M_i||\nu_i \wedge \alpha \in F(x)\}.$$

Let $k \in M_i||\nu_i$ map τ_i onto $J^E_{\tau_i}$. Then $k \in M_i^*$, since either $i = T(i+1)$ and $M^* \supset M_i||\nu_i$, or else $h = T(i+1) < i$, whence follows: $k \in J^{E^{M_i}}_{\lambda_h} = J^{E^{M_i^*}}_{\lambda_h} \subset M_i^*$. Set:

$$\tilde{F}_\alpha = \{\xi < \tau_i | k(\xi) \in F_\alpha\}.$$

Then $\tilde{F}_\alpha \subset \mathbb{P}(\tau_i)$ is $\underline{\Sigma}_1(M_i^*)$ by Lemma 3.4.7. Hence $F_\alpha = k''\tilde{F}_\alpha \in \underline{\Sigma}_1(M_i^*)$.

QED

We now prove Lemma 3.4.7. Suppose not. Let I be a counterexample of length $i+2$, where i is chosen minimally. Let $h = T(i+1)$. Then:

(1) $h < i$

Proof: Suppose not. Then $M_i^* = M_i||\mu$ where $\mu \geq \nu$. Hence $\underline{\Sigma}_1(M_i||\nu_i) \subset \underline{\Sigma}_1(M_i^*)$. Contradiction!

(2) $\nu_i = \mathrm{On}_{M_i}$ and $\rho^1_{M_i} \leq \tau_i$.

Proof: Suppose not. Let $A \subset \tau_i$ be $\underline{\Sigma}_1(M_i||\nu_i)$. Then $A \in \mathbb{P}(\tau_i) \wedge M_i \subset J^{E^{M_i}}_{\lambda_n}$, since $\lambda_h > \tau_i$ is inaccessible in ???. But $J^{E^{M_i}}_{\lambda_n} = J^{E^{M_i}}_{\lambda_n} \subset M_i^*$. Contradiction!

(3) i is not a limit ordinal.

Proof: Suppose not. Then $\sup\{\mathrm{crit}(\pi_{li})(\underset{T}{<} i\} = \sup\limits_{l<i}\lambda_l$, so we can pick $L \underset{T}{<} i$ such that $\mathrm{crit}(\pi_{l,i}) > \lambda_h > \tau_i$ and $\pi_{l,i}$ is a total function on M_l. Then $\pi_{l,i} : M_l \to_{\Sigma_1} M_i$, where $M_i = \langle J^{E_i}_{\nu_i}, F\rangle$, where $F \neq \emptyset$. Hence $M_l = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{F}\rangle$ where $\overline{F} \neq \emptyset$. Let $A \subset \tau_i$ be $\underline{\Sigma}_1(M_i)$ such that $A \notin \underline{\Sigma}_1(M_i^*)$. We can assume l to be chosen large enough that $p \in \mathrm{rng}(\pi_{li})$, where A is $\Sigma_1(M_i)$ in the parameter p. Thus $A \in \underline{\Sigma}_1(M_l)$. Clearly $\overline{\nu} > \nu_j$ for all $j < l$, since $\nu_j \in M_l = \langle J^{\overline{F}}_{\overline{\nu}}, \overline{F}\rangle$.

Extend $I|l+1$ to a potential iteration I' of cf length $l+2$ by setting $\nu_l = \overline{\nu}$. Since $\mathrm{crit}(\pi_{l,i}) > I_i$, it follows easily that $\tau_l' = \tau_i, \kappa_l' = \kappa_i$, where τ_l, κ_l' are defined in the usual way. But then $M_i^* = (M_l')^*$ and $A \in \underline{\Sigma}_1(M_i^*)$ by the minimality of i. Contradiction! QED (3)

Now let $i = j+1, \xi = T(i)$. Since $\pi_{\xi,i} : M_j^* \to_{\Sigma_1} M_i = \langle J^E_{\nu_i}, E_\nu\rangle$ where $E_{\nu_i} \neq \emptyset_i$ we have:

(4) $M_j^* = \langle J_{\overline{\nu}}^{\overline{E}}, \overline{E}_{\overline{\nu}} \rangle$ where $\overline{E}_{\overline{\nu}} \neq \emptyset$.

(5) $\tau_i < \kappa_j$

Proof: $\tau_i < \lambda_j$ since $\tau_i = \kappa_i^{+M_i}$ and $\kappa_i < \lambda_h \leq \lambda_j$, where λ_j is inaccessible in M_i. But obviously $\kappa_i, \tau_i \in \mathrm{rng}(\pi_{\xi,i})$ by (4) where $[\kappa_j, \lambda_j) \cap \mathrm{rng}(\pi_{\xi i}) = \emptyset$. QED (5)

(6) $\pi_{\xi i} : M_j^* \to_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower.

Proof: Suppose not. Then $\kappa_j < \rho^1_{M_j^*}$. Hence $\pi_{\xi,i}$ is $\Sigma_0^{(1)}$–preserving. Hence $\pi_{\xi i}{}''\rho^1_{M_i^*} \subset \rho^1_{M_i}$. Hence $\tau_i = \pi_{\xi i}(\tau_j) < \rho^1_{M_i}$, contradicting (2). QED (6)

But then:

(7) $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_j^*)$.

Proof: Let $A \subset \tau_i$ be $\Sigma_1(M_i)$ in the parameter p. Let $p = \pi_{\xi i}(f)(\alpha)$, where $f : \kappa_i \to M_i^*, f \in M_i^*$, and $\lambda < \lambda_j$. Then

$$A(\xi) \leftrightarrow \bigvee x A'(\zeta, x, p)$$

where A' is $\Sigma_0(M_i)$. Let $\overline{A}'$ be $\Sigma_0(M_j^*)$ by the same Σ_0 definition. Then, since $\pi_{\xi i}$ takes M_j^* cofinally to M_i by (6), we have

$$A(\zeta) \leftrightarrow \bigvee u \in M_j^* \bigvee x \in \pi_{\xi,i}(u) A'(\zeta, x, p).$$

By the minimality of i we know that $(E_{\nu_j})_\alpha \in \underline{\Sigma}_1(M_j^*)$ for $\alpha < \lambda_j$. But then:

$$A(\zeta) \leftrightarrow \bigvee u \in m_j^* \{\gamma < \kappa_j | \overline{A}'(\zeta, x, f(\gamma)\} \in (E_{\nu_i})_\alpha.$$

Hence A is $\underline{\Sigma}_1(M_j^*)$. QED (7)

Now extend $I|\xi + 1$ to a potential iteration I' of length $\xi + 2$ by setting $\nu'_\xi = \overline{\nu}$, where $M_j^* = M_\xi || \overline{\nu} = \langle J_{\overline{\nu}}^{\overline{E}}, \overline{E}_{\overline{\nu}} \rangle$. Then $\kappa_i = \kappa'_\xi$ and $\tau_i = \tau'_\xi$, since $\pi_{\xi i} \restriction \kappa_j = \mathrm{id}$. Hence $h = T(i+1) = T'(\xi + 1)$ and $M_i^* = (M_\xi^*)'$. By the minimal choice of i we conclude

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_j^*) \subset \underline{\Sigma}_1(M_i^*).$$

Hence $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_((M_i^*)$ by (7). Contradiction! QED (Lemma 3.4.7)

3.4.3 Padded iterations

Normal iterations are often used to "compare" two premice M and M'. The *comparison iteration* or *coiteration* consists of a pair $\langle I, I'\rangle$ of iteration I of M and I' of M'. When we have reached M_i, M_i', we proceed as follows: We look for the *least point of difference* — i.e. the least ν such that $M_i||\nu \neq M_i'||\nu$. Then $J_\nu^{E^{M_i}} = J_\nu^{E^{M_i'}}$ and $E_\nu^{M_i} \neq E_\nu^{M_i'}$. Then at least one of $E_\nu^{M_i}, E_\nu^{M_i'}$ is an extender. If both are extenders, we continue on the I–side with the index $\nu_i = \nu$. However, if, say, $E_\nu^{M_i}$ is an extender and $E_\nu^{M_i'} = \emptyset$, we iterate by $\nu_i = \nu$ on the I–side and on the I'–side do nothing. We then call i an *inactive point* on the I'–side and set: $M'_{i+1} = M_i', \pi'_{i,i+1} = \text{id}$ with $i = T'(i+1)$ in I. Thus i is active on one or the other side and we have achieved: $M_{i+1}||\nu = M'_{i+1}||\nu = \emptyset$. (This is called "iterating away the least point of difference".) At a limit λ we choose on either side a well founded branch and continue with that.

If all goes well, we eventually reach a point i such that $M_i = M_i'$ or one of M_i, M_i' is a proper segment of the other.

In order to carry this out we need a slightly more flexible definition of "normal iteration", which admits inactive points. We therefore define:

Definition 3.4.6. A *padded normal iteration of length* μ is a sequence:

$$I = \langle\langle M_i | i < \mu\rangle, \langle \nu_i | i \in A\rangle, \langle \pi_{ij} | i \leq_T j\rangle, T\rangle$$

such that:

(1) $A \subset \{i : j + 1 < \mu\}$ is called the set of *active points* in I.

(2) (a)-(h) of the previous definition hold, where (c) requires the assumption: $i, j \in A$ and (d), (f) require : $i \in A$.

(3) Let $h < j < \mu$ such that $[h, j) \cap A = \varnothing$. Then:

- $h \leq_T j, M_h = M_j, \pi_{hj} = \text{id}$.

It follows easily that if $i \leq j$, then $I_i = I_j$ if and only if $[i, j) \cap A = \emptyset$. (To see this, let $h = \min[i, j) \cap A$. Then ν_h is a cardinal in M_j but not in M_i.)

Note. This gives a new way of potentially extending I of length $i + 1$. Instead of appointing ν_i, we could set: $i \notin A, M_{i+1} = M_i$.

All previous results go through *mutatis mutandis*. We shall often use the term "normal iteration" so as to include padded normal iteration. We then call normal iterations in the sense of our previous definition *strict*. We can turn a padded iteration into a strict iteration simply by omitting the inactive points.

Conversely, we can turn a strict iteration into a padded iteration simply by inserting inactive points. The relevant lemmas are:

Lemma 3.4.8. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a (possibly padded) normal iteration of length μ. Let A be the set of active points in I. Set:*

$$A' =: \{i : i \in A \vee i + 1 = \mu\}$$

Let $B \subset \mu$ such that $A' \subset B$. Let f be the monotone enumeration of B. Then:

$$I' = \langle\langle M_{f(i)}\rangle, \langle\nu_{f(i)}\rangle, \langle\pi_{f(i),f(j)}\rangle, T'\rangle$$

is a normal iteration , where $T' = \{\langle i, j\rangle : f(i)Tf(j)\}$. (Moreover I' is strict if $B = A'$).

Proof. (a)-(i) are satisfied by I'.

Conversely:

Lemma 3.4.9. *Let I, μ be as above. Let $f : \mu \longrightarrow \mu'$ be monotone such that* $\text{lub}\, f"\mu = \mu'$ *if μ is a limit ordinal. Set: $\overline{f}(i) = \text{lub}\, f"i$ for $i < \mu$. For $i < \mu'$ set:*

$\xi_i =$ *that ξ such that either $\overline{f}(\xi) \leq i \leq f(\xi)$, or else $\xi + 1 = \mu$ and $f(\xi) < i$.*

Define:

$$I' = \langle\langle M'_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$$

by:

$$M'_i = M_{\xi_i}, \pi'_{ij} = \pi_{\xi_i,\xi_j}, T' = \{\langle i, j\rangle : \xi_i T \xi_j\}$$

and:

$$\nu'_i = \begin{cases} \nu_{\xi_i} & \text{if } i = f(\xi_i) \\ \text{otherwise undefined} & \end{cases}$$

Then I' is a normal iteration.

Proof: I' satisfies (a)-(i).

Note. Lemma 3.4.9 enables to recover I form the I' in Lemma 3.4.8.

We leave the proof to the reader.

3.4.4 n–iteration

In a normal iteration we always take Σ^* ultrapowers. For technical reasons, however, we may sometimes want to bound the degree of preservation of our ultraproducts. In a 0*–iteration* for instance, we would use the ordinary Σ_0 ultrapower to pass from M_i to M_{i+1}, as long as no $h \leq_T i+1$ is a truncation point. If, on the other hand, we have reached a truncation point $h \leq_T i+1$, we then revert to the full Σ^*–ultrapowers. More generally:

Definition 3.4.7. Let $n \leq \omega$. By a *normal n–iteration* of M of *length* μ we mean

$$\langle\langle M_i | i < \mu\rangle, \langle\nu_i | i+1 < \mu\rangle, \langle\pi_{ij} | i \leq_T\rangle, T\rangle,$$

where (a) – (e) and (g) ,(h) in the definition of "normal iteration" hold, and in addition:

(f) Let $h = T(i+1)$. If τ_i is a cardinal in M_h and π_{jh} is a total map on M_j for jTh, then $\pi_{h,i+1} : M_h \to^{(m)}_{E_{\nu_i}} M_{i+1}$, where $m \leq n$ is maximal such that $\kappa_i < \rho^m_{M_h}$.

Otherwise $\pi_{h,i+1} : M^*_i \to^{(m)}_{E_{\nu_i}} M_{i+1}$, where M^*_i is defined as before and $m \leq \omega$ is maximal such that $\kappa_i < \rho^m_{M^*_i}$.

Note. An ω–iteration is then the same as a normal iteration n the sense of our previous definition. We also call such iterations $*$*–iterations*, since we then always take the Σ^* ultrapowers. $*$–iterations are the ones we are interested in.

It is easily seen that the conclusions of Lemma 3.4.2 hold for normal n–iterations. Lemma 3.4.3 also holds for these iterations and Lemma 3.4.7 holds *mutatis mutandis*. We leave this to the reader. More suprising is:

Theorem 3.4.10. *Theoem 3.4.4 holds for normal n–iterations.*

Before proving this, we again note some consequences. It follows easily that:

Corollary 3.4.11. *Let I be a normal n–iteration. Let $h = T(i+1)$. Let m be maxiomal such that $\kappa_i < \rho^m_{M^*_i}$. Assume either that $m \leq n$ or that there is a $j \leq_T i+1$ which is a drop point. Then:*

$$\pi_{h,i+1} : M^*_i \to^*_{E_{\nu_i}} M_{i+1}.$$

In all other cases we have:

$$\pi_{h,i+1} : M^*_i \to^{(n)}_{E_{\nu_i}} M_{i+1}.$$

But then by induction on i we get:

Corollary 3.4.12. *Let I be as above. Let π_{ij} be a total map on M_i. If there is a drop point j such that jTi, then π_{ij} is Σ^*–preserving. Otherwise it is $\Sigma_0^{(n)}$–preserving.*

As before, we derive Lemma 3.4.10 from:

Lemma 3.4.13. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a potential n–iteration of length $i+2$. Then $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_i(M_i||\nu_i) \subset \underline{\Sigma}_1(M_i^*)$.*

The derivation of Lemma 3.4.10 from Lemma 3.4.13 is exactly as before. We prove Lemma 3.4.13. Almost all steps in the proof of Lemma 3.4.7 go through as before. The only difficulty occurs in the proof of (6), where we derived that $\pi_{\xi,i}$ is $\Sigma_0^{(1)}$–preserving from: $\kappa_j < \rho^1_{M_j^*}$. If $n \geq 1$, this is unproblematical. Now assume $n = 0$. If there is a drop point $l \leq_T i$, then $\pi_{\xi,i}$ is Σ^*–preserving and there is nothing to prove. Now suppose there is no such drop point.

By the definition of "0–iteration" we then have: $\pi_{\xi,i} : M_j^* \to^0_{E_{\nu_j}} M_i$, which was to be proven.

All other steps in the proof go through. QED (Lemma 3.4.13)

This proves Theorem 3.4.10.

The concept "padded n–iteration" is defined exactly as before. As before, every padded iteration can be converted into a strict iteration by omitting the inactive points, and every strict iteration can be expanded to a padded iteration by inserting inactive points. We leave this to the reader.

3.4.5 Copying an iteration

Suppose that I is a normal iteration of a premouse M and $\sigma : M \to_{\Sigma^*} M'$, where M' is a premouse. We can attempt to "copy" I onto an iteration I' of M' by repeating the same steps modulo σ. We define:

Definition 3.4.8. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a strict normal iteration of M. Let $\sigma : M \to_{\Sigma^*} M$, where M' is a premouse. We call $I' = \langle\langle M_i'\rangle, \langle\nu_i'\rangle, \langle\pi_{ij}'\rangle, T'\rangle$ a *copy of I induced by* $\langle\sigma, M'\rangle$ with *copying map* $\langle\sigma_i | i < lh(I)\rangle$ iff the following hold:

(a) $lh(I') = lh(I)$ and $T' = T$

(b) $\sigma_i : M_i \rightarrow_{\Sigma^*} M'_i$ and $\sigma_0 = \sigma$

(c) $\sigma_i \pi_{li} = \pi'_{li} \sigma_j$ for $l \leq_T i$

(d) $\sigma_i \upharpoonright \lambda_l = \sigma_l \upharpoonright \lambda_l$ for $l \leq i$

(e) $\nu'_i = \sigma_i(\nu_i)$ for $\nu_i \in M_i$. Otherwise $\nu'_i = \mathrm{On} \cap M'_i$.

Note. This definition can easily be extended to padded normal iterations. (b) – (e) are then stipulated for active points, and for inactive points we stipulate:

(f) If i is inactive in I, it is inactive in I' and $\sigma_{i+1} = \sigma_i$.

We shall often formulate our definitions and theorems for strict iteration, leaving it to the reader to discover — mutatis mutandis — the correct version for padded iterations. In particular, the remaining theorems in this section will assume strictness.

We also define:

Definition 3.4.9. $\langle I, I', \langle \sigma_i | i < lh(I) \rangle \rangle$ is a *duplication* iff I, I' are normal iterations and I' is a copy of I with copying maps $\langle \sigma_i \rangle$.

Lemma 3.4.14. *Let I' be a copy of I with copying maps $\langle \sigma_i \rangle$. Let $h = T(i+1)$.*

(i) If $i+1$ is a drop point in I, then it is a drop point in I' and ${M'}^*_i = \sigma_h(M^*_i)$.

(ii) If $i+1$ is not a drop point in I, it is not a drop point in I'. (Hence $M^*_i = M_h, {M'}^*_i = M'_h$.)

(iii) Let $F = E^{M_i}_{\nu_i}, F' = E^{M'_i}_{\nu'_i}$. Then:

$$\langle \sigma_h \upharpoonright M^*_i, \sigma_i \upharpoonright \lambda_i \rangle : \langle M^*_i, F \rangle \rightarrow \langle {M'}^*_i, F' \rangle$$

as defined in §3.2.

(iv) $\sigma_{i+1}(\pi_{h,i+1}(f)(\alpha)) = \pi'_{h,i+1} \sigma_h(f)(\sigma_i(\alpha))$ for $f \in \Gamma^*(\kappa_i, M^*_i) \alpha < \lambda_i$.

(v) $\sigma_j(\nu_i) = \nu'_i$ for $i < j$.

Proof:

(i) Let $h = T(i+1)$. Then $M_i^* = M_h||\mu$, where $\mu \in M_h$ is maximal such that τ_i is a cardinal in $M_h||\mu$. But $\tau_i' = \sigma_i(\tau_i) = \sigma_h(\tau_i)$ by (d), (e). Hence $\sigma_h(\mu) = \mu'$, where μ' is maximal such that τ_i' is a cardinal in M_h', and $\sigma_h(M_h||\mu) = M_h'||\mu'$.

(ii) If τ is a cardinal in M_h, then $\tau_i' = \tau_h(\tau)$ is a cardinal in M_h', since σ_h is Σ_1–preserving.

(iii) Clearly $\sigma_h \restriction M_i^* : M_i^* \to_{\Sigma^*} M_i'^*$ by (i) and (ii). Now let $x \in \mathbb{P}(\kappa_i) \cap M_i^*$ and $\alpha_1, \ldots, \alpha_n < \lambda_0$. Since $\sigma_i : M_i \longrightarrow M_i'$ is Σ^*-preserving we have:

$$\langle \vec{\alpha} \rangle \in F(x) \leftrightarrow \langle \sigma_i(\vec{\alpha}) \rangle \in F'(\sigma_i(x)).$$

But $\sigma_i(x) = \sigma_h(x)$, since by (d) we have: $\sigma_i \restriction J_{\lambda_n}^{E^{M_i}} = \sigma_h \restriction J_{\lambda_h}^{E^{M_h}}$.

(iv) If $f \in M_i^*$, then by (c):

$$\sigma_{i+1}\pi_{h,i+1}(f) = \pi'_{h,i+1}\sigma_h(f).$$

Otherwise $f(\xi) \simeq G(\xi, q)$ where $q \in M_i^*$ and G is a good $\Sigma_1^{(n)}(M_i^*)$ function for an n such that $\kappa_i < \rho_{M_i^*}^{n+1}$. But then:

$$\begin{aligned} \sigma_{i+1}\pi_{h,i+1}(f)(\xi) &\simeq G'(\xi, \sigma_{i+1}\pi_{h,i+1}(q)) \\ &\simeq G'(\xi, \pi'_{h,i+1}\sigma_h(q)) \\ &\simeq \pi'_{h,i+1}\sigma_h(f) \end{aligned}$$

where G' is $\Sigma_1^{(n)}(M_i'^*)$ by the same good definition.

(v) If $j > i+1$, then $\nu_i < \lambda_{i+1}$ and $\sigma_j(\nu_i) = \sigma_{i+1}(\nu_i)$. But letting $h = T(i+1)$, we have:

$$\sigma_{i+1}(\nu_i) = \sigma_{i+1}\pi_{h,i+1}(\tau_i) = \pi'_{h,i+1}\sigma_h(\tau_i),$$

where

$$\sigma_h(\tau_i) = \sigma_i(\tau_i) = \tau_i', \text{ since } \tau_i < \lambda_h.$$

Hence $\sigma_{i+1}(\nu_i) = \pi'_{h,i+1}(\tau_h') = \nu_i'$.

QED (Lemma 3.4.14)

It is apparent from Lemma 3.4.14 that there is only one way to extend a copy of $I|i+1$ to a copy of $I|i+2$. Moreover, the copying map σ_i is unique. Similarly, if η is a limit ordinal and I' is a copy of $I|\mu$ with copying maps $\langle \sigma_i | i < \eta \rangle$, ther is only one way to extend I' to a copy of $I|\eta+1$, for then:

$$M', \langle \pi'_{i,\eta} | iT\eta \rangle$$

is the direct limit of:

$$\langle M_i' | i < \eta \rangle, \langle \pi_{ij}' | i \leq_T j <_T \eta \rangle,$$

and σ_η is defined by:

$$\sigma_\eta \pi_{i\eta} = \pi_{i\eta}' \sigma_i \text{ for } i <_T \eta.$$

Hence, by induction on $lh(I)$ we get:

Lemma 3.4.15. *Let I be a normal iteration of M. Let $\sigma : M \to_{\Sigma^*} M'$. Then there is at most one copy I' of I induced by σ. Moreover, the copying maps σ_i are unique.*

Now suppose that I is a normal iteration of length $i+1$ and I' is a copy of I with copying maps $\langle \sigma_h \,|\, h \leq i \rangle$. Extend I to a potential iteration $\tilde{I}$ of length $i+2$ by appointing ν_i. Extend I' to a potential iteration $\tilde{I}'$ by appointing:

$$\nu_i' = \begin{cases} \sigma_i(\nu_i) \text{ if } \nu_i \in M_i \\ \mathrm{On} \cap M_i' \text{ if } \nu_i = \mathrm{On} \cap M_i. \end{cases}$$

We call $\langle \tilde{I}, \tilde{I}', \langle \sigma_j || \leq i \rangle \rangle$ a *potential duplication* of length $i+2$. The formal definition is:

Definition 3.4.10. Let I, I' be potential iteration of length $i+2$. $\langle \tilde{I}, \tilde{I}', \langle \sigma_j | j \leq i \rangle \rangle$ is a *potential duplication* of length $i+2$ iff

- $\langle \overline{I}, \overline{I}', \langle \sigma_j | j \leq i \rangle \rangle$ is a duplication of length $i+1$, where $\overline{I} = \tilde{I}|i+1, \overline{I}' = I'|i+1$.
- $\sigma_i(\nu_i) = \nu_i'$ if $\nu_i \in M_i$. Otherwise $\nu_i' = \mathrm{On} \wedge M_i'$.

Note. It is then easily seen that $T(i+1) = T'(i+1)$. We also know that $E_{\nu_i}^{M_i}$ is close to M_i^n and $E_{\nu_i'}^{M_i'}$ is clost to M_i'. The following theorem is an analogue of theorem 3.4.7

Lemma 3.4.16. *Let $\langle I, I', \langle \sigma_i \rangle \rangle$ be a potential duplication of length $i+2$. Let $h = T(i+1)$. Then:*

$$\langle \sigma_h \restriction M_i^*, \sigma_i \restriction \lambda_i \rangle : \langle M_i^*, F \rangle \to^* \langle M'^*_i, F' \rangle$$

(as defined in §3.2) where $F = E_{\nu_i}^{M_i}, F' = E_{\nu_i'}^{M_i'}$.

Before proving the theorem, we note some of its consequences. It gives us exact criteria for determining whether the copying process can be continued one step further.

Lemma 3.4.17. *Let I be a normal iteration of M of length $i+2$. Let $\sigma : M \to M'$ induce a copy I' of $I|0+1$ with copying maps $\langle \sigma_j | j \leq i \rangle$. Set:*

$$\nu'_i = \begin{cases} \sigma_i(\nu_i) \text{ if } \nu_i \in M_i \\ \mathrm{On} \cap M'_0 \text{ if } \nu_i = \mathrm{On} \cap M_i \end{cases}$$

Then σ induces a copy of I iff ${M'}^*_i$ is Σ^*–extendible by $E^{M'_i}_{\nu'_i}$.

Proof: If ${M'}^*_i$ is not extendible, then no such copy can exist. Now let ${M'}^*_i$ be extendible. Let $\pi'_{h,i+1} : {M'}^*_i \to^*_{E^{{M'}^*_i}_{\nu'_i}} {M'}^*_{i+1}$. By theorem 3.4.16 and Lemma 3.2.23 it follows that there is a unique $\sigma : M_{i+1} \to_{\Sigma^*} M'_{i+1}$ such that $\sigma\pi_{h,i+1} = \pi'_{h,i+1} \cdot \langle \sigma_h \restriction M^*_i \rangle$, where $h = T(i+1)$. Set: $\sigma_{i+1} =: \sigma$. This gives us the copy I'' of I with copying maps $\langle \sigma_j | j \leq 0+1 \rangle$.

QED (Lemma 3.4.17)

We also have:

Lemma 3.4.18. *Let I be a normal iteration of M of length $\eta+1$, where η is a limit ordinal. Let $\sigma : M \to_{E^*} M'$ induce a copy I' of $I|\eta$. We can extend I' to a copy of I induced by σ iff $b = T''\{\eta\}$ is a well founded branch in I'.*

The proof is left to the reader.

We also note:

Lemma 3.4.19. *Let I be a normal iteration of limit length. Let I' be a copy of I. If b is a cofinal well founded branch in I', then it is a cofinal well founded branch in I.*

The proof is left to the reader.

We now turn to the proof of theorem 3.4.16. As with theorem 3.4.7 we derive it from an even stronger lemma:

Lemma 3.4.20. *Let $\langle I, I', \langle \sigma_i \rangle \rangle$ be a potential duplication of length $i+2$. Let $A \subset \tau_i$ be $\Sigma_1(M_i||\nu_i)$ in a parameter p. Let $A' \subset \tau'_i$ be $\Sigma_1(M_i||\nu_i)$ in $\sigma_i(p)$ by the same definition. Then A is $\Sigma_1(M^*_i)$ in a parameter q and A' is $\Sigma_1({M'}^*_i)$ in $\sigma_h(q)$ by the same definition, where $h = T(i+1)$.*

The derivation of theorem 3.4.16 from lemma 3.4.20 is a virtual repetition of the proof of theorem 3.4.4 from lemma 3.4.7. We leave it to the reader.

Lemma 3.4.20 is proven by a virtual repetition of the proof of lemma 3.4.7, making changes as necessary. We give a brief sketch of the proof:

Suppose not. Let I, I', ν_i, ν'_i be counterexamples of length $i+1$, where i is chosen minimally. Let $h = T(i+1) = T'(i+1)$. Then:

(1) $h < i$.
Suppose not. Then $M_i||\nu_i \subset M_i^*$ and $M'_i||\nu'_i \subset M'^*_i$ as before. If $\nu_i \in M_i^*$, then $\sigma_i(M||\nu_i) = M'_i||\nu'_i$. Hence $A \in M_i^*$ and $\sigma_i(A) = A'$. Contradiction!

(2) $\nu_i = \mathrm{On}_{M_i}$ and $\rho^i_{M_i} \leq \tau_i$.
Otherwise, as before $A \in \mathbb{P}(\tau_i) \cap M_i^*, A' \in \mathbb{P}(\tau_i) \cap M'^*_i$ and $\sigma_h(A) = \sigma_i(A) = A'$. Contradiction!

(3) i is not a limit cardinal.
The proof of this is a virtual repetition of the argument given in the proof of lemma 3.4.7. We leave it to the reader.

Now let $i = j+1, \xi = T(i)$. Exactly as before we have:

(4) $M_j^* = \langle J_\nu^E, E_\nu \rangle, M'^*_j = \langle J_{\nu'}^{E'}, E'_{\nu'} \rangle$ where $E_\nu, E'_\nu \neq \emptyset$.

(5) $\tau_i < \kappa_j$.

(6) $\pi_{\xi,i} : M_j^* \to_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower (and therefore cofinal). Similarly for $\pi'_{\xi,i} : M'^*_j \to_{E_{\nu'_j}} M'_i$. By the minimality of σ we know that for all $\alpha < \lambda_j$, $(E^{M_j}{}_j)_\alpha$ is $\Sigma_1(M_j^*)$ in a parameter r and $(E^{M_i}{}_{\nu'_i})_{\sigma_i(\alpha)}$ is $\Sigma_1(M'^*_j)$ in $\sigma_\xi(r)$ by the same definition. Using this we can repeat the argument in the proof of Lemma 3.4.7 to get:

(7) A is $\Sigma_1(M_j^*)$ in a q and A' is $\Sigma_1(M'^*_j)$ in $\sigma_\xi(q)$ by the same definition.

Now extend $I|\xi+1$ to a potential iteration $\tilde{I}$ of length $\xi+2$ by setting $\tilde{\nu}_\xi = \nu$, where ν is as in (4). Extend $I'|\xi+1$ to $\tilde{I}'$ by setting $\tilde{\nu}_\xi = \nu'$ where ν' is as in (4). Then $\kappa_i = \tilde{\kappa}_\xi, \tau_i = \tilde{\tau}_\xi, \kappa'_i = \tilde{\kappa}_\xi, \tau'_i = \tilde{\tau}'_\xi$ as before. Hence $h = \tilde{T}(\xi+1) = \tilde{T}'(\xi+1)$ and $M_i^* = \tilde{M}_\xi^*, M'^*_i = \tilde{M}'^*_\xi$. By this minimality of i we conclude that A is $\Sigma_1(M_i^*)$ ia a q and A' is $\Sigma_1(M'^*_i)$ in $\sigma_h(q)$ by the same definition. Contradiction! QED (Lemma 3.4.20)

3.4.6 Copying an n–iteration

Definition 3.4.11. Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal n–iteration ($n \leq \omega$). Let $\sigma : M \to_{\Sigma_1^M}, M'$. We call:

$$I' = \langle \langle M'_i \rangle, \langle \nu'_i \rangle, \langle \pi'_{ij} \rangle, T' \rangle$$

a *copy* (or *n–copy*) of I *induced by* $\langle\sigma, M'\rangle$ iff I' is an n–iteration satisfying (a), (c), (d), (e) of the previous definition together with

(b') $\sigma_0 = \sigma$ and $\sigma : M_i \to_{\Sigma_1^{(n)}} M_i'$. Moreover, if some $h \leq_T i$ is a truncation point, then σ_i is Σ^*–preserving.

The notion "n–duplication" and "potential n–duplication" are defined as before. Lemma 3.4.14 goes through as before exept (iv) must be reformulated as:

(iv') If no $l \leq_T i+1$ is a truncation point and $\kappa_i < \rho^n_{M_h}$, then:

$$\sigma_{i+1}(\pi_{h,i+1}(f))(\alpha) = \pi'_{h,i+1}\sigma_i(f)(\sigma_i(\alpha))$$

for $f \in \Gamma^n_*(\kappa_i, M_h), \alpha < \lambda_i$. In all other cases the equation holds for

$$f \in \Gamma^*(\kappa_i, M_i^*), \alpha < \lambda_i.$$

Lemma 3.4.15 then holds as before. Theorem 3.4.16 and lemma 3.4.17 – 3.4.19 then go through as before. By theorem 3.4.16 we also get:

Lemma 3.4.21. *Let $\langle I, I', \langle\sigma_i\rangle\rangle$ be an n–duplication. Let $i <_T j$ in I such that π_{ij} is total on M_i.*

(a) If no $l \leq_T i$ is a truncation point and $\kappa_i < \rho^n_{M_i}$, then $\pi_{ij} : M_i \to_{\Sigma_1^{(n)}} M_j$.

(b) In all other cases π_{ij} is Σ^*–preserving.

These lemmas and theorems hold *mutatis mutandis* for padded n–iterations. The details are left to the reader.

3.5 Iterability

A mouse is a premouse which is iterable. Iterability is, however, as complex a notion as that of iterating itself. We begin with *normal iterability* which says that any normal iteration of M constructed accordig to an appropriate strategy, can be continued.

3.5.1 Normal iterability

Definition 3.5.1. A premouse M has the *normal uniqueness property* (NUP) iff every normal iteration of M of limit length has at most one cofinal well founded branch. The simplest mice, such as $0^{\#}, 0^{\#\#}$ etc. are easily seen to have this property. Unfortunately, however, there are mice which do not. If a premouse M does satisfy NUP, then normal iterability can be defined by:

Definition 3.5.2. Let M satisfy NUP, M is *normally iterable* iff every normal iteration of M can be continued — i.e.

- If I is a normal iteration of M of limit length, then it has a cofinal well founded branch.
- If I is a potential iteration of length $i+2$, then M_i^* is $*$–extendible by $E_{\nu_i}^{M_i}$.

If M does not satisfy NUP, we say that it is normally iterable if there exists a *strategy* for picking cofinal well founded branches such that any iteration executed in accordance with that strategy could be continued. We first define:

Definition 3.5.3. A *normal iteration strategy* is a partial function S on normal iterations of limit length such that $S(I)$, if defined, is a well founded cofinal branch in I. We call it a strategy *for* M if its domain is restricted to iterations of M.

Definition 3.5.4. A normal iteration $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle x_{ij}, T\rangle$ *conforms* to the iteration strategy S iff, whenever, $\eta < \mathrm{lh}\, I$ is a limit ordinal, then $T''\{\eta\} = S(I|\eta)$.

Definition 3.5.5. A normal iteration strategy S is *α–successful* for a premouse M iff every S–conforming iteration of M of length $< \alpha$ can be continued in an S–conforming way. In other words:

- If I is of limit length $< \alpha$, then $S(I)$ is defined
- If I is a potential normal iteration length $i+2 < \alpha$, then M_i^* is $*$–extendible by $E_{\nu_i}^{M_i}$.

Definition 3.5.6. M is *normally α–iterable* iff there exists an α–successful strategy for M.

Definition 3.5.7. M is *normally iterable* iff it is normally α–iterable for all α.

Note. It might seem more natural to take "normal iterable" as meaning that M is ∞–iterable, but that is a second order property, which we cannot express in ZFC.

Note. If M has NUP, then any two iteration strategies for M must coincide on their common domain. Hence, in this case, our initial definition of "normally iterable" is equivalent to the definition just given. It is then also equivalent to the second order statement that M is ∞–iterable.

Definition 3.5.8. M is *uniquely normally iterable* iff it is normally iterable and satisfies NUP.

Proving iterability is a central problem of inner model theory. There are large classes of premice for which it is unsolved. The success we have had to date depends strongly on NUP. Whenever we have been able to prove the iterability M, it is either because M satisfies NUP, or because we derive its iterability from that of another premouse which satisfies NUP.

Note. In the above definition we take "normal iteration" as meaning "padded normal iteration". One can, of course, define *strict iteration strategy, strictly α–successful* and *strictly α–iterable* in the obvious way. But in fact every strictly α–iterable premouse is α–iterable, since every strictly successful strategy S can be expanded to an α–successful S^* as follows. Let:

$$I = \langle\langle M_i\rangle, \langle\nu_i|i \in A\rangle, \langle\pi_{ij}, T\rangle$$

be padded iteration of limit length η. If A is cofinal in η, let $\langle\alpha_i|i < \mu\rangle$ be the monotone enumeration of A and set:

$$I' = \langle\langle M_{\alpha_i}\rangle, \langle\nu_{\alpha_0}\rangle, \langle\pi_{\alpha_i,\alpha_j}\rangle, \{\langle i, j\rangle|\alpha_i T\alpha_j\}\rangle.$$

Then I' is strict and we set:

$$S^*(\mathbb{I}) \simeq \{i|\bigvee j \in S(\mathbb{I}')iT_{\alpha_j}\}.$$

If A is not cofinal in η, let $j < \eta$ such that $[j, \eta] \cap A = \emptyset$. $S^*(I)$ is then defined to be the unique cofinal well founded branch:

$$\{i|iT_j \vee j \leq i < \eta\}.$$

3.5.2 The comparison iteration

As mentioned earlier, we can "compare" two normally iterable premice via a pair of padded normal iterations known as the *coiteration* or *comparison iteration*. We define:

Definition 3.5.9. Let M, N be premice. M is a *segment of* N (in symbols: $M \lhd N$) iff $M = N||\eta$ for an $\eta \leq \mathrm{On}_N$.

If neither of M^0, M^1 is a segment of the other, there is a *first point of difference* ν_0 defined as the least ν such that $M^0||\nu \neq M^1||\nu$. Then $J^{E^{M^0}}_{\nu_0} = J^{E^{M^1}}_{\nu_0}$ and $E^{M_0}{}_{\nu_0} \neq E^{M'}{}_{\nu_0}$. Set : $\pi^h_{0,1} : M^h \longrightarrow_{E_{\nu_0}} M^h_1$ if $E^{M^h}{}_{\nu_0} \neq \varnothing$. Otherwise set: $M^h_1 = M^h$, $\pi^h_{0,1} = \mathrm{id}$. Then $M^0_1||\nu_0 = M^*_1||\nu_0$. If M^0_1, M^1_1 have a point ν_1 of difference, then $\nu_1 > \nu_0$ and we can repeat the process to get M^h_2 etc. Suppose that $\mathrm{card}(M^h) < \Theta$ for $h = 0, 1$ where $\Theta + 1$ is regular and each M^h is $\Theta + 1$ iterable. The comparison process then continues until we have a pair of iterations of length $i + 1$, where either $i = \Theta$ of $i < \Theta$ and M^0_i, M^1_i have no point of difference. (Hence one is a segment of the other.) Using the initial segment condition we shall show that the comparison must terminate at an $i + 1 < \Theta$. The formal definition is:

Definition 3.5.10. Let $\Theta > \omega$ be a regular cardinal. Let M^0, M^1 be premice of height $< \Theta$ which are normally $\Theta + 1$-iterable. Let S^n be a successful Θth normal iteration strategy for M^n $(n = 0, 1)$. The *coiteration* of M^0, M^1 given by $\langle S^0, S^1 \rangle$ is a pair $\langle I^0, I^1 \rangle$ of padded normal iterations of common length $\mu + 1 \leq \Theta + 1$ with coindices $\langle \nu_i \mid i < \mu \rangle$ such that

$$I^n = \langle \langle M^n_i \rangle, \langle \nu_i \mid i \in A^n \rangle, \langle \pi^n_{i,j} \rangle, T^n \rangle$$

and:

- $M^n_0 = M^n$
- If M^0_i, M^1_i are given and $i < \Theta$, then

$$\nu_i \simeq \text{ the first point of difference } \nu \text{ such that } M^0_i||\nu \neq M^1_i||\nu.$$

- If ν_i exists and $E^n_{\nu_i} \neq \emptyset$, then $i \in A^n$ and:

$$\pi^n_{h,i+1} \colon M^*_i \longrightarrow^*_{E^n_{\nu_i}} M^n_{i+1}.$$

- If ν_i exists and $E^n_{\nu_i} = \emptyset$, then $i \notin A^n$ and $M^n_{i+1} = M^n_i$.
- If ν_i does not exist, then $\mu = i$.

Then the coiteration is uniquely determined by M^0, M^1, S^0, S^1. We prove the *Comparison* Lemma:

Lemma 3.5.1. *The comparison iteration terminates below* Θ.

Proof: Suppose not. Then M_i^h, $\pi_{i,j}^h$ are defined for all $i \leq \Theta$ and $i \leq_{T^h} j \leq \Theta$. By induction we have: $M_i^h, \pi_{i,j}^h \in H_\Theta$ for $i \leq_{T^h} j < \Theta$. Hence $I^h \in H_{\Theta^+}$. Set: $Q = H_{\Theta^+}$. By a Löwenheim–Skolem argument, there is $X \prec Q$ such that:

$$\mathrm{card}(X) < \Theta, X \cap \Theta \text{ is transitive }, I^0, I^1 \in X.$$

Let $\sigma \colon \bar{Q} \overset{\sim}{\longleftrightarrow} X$ where $\bar{Q}$ is transitive. Then $\sigma \colon \bar{Q} \prec Q$. Let $\sigma(\bar{I}^h) = I^h$ $(h = 0, 1)$. Let:

$$\bar{I}^h = \langle \langle \bar{M}_i^h \rangle, \langle \bar{\nu}_i \mid i \in A^h \rangle, \langle \bar{\pi}_{i,j}^h \rangle, \bar{T}^h \rangle$$

for $h = 0, 1$. Clearly $\bar{Theta} = \Theta \cap X$ and $\sigma \restriction \bar{\Theta} = \mathrm{id}$. Hence:

(1) (a) $i \leq_{\bar{T}^h} j \longleftrightarrow i \leq_{T^h} j$ for $i, j < \bar{\Theta}$

(b) $i <_{\bar{T}^h} \bar{\Theta} \longleftrightarrow i <_{T^h} \Theta$ for $i < \bar{\Theta}$.

Hence:

(1)(c) $\bar{\Theta} <_{T^h} \Theta$,

since $\bar{\Theta}$ is a limit point of the club set T^h"Θ.

Set: $\bar{H} =: (H_{\bar{\Theta}})^Q$. Then:

(2) $\sigma \restriction \bar{H} = \mathrm{id}$.

Proof. Since $\sigma \restriction \bar{\Theta} = \mathrm{id}$, we have: $\sigma(a) = a$ for $a \subset < \alpha < \bar{\Theta}$ such that $a \in \bar{H}$. Similarly $\sigma(r) = r$ for $r \subset \alpha^2$ such that $\alpha < \bar{Theta}$, $r \in \bar{H}$. Let $x \in \bar{H}$. Let $u = TC(x)$. Then $u \in \bar{H}$ and there is $\alpha < \bar{H}$, $f \in \bar{H}$ such that $f \colon \alpha \xrightarrow{\text{onto}} u$. Hence:

$$\sigma(r) = r \text{ where } r = \{\langle i, j \rangle \mid i, j < \alpha \wedge f(i) \in f(j)\}.$$

Hence $f = \sigma(f)$, since both are defined by the recursion:

$$f(i) = \{f(j) \mid jri\} \text{ for } i < \alpha.$$

Hence $\sigma(a) = a$ where $a = f^{-1}$"x. Hence $x = \sigma(x) = f$"a. QED (2)

Hence:

(3) $\bar{M}_i^h = M_i^h$, $\bar{\pi}_{i,j}^h = \pi_{i,j}^h$ for $i \leq_{\bar{T}^h} j < \bar{\Theta}$.

But then:

(4) $\bar{M}_{\bar{\Theta}}^h$, $\langle \bar{\pi}_{i,\bar{\Theta}}^h \mid i <_{\bar{T}^h} \bar{\Theta} \rangle$ is the direct limit of:

$$\langle M_i^h \mid i <_{T^h} \bar{\Theta} \rangle, \langle \pi_{i,j}^h \mid i \leq_{T^h} j <_{T^h} \bar{\Theta} \rangle.$$

Hence:

(5) $\bar{M}^h_{\bar\Theta} = M^h_{\bar\Theta}$, $\bar\pi^h_{i,\bar\Theta} = \pi^h_{i,\bar\Theta}$ for $i < \bar\Theta$.

Using this we get:

(6) $\pi^h_{\bar\Theta,\Theta} = \sigma \restriction M^h_{\bar\Theta}$

Proof. Let $x \in M^h_{\bar\Theta}$, $x = \pi^h_{i,\bar\Theta}(z)$ where $i <_{T^n} \bar\Theta$. Then;

$$\sigma(x) = \sigma(\pi^h_{i,\bar\Theta}(z)) = \pi^h_{i,\Theta}(z) = \pi^h_{\bar\Theta,\Theta}\pi^h_{i,\bar\Theta}(z) = \pi^h_{\bar\Theta,\Theta}(x).$$

QED (6)

(7) There is $i \in [\bar\Theta, \Theta)_{T^h}$ such that $M_{\bar\Theta} \neq M_i$.

Proof. Suppose not. Then $M_i = M_{\bar\Theta}$ and hence $[\bar\Theta, i) \cap A^h = \emptyset$ for $i <_{T^h} \Theta$. Hence $M_\Theta = M_{\bar\Theta}$. But then $[\bar\Theta, \Theta) \subset A^{1-h}$. Let $j \in [\bar\Theta, \Theta)$ such that $\nu_j > \mathrm{ht}(M_{\bar\Theta})$. ν_j is a point of difference. Hence $\nu_j \leq \mathrm{ht}(M_{\bar\Theta})$. Contradiction! QED (7)

Now let i_h be least such that $\bar\Theta \leq_{T^h} i <_{T^h} \Theta$ and $M_i \neq M_{\bar\Theta}$. By minimality, $i_h = j_h + 1$ for some j_h. But then $j_h \in A^h$, since otherwise $M_j = M_i$ and i was not minimal. Let $t_h = T^h(i_h)$. Then $\bar\Theta \leq_{T^h} t_h < i_h$. Hence $M^h_{t_h} = M^h_{\bar\Theta}$ and $\pi_{\bar\Theta,t_h} = \mathrm{id}$. Set:

$$F_h =: E^{M^h_{j_h}}_{\nu_{j_h}}, \kappa_h =: \mathrm{crit}(F_h).$$

We know: $\pi_{\bar\Theta,\Theta} = \pi_{\bar\Theta,t_h}\pi_{t_h,i_h}\pi_{i_h,\Theta}$, where $\pi^h_{\bar\Theta,t_h} = \mathrm{id} \restriction M^h_{\bar\Theta}$ and $\pi^h_{i_h,\Theta} \restriction \lambda_{j_h} = \mathrm{id}$. From this it follows easily that:

$$\kappa_h = \mathrm{crit}(\pi^h_{t_h,i_h}) = \mathrm{crit}(\pi^h_{\bar\Theta,i_h})$$

and:

(8) $F_h(X) = \sigma(X) \cap \lambda_{j_h}$ for $h = 0, 1$, $X \in \mathbb{P}(\bar\Theta) \cap M^h_{\bar\Theta}$.

But then:

(9) $j_0 \neq j_1$,

since otherwise $E^{M^0}_{\nu_{j_0}} = E^{M^1}_{\nu_{j_1}}$ and ν_{j_h} is not a point of difference.

Now suppose e.g. that $j_0 < j_1$. ν_{j_0} is then a cardinal in $M^0_{j_1}$. But $E^0_{j_0} = E^1_{j_1}|\lambda_{j_0} \in M^1_{j_1}||\nu_{j_1}$. Hence ν_{j_0} is not a cardinal in $M^0_{j_1}$, since: $M^0_{j_1}||\nu_{j_1} = M^1_{j_1}||\nu_{j_1}$. Contradiction!

QED (Lemma 3.5.1)

3.5.3 n–normaliterability

By an *n–normal iteration strategy* we mean a partial function s on normal n–iterations of limit length such that $S(I)$, if defined, is a well founded cofinal branch in $\mathbb{I}$. The concepts *α–successful n–normal strategy* and *n–normally α–iterable* are then defined in the obvious way. M is called *n–normally iterable* iff it is n–normally α–iterable for all α. If M^0, M^1 are premice of cardinals $1 < \Theta$, where Θ is regular, and S^h is a $\Theta + 1$–successful n_h–normal iteration strategy for $M^h (h = 0, 1)$, we can define the $\langle n_0, n_1 \rangle$–coiteration of M^0, M^1 *given* by $\langle S^0, S^1 \rangle$ exactly as before. But then the comparison lemma holds for this coiteration by exactly the same proof as before.

3.5.4 Iteration strategy and copying

Lemma 3.5.2. *Let M be normally α–iterable. Let $\sigma : \overline{M} \to_{\Sigma^*} M$. Then $\overline{M}$ is normally α–iterable.*

Proof: Let S be an α–successful strict normal iteration strategy for M. We use the copying procedure and Lemma 3.4.19 to define an α–successful strategy $\overline{S}$ for $\overline{M}$. $\overline{S}$ is defined on the set of strict iterations $\overline{I}$ of $\overline{M}$ having limit length such that σ induces a copy I of $\overline{I}$ onto M with copying maps $\langle \sigma_0 | i < \text{lh}(\overline{I}) \rangle$ which conforms to S. We then set: $\overline{S}(\overline{I}) = S(I)$. $\overline{S}(\overline{I})$ is then a cofinal well founded branch in $\overline{I}$ by Lemma 3.4.19. By induction on $\mu = \text{lh}(\overline{I})$ it then follows that, if $\overline{I}$ is $\overline{S}$–conforming, then σ induces an S–conforming copy I with copying maps $\langle \sigma_i | i < \mu \rangle$. For $\mu = 1$ or limit μ this is trivial. For $\mu = \eta + 1$ where η is a limit, we use the definition of $\overline{S}$. If $\mu = \eta + 1$, we use Lemma 3.4.18 By a virtual repitition of this proof:

Lemma 3.5.3. *Let M be n–normally α–iterable. Let $\sigma : \overline{M} \to_{\Sigma_1^{(n)}} M$. Then $\overline{M}$ is n–normally α–iterable.*

The details are left to the reader.

3.5.5 Full iterability

Normal iterability is too weak a property for many purposes. For instance, we do not kknow, in general, that a normal iterate N of a normally iterable M is itself normally iterable. We therefore introduce the notion of *full iterability*, which is often more useful but, unfortunately, harder to verify.

The process of taking a normal iteration of M can itself be iterated, as can the process of taking a segment of a normal iterate of M. This suggests an expande notion of iteration: Not only normal iterations are allowed, but also (finite or infinite) successions of normal iteration, where the $i+1$ set iteration is applied to a segment of the iterate given by stage i. The formal definition is:

Definition 3.5.11. Let M be a premouse. By a *full iteration I of M of length μ* we mean a sequence $\langle I^i | i < \mu\rangle$ of normal iteration:

$$I^i = si\langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_{h,j}\rangle, T^i\rangle$$

inducing a sequence $M_i = M_i^{M,I}(i < \mu)$ of premice and a commutative sequence of partial maps $\pi_{hj} = \pi_{hj}^{(M,I)}(h \leq j < \mu)$ such that the following hold:

(a) $M_0 = M$.

(b) $M^i_0 \lhd M_i$ for $i < \mu$.

(c) If $i + 1 < \mu$, then I^i has length $l_i + 1$ for some l_i and:

$$M_{i+1} = M^i_{l_i}, \pi_{i,i+1} = \pi^i_{0,l_i}.$$

Call $i < \mu$ a *drop point* in I iff either $M^i_0 \neq M_i$ or $i + 1 < \mu$ and I^i has a truncation on its main branch.

(d) Let $\alpha < \mu$. Then the set of drop points $i < \alpha$ is finite. Moreover, $\pi_{i,\alpha}$ is a total function on M_i whenever $[i, \alpha)$ has no drop point. If α is a limit ordinal then:

$$M_\alpha, \langle\pi_{i\alpha} | i < \mu\rangle$$

is the transitivized direct limit of:

$$\langle M_i | i < \alpha\rangle, \langle\pi_{ij} | i \leq j < \mu\rangle.$$

It is clear that the sequence $\langle M_i | i < \mu\rangle, \langle\pi_{ij} | i \leq j < \mu\rangle$ are uniquely determined by the pair $\langle M, I\rangle$.

Definition 3.5.12. $I = \langle I^i | i < \mu\rangle$ is a full iteration iff it is a full iteration of some M.

Note. We have not excluded the case $\mu = 0$. In this case $I = \emptyset$ is a full iteration of every premouse. We then have: $M^{(N,\emptyset)} = N, \pi^{(N,\emptyset)} = \mathrm{id} \restriction N$.

Definition 3.5.13. Let $I = \langle I^i | i < \mu\rangle$ be a full iteration. The *total length* of I is $\Sigma_{i<\mu} \operatorname{lh}(I^i)$.

Definition 3.5.14. Let I be a full iteration of M. $i < \mu$ is a *truncation point (or drop point) v* with M, I, iff either I^σ is of length $l_i + 1$ and has a truncation on its main branch $T^{i\prime\prime}\{l_i\}$, or else $M_0^i \neq M_i$.

By (d) the set of truncation points $i < \alpha$ is always finite if $\alpha < \mu$ is a limit ordinal.

Definition 3.5.15. I is a full iteration of M *to* M' iff I is a full iteration of M and one of the following holds:

(i) $I = \emptyset$ and $M' = M$

(ii) I has length $\mu = \eta + 1$ and I^η has length $\gamma + 1$, where $M' = M_\gamma^\eta$.

(iii) I has limit length mu, the set of truncation points $i < \mu$ is finite, and:

$$\langle M_i < i < \mu\rangle, \langle \pi_{ij} | i \leq j < \mu\rangle$$

is as the transitive direct limit:

$$M', \langle \pi_i | i < \mu\rangle.$$

Definition 3.5.16. Let M, M', I be as above. The *iteration map* $\pi = \pi^{(M,I)}$ from M to M' given by the pair (M, I) is defined as follows:

(i) $\pi = \operatorname{id} \restriction M$ if $I = \varnothing$

(ii) If I, I^ξ are as in (ii) we set $\pi = \pi_{0,l_\eta}^\eta \circ \pi_{0,\eta}^{(M,I)}$

(iii) If case (iii) holds, we set: $\pi = \pi_0$.

Definition 3.5.17. Let $I = \langle I^i | i < \mu\rangle, I' = \langle I'^i | i < \mu'\rangle$ be full iterations. the *concatenation* $I^\frown I'$ of I, I' is the sequence $\langle \tilde{I}^i | i < \mu + \mu'\rangle$ such that $\tilde{I}^i = I^i$ for $i < \mu$ and $\tilde{I}^{\mu+i} = I'^i$ for $i < \mu'$.

$I^\frown I'$ is not necessarily a full iteration. However, it is easily seen that

Lemma 3.5.4. *If I is a full iteration from M to M' and I' is a full iteration of M', then*

(a) $I^\frown I'$ is a full iteration of M.

(b) If $I' \neq \emptyset$, then $\pi^{(M,I)} = \pi_{0\mu}^{(M,I^\frown I')}$, where $\mu = \operatorname{lh}(I)$.

(c) *If I' is an iteration of M' to M'', then $I^\frown I'$ is an iteration of M to M'' and* $\pi^{(M,I^\frown I')} = \pi^{(M',I')} \circ \pi^{(M,I)}$.

Definition 3.5.18. Let I be a full iteration of M. By a *lenthening* of I we mean any $I^\frown I'$ which is a full iteration.

(Hence we *cannot* lengthen $\langle I^i | i \leq \eta\rangle$ by extending its last normal iteration I^η, but only by starting a new normal iteration.)

Note. Lemma 3.5.4 (b) then says that, if I is an iteration from M to M' and I' is a *proper* lenghtening of I (i.e. $\mu = \mathrm{lh}(I) < \mu' = \mathrm{lh}(I')$, then $\pi^{(M,I)} = \pi_{0\mu}^{(M,I')}$.

We now define the concept of *full iterability*:

Definition 3.5.19. A *full iteration strategy* is a partial function on full iterations I of length $\eta + 1$ such that I^η is of limit length. $S(I)$, if defined is then a cofinal well founded branch in I^η (we refer such full iterations I as *critical*).

Definition 3.5.20. A full iteration $I = \langle I^i | i < \mu\rangle$ *conforms* to the strategy S iff whenever $i < \mu$ and $\gamma < \mathrm{lh}(I^i)$ is a limit ordinal, then $T^{0''}\{\gamma\}$ is the branch $S((I \restriction i)^\frown(I^i|\gamma))$ given by S.

Definition 3.5.21. A strategy S is *α–successful* for M iff whenever $I = \langle I^i | i < \mu\rangle$ is an S–conforming full iteration of M of total length $\Sigma_{i<\mu} \mathrm{lh}(I^i) < \alpha$, then I can be extended one step further in an S–conforming way:

(a) If $\mu = i + 1$ and I^i is of limit length, then $S(I)$ exists.

(b) Let $\mu = i + 1$ and $\mathrm{lh}(I^i) = h + 1$. Extend I^i to a potential normal iteration by appointing ν_h. This gives E_{ν_h} and M_i^*. Then M_h^* is $*$–extendible by E_{ν_h}.

(c) If μ is a limit ordinal, then there are at most finitely many truncation points below μ. Moreover:

$$\langle M_i^{(M,I)} | i < \mu\rangle, \langle \pi_{i,j}^{(M,I)} | i \leq j < \mu\rangle$$

has a well founded limit.

Definition 3.5.22. M is *fully α–iterable* iff it has an α–successful full iteration strategy.

Definition 3.5.23. M is *fully iterable* iff it is fully α–iterable for every α.

3.5.6 The Dodd–Jensen Lemma

We now prove a theorem about normal iteration of premice which are fully iterable and have the normal unique new property.

Theorem 3.5.5. (The Dodd–Jensen Lemma)
Suppose that M has the normal uniqueness property and is fully Θ–iterable, where $\Theta > \omega$ is regular. Let:

$$I^0 = \langle\langle M_i^0\rangle, \langle\nu_i^0\rangle, \langle\pi_{ij}^0\rangle, T^0\rangle$$

be a normal iteration of M with length $\eta + 1$. Let $\sigma : M \to_{\Sigma^*} N$ where $N \lhd M_\eta^0$. Then:

(a) $N = M_\eta^0$.

(b) There is no truncation point on the main branch $T^{0\prime\prime}\{\eta\}$ of I^0.

(c) $\sigma(\xi) \geq \pi_{0,}(\xi)$ *for all* $\xi \in \mathrm{On} \cap M$.

Note. Let $M' = M_\eta^0, \pi = \pi_{0,\eta}$. Then π is the unique Σ^*–preserving map of M to M' such that $\pi(\xi) =$ the least ξ' such that $\xi' = \sigma(\xi)$ for some $\sigma : M \to M'$ which is Σ^*–preserving. Thus π depends only on the models M, M' and not on the iteration I^0.

We now prove the theorem. Fix a Θ–successful strategy S for M. By induction on $i < \omega$ we construct I^i, N^i, σ^i such that

- $I^i = \langle\langle M_h^i\rangle, \langle\nu_h^i\rangle, \langle\pi_{hj}^i, T^i\rangle$ is a normal iteration.
- $N^i \lhd M_\eta^i$ and $\sigma^i : M \to_{\Sigma^*} N^i$.
- $\langle I^0, \ldots, I^i\rangle$ is S–conforming.
- If $i = h + 1$, then I^i is the copy of I^0 onto N^h by σ^h.

Case 1 $i = 0$
I^0 is given. Set: $N^0 = N, \sigma^0 = \sigma$.

Case 2 $i = h + 1$
We first construct I^i. We construct $I^i|\gamma + 1$ and copying maps

$$\sigma_l^h : M_l^0 \to_{\Sigma^*} M_l^0 (l \leq \gamma)$$

by induction on γ, ensuring at each stage that $\langle I^0, \ldots, I^h, I^i|\gamma + 1\rangle$ is S–conforming.

For $\gamma = 0$ set $I^i|\gamma+1 = \langle\langle N^h\rangle, \varnothing, \langle \mathrm{id}\rangle, \varnothing\rangle$. We set $\sigma^h_0 = \sigma^h$. If $\gamma = l+1$, we follow the usual procedure.

Now let γ be a limit ordinal. We are given $I^i|\gamma$ and copying maps $\langle\sigma^h_l | l < \gamma\rangle$, where $I^i|\gamma$ is the copy of $I^0|\gamma$ onto $M^i_0 = N^h$ by σ^h. Then $I' = \langle I^0, \ldots, I^h, I^i|\gamma\rangle$ is S–conforming. Hence S gives us a cofinal well founded branch $b = S(I')$ in $I^i|\gamma$ and we extend $I^i|\gamma$ to $I^i|\gamma+1$ by setting $T^{i\prime\prime}\{\gamma\} = B$. But by Lemma 3.4.19, b is a well founded cofinal branch in $I^0|\gamma$. Hence $b = T^{0\prime\prime}\{\gamma\}$ by uniqueness. But then $\sigma^i_{\gamma+1} : M^0_\gamma \to M^i_\gamma$ can be defined as usual. This gives $\langle I^0, \ldots, I^i\rangle$, which is S–conforming. But $\sigma^h_\eta : M^0_\eta \to_{\Sigma^*} M^i_\eta$, where $N^0 \lhd M^0_\eta$. If $N^0 = M^0_\eta$, set $N^i = M^i_\eta$. Otherwise set: $N^i = \sigma^h_\eta(N^0)$. In either case $\sigma^h_\eta \cdot \sigma^0 : M \to_{\Sigma^*} N^i$, and we set: $\sigma^i = \sigma^h_\eta \cdot \sigma^0$. QED (Case 2)

Thus $\langle I^i | i < \omega\rangle$ is an S–conforming full iteration of M. Using this we prove (a) – (c):

(a) Suppose not. Then $N^i \neq M^i$ for $i < \omega$. But $M_0 = M, M_{n+1} = M^n_\eta$ and $M^{n+1}_0 = N^n \neq M_{n+1}$. Hence every $n+1 < \omega$ is a truncation point in $I = \langle I^n | n < \omega\rangle$.
Contradiction!

(b) Suppose not. Let $i+1$ be a truncation point on the main branch $T^{0\prime\prime}\{\eta\}$ of I^0. By our construction $i+1$ is a truncation point in $T^{n\prime\prime}\{\eta\}$ for $n < \omega$. Hence each $n+1$ is a truncation point in I.
Contradiction!

(c) By (a), (b), $\pi_{nm} : M_n \to M_m$ is a total function on M_n for $n \leq m < \omega$. Suppose (c) to be false. Let $\sigma^0(\xi) < \pi^0_0(\xi)$. Then $\sigma^{i+1}(\xi) = \sigma^i_\eta(\sigma^0(\xi) < \sigma^i_\eta(\pi^0_{0\eta}(\xi)) = \pi^i_{0\eta}(\sigma^i(\xi)) = \pi^{(M,I)}_{i,i+1}(\sigma^i(\xi))$. Hence $\pi_{i+1,\omega}\sigma^{i+1}(\xi) < \pi_{i,\omega}\sigma^i(\xi)$ for $i < \omega$.
Contradiction! QED (Theorem 3.5.5)

Lemma 3.5.6. *Let $\omega < \Theta \leq \alpha$ where Θ is a regular cardinal. Let S be an α–successful strategy for M. Let I be an S–conforming iteration from M to M' with total length $< \Theta$. Define an iteration strategy S' for M' by*

$$S'(I') \simeq S(I^\frown I')$$

for full iteration I' of M'. Then S' is an α–successful strategy for M'.

The proof is left to the reader. Similarly, we obtain a normal iteration strategy S'' for M by setting S'' for M by setting $S''(I) \simeq S'(\langle I\rangle)$ where I is a normal iteration of limit length $< \alpha$ and $\langle I\rangle$ is the full iteration $\tilde{I}$ of length 1 such that $\tilde{I}^0 = I$.

3.5.7 Copying a full iteration

Definition 3.5.24. Let $\sigma : M \to_{\Sigma^*} M'$ where M, M' are premice. Let $I = \langle I^i | i < \mu\rangle$ be a full iteration of M. $I' = \langle I'^i | i < \mu\rangle$ is the *copy* of I *onto* M' by σ with copying maps $\langle \sigma^i < i < \mu\rangle$ iff

(a) I' is a full iteration of M' inducing

$$\langle M'_i | i < \mu\rangle, \langle \pi'_{ij} | i \leq j < \mu\rangle$$

(b) $\sigma_i : M_i \to_{\Sigma^*} M'_i$ such that $\sigma_i \pi_{ij} = \pi'_{ij}\sigma_i$

(c) $\sigma_0 = \sigma$

(d) I'^i is the copy of I^i induced by $\sigma_i \restriction M_0^i$ with copying maps $\langle \sigma_h^i | h < \mathrm{lh}(I^i)\rangle$

(e) If $M_i = M_0^i$, then $M'_i = M'^i_0$ and $\sigma^i = \sigma_0^i$.

(f) If $M_i \neq M_0^i$, then $M'^i_0 = \sigma_i(M_0^i)$ and $\sigma_0^i = \sigma_i \restriction M_0^i$

(g) If $i + 1 < \mu$, then $\sigma_{i+1} = \sigma_{li}^i$ where $\mathrm{lh}(I^i) = l_i$.

Clearly I' and the copying maps $\langle \sigma_i | i < \mu\rangle, \langle \sigma_h^i | i < \mu, h < \mathrm{lh}(I^i)\rangle$ are unique, if they exist. (Note that if $\eta < \mu$ is a limit ordinal, then σ_η is uniquely defined by: $\sigma_\eta \pi_{i\eta} = \pi'_{i\eta}\sigma_i$ for $i < \eta$.)

Lemma 3.5.7. *Let $\sigma : M \to_{\Sigma^*} M'$, where M' is fully α–iterable. Then M is fully α–iterable.*

Let S' be an α–successful strategy for M'. We define a strategy S for M as follows: If $I = \langle I^i | i \leq \eta\rangle$ is a full iteration of M such that I^η is of limit length, we ask whether σ induces a copy I' of I onto M'. If so we set: $S(I) \simeq S'(I')$. If not, $S(I)$ is undefined. ($S(I)$, if defined, is a cofinal well founded branch in I^η by Lemma 3.4.19.) It follows that if I is S–conforming, then σ induces a copy I' which is S'–conforming. (We prove this by induction on μ, where $I = \langle I^i | i < \mu\rangle$ and for $\mu = \eta + 1$ by induction on the length of I^η.) Using Lemma 3.4.18 and 3.4.19 it then follows that I can be extended in an S–conforming way, since I' can be extended in an S'–conforming way.

QED (Lemma 3.5.7)

3.5.8 The Neeman–Steel lemma

The usefulness of the Dodd–Jensen Lemma is limited by the fact that it applies only to premice with the normal uniqueness property. In the absence of normal uniqueness we have the following subtleties:

Theorem 3.5.8 (The Neeman–Steel Lemma). *Let M be a countable premouse which is fully $\omega+1$ iterable. Let $\langle \xi_n | n < \omega \rangle$ be an enumeration of $\mathrm{On} \cap M$. There is an ω_1-successful full iteration strategy S for M such that whenever $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$ is an S-conforming normal iteration of M of length $\eta+1 < \omega_1$ and $\sigma : M \longrightarrow_{\Sigma^*} M'$, where $M' \lhd M_\eta$, then:*

(a) $M' = M_\eta$.

(b) *There is no truncation point on the main branch $\{i : iT\eta\}$.*

(c) *If $\sigma(\xi_i) = \pi_{0,\eta}(\xi_i)$ for $i \leq n < \omega$, then $\sigma(\xi_n) \geq \pi_{0,\eta}(\xi_n)$.*

Then $\pi_{0,\eta}$ is the unique $\pi : M \longrightarrow_{\Sigma^*} M'$ such that $\pi(\xi_n) =$ the least ξ' such that $\sigma(\xi_n) = \xi'$ for a σ such that $\sigma : M \longrightarrow_{\Sigma^*} M'$ and $\sigma(\xi_i) = \pi(\xi_i)$ for $i < n$. Then π depends only on M, M' and the enumeration $\langle \xi_i : i < \omega \rangle$, rather than on the iteration I.

Note. When we say that a normal iteration is S-conforming, we mean that the full iteration $\langle I \rangle$ of length 1 is S-conforming.

We shall derive Theorem 3.5.8 from a stronger statement:

Lemma 3.5.9. *Let $M, \langle \xi_i : i < \omega \rangle$ be as above. There is a $\omega_1 + 1$-successful full iteration strategy S for M such that whenever I is an S-conforming full iteration from M to M' and $\sigma : M \longrightarrow_{\Sigma^*} M'$, then:*

(a) *No $i < \mathrm{lh}(I)$ is a drop point in I (hence the iteration map π from M to M' is a total function on M).*

(b) *If $\sigma(\xi_i) = \pi(\xi)$ for $i < n$, then $\sigma(\xi_n) \geq \pi(\xi_n)$.*

This clearly implies Theorem 3.5.8 since if $I = \langle \langle M_i \rangle, m \rangle$, M' are as in the theorem, then $\langle I, \langle M' \rangle \rangle$ is an S-conforming full iteration from M to M' of length 2. (Here $\langle M \rangle$ denotes the minimal normal iteration of M of length 1: $\langle M \rangle, \varnothing, \langle \mathrm{id} \restriction M \rangle, \varnothing \rangle$.)

Proof. We prove Lemma 3.5.9. In the following we use the term "iteration" to mean a full iteration of total length $< \omega_1$. By a *lengthening* of an iteration

I we mean an iteration of the form $I^\frown I'$. Fix an ω_1+1-successful iteration strategy for M. We write "S-iteration" to mean "S-conforming iteration".

(1) There is an iteration I_0 from M to an N_0 such that:

- There is $\sigma_0 : M \longrightarrow_{\Sigma^*} N_0$.
- Let I be any lengthening of I_0 which is an S-iteration from M to M'. Let $\sigma' : M \longrightarrow_{\Sigma^*} M'$. Then I has no truncation point in $\mathrm{lh}(I) \setminus \mathrm{lh}(\hat{I})$.

Proof. Suppose not. Recall that $\varnothing$ is an S-iteration of M to M. There is then a sequence of $\langle I_i, N_i, \sigma_i\rangle(\sigma < \omega)$ such that:

- $I_0 = \varnothing, N_0 = M, \sigma_0 = \mathrm{id}\, M$.
- $I_i + 1$ is an S-iteration of M to $N_i + 1$ which lengthen Ii.
- $I_i + 1$ has a truncation point in $\mathrm{lh}(I_i + 1) \setminus \mathrm{lh}(I_i)$.
- $\sigma_i : M \longrightarrow_{\Sigma^*} N_i$.

Set $I = \bigcup_i I_i$. Then I is an S-iteration with infinitely many truncation points below $\mathrm{lh}(I)$. Contradiction!

QED (1)

Fix I_0, N_0, σ_0.

(2) We can extend $\langle I_0, N_0, \sigma_0\rangle$ to an infinite sequence $\langle I_i, N_i, \sigma_i\rangle$ $(i < \omega)$ such that:

- $I_i = I_h^\frown I_{h,i}$ is an S-iteration which lengthen I_h for $h < i$.
- $I_{h,i}$ is an iteration from N_h to N_i with iteration map $\pi_{h,i} = \pi^{(N_h, I_{h,i})}$.
- $\pi_{ij}\pi_{hi} = \pi_{hi}$ for $h \leq i \leq j < \omega$.
- $\sigma_i : M \longrightarrow_{\Sigma^*} N_i$
- $\pi_{ij}\sigma_i(\xi_h) = \xi_h$ for $h < i < j$.
- Let $j = i+1$ and let $I_j^\frown I$ be any S-iteration, where I is from N_j to N. Let $\sigma : M \longrightarrow_{\Sigma^*} N$ such that $\sigma(\xi_h) = \pi\sigma_j(\xi_h)$ for $h < j$, where $\pi = \pi^{(N_j, I)}$ is the iteration map. Then $\sigma(\xi_i) \geq \pi\sigma_j(\xi_i)$.

Proof. Suppose not. Consider the tree of finite sequences $\langle\langle I_i, N_i, \sigma_0\rangle : i \leq n\rangle$ such that the above holds for all $h, i, j \leq n$. This tree has no infinite branch. Hence there is a finite sequence $\langle\langle I_i, N_i, \sigma_i\rangle : i \leq n\rangle$ which has no successor in the tree. Nut then we can form a sequence

$$\langle \tilde{I}_i, \tilde{N}_i, \tilde{\sigma}_i\rangle,\ i \leq \omega$$

with the properties:

- $\tilde{I}_0 = I_n, \tilde{N}_0 = N_n, \tilde{\sigma}_0 = \xi_n$.
- $\tilde{I}_{i+1} = \tilde{N}_i^\frown \tilde{I}_i'$ is an S-iteration from M to $\tilde{N}_{i+1}$ which properly lengthens $\tilde{N}_i$.
- $\tilde{I}_i'$ is an iteration from $\tilde{N}_i$ to $\tilde{N}_{i+1}$ with iteration map $\pi_i = \pi^{(\tilde{N},\tilde{I}_i')}$.
- $\tilde{\xi}_{i+1} : M \longrightarrow_{\Sigma^*} \tilde{N}_{i+1}$ is such that $\tilde{\xi}_{i+1}(\xi_h) = \pi\tilde{\xi}_i(\xi_h) = \pi_i\tilde{\xi}_i(\xi_h)$ for $h < n$ but $\tilde{\xi}_{i+1}(\xi_n) < \pi_i(\tilde{\xi}_i(\xi_n))$.

Set $\mu_i = \mathrm{lh}(\tilde{I}_i)$, $\tilde{I} = \bigcup_i I_i$. Then $\mu_i < \mu_{i+1}$ and $\tilde{I}$ is of limit length $\mu = \sup_i \mu_i$ since $\tilde{I}_i$ lengthens I_0 and $\tilde{\sigma}_i : M \longrightarrow_{\Sigma^*} \tilde{N}_i$. Let $M_l = M_l^{(M,\tilde{I})}, \tilde{\pi}_{l,j} = \pi_{l,j}^{(M,\tilde{I})}$ for $l \leq i < \mu$, it follows easily that $\pi_i = \tilde{\pi}_{\mu_i,\mu_{i+1}}$ and $\tilde{N}_i = M_i$. Moreover $\tilde{\pi}_{\mu_i,j}$ is a total function on M_i for $\mu_i \leq j < \mu$. Since $\tilde{I}$ is S-conforming we can form the transitive limit $\tilde{M}, \langle \tilde{\pi}_i : i < \mu\rangle$ of:

$$\langle M_i : i < \mu\rangle, \langle \pi_{i,j} : i \leq j < \mu\rangle.$$

But then $\tilde{\pi}_{\mu_i+1}\tilde{\sigma}_{i+1}(\xi_n) < \tilde{\pi}_i\tilde{\sigma}_i(\xi_n), i < \omega$. Contradiction!

QED(2)

Now let $\langle I_i, N_i, \sigma_i\rangle, i < \omega$ be as in (2). Let $\mu_i =: \mathrm{lh}(I_i)$. We assume without lose of generality that $\mu_i < \mu_j$ for $i < j$. If I_i' is an S-iteration from M to M', then so if $I'^\frown\langle M'\rangle$. Set $I^* = \bigcup_i I_i$. I^* is an S-iteration of length $\mu^* = \sup_i \mu_i$. We know by (1) that I^* has no truncation point in $\mu^* \setminus \mu_0$. Letting $M^* = M_i^{M,I^*}, \pi_{i,j}^* = \pi_{i,j}^{(M,I^*)}$, we have:

$$N_i = M_{\mu_i}^* \text{ and } \pi_{ij} = \pi_{\mu_i,\mu_j}^*$$

where N_i, π_{ij} are as in (2). Since I^* is an S-iteration, we can form the limit:

$$M^*, \langle \pi_i^* : i < \mu^*\rangle$$

of $\langle M_i^* : i < \mu^*\rangle, \langle \pi_{ij}^* : i \leq j < \mu^*\rangle$. But $\pi_{\mu_{i+1}}^*(\sigma_{i+1}(\xi_h)) = \pi_{\mu_{j+1}}^*(\sigma_{j+1}(\xi_h))$ for $h \leq i \leq j < \omega$, where $\sigma_{i+1} : M \to M$ and $\pi_{\mu_{i+1}}^* : M_{\mu_{i+1}} \to M^*$ are Σ^*–preserving. But then we can define a $\sigma^* : M \to_{\Sigma^*} M^*$ by:

$$\sigma^*(\xi_n) = \pi_{\mu_{i+1}}(\sigma_{i+1}(\xi_h)) \text{ for } h \leq i < \omega.$$

Let S^* be the $\omega_1 + 1$–successful strategy for M^* defined by:

$$S^*(I) \simeq S(I^{*\frown} I)$$

where I is any full iteration of M^*. Following the prescription in the proof of Lemma **??** we can then define a strategy $\overline{S}$ for M by: If $\overline{I}$ is an iteration of M, we first ask wheter σ^* induces a copy I of $\overline{I}$ onto M^*. If so we set:

$$\overline{S}(\overline{I}) \simeq S^*(I) \simeq S(I^{*\frown} I).$$

If $\overline{I}$ is $\overline{S}$–conforming, it follows that I is S^*–conforming, hence that $I^* \frown I$ is S–conforming. Using this, we show that $\overline{S}$ satisfies (a), (b). Let $\overline{I}$ be an iteration from M to $\overline{M}$ and let $\overline{\sigma} : M \to_{\Sigma^*} \overline{M}$. σ^* induces an iteration I from M^* to M' with copying map $\sigma' : \overline{M} \to M'$. Thus $\sigma'\overline{\sigma} : M \to_{\Sigma^*} M'$. Let $\overline{\pi} = \pi^{(M,\overline{I})}$ be the iteration map from M to $\overline{M}'$. Let $\pi = \pi^{(M^*,I)}$ be the iteration map from M^* to M'. Then $\sigma'\overline{\pi} = \pi\sigma^*$, since σ' is a copying map.

(3) There is no truncation point $i < \mathrm{lh}(\overline{T})$.

Proof. Suppose not. Then i is a truncation point in I and $\mu^* + i$ is a truncation point in $I^* \frown I$, contradicting (1), since $\sigma'\overline{\sigma} : M \to_{\Sigma^*} M'$.

QED (3)

(4) Let $\overline{\sigma}(\xi_h) = \overline{\pi}(\xi_h)$ for $h < i$. Then $\overline{\sigma}(\xi_i) \geq \overline{\pi}(\xi_i)$.

Proof. Suppose not. Note that

$$\sigma'\overline{\pi}(\xi_h) = \pi\sigma^*(\xi_h) = \pi\pi_{\mu^*_{i+1}}\sigma_{i+1}(\xi_h)$$

for $h \leq i$. But $I^* \frown I = I_{i+1} \frown \tilde{I}$ where $\tilde{I}$ is an iteration from N_{i+1} to N with iteration map $\tilde{\pi} = \pi^{(N_{i+1},\tilde{I})}$. It is easily seen that $\tilde{\pi} = \pi\pi_{\mu^*_{i+1}}$, hence

$$\sigma'\overline{\pi}(\xi_h) = \tilde{\pi}\sigma_{i+1}(\xi_h) \text{ for } h \leq i.$$

Hence $\sigma'\overline{\sigma}(\xi_h) = \tilde{\pi}\sigma_{i+1}(\xi_h)$ for $h < i$, but

$$\sigma'\overline{\sigma}(\xi_i) < \sigma'\overline{\pi}(\xi_i) = \tilde{\pi}\sigma_{i+1}(\xi_i).$$

This contradicts (2).

QED(4)

This proves Lemma 3.5.9 and with it Theorem 3.5.8.

QED(Lemma 3.5.9)

QED(Theorem 3.5.8)

The fact that the Neeman-Steel lemma holds only for countable mice is a less serious limitation than one might suppose. In practice, both the Dodd–Jensen lemma and the Newman–Steel lemma are used primarily to establish properties of mice which - by a Löwenheim-Skolem argument - hold generally if they hold for countable mice.

3.5.9 Smooth iterability

Definition 3.5.25. By a *smooth iteration of M* we mean a full iteration I of M such that $M_i = M_0^i$ for $i < \mathrm{lh}(I)$.

The concepts "smooth iteration strategy", "i–successful smooth iteration strategy" and "smooth α–iterable" are defined accordingly. We shall eventually prove that every smoothly iterable premouse is fully iterable. The proof will depend on enhanced copying procedures.

3.5.10 n–full iterability

We said at the outset that a "mouse" will be defined to be a premouse which is iterable. But what is the right notion of iterability? full iterability feels right. An, indeed, we shall ultimately show that, if there is no inner model with a Woodin cardinal, then every normally iterable premouse is fully iterable. However, it will take a long time to reach that point, and in the meantime we must make do with weaker forms of iterability which are easier to verify. The main problem will be this. Our procedure for verifying that a premouse M is normally iterable will not show that normal iterates of M are themselves iterable. What it will show is weaker: If, by an appropriate strategy, I is a normal iteration of M to M' of length $\eta + \delta$ *and* if $\rho^n_{M'} > \lambda_i$ for $i < \eta$, then M' ia n–normally iterable. For this reason we will often be forced to work with n–iteration rather than $*$–iterations, and we must employ a sharply restricted notion of "full iteration". We define:

Definition 3.5.26. Let I be an m–normal iteration of length $\eta+1$ for some $m \leq \omega$. Let $n \leq \omega$. I is *n–bounded* iff $\lambda_i < \rho^n_{M_2}$ for all $i < \eta$.

Definition 3.5.27. I is an *m to n–normal iteration* iff I is an n–bounded m–normal iteration.

We shall be mainly interested in n to n iterations.

Definition 3.5.28. Let M be a premouse. Let $n \leq \omega$ by an *n–full iteration i of length μ* we mean a sequence $\langle I^i | i < \mu\rangle$ of n–normal iterations such that I^i is n to n normal for $i + 1 < \mu$, inducing a sequence $M_i = M_i^{(M,I)} (i < \mu)$ of premice and a commutative sequence $\pi_{ij} = \pi_{ij}^{(M,I)}$ of partial maps from M_i to $M_j (i \leq j < \mu)$ satisfying (a) – (d) of our previous definition.

Note. If $I = \langle I^i | i \leq \eta\rangle$ is an n–full iteration of length $\eta + 1$, then the final n–normal iteration I^η is not neccessarily n to n, though the previous ones are. However, if I^η is not n to n, then there is no possibility of lengthening the sequence I, thouch I^η itself could be lengthened.

We can take over our previous definitions — in particular the definition of "n–full iteration from M to N" and "n–full iteration map" $\pi^{M,I}$.

Definition 3.5.29. $I = \langle I^i | i < \eta \rangle$ is an *n to n full* iteration if I is n–full and each I^i is an n to n–normal iteration.

The definition of "concatenation" is as before. It is cler that if I is an n to n–full iteration from M to M' and I' is an n–full iteration of M', then $I^\frown I'$ is an n–full iteration of M.

Lemma 3.5.4 holds as before, on the assumption that I is an n to n–full iteration from M to M' and I is an n–full iteration of M. THe concepts *n–full iteration strategy* is defined as before, as is the concept of an *S–conforming* n–full iteration, *α–successful* n–full strategy, and n–full *α–iterability.*

The Dodd–Jensen lemma then holds in the form:

Theorem 3.5.10. *Suppose that M has the n–normal uniqueness property and is n–fully Θ–iterable, where $\Theta > \omega$ is regular. Let:*

$$I = \langle\langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T\rangle$$

be an n to n–normal iteration of M with length $\eta + 1$. Let $\sigma : M \rightarrow_{\Sigma^*} N$ where $N \lhd M_\eta$. Then:

(a) $N = M_\eta$.

(b) There is no truncation point on the main branch $T''\{\eta\}$ of I.

(c) $\sigma(\xi) \geq \pi_{o,\eta}(\xi)$ for all $\xi \in \mathrm{On} \cap M$.

The proof is a virtual repetition of the previous proof.

Lemma 3.5.6 holds *mutatis mutandis* just as before. We define what it means for $\sigma : M \rightarrow_{\Sigma^{(n)}} M'$ to *induce a copy I' of I onto M' with copying maps $\langle \sigma^i \rangle$* just as before, writing $\Sigma^{(n)}$ instead of Σ^* everywhere.

Theorem 3.5.11. *Let M be a countable premouse which is n–fully $\omega_1 + 1$ iterable. Let $\langle \xi_n | n < \omega \rangle$ be an enumeration of $\mathrm{On} \cap M$. There is an $\omega_1 + 1$–successful n–full iteration strategy S for M such that whenever $I = \langle\langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, \tau\rangle$ is an S–conforming n to n–normal iteration of M of length $\eta + 1 < \omega_1$ and $\sigma : M \rightarrow_{\Sigma^{(n)}} M'$ where $M' \lhd M_\eta$, then:*

(a) $M' = M_\eta$.

(b) There is no truncation point on the main branch $\{i | iT_\eta\}$.

(c) If $\sigma(\xi_i) = \pi_{0,\eta}(\xi_i)$ for $i < n < \omega$, then $\sigma(\xi_n) \geq \pi_{0,\eta}(\xi_n)$.

As before, this follows from:

Lemma 3.5.12. *Let $M, \langle \xi_i | i < \omega \rangle$ be as above. There is an ω_1+1–successful n–full iteration strategy S to M such that whenever I is an S–conforming n to n–full iteration from M to M' and $\sigma : M \to_{\Sigma^{(n)}} M'$, then:*

(a) No $i < \mathrm{lh}(I)$ is a truncation point. (Hence the map $\pi = \pi^{(M,I)}$ is a total function on M.)

(b) If $\sigma(\xi_i) = \pi(\xi_i)$ for $i < n$, then $\sigma(\xi_n) \geq \pi(\xi_n)$.

The proofs are virtually unchanged.

3.6 Verifying full iterability

3.6.1 Introduction

As we said, full iterability is a difficult property to verify. A theorem that every normally iterable mouse is fully iterable would be useful, if true, but seems unlikely. We can, however, prove the following pair of theorems:

Theorem 3.6.1. *If M is smoothly α–iterable, then it is fully α–iterable.*

Theorem 3.6.2. *Let $\kappa > \omega$ be regular and let M be uniquely normally $\kappa+1$ iterable. Then M is smoothly $\kappa+1$–iterable.*

The proofs of these theorems are quite complex. To prove theorem 3.6.1, we redo much of chapter 2, developing a theory of embeddings which are Σ^*–preserving *modulo pseudo projecta*, which may not be the real projecta, but behave simiarly. The proof of theorem 3.6.2 requires us, in addition, to delve rather deeply into the combinatorics of normal iteration, using technique which, essentially, were developed by John Steel and Farmer Schlutzenberg.

This section (§3.6) is devoted to the proof of theorem 3.6.1. The following section brings the proof of theorem 3.6.2. In later chapters we shall make frequent use of both these theorems, but will seldom, if ever, refer to their proofs. Hence it would be justifiable for a first time reader of this this book to skip §3.6 and §3.7, taking the above theorems for granted and deferring their proofs until later.

3.6.2 Pseudo projecta

In order to prove theorem 3.6.1, we must redo §2.6, allowing "pseudo projecta" to play the role of the real projecta.

Definition 3.6.1. Let $M = \langle J_\alpha^A, B\rangle$ be acceptable. Then $\rho = \langle \rho_i | i < \omega\rangle$ is a *good sequence of pseudo projecta for M* iff the following hold:

(a) ρ_i is p.r. closed if $i > 0$.

(b) $\omega \leq \rho_{i+1} \leq \rho_i \leq \rho_M^i$ for $i < \omega$.

(c) $J_{\rho_i}^A$ is cardinally absolute in M (i.e. if $\gamma \in J_{\rho_i}^A$ is a cardinal in $J_{\rho_i}^A$, then it is a cardinal in M).

Note. $\rho_0 < \rho_M^0 = \mathrm{On}_M$ is not excluded. Moreover, ρ_i itself need not be a cardinal in M.

We shall generally write "ρ is good for M" instead of "ρ is a good sequence of pseudo projecta for M^i".

Definition 3.6.2. Let ρ be good for $M = J_\alpha^A$. $H_i = H_i(M, \rho) =: |J_{\rho_i}^A|$ for $i < \omega$.

We adopt the same language with typed variables $v^i (i < \omega)$ as before. The formula classes $\Sigma_h^{(n)} (h, n < \omega)$ are defined exactly as before. The satisfaction relation:

$$M \models \varphi[x_1, \ldots, x_n] \mod \rho$$

is defined as before except that the variables v^i now range over $H_i = H_i(M, \rho)$ instead of $H^i = H_M^i$. A relation $R(x_1^{i_1}, \ldots, x_n^{i_n})$ is $\Sigma_j^{(n)}(M, \rho)$ (or $\Sigma_j^{(n)}(M)$ mod ρ) iff it is M–definable mod ρ by a $\Sigma_j^{(n)}$ formula.
Similarly for $\underline{\Sigma}_j^{(n)}, \Sigma^*, \underline{\Sigma}^*$. We then define:

Definition 3.6.3. $\sigma : M \to_{\Sigma_j^{(n)}} M' \mod (\rho, \rho')$ iff the following hold:

(a) ρ is good for M and ρ' is good for M'.

(b) $\sigma'' H_i \subset H_i'$ for $i < \omega$, where $H_i = H_i(M, \rho), H_i' = H_i(M', \rho')$.

(c) Let φ be $\Sigma_i^{(n)}, \varphi = \varphi(v_1^{i_1}, \ldots, v_p^{i_p})$ where $i_1, \ldots, i_p \leq n$. Then:

$$M \models \varphi[\vec{x}] \mod \rho \leftrightarrow M' \models \varphi[\sigma(\vec{x})] \mod \rho'$$

for all $x_1, \ldots, x_p \in M$ such that $x_i \in H_{i_l} (l = 1, \ldots, p)$.

We also define:

Definition 3.6.4. $\sigma : M \to_{\Sigma^*} M' \mod (\rho, \rho')$ iff

$$\sigma \text{ is } \Sigma_0^{(n)}\text{–preserving} \mod (\rho, \rho') \text{ for } n < \omega.$$

As before, this is equivalent to:

$$\sigma \text{ is } \Sigma_1^{(n)}\text{–preserving} \mod (\rho, \rho') \text{ for } n < \omega.$$

We also write:

$$\sigma : M \to_{\Sigma_j^{(n)}} M' \mod \rho'$$

to mean

$$\begin{cases} \sigma : M \to_{\Sigma_j^{(n)}} M' \mod (\rho, \rho'), \\ \text{where } \rho_i = \rho_M^i \text{ for } i < \omega. \end{cases}$$

(Similarly for $\sigma : M \to_{\Sigma^*} M' \mod \rho'$.)

Lemma 3.6.3. *Let $\sigma : M \to_{\Sigma_j^{(n)}} M'$. Let ρ be good for M and define ρ' by:*

$$\rho'_i = \begin{cases} \sigma(\rho_i) & \text{if } \rho_i < \rho_M^i \\ \rho_M^i & \text{if not.} \end{cases}$$

Then $\sigma : M \to_{\Sigma_j^{(n)}} M' \mod (\rho, \rho')$.

(Hence, if σ is fully Σ^*–preserving, it is also Σ^*–preserving modulo (ρ, ρ').)

Proof: Clearly ρ' is good for M'. Now let $R(x_1^{i_1}, \ldots, x_p^{i_p})$ be $\Sigma_j^{(n)}(M, \rho)$, where $i_1, \ldots, i_p \leq n$. By an induction on n, R is uniformly $\Sigma_j^{(n)}(M)$ in the parameter $u = \langle \rho_i : l \leq n \wedge \rho_l < \rho_M^l \rangle$. (We leave the detail to the reader.)

But then, if R' is $\Sigma_i^{(n)}(M', \rho')$ by the same definition, it is $\Sigma_j^{(n)}(M')$ in $\sigma(u)$ by the same definition. QED (Lemma 3.6.3)

Lemma 3.6.4. *Let $\sigma : M \to_{\Sigma^*} M'$ and let ρ, ρ' be as in lemma 3.6.3. Let $\kappa = \mathrm{crit}(\sigma)$, where $\rho_{i+1} \leq \kappa < \rho_i$. Define ρ'' by:*

$$\rho''_j =: \rho'_j \textit{ for } j \neq i, \rho''_i =: \sup \sigma''\rho_i.$$

Then:

$$\sigma : M \to_{\Sigma^*} M' \mod (\rho, \rho'').$$

Proof: ρ'' is still good for M'. By induction on n it then follows that σ is $\Sigma_1^{(n)}$–preserving modulo (ρ, ρ''). QED (Lemma 3.6.4)

One might expect that most of §2.6 will not go through with pseudo projecta in place of projecta, since $\langle H_i, B\rangle$ is *not* necessarily amenable when B is $\Sigma_0^{(i)}(M, \rho)$. As it turns out, however, a great many proofs in §2.6 do not use this property (in contrast to the treatment in §2.5). In particular, lemmas 2.6.3 – 2.6.16 go through without change. Similarly, the definition of a good function can be relativized to a good ρ in place of $\langle \rho_M^n | n < \omega\rangle$. We define

$$\mathbb{G}_n = \mathbb{G}_n(M, \rho); \mathbb{G}^* = \mathbb{G}^*(M, \rho)$$

exactly as before with ρ in place of $\langle \rho_M^i | i < \omega\rangle$. Lemma 2.6.22 — 2.6.25 then go through exactly as before. Leaving the definition of *good* $\Sigma_1^{(n)}$ *definition* unchanged, we get the following version of Lemma 2.6.27: Let F be a good $\Sigma_1^{(n)}$ function mod ρ. There is a good $\Sigma_1^{(n)}$ definition which defines F mod ρ.

Even some of §2.7 remains valid for pseudo projecta. In §2.7.1 we define $\Gamma^0(\tau, M)$ (τ being a cardinal in M) as the set of maps $f \in M$ such that $\mathrm{dom}(f) \in H = H_\tau^M$. In §2.7.2 we then introduce $\Gamma^n = \Gamma^n(\tau, M)$ for the case that $n > 0$ and $\tau \leq \rho_M^n$, defining Γ^n to be the set of f such that:

(a) $\mathrm{dom}(f) \in H = H_\tau^M$.

(b) For some $i < n$ there is a good $\Sigma_1^{(i)}(M)$ function G and a parameter $p \in M$ such that:

$$f(x) = G(x, p) \text{ for all } x \in \mathrm{dom}(f).$$

Lemma 2.7.10 then told us that, whenever $\pi : M \to_{\Sigma_0^{(n)}} M'$, there is a canonical way of assigning to each $f \in \Gamma^n$ a definable partial map $\pi'(f)$ on M'. This continues to hold if $\pi : M \to_{\Sigma_0^{(n)}} M'$ mod ρ. The extended version of 2.7.10 reads:

Lemma 3.6.5. *Let $\pi : M \to_{\Sigma_0^{(n)}} M'$ mod ρ. There is a unique map π' which assigns to each $f \in \Gamma^n(\tau, M)$ a function $\pi'(f)$ with the following property:*

(*) *$\pi'(f) : \pi(\mathrm{dom}(f)) \to M'$. Moreover, if $f(x) = G(x, p)$ for all $x \in \mathrm{dom}(f)$, where G is a good $\Sigma_1^{(i)}(M)$ function for an $i < n$ and $p \in M$, then*

$$\pi'(f)(x) = G'(x, \pi(p)) \text{ for } x \in \pi(\mathrm{dom}(f)),$$

where G' is a good $\Sigma_1^{(i)}(M', \rho)$ function by the same good definition.

The proof is exactly as before. As before we get:

Lemma 3.6.6. *Let u, τ, π, π' be as above. Then $\pi'(f) = \pi(f)$ for $f \in \Gamma^0(\tau, M)$.*

Thus, again, we could unambiguously write $\pi(f)$ instead of $\pi'(f)$ for f. *However*, this is only unambiguous if we have previously specified the good sequence ρ. π' depends not only on π but also on the good sequence ρ. For this reason we shall write: $\pi_\rho(f)$ for $\pi'(f)$. We can omit the subscript ρ if the good sequence is clear from the context.

In §3.2 we then considered the special case that $\tau = \kappa^{+M}$ where κ is a cardinal in$M$. (This is mainly of interest when there is an extender F on M at κ.) We then set:

$$\Gamma^n_*(\kappa, M) =: \{f \in \Gamma^n(\kappa, M) | \operatorname{dom}(f) = \kappa\}.$$

We also set:

$$\Gamma^*(\kappa, M) =: \Gamma^n_*(\kappa, M) \text{ where } n \leq \omega \text{ is maximal such that } \kappa < \rho^n_M.$$

Let us call p a *defining parameter* for $f \in \Gamma^*(\kappa, M)$ iff either $p = f$ or else:

$$f(\xi) = G(\xi, p) \text{ for all } \xi < \kappa$$

where G is a good $\Sigma_1^{(i)}(M)$ function for an $i < n$. By lemma 2.6.25 we can then conclude:

Fact 1 Let $R(\vec{x}, y_1, \ldots, y_r)$ be a $\Sigma_0^{(n)}(M)$ relation. Let $f_i \in \Gamma^n_*(\kappa, M)$ have a defining parameter p_i for $i = 1, \ldots, r$. Then the relation:

$$Q(\vec{x}, \vec{\xi}) \longleftrightarrow: R(\vec{x}, f_1, (\xi_1), \ldots, f_r(\xi)$$

is $\Sigma_0^{(n)}(M)$ in the parameters $\kappa, p_1, \ldots, p_r$.

Moreover, if:

$$\sigma : M \to_{\Sigma_0^{(n)}} M' \mod \rho.$$

and R' has the same $\Sigma_0^{(n)}(M, \rho)$ definition, then the relation:

$$Q'(\vec{x}, \vec{\xi}) \leftrightarrow: R'(\vec{x}, \sigma_\rho(f_1)(\xi_1), \ldots, \sigma_\rho(f_r)(\xi_r))$$

is $\Sigma_1^{(n)}(M', \rho)$ in $\kappa, \sigma(p_1), \ldots, \sigma(p_r)$ by the same definition as Q.

Now let $a_1, \ldots, a_m \in M$ and set:

$$X = \{\langle \vec{\xi} \rangle | R(\vec{a}, \vec{f}(\xi))\}.$$

Then $X \in H^n_M$ and $\langle H^n_M, Q \rangle$ is amenable.

Fact 2 Let $R, R', Q, Q', f_1, \ldots, f_r, \sigma, M, M'$ be as in Fact 1. Let $\vec{a}, X$ be as above. Then:

$$\sigma(X) = \{\prec \vec{\xi} \succ \in \sigma(\kappa) | R'(\sigma(\vec{a}), \sigma_\rho(\vec{f})(\vec{\xi}))\}.$$

Proof (sketch)

We know:

$$\bigwedge \vec{\xi} < \kappa(\prec \vec{\xi} \succ \in X \leftrightarrow Q(\vec{a}, \vec{\xi}))$$

which is $\Pi_0^{(n)}(M)$ in the parameters $H_\kappa^M, \vec{a}, \vec{p}$. (We use here the fact that κ and the Gödel ν–tuple function on κ are H_κ^M–definable.) But then the corresponding $\Pi_0^{(n)}(M', \rho)$ statement holds of $H_n(M', \rho), \sigma(\vec{a})$, $\sigma(\vec{\alpha}), \sigma(\vec{p})$. QED (Fact 2)

Note. σ is Σ_1 preserving $\mod \rho$, if $n > 0$. But then $\kappa' = \sigma(\kappa)$ is a cardinal in M', since it is a cardinal in $H_0 = H_0(M', \rho)$ and ρ_0 is cardinally absolute in M'.

We now recall the Q–quantifier:

$$Qz^i \varphi(z^i) =: \bigwedge u^i \bigvee v^i (v^i \supset u^i \wedge \varphi(v^i)).$$

By a $Q^{(i)}$ formula we mean any formula of the form $Qz' \varphi(z^i)$, where $Q(\nu^i)$ is $\Sigma_1^{(i)}$. We write:

$$\sigma : M \to_{Q^*} N \mod (\rho, \rho')$$

to mean that σ is elementary $\mod (\rho, \rho')$ with suspect to $Q^{(n)}$ formulae for all $n < \omega$. Clearly, if σ is Q^* preserving $\mod (\rho, \rho')$, then it is Σ^*–preserving $\mod (\rho, \rho')$. If $\rho = \langle \rho_M^i | i < \omega \rangle$, we write:

$$\sigma : M \to_{Q^*} N \mod \rho.$$

In the following assume:

(1) $\sigma : M \to_{\Sigma^*} N \mod \rho'$.

We define a *minimal* good sequence:

$$\rho = \min \rho' = \min(\sigma, N, \rho')$$

with the following properties:

(a) $\sigma : M \to_{Q^*} N \mod \rho$.

(b) $\sup \sigma'' \rho_M^i \leq \rho_i \leq \rho_i'$ for $i < \omega$.

(c) Let φ be $\Sigma_0^{(i)}$. Let $x \in M, z_1, \ldots, z_p \in H_i(N, \rho)$. Then:

$$N \models \varphi[\vec{z}, \sigma(x)] \mod \rho \leftrightarrow N \models \varphi[\vec{z}, \sigma(x)] \mod \rho'.$$

(d) $\rho = \min \rho$.

We define ρ as follows:

Definition 3.6.5. Let $\sigma : M \to_{\Sigma^*} N \mod \rho'$. We define:

- $\rho_i(0) =: \sup \sigma'' \rho^i_M$.
- $\rho_i(n+1) =:$ the supremum of all $F(\eta)$ such that $\eta < \rho_{i+1}(n)$ and F is a $\Sigma_1^{(i)}(N, \rho')$ map to ρ'_i in parameters from $\mathrm{rng}(\sigma)$.
- $\rho_i =: \sup\limits_{n<\omega} \rho_i(n)$.
- $\rho = \langle \rho_i | i < \omega \rangle$.

Lemma 3.6.7. $\rho_i(n) \leq \rho_i(n+1)$.

Proof: We show by induction on n that it holds for all $i \leq \omega$.

Case 1 $n = 0$.
If $\xi < \rho^i_M$, then $\sigma(\xi) = F(0)$, where $F =$ the constant function $\sigma(\xi)$. But then F is $\Sigma_1^{(i)}(N, \rho')$ in $\sigma(\xi)$. Hence $\sigma(\xi) < \rho_i(1)$.

Case 2 $n > 0$.
Then $\rho_{i+1}(n) \geq \rho_{i+1}(n-1)$. Hence:

$$F'' \rho^{(n)}_{i+1} \supset F'' \rho^{(n-1)}_{i+1}$$

for all F which is a $\Sigma_1^{(i)}(N, \rho')$ map to ρ'_i.

The conclusion is immediate. QED (Lemma 3.6.7)

Lemma 3.6.8. *$\rho_i(n)$ is p.r. closed for $i > 0$.*

Proof: We show by induction on n that it holds for all $i > 0$.

Case 1 $n = 0$.
$\sigma \upharpoonright J^A_{\rho^i_M} : J^A_{\rho^i_M} \to_{\Sigma_0} J^A_{\rho_i}$ cofinally, where ρ^i_M is p.r. closed.

Case 2 $n > 0$. Let $n = m + 1$.
Then $\rho_i(m)$ is p.r. closed. Let f be a monotone p.r. function on On. It suffices to show:

Claim $f``\rho_i(n) \subset \rho_i(n)$.

Let $\nu < \rho_i(n)$. Then $\nu < F(\eta)$ where $\eta < \rho_{i+1}^{(m)}$ and F is $\Sigma_1^{(i)}(N, \rho')$ to ρ'_i in $\sigma(x)$. But then $f \circ F$ is $\Sigma_1^{(i)}(N, \rho')$ to ρ'_i, since ρ'_i is p.r. closed. Hence $f(\nu) < f \cdot F(\eta) < \rho_i(n)$. QED (Lemma 3.6.8)

Corollary 3.6.9. *ρ_i is p.r. closed for $i > 0$.*

Definition 3.6.6.

$$H_i(n) = H_i(N, \sigma, \rho_i(n)) =: |J^{A^N}_{\rho_i(n)}|$$
$$H_i = H_i(N, \rho) =: |J^{A^N}_{\rho_i}|$$

Lemma 3.6.10. *(a) $H_i(0) = \bigcup \sigma'' H^i_M$.*

(b) $H_i(n+1) =$ the union of all $F(x)$ such that $x \in H^{(n)}_{i+1}$ and F is $\Sigma_1^{(i)}(n, \rho')$ to ρ'_i in parameters from $\mathrm{rng}(\sigma)$.

(c) $H_i = \bigcup\limits_n H_i(n)$.

Proof: (c) is immediate. (a) is immediate since:

$$\sigma \restriction H^i_M : H^i_M \to_{\Sigma_0} H_i(0) \text{ cofinally.}$$

We prove (b). Let $y = F(x)$, where F, x are as in (b).

Claim $y \in H_i(n+1)$.

Proof: We recall the function $\langle S^A_\nu | \nu < \infty \rangle$ such that for all limit α:

$$J^A_\alpha = \bigcup\nolimits_{\nu<\alpha} S^A_\nu \text{ and } \langle S^A_\nu | \nu < \alpha \rangle \text{ is}$$
$$\text{uniformly } \sigma_1(J^A_\alpha).$$

Since $\rho_{i+1}(n)$ is p.r. closed, there is a $\Sigma_1(H_{i+1}(n))$ map f of $\rho_{i+1}(n)$ onto $H_{i+1}(n)$. Set:

$$g(x) =: \text{ the least } \nu \text{ sucht that } x \in S_\nu.$$

Then $\tilde{F}(\xi) \simeq gFf(\xi)$ is a $\Sigma_1^{(i)}(N, \rho')$ map to ρ'_i in parameters from $\mathrm{rng}(\sigma)$. Hence, where $f(\eta) = x$, we have $y \in S^A_{\tilde{F}(\eta)} \subset H_i(n+1)$.

QED (Lemma 3.6.10)

By the definition 3.6.5 and Lemma 3.6.7:

Lemma 3.6.11. *Let $\rho = \min \rho'$. Then:*

- $\sigma" \rho^i_M \subset \rho_i \leq \rho'_0 \leq \rho^0_N$.
- $\rho_i = \sup X$, *where* X *is the set of all* $F(\nu)$ *such that* $\nu < \rho_{i+1}$ *and* F *is a* $\Sigma_1^{(i)}(N, \rho')$ *map to* ρ'_0 *in some* $\sigma(x)$.

Similarly by Lemma 3.6.10.

Lemma 3.6.12. *Let* $\rho = \min \rho'$. *Then:*

- $\sigma'' H^i_M \subset H_i \subset H'_i \subset H^i_N$.
- $H_i = \bigcup X$ *where* S *is the set of all* $F(x)$ *such that* $z = H_{i+1}$ *and* F *is a* $\Sigma_1^{(i)}(N, \rho')$ *map to* H'_i *in some* $\sigma(x)$.

We now can show:

Lemma 3.6.13. ρ *is good for* N.

Proof: By Lemma 3.6.11 we have:

$$\omega \leq \rho_{i+1} \leq \rho_i \leq \rho'_i \leq \rho^i_N.$$

Moreover, ρ_i is p.r. closed for $i > 0$ by Lemma 3.6.8.

It remains only to show:

Claim H_i is cardinally absolute with respect to N.

Proof: We know: $H_i = \bigcup X$, where $X =$ the set of $F(z)$ such that $z \in H_{i+1}$ and F is a $\Sigma_1^{(i)}(N, \rho')$ map to $H'_i = H_i(N, \rho')$. Moreover H'_i is cardinally absolute in N.

(1) Let $\alpha \in X$. Then $\overline{\overline{\alpha}}^N \in X$ and there is $f \in X$ such that $f : \overline{\overline{\alpha}}^N \xrightarrow{\text{onto}} \alpha$.

Proof: Suppose not.

Define a $\Sigma_1(H_i)$ map by:

$$F(\beta) \simeq \text{ the } <_{SA}\text{-least pair } \langle \gamma, f \rangle \text{ such that } \gamma < \beta \text{ and } f : \gamma \xrightarrow{\text{onto}} \beta.$$

Then $F''X \subset X$. Set:

$$\alpha_0 = \alpha_i \alpha_{i+1} \simeq (F(\alpha_i))_0.$$

By induction on i it follows that α_i exists and $\alpha_i \in X$. But then $\alpha_{i+1} < \alpha_i$ for $i < \omega$. Contradiction! QED (1)

Now let α be a cardinal in H_i but not in N. Then $\alpha \notin X$ by (1). But $\alpha < \beta$ for a $\beta \in X$. Hence $\overline{\overline{\beta}}^N > \alpha$. (Otherwise, letting $\gamma = \overline{\overline{\beta}}^N < \alpha$, we have $\gamma \in X \subset H_i$ and there is $f \in X \subset H_i$ such that $f : \gamma \xrightarrow{\text{onto}} \beta$. Hence there is $g \in H_i$ such that $g : \gamma \xrightarrow{\text{onto}} \alpha$, since $0 < \alpha < \beta$. Hence α is not a cardinal in H_i.) But then, letting $\gamma = \overline{\overline{\beta}}^N$, α is a cardinal in J^A_γ and γ is a cardinal in N. Hence α is a cardinal in N by acceptability. QED (Lemma 3.6.13)

We now verify property (c) for $\rho = \min \rho'$.

Lemma 3.6.14. *Let $\overline{B}(\vec{w}^i)$ be $\Sigma^{(i)}_0(M)$ in the parameter $x \in M$. Let $B'(\vec{w}^i)$ be $\Sigma^{(i)}_0(N, \rho')$ in $\sigma(x)$ and $B(\vec{w}^i)$ be $\Sigma^{(i)}_0(N, \rho)$ in $\sigma(x)$ by the same definition. Then:*

$$\bigwedge \vec{z} \in H_i (B(\vec{z}) \leftrightarrow B'(\vec{z})).$$

Proof: By induction on i. The case $i = 0$ is trivial. Now let it hold for h where $i = h + 1$. It suffices to prove the claim for $\overline{B}$ which is $\Sigma^{(h)}_1(M)$ in x. We than have:

$$\overline{B}(\vec{z}) \leftrightarrow \bigvee a^h D(a^h, \vec{z})$$

where $\overline{D}$ is $\Sigma^{(h)}_0(M)$ in x;

$$B'(\vec{z}) \leftrightarrow \bigvee a^h D'(a^h, \vec{z})$$

where D' is $\Sigma^{(h)}_0(N, \rho')$ in $\sigma(x)$ by the same definition, and:

$$B(\vec{z}) \leftrightarrow \bigvee a^h D(a^h, \vec{z})$$

where D is $\Sigma^{(h)}_0(N, \rho)$ in $\sigma(x)$ by the same definition.

Define a map F to ρ'_h which is $\Sigma^{(h)}_1(N, \rho')$ in $\sigma(x)$ by:

$$\begin{aligned} \xi = F(\vec{z}) \quad & \leftrightarrow (\vee u \in S_\xi D'(u\vec{z}) \cap \\ & \qquad \wedge \xi' < \xi \wedge u \in S_\xi, \neg D'(u, \vec{z}) \end{aligned}$$

Hence for $\vec{z} \in H_i$:

$$\begin{aligned} B'(\vec{z}) \quad & \leftrightarrow \vee u \in H_h D'(u, \vec{z}) \\ & \leftrightarrow \vee u \in S_{F(\vec{z})} D'(u, \vec{z}) \\ & \leftrightarrow \vee u \in H_h D'(u, \vec{z}) \\ & \leftrightarrow \vee u \in H_h D(u, \vec{z}) \leftrightarrow B(\vec{z}) \end{aligned}$$

(by the induction hypothesis). QED (Lemma 3.6.14)

Since $\sigma : M \to_{\Sigma(i)} N \mod \rho'$, we conclude that $\sigma : M \to_{\Sigma(i)} N \mod \rho$.

Since this holds for all $i < \omega$, we conclude:

Corollary 3.6.15. $\sigma : M \to_{\Sigma^*} N \mod \rho$.

Another immediate corollary is:

Corollary 3.6.16. $\rho = \min(N, \sigma, \rho)$.

It remains only to prove:

Lemma 3.6.17. $\sigma : M \to_{Q^*} N \mod \rho$.

Proof:

Assume: $M \models Qu^i\varphi(u^i, x)$ where φ is $\Sigma_1^{(i)}$.

Claim $N \models Qu^i\varphi(u^i, x) \mod \rho$.

Let $v \in H_i$. Then $v \subset w = G(\overline{w})$, where $\overline{w} \in H_{i+1}$. Then $v \subset w = G(\overline{w})$, where $\overline{w} \in H_{i+1}$ and G is $\Sigma_1^{(i)}(N, \rho)$ map to H_i in parameter from $\operatorname{rng} \sigma$. Let:

$$\varphi = \bigvee z^i \psi(z^i, u^i, x) \text{ where } \psi \text{ is } \Sigma_0^{(i)}.$$

Define a $\Sigma_1^{(i)}(N, \rho)$ map to H_i in $\sigma(x)$ by:

$$F(w) \simeq \text{ the } N\text{–least } \langle z, u\rangle \in H^i \text{ such that}$$
$$z \subset u \wedge \psi(z, u, \sigma(x)).$$

The $\Pi_1^{(i+1)}$–statement:

$$\bigwedge a^{i+1}(a^{i+1} \in \operatorname{dom}(G) \to a^{i+1}) \in \operatorname{dom}(F \circ G))$$

holds in N, since the corresponding statement holds in M by our assumption. Let $\langle z, u\rangle = FG(\overline{w}) = F(w)$. Then $v \subset w \subset u$ and $\psi(z, u, \sigma(x))$. Hence:

$$N \models Qu\varphi(u, \sigma(x)) \mod \rho.$$

QED (Lemma 3.6.17)

Then $\rho = \min \rho'$ possess all the properties that we ascribed to it.

As a corollary of Lemma 3.6.17 we get:

Corollary 3.6.18. *Let B be $\Sigma_1^{(i)}(N, \rho)$ in parameters from $\operatorname{rng} \sigma$. Then $\langle H_i, B \rangle$ is amenable.*

Proof: Let $\overline{B}$ be $\Sigma_1^{(i)}(M)$ in x and B be $\Sigma_1^{(i)}(N, \rho)$ in the same definition. Since $\langle H_M^i, \overline{B} \rangle$ is amenable, we have:

$$Qu^i \bigvee y^i \; y^i = u^i \cap \overline{B} \text{ in } M.$$

But then:

$$Qu^i \bigvee y^i \; y^i = u^i \cap B \text{ in } N \mod \rho.$$

Let $u \in H_i$. There is then $v \supset u, v \in H_i$ such that $v \cap B \in H_i$. Hence $u \cap B = u \cap v \in H_i$. QED (Corollary 3.6.18)

Definition 3.6.7. $\sigma : M \to_{\Sigma^*} N \min \rho$ iff

$$[\sigma : M \to_{\Sigma^*} N \mod \rho] \wedge [\rho = \min(N, \sigma, \rho)].$$

(Similarly for $\Sigma_j^{(n)}, Q_j^{(n)}, Q^*$ etc.)

In the following we shall always assume that M is acceptable, $\kappa \in M$ is inaccessable in M, and that $\tau = \kappa^{+M} \in M$.

Lemma 3.6.19. *Let $\pi : M \to_{\Sigma^*} M'$. Let $\kappa = \operatorname{crit}(\pi)$, $\lambda \leq \pi(\kappa)$, and suppose an extender F at κ, λ on M to be defined by:*

$$F(X) = \lambda \cap \pi(X) \text{ for } X \in \mathbb{P}(\kappa) \cap M.$$

Let $\sigma : \overline{M} \to_{\Sigma^*} M \min \rho$, where $\sigma(\overline{\kappa}) = \kappa$. Let F be a weakly amenable extender at $\overline{\kappa}, \overline{\lambda}$ on $\overline{M}$. Assume:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to \langle M, F \rangle, \text{ where } g : \overline{\lambda} \to \lambda.$$

Let $n \leq w$ be maximal such that $\overline{\kappa} < \rho_{\overline{M}}^n$.

Define a good sequence ρ^* for M' by:

$$\rho_i^* = \begin{cases} \sup \pi''\rho_n \text{ if } i = n \\ \pi(\rho_i) \text{ if } i \neq n \text{ and } \rho_i < \rho_M^i \\ \rho_{M'}^i \text{ if } i \neq n \text{ and } \rho_i = \rho_M^i. \end{cases}$$

(Hence $\pi : M \to_{\Sigma^*} M' \mod (\rho, \rho^*)$ by Lemma 3.6.3 and 3.6.4.) Then:

(a) $\overline{M}$ is n–extendible by $\overline{F}$.

(b) Let $\overline{\pi} : \overline{M} \to_{\overline{F}}^{(n)} \overline{M}'$. There is a map σ' such that

$$\sigma' : \overline{M}' \to_{\Sigma_0^{(n)}} M' \mod \rho^* \text{ and } \sigma'\overline{\pi} = \pi\sigma, \sigma' \restriction \overline{\lambda} = g.$$

Moreover, σ' is defined by:

$$\sigma'(\overline{\pi}(f)(\alpha)) = ((\pi\sigma)_{\rho^*}(f))(g(\alpha))$$

for $f \in \Gamma^*(\overline{\kappa}, \overline{M})$, $\alpha < \lambda$.

Proof: We obviously have:

$$\pi\sigma : \overline{M} \to_{\Sigma^*} M' \mod \rho^*.$$

It is also clear that n is maximal such that $\kappa < \rho_n$ and also maximal such that $\kappa' = \pi(\kappa) < \rho_n^*$.

We now prove (a). We must show that the $\in$–relation $\in^*$ of $\mathbb{D}^*(\overline{F}, \overline{M})$ is well founded. Let $\langle f, \alpha\rangle, \langle f', \alpha'\rangle \in \mathbb{D}^*$. Set:

$$e = \{\prec \xi, \zeta \succ < \overline{\kappa} | f(\xi) \in f'(\zeta)\}.$$

Then:

$$\begin{aligned}
\langle f, \alpha\rangle \in^* \langle f', \alpha'\rangle &\longleftrightarrow \langle a, \alpha'\rangle \in \overline{F} \\
&\longleftrightarrow \prec g(\alpha), g(\alpha') \succ \in F(\sigma(e)) \\
&\longleftrightarrow \prec g(\alpha), g(\alpha') \succ \in \pi\sigma(e) \\
&\longleftrightarrow (\pi\sigma)_{\rho^*}(f)(g(\alpha)) \in (\pi\sigma)_{f^*}(f')(g(\alpha))
\end{aligned}$$

(The second line rises the assumption: $\langle \sigma, g\rangle : \langle \overline{M}, \overline{F}\rangle \to \langle M, F\rangle$. The third uses: $F(X) = \lambda \cap \pi(X)$. The fourth uses Fact 2, which we established earlier in the section. QED (a)

We now prove (b). Let $\overline{R}'$ be a $\Sigma_0^{(n)}(\overline{M}')$ relation and let R' be $\Sigma_0^{(n)}(M')$ by the same definition. We claim that: $\sigma' : \overline{M}' \to_{\Sigma_0^{(n)}} M'$ where σ' is defined by:

$$\sigma'(\overline{\pi}(f)(\alpha)) = (\pi\sigma)_{\rho^*}(f)(g(\alpha))$$

for $f \in \Gamma^*(\overline{u}, \overline{M}), \alpha < \lambda$.

Let $\overline{R}'$ be a $\Sigma_0^{(n)}(\overline{M}')$ relation and let R' be $\Sigma_0^{(n)}(M', \rho^*)$ by the same definition. Let $\alpha_1, \ldots, \alpha_m < \overline{\lambda}$ and $f_1, \ldots, f_m \in \Gamma^*(\overline{u}, \overline{M})$. Writing e.g. $\vec{f}(\vec{\alpha})$ for $f_1(\alpha_1), \ldots, (\alpha_m)$, it suffices to show:

Claim $\overline{R}'(\overline{\pi}(\vec{f})(\vec{\alpha})) \leftrightarrow R'(\pi\sigma(\vec{f}), g(\vec{\alpha}))$.

Proof: Let $\overline{R}$ be $\Sigma_0^{(n)}(\overline{M})$ and R be $\Sigma_0^{(n)}(M, \rho)$ by the same definition. Set:

$$e = \{\prec \vec{\xi} \succ | \overline{R}(\vec{f}(\vec{\overline{\xi}})\}.$$

Then:

$$\begin{aligned}
\overline{R}'(\overline{\pi}(\vec{f})(\vec{\alpha})) &\longleftrightarrow \prec \vec{\alpha} \succ \in \overline{F}(e) \\
&\longleftrightarrow \prec g(\vec{\alpha}) \succ \in F(\sigma(e)) \\
&\longleftrightarrow \prec g(\vec{\alpha}) \succ \in \pi\sigma(e) \\
&\longleftrightarrow R'((\pi\sigma)_{\rho^*}(\vec{f})(g(\vec{\alpha})))
\end{aligned}$$

QED (Lemma 3.6.19)

We would like to prove something stronger namely that $\overline{M}$ is $*$–extendible by $\overline{F}$ and that:

$$\sigma' : \overline{M}' \to_{\Sigma^*} M' \quad \text{mod } \rho^*.$$

For this we must strengthen the condition:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to \langle M, F \rangle.$$

In §3.2 we helped ourselves in a similar situation by strengthening the relation $\to$ to $\to^*$. However $\to^*$ is too strong for our purposes and we adopt the following weakening:

Definition 3.6.8. $\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle$ mod ρ iff the following hold:

(a) $\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to \langle M, F \rangle$

(b) $\sigma : \overline{M} \to_{\Sigma_0} M$ mod ρ

(c) Let $\overline{\alpha} < \text{lh}(\overline{F}), \alpha = g(\overline{\alpha})$. There are $\overline{G}, G, \overline{H}, H$ such that letting

$$\overline{\kappa} = \text{crit}(\overline{F}), \kappa = \text{crit}(F)$$

we have:

(i) $\overline{G}, \overline{H}$ are $\Sigma_i(\overline{M})$ in a $\overline{q} \in \overline{M}$ and G, H are $\Sigma_1(M, \rho)$ in $q = \sigma(\overline{q})$ by the same definition.

(ii) $\overline{G} = \overline{F}_{\overline{\alpha}}, \overline{H} = \overline{M} \cap ({}^{\overline{\kappa}}\mathbb{P}(\overline{u}))$

(iii) $G \subset F_\alpha$

(iv) $H \subset \{X \in {}^{\kappa}\mathbb{P}(u) | \bigwedge \xi < \kappa (X_\xi \text{ or } \kappa \setminus X_\xi \in G)\}$

Note. Actually, only the first pseudo projectum ρ_0 is relevant in this definition. (b)says merely that ρ is good for M and that σ is a Σ_0-preserving map into M with $\sigma'' \mathrm{On}_{\overline{M}} \leq \rho_0$. In (c) the statement "$G, H$ are $\Sigma_1(M, \rho)$ in q by the same definition" can be rephrased as: "G, H are $\Sigma_1(M|\rho_0)$ in q by the same definition", where $M|\eta =: \langle J^A_\eta, B \cap J^A_\eta \rangle$ for $M = \langle J^A_\alpha, B \rangle$.

(Note that $M|\eta$ is not necessarily amenable.) We set:

Definition 3.6.9. $\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle$ iff:

$$\langle X, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod (\langle \rho^n_M | n < w \rangle).$$

Note. This always holds if $\rho_0 = \mathrm{On}_M$.

Note. Let $\sigma : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod \rho$. Let $\overline{X} \in \overline{M} \cap ({}^{\overline{\kappa}}\mathbb{P}(\overline{\kappa}))$. If $X = \sigma(\overline{X})$, then $X \in M$ and hence $\bigwedge \xi < \kappa (X_\xi \text{ or } (\kappa \setminus X_\xi) \in G)$.

Note. Let $\sigma : \langle \overline{M}, \overline{F} \rangle \to^{*} \langle M, F \rangle$. It follows easily that:

$$\sigma : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle.$$

Note. Suppose that $\sigma : \overline{M} \to_{\Sigma^*} M \min \rho$. Set $M|\rho_0 = \langle J^A_{\rho_0}, B \cap J^A_{\rho_0} \rangle$, where $M = \langle J^A_\gamma, B \rangle$. Then $M|\rho_0$ is amenable by Corollary 3.6.18. Clearly $\tau = \kappa^{+M} \in M|\rho_0$ since $\overline{\tau} = \kappa^{+\overline{M}} \in \overline{M}$. Hence $\mathbb{P}(\kappa) \cap M \subset M|\rho_0$. But then F is an extender at κ on $M|\rho_0$ and it makes sense to write:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M|\rho_0, F \rangle.$$

But this means exactly the same thing as:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod \rho.$$

We are now ready to prove:

Lemma 3.6.20. *Let $\pi, \sigma, \overline{M}, M, \overline{M}', M', \rho, \rho^*, \overline{\tau}, \tau, \overline{\pi}, \sigma', g$ be as in lemma 3.6.19. Assume:*

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod \rho.$$

Then $\overline{M}$ is $*$–extendible by $\overline{F}$ and:

$$\sigma' : \overline{M}' \to_{\Sigma^*} M' \mod \rho^*.$$

Proof: $\overline{F}$ is then close to $\overline{M}$. Hence $\overline{M}$ is $*$–extendible by $\overline{F}$. By induction on i we now show:

Claim $\sigma' : \overline{M}' \to_{\Sigma_1^{(i)}} M'$ mod ρ^*.

For $i < n$ this is given. Now let $i = n$. We prove a somewhat stronger claim:

Subclaim 1 Let $\overline{A} \subset \overline{\kappa}$ be $\Sigma_1^{(n)}(\overline{M}')$ in $\overline{a} \in \overline{M}'$ and $A \subset \kappa$ be $\Sigma_1^{(n)}(M', \rho^*)$ in $a = \sigma'(\overline{a})$ by the same definition. There is $\overline{r} \in \overline{M}$ such that $\overline{A}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{r}$ and A is $\Sigma_1^n(M, \rho)$ in $r = \sigma(\overline{r})$ by the same definition.

(As we shall see, this proves the claim for the case $i = n$.)

We now prove the subclaim. Let:

$$\overline{A}(i) \leftrightarrow \bigvee y \overline{P}'(y, i, \overline{a}),$$
$$A(i) \leftrightarrow \bigvee y P'(y, i, a)$$

where $\overline{P}'$ is $\Sigma_0(\overline{M}')$ and P' is $\Sigma_0(M', \rho^*)$ by the same definition.

Let $\overline{P}$ be $\Sigma_0^{(n)}(\overline{M})$ and P be $\Sigma_0^{(n)}(M)$ by the same definition. Let $\overline{a} = \overline{\pi}(f)(\overline{\alpha})$ and $a = \overline{\pi}\sigma(f)(\alpha)$, where $\alpha = g(\overline{\alpha})$. Let $\overline{p}$ be a "defining parameter" for f (i.e. either $\overline{p} = f$ or else $f(\xi) = B(\xi, \overline{p})$ where B is a good $\Sigma_1^{(i)}(\overline{M})$ function for an $i < n$.) Then $p = \sigma(\overline{p})$ is in the same sense a defining parameter for $\sigma(f)$ and $p' = \pi\sigma(\overline{p})$ is a defining parameter for $\pi\sigma(f)$. (The good definition of B remaining unchanged.)

Finally, let $\overline{G}, G, \overline{H}, H$ be as given for $\overline{\alpha}, \alpha = g(\overline{\alpha})$ by the principle:

$$\langle \sigma, q \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \quad \text{mod } \rho^*.$$

Since $\langle \overline{M}', \overline{\pi} \rangle$ is the extension of $\langle \overline{M}, \overline{F} \rangle$, we know that: $\overline{\pi}``H^n_{\overline{M}}$ is cofinal in H^n_M.

Thus:

(1)

$$\begin{aligned} \overline{A}(i) \; &\leftrightarrow \bigvee u \in H^n_{\overline{M}} \bigvee y \in \overline{\pi}(u) \overline{P}'(g, i, \overline{\pi}(f)(\overline{\alpha})) \\ &\leftrightarrow \bigvee u \in H^n_{\overline{M}} \overline{\alpha} \in \overline{\pi}(\overline{X}(i, u)) \\ &\leftrightarrow \bigvee u \in H^n_{\overline{M}} \overline{X}(i, u) \in \overline{G}, \end{aligned}$$

where $\overline{X}(i, u) = \{\xi < \overline{u} | \overline{P}(y, i, f(\xi))\}$.

Thus $\overline{A}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{p}, \overline{q}, \overline{\kappa}$. We now show that A is $\Sigma_1^{(n)}(M)$ in p, q, κ by the same definition. Set:

$$H_n = H_n(M, \rho), \; H'_n = H_n(M', \rho^*).$$

It is easily seen that the relation:

$$Q(u, i, \xi) \longleftrightarrow: (u \in H_n \wedge \bigvee y \in u P(y, i, \sigma_\rho(f)(\xi))$$

is $\Sigma_0^{(n)}(M,\rho)$ in p and the relation:

$$Q'(u,i,\xi) \longleftrightarrow: (u \in H'_n \wedge \bigvee y \in uP'(y,i,(\pi\sigma)_{\rho^*}(\xi))$$

is $\Sigma_0^{(n)}(M',\rho^*)$ in p' by the same definition. Set: $X(u,i) = \{\xi < u | Q(u,i,\xi)\}$. Then $X(u,i) \in H_n$, since $\langle H_n, Q\rangle$ is amenable by lemma 3.6.14 and hence is rud closed. Since $\rho_n^* = \sup \sigma"\rho_n$, we know that $\pi"H_n$ is cofinal in H'_n. Thus:

(2)

$$\begin{aligned} A(i) &\leftrightarrow \bigvee u \in H_n \bigvee y \in \pi(u) P'(y,i,((\pi\sigma)_{\rho^*}(f)(\alpha)) \\ &\leftrightarrow \bigvee u \in H_n Q(\pi(u),i,\alpha) \\ &\leftrightarrow \bigvee u \in H_n \alpha \in \pi(X(u,i)) \cap X \\ &\leftrightarrow \bigvee u \in H_n \alpha \in F(X(u,i)) \\ &\leftrightarrow \bigvee u \in H_n X(u,i) \in F_\alpha. \end{aligned}$$

If $F_\alpha = G$, we would be finished, but G might be a proper subset of F_α. (Moreover, we don't even know that F_α is M–definable in parameters.) However, we can prove:

(3) $A(i) \leftrightarrow \bigvee u \in H_n X(u,i) \in G$,

which establishes subclaim 1. The direction $(\leftarrow)$ is trivial by (2), since $G \subset F_\alpha$. We prove $(\rightarrow)$. Assume $A(i_0)$, where $i_0 < \kappa$. We must show that $u \in H_n$ can be chosen large enough that $X(u,i_0) \in G$. We know that it can be chosen large enough that $X(u,i_0) \in F_\alpha$. Since $\rho = \min(M,\sigma,\rho)$, we also know that the set of $S(\xi)$ such that S is a partial $\Sigma_1^{(n)}(M,\rho)$ map to H_n in a parameter $s = \sigma(\overline{s})$ and $\xi < \rho_{n+1}$ is cofinal in H_n. (This uses Lemma 3.6.12.) Hence we can assume w.l.o.g. that $u = S(\xi_0)$ for a $\xi_0 < \rho_{n+1}$. Now set:

$$Y(v) =: \{x(v,i) | i < u\} \text{ for } v \in H_n.$$

Then $Y(v) \in H_n$ by the rud closure of $\langle H_n, Q\rangle$. Moreover, the function Y is $\Sigma_1(\langle H_n, Q\rangle)$ and hence is a $\Sigma_1^{(n)}(M,\rho)$ function. Hence $Y \circ S$ in $\Sigma_1^{(n)}(M,\rho)$ in s. Let $\overline{S}$ be $\Sigma_1^{(n)}(M)$ is $\overline{s}$ and $\overline{Y}$ be $\Sigma_1^{(n)}(\overline{M})$ by the same definition. The $\Pi^{(n+1)}(M,\rho)$ statement:

$$\bigwedge \zeta < \rho_{n+1}(\zeta \in \mathrm{dom}(Y \cdot S) \rightarrow Y \cdot S(\zeta) \in H)$$

is true, since the corresponding statement:

$$\bigwedge \zeta < \rho_M^{n+1}(\zeta \in \mathrm{dom}(\overline{Y} \cdot \overline{S}) \rightarrow \overline{Y} \cdot \overline{S}(\zeta) \in \overline{H})$$

is true in $\overline{M}$. Since $u = S(\zeta_0)$, it follows that: $Y(u) \in H$ and:

$$X(\kappa, i_0) \in G \vee (\kappa \setminus X(u,i_0)) \in G.$$

But $G \subset F_\alpha(\kappa \setminus X(u, i_0)) \in G$ is therefore impossible, since we would then have:

$$X(\kappa, i_0) \cap (\kappa \setminus X(u, i_0)) = \emptyset \in F_\alpha.$$

Hence, $X(U, i_0) \in G$. QED (Subclaim 1)

Subclaim 2 $\sigma' : \overline{M}' \rightarrow_{\Sigma_1^{(n)}} (\overline{M}') \mod \rho^*$.

Proof. Let Q be $\Sigma_1^{(n)}(M', \rho^*)$ and $\overline{Q}$ be $\Sigma_1^{(n)}(\overline{M}')$ by the same definition. Set:

$$P(i, x) \leftrightarrow (i = 0 \wedge Q(x)),$$
$$\overline{P}(i, x) \leftrightarrow (i = 0 \wedge \overline{Q}(x)).$$

Set:

$$A(x) = \{i | P(i, x)\}, \overline{A}(x) = \{i | \overline{P}(i, x)\}.$$

Then A is the characteristic function of Q and $\overline{A}$ is the characteristic function of $\overline{Q}$. But $A(\sigma'(x)) = \overline{A}(x)$ for $x \in \overline{M}$ by Subclaim 1.

QED (Subclaim 2)

A slight reformulation of Subclaim 1 yields:

Subclaim 3 Let A be $\Sigma_1^{(n)}(M', \rho^*)$ i $p = \sigma'(\overline{p})$. Let $\overline{A}$ be $\Sigma_1^{(n)}(\overline{M}')$ in $\overline{p}$ by the same definition. Set: $\overline{H} = H_{\overline{\kappa}}^{\overline{M}}, H = H_\kappa^M$. Then $A \cap H$ is $\Sigma_1^{(n)}(M, \rho)$ in a $q = \sigma(\overline{q})$ and $\overline{A} \cap \overline{H}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{q}$ by the same definition.

Proof: $H = J_\kappa^E$, where $E = E^M$ and $\overline{H} = J_{\overline{\kappa}}^{\overline{E}}$ where $\overline{E} = E^{\overline{M}}$. But $\kappa, \overline{\kappa}$ are preclosed. Let $f : \kappa \overset{\text{onto}}{\longrightarrow} H$ be primitive recursive in E and let $\overline{f} : \overline{\kappa} \overset{\text{onto}}{\longrightarrow} \overline{H}$ be primitive recursive in $\overline{E}$ by the same definition. Apply subclaim 1 to

$$B = f^{-1\prime\prime} A, \overline{B} = \overline{f}^{-1\prime\prime} \overline{A}.$$

Then $B \subset \overline{\kappa}$ is $\Sigma_1^{(n)}(M, \rho)$ in a $q = \sigma(\overline{q})$ and $\overline{B} \subset \overline{\kappa}$ is $\Sigma_1^{(n)}(\overline{M})$ in $\overline{q}$. But then the same holds for $A = f'' B, \overline{A} = \overline{f}'' \overline{B}$.

QED (Subclaim 3)

For $i > n$, we know: $\rho_{\overline{M}}^i = \rho_M^i$, so we can write $\rho^i =: \rho_{\overline{M}}^i$. By the definition of ρ^*, we know: $\rho_i = \rho_i^*$ for $i > n$. We can also set:

$$\overline{H}^i = H_{\overline{M}}^i = H_{\overline{M}}^i, H_i = H_i(M, \rho) = H_i(M', \rho^*).$$

We now prove:

Subclaim 4 Let $i > n$. Let $\overline{A}$ be $\Sigma_1^{(i)}(\overline{M}')$ in $\overline{a} \in \overline{M}'$ and let A be $\Sigma_1^{(i)}(M', \rho^*)$ in $a = \sigma'(\overline{a})$ by the same definition. Then there are $\overline{B}, B$, $\overline{q}, q$ such that

(a) $\overline{B}$ is $\Sigma_0^{(i)}(\overline{M})$ in $\overline{q} \in M$.

(b) B is $\Sigma_0^{(i)}(M, \rho)$ in $q = \sigma(\overline{q})$ by the same definition.

(c) $\overline{A} \cap \overline{H}^i = \overline{B} \cap \overline{H}^i$.

(d) $A \cap H_i = B \cap H_i$.

Proof: By induction on i. Let it hold below i. Then w.l.o.g. we can assume:

(1) $\overline{A}(x) \longleftrightarrow \langle \overline{H}^i, \overline{P} \cap \overline{H}^i \rangle \models \varphi[x]$ for $x \in \overline{H}^i$ where φ is Σ_1 and $\overline{p}$ is $\Sigma_0^{i-1}(\overline{M}')$ in $\overline{a}$.

(2) $A(x) \longleftrightarrow \langle H', P \cap H_i \rangle \models \varphi[x]$ for $x \in H_i$ where φ is the same Σ_1 formula and P is $\Sigma_0^{i-1}(M', \rho^*)$ in a by the same definition.

But then there are $\overline{Q}, Q, \overline{q}, q$ such that

(3) $\overline{P} \cap H^i = \overline{Q} \cap H^i$, where $\overline{Q}$ is $\Sigma_1^{i-1}(\overline{M})$ in $\overline{q} \in \overline{M}$.

(4) $P \cap H_i = Q \cap H_i$, where $\overline{Q}$ is $\Sigma_1^{i-1}(M, \rho)$ in $q = \sigma(q)$ by the same definition.

This is by subclaim 3 if $i = n + 1$, and otherwise by the induction hypothesis. QED (Sublemma 4)

The claim then follows easily, since σ is Σ^*–preserving mod ρ^*.
QED (Lemma 3.6.20)

We can then go on further and set:

$$\rho' = \min(M', \sigma', \rho^*).$$

It then follows that:

$$\pi``\rho_i \subset \rho'_i \leq \rho^*_i \text{ for } i < \omega.$$

To see that $\pi''\rho_i \subset \rho'_i$, we recall that $\rho'_i = \sup\{\rho'_i(n) : n < \omega\}$ where the sequence $\langle \rho'_i(n) | i < w \rangle$ is defined from ρ^*, M', σ' by a canonical recursion on n (cf. Definition 3.6.5).

But since $\rho = \min(M, \sigma, \rho)$, we have: $\rho_i = \sup\limits_{n<w} \rho_i(n)$, where $\langle \rho_i(n) | i < w \rangle$ is defined from ρ, M, σ by the same induction on n. Since $\pi'\sigma = \pi\sigma$, it follows easily by induction on n that:

$$\pi``\rho_i(n) \subset \rho'_i(n) \text{ for } i < w.$$

The details are left to the reader.

Putting all of this together:

Theorem 3.6.21. *Let $\pi : M \to_{\Sigma^*} M'$ with critical point κ. Let $\lambda \leq \pi(\kappa)$ and let the extender F at κ, λ on M be defined by:*

$$F(X) = \pi(X) \cap \lambda.$$

Let $\sigma : \overline{M} \to_{\Sigma^*} M \min \rho$ with $\sigma(\overline{\kappa}) = \kappa$. Assume:

$$\langle \sigma, g \rangle : \langle \overline{M}, \overline{F} \rangle \to^{**} \langle M, F \rangle \mod \rho$$

where $\overline{F}$ is a weakly amenable extender at $\overline{\kappa}, \overline{\lambda}$ on $\overline{M}$. Then

(a) *$\overline{M}$ is $*$-extendable by $\overline{F}$, giving $\overline{\pi} : \overline{M} \to^*_{\overline{F}} \overline{M}'$.*

(b) *There are σ', ρ' such that*

  (i) *$\sigma' : \overline{M}' \to_{\Sigma^*} M' \min \rho'$*

  (ii) *σ' is defined by:*

$$\sigma'(\overline{\pi}(f)(\alpha)) = (\pi\sigma)_\rho(f)(g(\alpha))$$

  for $\alpha < \lambda^-, f \in \Gamma^*(\overline{\kappa}, \overline{M})$. (Hence $\sigma'\overline{\pi} = \pi\sigma$ and $\sigma' \restriction \overline{\lambda} = g$.)

  (iii) *$\pi''\rho_i \subset \rho'_i \leq \pi(\rho_i)$ for $i < w$ (taking $\pi(\rho_i) = \mathrm{On}_M$, if $\rho_i = \mathrm{On}_M$).*

(c) *The above, in fact, holds for:*

$$\rho' =: \min(\rho^*) = \min(M', \sigma'\rho^*).$$

where ρ^* is defined by:

$$\rho_0^* = \begin{cases} \sup{}''\rho_i \text{ if } \rho_{i+1} \leq \kappa_i \\ \pi(\rho_i) \text{ if } \kappa_i < \rho_{i+1} \text{ and } \rho_i < \rho_M^i \\ \rho_M^i, \text{ if } \kappa_i < \rho_{i+1} \text{ and } \rho_i = \rho_M^i. \end{cases}$$

This is the most important result on pseudo projecta.

The argumentation used in the proof of Lemma 3.6.35, Lemma 3.6.36 and Lemma 3.6.37 actually establishes a more abstract result which is useful in other contexts:

Lemma 3.6.22. *Assume that M_i, M'_i are amenable for $i < \mu$, where μ is a limit ordinal. Assume further than:*

(a) *$\pi_{i,j} : M_i \longrightarrow_{\Sigma^*} M_j$ $(i \leq j < \mu)$, where the $\pi_{i,j}$ commute.*

(b) $\pi'_{i,j} : M'_i \longrightarrow_{\Sigma^*} M'_j \ (i \leq j < \mu)$, *where the* $\pi'_{i,j}$ *commute.*
Moreover:
$$\langle M'_i : i < \mu\rangle, \langle \pi'_{i,j} : i \leq j < \mu\rangle$$
has a transitivized direct limit $M', \langle \pi'_{i,j} : i \leq j < \mu\rangle$.

(c) $\sigma_i : M'_i \longrightarrow_{\Sigma^*} M'_j \min \rho^i \ (i \leq j < \mu)$.

(d) $\sigma_j \pi_{i,j} = \pi'_{i,j}\sigma_i$.

(e) $\pi'_{i,j}$ "$\rho^i_n \subset \rho^i_n \leq \pi'_{i,j}(\rho^i_n)$ *for* $i \leq j < \mu, n < \omega$.

Then:
$$\langle M_i : i < \mu\rangle, \langle \pi_{i,j} : i \leq j < \mu\rangle$$
has a transitivized direct limit $M, \langle \pi_{i,j} : i < \mu\rangle$.

There is then $\sigma : M \longrightarrow M'$ defined by: $\sigma\pi_i = \pi'_i\sigma_i(i < \mu)$. Moreover:

(1) There is a unique ρ such that $\sigma : M \longrightarrow_{\Sigma^*} M' \min \rho$ and:
$$\pi' \text{“}\rho^i_n \subset \rho_n \leq \pi'_i(\rho^i_n) \text{ for } i < \mu, n < \omega.$$

(2) There is $i < \mu$ such that $\rho_n = \pi'_j(\rho^i_n)$ for $i \leq j < \mu, n < \omega$.

3.6.3 Mirrors

Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration of length η. By a *mirror* of I we shall mean a sequence:
$$I' = \langle\langle M'_i\rangle, \langle\pi'_{ij}\rangle, \langle\sigma_i\rangle, \langle\rho^i\rangle\rangle$$
such that $\sigma_i : M_i \rightarrow_{\Sigma^*} M'_i \min \rho^i$ for $i < \eta$ and the sequence:
$$I'' = \langle\langle M'_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T\rangle$$
"mirrors" the action of I, where $\nu'_i =: \sigma_i(\nu_i)$. However, I'' will not necessarily be an iteration. If $i+1$ is not a drop point in I and $h = T(i+1)$, we will, indeed, have:
$$\pi'_{h,i+1} : M'_h \rightarrow_{\Sigma^*} M'_{i+1},$$
but M'_{i+1} is *not* necessarily an ultrapower of M'_h. None the less $\kappa'_i =: \sigma_i(\kappa_i)$ will still be the critical point and we shall have:
$$\mathbb{P}(\kappa'_i) \cap M'_h = \mathbb{P}(\kappa'_i) \cap J^{E^{M'_i}}_{\nu_i}$$

and:

$$\alpha \in E_{\nu_i}^{M'_i}(X) \leftrightarrow \alpha \in \pi'_{h,i+1}(X) \text{ for}$$
$$X \in \mathbb{P}(\kappa'_i) \cap M'_h \text{ and } \alpha < \lambda'_i,$$

where $\lambda'_i =: \sigma_i(\lambda_i)$.

We shall also require a measure of agreement among the maps σ_i. In particular, if $h = T(i+1)$ is as above, then:

$$\sigma_{i+1}\pi_{h,i+1} = \pi'_{h,i+1}\sigma_h;\ \sigma_i \restriction \lambda_i = \sigma_{i+1} \restriction \lambda_i.$$

Note. that this gives:

$$\langle \sigma_h, \sigma_i \restriction \lambda_i \rangle : \langle M_h, E_{\nu_i}^{M_i} \rangle \to \langle M'_h, E_{\nu_i}^{M'_i} \rangle.)$$

The formal definition is:

Definition 3.6.10. Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of length η. By a *mirror* of I we mean a sequence:

$$I' = \langle \langle M'_i | i < \eta \rangle, \langle \pi'_{ij} | i \leq_T i \rangle, \langle \sigma_i < | i < \eta \rangle, \langle \rho^i | i < \eta \rangle \rangle$$

satisfying the following conditions:

(a) M'_i is a premouse and $\sigma_i : M_i \to_{\Sigma^*} M'_i \min \rho^i$.

(b) π'_{ij} is a partial structure preserving map from M'_i to M'_j. Moreover the π'_{ij} commute and $\pi_{ii} = \mathrm{id} \restriction M_i$. If $\lambda < \eta$ is a limit, then $M'_\lambda = \bigcup\limits_{i \top \lambda} \mathrm{rng}(\pi'_{i\lambda})$.

(c) $\sigma_i \pi_{ij} = \pi'_{ij}\sigma_i$ for $i \leq_\top j$.

(d) $\sigma_i \restriction \lambda_i = \sigma_j \restriction \lambda_i$ for $i < j < \eta$.

In order to state the further clauses we need some notation. Set:

$$\nu'_i = \sigma_i(\nu_i) =: \begin{cases} \sigma_i(\nu_i) \text{ if } \nu_i \in M_i \\ \mathrm{On} \cap M'_i \text{ if not} \end{cases}$$
$$\kappa'_i = \sigma_i(\kappa_i), \tau'_i = \sigma_i(\tau_i), \lambda'_i = \sigma_i(\lambda_i)$$

For $h = T(i+1)$ set:

$$M'^*_i = \begin{cases} \sigma_h(M^*_i) \text{ if } M^*_i \in M_h \\ M'_h \text{ if not.} \end{cases}$$

Noting that $\tau'_i = \sigma_h(\tau_i)$ by (d) we can easily see that:

$M_i'^* = M_h'||\mu$, where $\mu \leq \mathrm{On}_{M_h'}$ is maximal such that $\tau_o' < \mu$ and τ_i' is a cardinal in $M_h'||\mu$.

(To see that this holds for $M_i'^* = M_h'$, we note that $\tau_i' = \sigma_h(\tau_i)$ is a cardinal in $M_h'||\rho_0^h$ and ρ_0^h is cardinally absolute in M_h'.)

We now complete the definition of *mirror*:

(e) Let $h = T(i+1), i+1 \leq_T i$, and assume that there is no drop point in $(i+1, j)_T$. Then:

(i) $\pi_{h,i}' : M_i'^* \to_{\Sigma^*} M_j'$.

(ii) $\kappa_i' = \mathrm{crit}(\pi_{hj}')$.

(iii) If $X \in \mathbb{P}(\kappa_i') \cap J_{\tau_i'}^{E^{M_i}}$, then $X \in M_i'^*$ and $E_{\nu_i'}^{M_i'}(X) = \lambda_i' \cap \pi_{h,j}'(X)$.

(iv) Set:

$$\hat{\rho}^i = \begin{cases} \rho^h \text{ if } M_i'^* = M_h' \\ \min(M_i'^*, \rho_h \restriction M_i^*, \langle \rho_{M_i'^*}^n | n < w \rangle) \text{ if not.} \end{cases}$$

Then:

$$\pi_{h,j}' \text{``} \hat{\rho}_M^i \subset \rho_n^j \leq \pi_{h,j}'(\hat{\rho}_n^i) \text{ for } n < w$$

(where $\pi_{hj}'(\hat{\rho}_n^i) =: \mathrm{On}\, M_j'$ if $\hat{\rho}_n^i = \mathrm{On}_{M_i'^*}$).

(Hence, if $h \leq_T j$ and $[h, j]_T$ has no drop point, then $\pi_{h,j}' \text{``} \rho_n^h \subset \rho_n^j \leq \pi_{h,j}'(\rho_n^h)$.)

This completes the definition.

Lemma 3.6.23. $J_{\lambda_i'}^{E^{M_i'}} = J_{\lambda_i'}^{E^{M_{i+1}'}}$ *for* $i + 1 < \eta_i$.

Proof: λ_i' is an inaccessible cardinal in $J_{\nu_i}^{E^{M_i}}$. Hence there are arbitrarily large primitive recursive closed ordinals $\alpha < \lambda_i'$ and it suffices to show:

Claim $J_\alpha^{E^{M_i'}} = J_\alpha^{M_{i+1}'}$ for primitive recursive closed $\alpha < \lambda_i'$.

Proof: Let $h = T(i+1)$. Since $x \in J_\alpha^E$ is J_α^E–definable from parameters $\beta_1, \ldots, \beta_n < \alpha$, it suffices to show:

Subclaim Let $\beta_1, \ldots, \beta_n < \alpha$. Let φ be a first order formula. Then:

$$J_\alpha^{E^{M_i'}} \models \varphi[\vec{\beta}] \longleftrightarrow J_\alpha^{E^{M_{i+1}'}} \models \varphi[\vec{\beta}].$$

Proof: Set: $X = \{\prec \vec{\xi}, \zeta \succ < \kappa_i' | J_\zeta^{E^{M_i'}} \mod \varphi[\vec{\xi}]\}$. Then $X \in \mathbb{P}(\kappa_i') \cap J_{\nu_i'}^{E^{M_i'}} \subset M_i'^*$ by (e) (iii). But $J_{\kappa_i'}^{E^{M_i'}} = J_{\kappa_i'}^{E^{M_i'^*}} = J_{\kappa_i'}^{E^{M_h'}}$, by (e) (i), (ii). Then:

$$\bigwedge \vec{\xi}, \zeta < \kappa_i' (\prec \vec{\xi} \succ \in X \leftrightarrow J_\zeta^E \models \varphi[\vec{\xi}]),$$

which is a first order statement in $\langle J^E_{\kappa'_i}, X\rangle$, where $E = E^{M'^*_i}$. But then the same first order statement holds in $\langle \pi'(J^E_{\kappa'_i}), \pi'(X)\rangle$, where $\pi' = \pi'_{h,i+1}$. Clearly $\pi'(J^E_{\kappa'_0}) = J^{E^{M'_{i+1}}}_{\pi'(\kappa'_i)}$. Thus:

$$\pi'(X) = \{\prec \vec{\xi}, \zeta \succ < \pi(\kappa'_i) | J^{E^{M'_{i+1}}}_{\zeta} \models \varphi[\vec{\xi}]\},$$

and we have:

$$\begin{aligned} J^{E^{M'_{i+1}}}_{\alpha} \models \varphi[\vec{\beta}] \quad &\longleftrightarrow \prec \vec{\beta}, \alpha \succ \in \pi'(X) \\ &\longleftrightarrow \prec \vec{\beta}, \alpha \succ \in E^{M'_i}_{\nu'_i}(X) \text{ by (e) (iii)} \\ &\longleftrightarrow J^{E^{M'_i}}_{\alpha} \models \varphi[\vec{\beta}]. \end{aligned}$$

QED (Lemma 3.6.23)

We know that $\lambda'_i = E^{M'_i}_{\nu'_i}(\kappa'_i) \leq \pi'(\kappa'_i)$, where $h = T(i+1)$, $\pi' = \pi_{h,i+1}$ (by (e) (iii)). Set:

$$\lambda^*_i =: \pi'_{h,i+1}(\kappa'_i) \text{ where } h = T(i+1), \text{ for } i+1 < \eta.$$

Lemma 3.6.24. *Let $i+1 < \eta$. Then $\lambda'_i \leq \lambda^*_i = \sigma_j(\lambda_i)$ for $i < j < \eta$.*

Proof: $\lambda'_i \leq \lambda^*_i$ is trivial. But then:

$$\begin{aligned} &\sigma_{i+1}(\lambda_i) = \sigma_{i+1}\pi_{h,i+1}(\kappa_i) = \pi'_{h,i+1}\sigma_h(\kappa_i) \\ &= \pi'_{h,i+1}(\kappa'_i) = \lambda^*_i. \end{aligned}$$

Hence $\sigma_j(\lambda_i) = \sigma_{i+1}(\lambda_i)$ for $j > i$, since $\lambda_i < \lambda_{i+1}$. QED (Lemma 3.6.24)

Note. The main difference between a *mirror* of I and a simple *copy* of I in our earlier sense is that we can have: $\lambda'_i < \lambda^*_i$.

Corollary 3.6.25. *$\lambda'_i < \lambda'_j$ for $i < j$, $j+1 < \eta$.*

Proof: $\lambda'_i \leq \lambda^*_i = \sigma_j(\lambda_i) < \sigma_j(\lambda_j) = \lambda'_j$. QED (Corollary 3.6.25)

Corollary 3.6.26. *If $h = T(i+1), h+1 \leq_T j$, then $\kappa'_i < \lambda'_h \leq \lambda^*_h \leq \kappa'_j$ (since $\kappa_j \geq \lambda_h$).*

Lemma 3.6.27. *$J^{E^{M'_i}}_{\lambda'_i} = J^{E^{M'_j}}_{\lambda'_i}$ for $i \leq j < \eta$.*

Proof: By induction on j

Case 1 $j = i$ trivial.

Case 2 $j = l + 1$. Then it holds at l. But $J^{E^{M_l}}_{\lambda'_l} = J^{E^{M_j}}_{\lambda'_l}$ where $\lambda'_i \leq \lambda'_l$. The conclusion is immediate.

Case 3 $j = \mu$ is a limit ordinal.

By 3.6.26 we have: $\kappa'_i < \kappa'_j$ for $i + 1 \leq_T j + 1 \leq_T \mu$. Moreover $\sup \kappa'_i = \sup \lambda'_i$ by 3.6.26, 3.6.25. Pick an $l+1 \leq_T \mu$ such that $\kappa'_l > \lambda'_i$. Then $J^{E^{M'_l}}_{\kappa'_l} = J^{E^{M'_\mu}}_{\kappa'_l}$ by axiom e (i), (ii) and $J^{E^{M'_i}}_{\lambda'_i} = J^{E^{M'_l}}_{\lambda'_i}$, where $\lambda'_i < \kappa'_l$.

The conclusion is immediate. QED (Lemma 3.6.27)

Lemma 3.6.28. $J^{E^{M'_{i+1}}}_{\lambda^*_i} = J^{E^{M'_j}}_{\lambda^*_i}$ *for* $i < j < \eta$.

Proof: For $j = i+1$ it is trivial. For $j > i+1$, we have $\lambda'_{i+1} = \sigma_{i+1}(\lambda_{i+1}) > \sigma_{i+1}(\lambda_i) = \lambda^*_i$ and $J^{E^{M'_{i+1}}}_{\lambda'_{i+1}} = J^{E^{M'_j}}_{\lambda'_{i+1}}$. The conclusion is immediate. QED (Lemma 3.6.28)

Lemma 3.6.29. λ^*_i *is a limit cardinal in* M'_j *for all* $j > i$.

Proof: $\lambda^*_i = \sigma_j(\lambda_i)$ is a cardinal in M'_j, since λ_i is a cardinal in M_j. (This uses that ρ^j_0 is cardinally absolute if $\rho^i_0 < \mathrm{On}_{M'_i}$.) But then λ^*_i is cardinally absolute in M'_j and:

$$J^{E^{M'_i}}_{\lambda^*_i} \models \text{there are arbitrarily large cardinals,}$$

since the same is true in $J^{E^{M_i}}_{\lambda_i}$. QED (Lemma 3.6.29)

Lemma 3.6.30. λ'_i *is cardinally absolute in* M'_j *for* $j \geq i$.

Proof: Let α be a cardinal in $J^E_{\lambda'_i} = J^{E^{M'_i}}_{\lambda'_i} = J^{E^{M'_j}}_{\lambda'_i}$. Let $h = T(i+1)$ and let:

$$X = \{\xi < \kappa'_i) J^E_{\kappa'_i} \models \xi \text{ is a cardinal}\}.$$

Then: $\alpha \in E^{M'_{i+1}}_{\nu'_i}(X) \subset \pi'_{h,i+1}(X)$. Hence:

$$J^{E^{M'_{i+1}}}_{\lambda^*_i} \models \alpha \text{ is a cardinal.}$$

But $J^{E^{M'_{i+1}}}_{\lambda^*_i} = J^{E^{M'_j}}_{\lambda^*_i}$ and λ^*_i is cardinally absolute in M'_j.

QED (Lemma 3.6.30)

But there are arbitrarily large cardinals in the sense of $J^{E^{M'_i}}_{\lambda'_i}$. Hence:

Corollary 3.6.31. *λ'_i is a limit cardinal in M'_j for $i < j$.*

Lemma 3.6.32. *Let $h = T(i+1)$. Then $J^{E^{M'_h}}_{\tau'_i} = J^{E^{M'_i}}_{\tau'_i}$.*

Proof: For $h = i$ it is trivial. Let $h < i$. Then $J^{E^{M'_h}}_{\lambda'_h} = J^{E^{M'_i}}_{\lambda'_h}$, so we need only show that $\tau'_i < \lambda'_h$. But λ'_h is a limit cardinal in M'_i and $\kappa'_i < \tau'_i$. Hence in M'_i we have: $\tau'_i \leq \kappa'^{+}_i < \lambda'_h$. QED (Lemma 3.6.32)

Corollary 3.6.33. $\mathbb{P}(\kappa'_i) \cap M'^*_i = \mathbb{P}(\kappa'_i) \cap J^{E^{M'_i}}_{\nu'_i}$.

Proof: Since $\tau'_i > \kappa'_i$ is a cardinal in M'^*_i, we have by acceptability:

$$\begin{aligned}\mathbb{P}(\kappa'_i) \cap M'^*_i &= \mathbb{P}(\kappa'_i) \cap J^{E^{M'_h}}_{\tau'_i} = \mathbb{P}(\kappa'_i) \cap J^{E^{M'_i}}_{\tau'_i} \\ &= \mathbb{P}(\kappa'_i) \cap J^{E^{M'_h}}_{\nu'_i}\end{aligned}$$

QED(Corollary 3.6.33)

Lemma 3.6.34. *Let $h = T(i+1), F = E^{M_i}_{\nu_i}, F' = E^{M'_i}_{\nu'_i}$. Then*

$$\langle \sigma_h \restriction M^*_i, \sigma_i \restriction \lambda_i \rangle : \langle M^*_i, F \rangle \longrightarrow \langle M'^*_i, F' \rangle.$$

Proof. Clearly $(\sigma_h \restriction M^*_i) : M^*_i \longrightarrow_{\Sigma_0} M'^*_i$. Moreover, $\mathrm{rng}(\sigma_i \restriction \lambda_i) \subset \lambda'_i$. Now let $X \subset \kappa_i, X \in M^*_i, \alpha_i, \ldots, \alpha_n < \lambda_i$. Then:

$$\begin{aligned}&\prec \vec{\alpha} \succ \in F(X) = \pi_{h,i+1}(X) \\ &\longleftrightarrow \prec \sigma_{i+1}(\vec{\alpha}) \succ \in \sigma_{i+1}\pi_{h,i+1}(X) = \pi'_{h,j+1}\sigma_h(X) \\ &\longleftrightarrow \prec \sigma_i(\vec{\alpha}) \succ \in F'(\sigma_h(X)),\end{aligned}$$

since $\sigma_i \restriction \lambda_i = \sigma_{i+1} \restriction \lambda_i$ and $F'(\sigma_h(X)) = \lambda'_i \cap \pi'_{h,i+1}(\sigma_h(X))$.

QED(Lemma 3.6.34)

We also note:

Lemma 3.6.35. *Let $\lambda < \eta$ be a limit ordinal. Then for sufficiently large $i <_T \lambda$ we have:*

$$\rho^\lambda = \pi'_{i,\lambda}(p^i_n) \ \text{ for } n < \omega$$

Proof. Pick $\xi < \lambda$ such that $[\xi, \lambda)_T$ has no drop points. For each $n < \omega$ and each i, j such that $\xi \leq_T i \leq_T j \leq_T \lambda$ we have:

$$\pi'_{i,j}``\rho^i_n \subset \rho^j_n \leq \pi'_{ij}(\rho^i_n).$$

(1) For each $n < \omega$ there is $i_n \in [\xi, \lambda)_T$ such that:

$$\pi'_{i,j}(\rho^i_n) = \rho^i_n \text{ for } i_n \leq_T i \leq_T j <_T \lambda.$$

Proof. Suppose not. Then there exist $i_r (r < \omega)$ such that $\xi <_T i_r <_T i_{r+1}$ and $\rho^{i_{r+1}}_n < \pi'_{i_{r+1},\lambda}(\rho^{i_{r+1}}_n) < \pi'_{i_r,\lambda}(\rho^{i_r}_n)$. Hence: $\pi'_{i_{r+1},\lambda}(\rho^{i_{r+1}}_n) < \pi'_{i_r,\lambda}(\rho^i_n)$ for $r < \omega$. Contradiction!

QED(1)

(2) $\pi'_{i,\lambda}(\rho^i_n) = \rho^\lambda_n$ for $i_n \leq_T <_T \lambda$.

Proof. Since $M, \langle \pi'_{i,\lambda} : i_n \leq_T i <_T \lambda \rangle$ is a direct limit, we have:

$$\pi'_{i,\lambda}(\rho^i_n) = \bigcup_{i_n \leq_T i <_T \lambda} \pi'_{i,\lambda} \text{``} \rho^i_n \subset \rho^\lambda_n \leq \pi'_{i,\lambda}(\rho^i_n).$$

QED(2)

(3) If $\rho^\lambda_n = \rho^n_{M_\lambda}$ then $i_n = \xi$.

Proof. If not, there is $i \in [\xi, \lambda)_T$ such that $\rho^i_n < \rho^n_{M_i}$. Hence $\rho^\lambda_n \leq \pi'_{i,\lambda}(\rho^i_n) < \rho^n_{M_\lambda}$. Contradiction!

QED(3)

But then the set $\{n : i_n > \xi\}$ is finite. Set: $i = \max\{i_n : i_n > \xi\}$. This has the desired property.

QED(Lemma 3.6.35)

Corollary 3.6.36. *Let λ be a limit ordinal. Then*

$$\pi'_{i,\lambda} : M'_i \longrightarrow_{\Sigma^*} M'_\lambda \mod (\rho^i, \rho^\lambda)$$

for sufficiently large $i \leq_T \lambda$.

Proof. Let $i_0 \leq_T i <_T \lambda$ such that $\pi'_{i,\lambda}(\rho^i_n) = \rho^\lambda_n$ for $i_0 \leq_T i < \lambda, n < \omega$. By Lemma 3.6.3 we need only show:

(1) $\rho^i_n < \rho^n_{M_i} \longrightarrow \rho^\lambda_n = \pi'_{i,\lambda}(\rho^i_n)$

(2) $\rho^i_n = \rho^n_{M_i} \longrightarrow \rho^\lambda_n = \rho^n_{M_\lambda}$

(1) is immediate. To prove (2) we note:

$$\rho^\lambda_n = \pi'_{i,\lambda}(\rho^i_n) = \pi_{i,\lambda}(\rho^n_{M_i}) \geq \rho^n_{M_\lambda} \geq \rho^\lambda_n$$

QED Corollary 3.6.36

Definition 3.6.11. By a *mirror pair* of length η we mean a pair $\langle I, I'\rangle$ such that I is a normal iteration of length η and I' is a mirror of I.

It is natural to ask whether, and in what circumstances, a mirror pair of length η can be extended to one of length $\eta+1$. For limit η the answer is fairly straightforward:

Lemma 3.6.37. *Let $\langle I, I'\rangle$ be a mirror pair of limit length. Let b be a cofinal branch in $T = T_I$. Let the sequence:*

$$\langle M_i' : i \in b\rangle,\ \langle \pi_{ij}' : i \leq j \text{ in } b\rangle$$

have a well founded direct limit. Then $\langle I, I'\rangle$ extends uniquely to a mirror pair $\langle \hat{I}, \hat{I}'\rangle$ of length $\eta+1$ with $b = \hat{T}\text{"}\{\eta\}$ (where $\hat{T} = T_{\hat{I}}$).

Proof. Let $M_\eta', \langle \pi_{i,\eta}' : i \in b\rangle$ be the transitivized direct limit.

Note. By our convention this means that for some $j_0 \in b$, $b \setminus j_0$ is drop free and:

$$\langle M_i' : i \in b \setminus j_0\rangle, \langle \pi_{i,j}' : j_0 \leq i \leq j \text{ in } b\rangle$$

in the usual sense, and we define:

$$\pi_{i\eta}' = \pi_{j_0,\eta}' \circ \pi_{i,j_0}' \text{ for } i < j_0 \text{ in } b$$

In the same sense the sequence:

$$\langle M_i : i \in b\rangle, \langle \pi_{i,j} : i \leq j \text{ in } b\rangle$$

has a transitivized limit:

$$M, \langle M_{i\eta} : i \in b\rangle$$

The maps $\pi_{i,\eta}$, $\pi_{i,\eta}'$ are easily seen to be Σ^*−preserving for $j_0 \leq i \in b$. We extend T to $\hat{T}$ by setting $\hat{T}\text{"}\{\eta\} = b$. We define the map $\sigma_\eta : M_\eta \longrightarrow M_\eta'$ by: $\sigma_\eta \pi_{i\eta} = \pi_{i\eta}' \sigma_i$ for $i < \eta$. We must then define a good sequence $\hat{\rho} = \rho^\eta$ for M_η'. We first imitate the proof of Lemma 3.6.35 by showing that there is $i_0 \in b$ such that $b \setminus i_0$ has no drop points and for all $j \in b \setminus i_0$:

$$\pi_{i,j}'(\rho_n^i) = \rho_n^j \text{ for } n < \omega$$

Thus, setting: $\hat{\rho}_n =: \pi_{i_0,\eta}'(\rho_n^{i_0})$, we have:

$$\hat{\rho}_n = \pi_{j,\eta}'(\rho_n^j) \text{ for } n < \omega, i_0 \leq_T j \in b$$

It is easily shown that $\hat{\rho} = \langle \hat{\rho}_n : n < \omega\rangle$ is a good sequence for M_η'. Repeating the proof of Lemma 3.6.36 we then have:

(1) $\pi'_{j\eta} : M'_j \longrightarrow_{\Sigma^*} M'_\eta \mod (\rho^i, \hat{\rho})$ for $i_0 \leq_T j \leq_T \eta$.

Using this we show:

Claim 1. $\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_\eta \mod \hat{\rho}$.

Proof. Let $x_1, \ldots, x_n \in M_\eta$. Then $\vec{x} = \pi_{i\eta}(\vec{z})$ for an $i \in [i_0, \eta)$. Hence for any $\Sigma_0^{(n)}$ formula:

$$\begin{aligned} M_\eta \models \varphi[\vec{x}] &\longleftrightarrow M_i \models \varphi[\vec{z}] \\ &\longleftrightarrow M'_i \models \varphi[\sigma_i(\vec{z})] \mod \rho^i \\ &\longleftrightarrow M'_i \models \varphi[\pi'_{i,\eta}\sigma_i(\vec{z})] \mod \hat{\rho} \end{aligned}$$

where $\pi'_{i,\eta}\sigma_i(\vec{z}) = \sigma_\eta \pi_{i,\eta}(\vec{z}) = \sigma_\eta(\vec{x})$.

QED(Claim 1)

We must also show:

Claim 2. $\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_\eta \min \hat{\rho}$.

Proof. We must show:

$$\hat{\rho} = \min(M_\eta, \sigma_\eta, \tilde{\rho})$$

Let $\langle \hat{\rho}_l(n) : l < \omega \rangle$ be defined by induction on $n < \omega$ as in Definition 3.6.5. We must show: $\hat{\rho}_l = \bigcup_{n<\omega} \hat{\rho}_l(n)$. Let $\xi < \hat{\rho}_l$. Then $\xi = \pi'_{i,\eta}(\bar{\xi})$ where $i_0 \leq_T <_T \eta$ and $\bar{\eta} < \rho_l^i$. But $\rho_l^i = \bigcup_{n<\omega} \rho_l^i(n)$. Thus $\bar{\xi} < \rho_l^i(n)$ for some n. Using (1) and Definition 3.6.5, we easily get:

$$\pi'_{i,\eta}{}^{"} \rho_l^i(n) \subset \hat{\rho}_l(n) \text{ by induction on } n$$

But then $\xi = \pi'_{i,\eta}(\bar{\xi}) \in \hat{\rho}_l(n)$.

QED(Claim 2)

Using these facts it is easy to see that the extension $\langle \hat{I}, \hat{I}' \rangle$ we have defined satisfies the axiom (a)-(e) and is, therefore a mirror pair of length $\eta + 1$. (We leave the detail to the reader). The uniqueness of the maps $\pi_{i,\eta}, \pi'_{i,\eta}, \sigma_\eta$ is immediate from our construction. Finally, we must show that $\hat{\rho} = \rho^\eta$ is unique. This is because $\hat{\rho}_n = \pi'_{i_0,\lambda}(\rho_n^{i_0})$ where $\pi'_{i_0,\lambda}$ is unique.

QED(Lemma 3.6.37)

We now ask how we can extend a mirror pair of length $\eta+1$ to one of length $\eta + 2$. This will turn out to be more complex.

If $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ is a normal iteration of length $\eta+1$, we can turn it into a *potential iteration* of length $\eta+2$ simply by appointing a ν_η such that $E^{M_\eta}_{\nu_\eta} \neq \varnothing$ and $\nu_\eta > \nu_i$ for $i < \eta$. This then determines $h = T(\eta+1)$ and M^*_η. (The notion of potential iteration was introduced in §3.4, where we gave a more formal definition). If $\langle I, I'\rangle$ is a mirror pair of length $\eta+1$, we can then form a *potential mirror pair* of length $\eta+2$ by appointing $\nu'_\eta =: \sigma_\eta(\nu_\eta)$. This determines M'^*_η. Our main lemma on "1-step extension" of mirror pair reads:

Lemma 3.6.38. *Let $\langle I, I'\rangle$ be a mirror pair of length $\eta+1$. Form a potential pair of length $\eta+2$ by appointing ν_η and $\nu'_\eta = \sigma_\eta(\nu_\eta)$. Let:*

$$\pi' : M'^*_\eta \longrightarrow_{\Sigma^*} M' \ \textit{ such that } \kappa'_\eta = \mathrm{crit}(\pi')$$

and

$$E^{M'_\eta}_{\nu_\eta}(X) = \lambda'_\eta \cap \pi'(X) \textit{ for } X \in \mathbb{P}(\kappa'_\eta) \cap J^{E^{M'_\eta}}_{\nu'_\eta}$$

Our potential pair then extends to a full mirror pair with:

$$M' = M'_{\eta+1},\ \pi' = \pi'_{h,\eta+1} \textit{ where } h = T(\eta+1)$$

In order to prove this, we must first form a $*$-ultrapower:

$$\pi : M^*_\eta \longrightarrow^*_F M \text{ where } F = E^{M_\eta}_{\nu_\eta}$$

We must then define σ, ρ such that:

$$\pi' \text{``}\hat{\rho}_n \subset \rho_n \leq \pi'(\hat{\rho}_n) \text{ for } n < \omega$$

where $\hat{\rho}$ is defined as in axiom (e)(iv). If we then set:

$$M_{\eta+1} =: M, M'_{\eta+1} =: M', \pi_{h,\eta+1} =: \pi, \pi'_{h,\eta+1} =: \pi', \sigma_{\eta+1} = \sigma, \rho^{\eta+1} = \rho$$

we will have defined the desired extension. (We leave it to the reader to verify the axioms (a)-(e)). By the proof of Lemma 3.6.34 we have:

$$\langle \sigma_h \restriction M^*_\eta, \sigma_\eta \restriction \lambda_\eta\rangle : \langle M^*_i, F\rangle \longrightarrow \langle M^*_i, F'\rangle$$

where $F = E^{M_\eta}_{\nu_\eta}$, $F' = E^{M'_\eta}_{\nu'_\eta}$.

Lemma 3.6.19 then points us in the right direction. In order to get the full result, however, we must use Theorem 3.6.21 together with:

Lemma 3.6.39. *Let $\langle I, I'\rangle, \nu_\eta, \nu'_\eta, \pi'$ be as in Lemma 3.6.38. Set: $\xi = T(\eta+1), F = E^{M_\eta}_{\nu_\eta}$, $F' = E^{M'_\eta}_{\nu'_\eta}$. Set:*

$$\hat{\rho} = \begin{cases} \rho^\xi & \textit{if } M'^*_\eta = M'_\xi \\ \min(M'^*_\eta, \sigma_h \restriction M'^*_\eta, \langle \rho^n_{M'^*_\eta} : n < \omega\rangle) & \textit{if not} \end{cases}$$

Then:

$$\sigma_h \restriction M^*_h, \sigma_\eta \restriction \lambda_\eta : \langle M^*_\eta, F\rangle \longrightarrow^{**} \langle M'^*_\eta, F'\rangle \quad \bmod \hat{\rho}$$

We leave it to the reader to see that Theorem 3.6.21 and Lemma 3.6.39 give the desired result.

Note. It is clear that $\pi_{h,\eta+1}, \pi'_{h,\eta+1}, \sigma_{\eta+1}$ are uniquely determined by the choice of $\nu_\eta, \nu'_\eta, \pi'$. If we wished, we could use clause (c) of Theorem 3.6.21 to make $\rho^{\eta+1}$ unique.

We are actually in familiar territory here. The notion of mirror is clearly analogous to that of *copy* developed in §3.4.2. The analogue of mirror pair was there called a *duplication*. The role of Lemma 3.4.16 is now played by Lemma 3.6.38 and that of Theorem 3.4.16 by Lemma 3.6.39, which verifies the weaker principle $\longrightarrow^{**}$ in place of $\longrightarrow^{*}$ (which was, in turn, patterned on the proof of Theorem 3.4.3), which said that, if I is a potential normal iteration of length $\eta+2$, then $E_\eta^{M_\eta}$ is close to M^*_η).

We now turn to the proof of lemma 3.6.39. Just as in §3.4.2 we derive it from a stronger lemma. In order to formulate this properly we define:

Definition 3.6.12. Let M be acceptable. Let $\kappa \in M$ be inaccessible in M such that $\mathbb{P}(\kappa) \cap M \in M$. $A \subset \mathbb{P}(\kappa) \cap M$ is *strongly* $\Sigma_1(M)$ *in the parameter* p iff there is $B \subset M$ such that B is $\Sigma_0(M)$ and:

- $x \in A \longleftrightarrow \bigvee z B(z,x,p)$
- If $u \in M$ such that $u \subset \mathbb{P}(\kappa)$ and $\overline{\overline{u}}^M \leq \kappa$, then:

$$\bigvee v \in M \bigwedge X \in u \bigvee z \in v (B(z,X,p) \vee B(z,\kappa \setminus X,p))$$

We shall derive:

Lemma 3.6.40. *Let $\langle I, I'\rangle, \eta, \xi, \nu_\eta, \nu'_\eta, \pi'$ be as in Lemma 3.6.39. Let $A \subset \mathbb{P}(\kappa_\eta)$ be strongly $\Sigma_1(M_\eta||\nu_\eta)$ in p. Let $A' \subset \mathbb{P}(\kappa'_\eta)$ be $\Sigma_1(M'_\eta||\nu'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition. Then there is $q \in M^*_\eta$ such that*

- *A is strongly $\Sigma_1(M^*_\eta)$ in q.*
- *Let A'' be $\Sigma_1(M'^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same definition. Then $A'' \subset A'$.*

Before proving this, we show that it implies Lemma 3.6.39:

Lemma 3.6.41. *Assume Lemma 3.6.40. Let ρ^* be good for M'^* and let:*

$$\sigma_\xi \restriction M^*_\eta : M^*_\eta \longrightarrow_{\Sigma^*} M'^*_\eta \quad \text{mod } \rho^*.$$

Then:

$$\langle \sigma_\xi \restriction M^*_\eta, \sigma_\eta \restriction \lambda_\eta \rangle : \langle M^*_\eta, F\rangle \longrightarrow^{**} \langle M'^*_\eta, F'\rangle \quad \text{mod } \rho^*.$$

Proof. Let $\alpha < \lambda_\eta, \alpha' = \sigma_\eta(\alpha)$. Then F_α is $\Sigma_1(J^{E^{M_\eta}}_{\nu_\eta})$ in α, since:

$$X \in F_\alpha \longleftrightarrow \bigvee Y(Y = F(X) \wedge \alpha \in Y)$$

We know, however, that if $u \in J^{E^{M_\eta}}_{\nu_\eta}, u \subset \mathbb{P}(\kappa)$, and $\overline{\overline{u}} \leq \kappa$ in $J^{E^{M_\eta}}_{\nu_\eta}$, then:

$$\bigvee v \in J^{E^{M_\eta}}_{\nu_\eta} \wedge X \in u \bigvee Y \in v(Y = F(X) \wedge (\alpha \in Y \vee \alpha \in (\kappa \setminus Y)))$$

Hence F_α is strongly $\Sigma_1(J^{E^{M_\eta}}_{\nu_\eta})$ in α. Obviously $F^{\alpha'}_{\alpha'}$ is $\Sigma_1(J^{E^{M'_\eta}}_{\nu'_\eta})$ in $\alpha' = \sigma_\eta(\alpha)$ by the same definition. Hence $\overline{G} = F_\alpha$ is strongly $\Sigma_1(M^*_\eta)$ in a parameter q. Moreover, if G' in $\Sigma_1(M'^*_\eta)$ in $\sigma_\xi(q)$ by the same definition, then $G' \subset F'_{\alpha'}$. Now let G be $\Sigma_1(M'^*_\eta, \rho^*)$ in $\sigma_\xi(q)$ by the same definition. Then $G \subset G' \subset F'_{\alpha'}$. Now let:

$$X \in \overline{G} \longleftrightarrow \bigvee z\overline{B}(z, X, q)$$

be the strongly $\Sigma_1(M^*_\eta)$-definition of G in q. Then:

$$X \in G \longleftrightarrow \bigvee zB(z, X, q')$$

where $q' = \sigma_\eta(q)$ and B is $\Sigma_0(M^*_\eta, \rho^*)$ by the same definition. (In other words, B is $\Sigma_0(M'^*_\eta|\rho^*_0)$ by the same definition). Now let $\overline{H}$ be the set of $f \in M^*_\eta \cap {}^\kappa\mathbb{P}(\kappa)$ such that

$$\bigvee z \bigwedge i < \kappa(\overline{B}(z, f(i), q) \vee \overline{B}(z, \kappa \setminus f(i), q))$$

Then $\overline{H} = M^*_\eta \cap {}^\kappa\mathbb{P}(\kappa)$ by the strongness of our definition. But if H has the same $\Sigma_1(M^*_\eta, \rho^*)$ definition in q', then we obviously have:

$$f \in H \longrightarrow \bigwedge i < \kappa'(f(i) \in G \vee \kappa \setminus f(i) \in G)$$

QED(Lemma 3.6.41)

(In the application we, of course, take $\rho^* = \hat{p}$, where $\hat{p}$ is defined as in Lemma 3.6.39).

We now turn to the proof of Lemma 3.6.40. Suppose not. Let η be the least counterexample. We again have fixed ν_η and $\nu'_\eta = \sigma_\eta(\nu_\eta)$, which gives us $\kappa_\eta, \kappa'_\eta\tau_\eta, \tau'_\eta, \lambda_\eta, \lambda'_\eta, \xi = T(\eta+1), M^*_\eta, M'^*_\eta$ and ρ^*.

(1) $\xi < \eta$.

Proof. Suppose not. Let $A \subset \mathbb{P}(\kappa)$ be strongly $\Sigma_1(M_\eta||\nu_\eta)$ in p and let $A' \subset \mathbb{P}(\kappa'_\eta)$ be $\Sigma_1(M'_\eta||\nu'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition.

Clearly τ_η is a cardinal in $M_\eta||\nu_1$, so $M^*_\eta = M_\eta||\mu$ for a $\mu \geq \nu_\eta$. Similarly $M'^*_\eta = M'_\eta||\mu'$ where:

$$\mu' = \begin{cases} \sigma_\eta(\mu) & \text{if } \mu \in M_\eta \\ \mathsf{ON} \cap M_\eta & \text{if not} \end{cases}$$

Now suppose $\nu_\eta \in M^*_\eta$ (i.e. $\mu > \nu_\eta$). Then $A \in M^*_\eta$ and $A' \in M'^*_\eta$ where $\sigma_\eta(A) = A'$. Then A is trivially strongly $\Sigma_1(M^*_\eta)$ in the parameter A and A' is $\Sigma_1(M^{*'}_\eta)$ in $A' = \sigma_\eta(A)$ by the same definition, where $A' \subset A'$. Contradiction!

Now let $M^*_\eta = M_\eta||\nu_\eta$. Then $M'^*_\eta = M'_\eta||\nu'_\eta$ and A' is $\Sigma_1(M'^*_\eta)$ definable in $p' = \sigma_\eta(p)$ by the same definition. But A is strongly $\Sigma_1(M^*_\eta)$ in p, since $M^*_\eta = M_\eta|\nu_\eta$. Contradiction!

QED(1)

(2) $\nu_\eta = \mathsf{ON} \cap M_\eta$.

Proof. Suppose not. Then $\lambda_\xi > \tau_\eta$ is inaccessible in M_η. Hence $A \in J^{E^{M_\eta}}_{\lambda_\xi} = J^{E^{M_\xi}}_{\lambda_\xi} \subset M^*_\eta$. Similarly $A' \in J^{E^{M'_\eta}}_{\lambda'_\xi} = J^{E^{M'_\xi}}_{\lambda'_\xi} \subset M'^*_\eta|\rho^*_0$. Then A is strongly $\Sigma_1(M^*_\eta)$ in $A' = \sigma_\xi(A)$ by the same definition. Contradiction!

QED(2)

(3) $\tau_\eta \geq \rho^1_{M_\eta}$.

Proof. Suppose not. Then $\tau_\eta < \rho^1_{M_\eta}$. Hence $A \in J^{E^{M_\eta}}_{\rho^1_{M_\eta}}$ since $A \subset J^{E^{M_\eta}}_{\tau_\eta}$. Hence $A \in J^{E^{M_\eta}}_{\lambda_\xi} = J^{E^{M_\xi}}_{\lambda_\xi} \subset M^*_\eta$. Hence A is strongly $\Sigma_1(M^*_\eta)$ in the parameter A_r. Now let A'' be $\Sigma_1(M'_\eta|\rho^\eta_0)$ in $p' = \sigma_\eta(p)$ by the same definition. Then $A'' \subset A'$. But since

$$\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_\eta \min(\rho^\eta),$$

we have: $A'' = \sigma_\eta(A)$. But λ''_ξ is inaccessible in M'_η; hence $A'' \in J^{E^{M_\eta}}_{\lambda'_\xi} = J^{E^{M_\xi}}_{\lambda'_\xi} \subset M'^*_\eta$. Hence $A'' = \sigma_\xi(A)$ is $\Sigma_1(M'^*_\eta)$ in $A'' = \sigma_\xi(A)$ by the same definition. Contradiction!

QED(3)

(4) η is not a limit ordinal.

Proof. Suppose not. Pick $\overline{\eta} <_T \eta$ such that $\overline{\eta} = \mu + 1$. $\pi_{\overline{\eta}\eta}$ is total on $M_{\overline{\eta}}, \kappa = \text{crit}(\pi_{\overline{\eta},\eta}) > \lambda_\eta$ and $p \in \text{rng}(\pi_{\overline{\eta},\eta})$. Then $\pi'_{\overline{\eta},\eta}$ is total in $M'_{\overline{\eta}}$, $\kappa' = \text{crit}(\pi'_{\overline{\eta},\eta}) > \lambda'_\eta$ and $p' \in \text{rng}(\pi'_{\overline{\eta},\eta})$, where $p' = \sigma_\eta(p)$. Set $\overline{p} = \pi^{-1}_{\overline{\eta},\eta}(p), \overline{p}' = \pi^{-1}_{\overline{\eta},\eta}(p')$. Then $\sigma_{\overline{\eta}}(\overline{p}) = p$. Then $M_{\overline{\eta}} =$

$\langle J_{\overline{\nu}}^{E^{M_{\overline{\eta}}}}, \overline{F}\rangle, M'_{\overline{\eta}} = \langle J_{\overline{\nu}'}^{E^{M'_{\overline{\eta}}}}, \overline{F}\rangle$. Extend the mirror $\langle I|\overline{\eta}+1, I'|\overline{\eta}+1\rangle$ to a potential mirror $\langle \overline{I}, \overline{I}'\rangle$ of length $\overline{\eta}+2$, by setting: $\overline{\nu}_{\overline{\eta}} = \overline{\nu}, \overline{\nu}'_{\overline{\eta}} = \overline{\eta}'$. Then $\overline{M}^*_{\overline{\eta}} = M^*_\eta, \overline{M}'^*_{\overline{\eta}} = M'^*_{\overline{\eta}} = M'^*_\eta, \xi = \overline{T}(\overline{\eta}+1) = T(\eta+1)$ and $\sigma_\xi \restriction M^*_{\overline{\eta}} : \overline{M}^*_{\overline{\eta}} \longrightarrow_{\Sigma^*} \overline{M}'^*_{\overline{\eta}} \min \rho^*$. It is easily seen that A is $\Sigma_1(M_{\overline{\eta}})$ in $\overline{p}'$ by the same definition. By the minimality of η we conclude that there is $q \in M^*_\eta = \overline{M}^*_{\overline{\eta}}$ such that A is strongly $\Sigma_1(M^*_\eta)$ in q and A is $\Sigma_1(M'^*_\eta)$ in $q' = \sigma_\xi(q)$ by the same definition. Contradiction!

QED(4)

Now let $\eta = \mu + 1$. Let $\zeta = T(\mu+1)$. Then $\pi_{\zeta,\eta} : M^*_\mu \longrightarrow_{\Sigma^*} M_\eta$ and $\kappa_\mu = \mathrm{crit}(\pi_{\zeta,\eta})$. Hence M^*_μ has the form $\overline{M} = \langle J_{\overline{\nu}}^{\overline{E}}, \overline{F}\rangle$ where $\overline{F} \neq \varnothing$. Set: $\overline{\kappa} = \mathrm{crit}(\overline{F}), \overline{\tau} = \tau(\overline{F}) =: \overline{\kappa}^{+\overline{M}}, \overline{\lambda} = \lambda(\overline{F}) =: \overline{F}(\overline{\kappa})$. Similarly M'^*_μ has the form $\overline{M}' = \langle J_{\overline{\nu}'}^{\overline{E}'}, \overline{F}'\rangle$ and we define $\overline{\kappa}', \overline{\tau}', \overline{\lambda}'$ accordingly.

Set: $\pi = \pi_{\zeta,\eta}, \pi' = \pi'_{\zeta,\eta}$.

(5) $\kappa_\mu > \overline{\kappa}$,

since otherwise $\kappa_\eta = \pi(\overline{\kappa}) \geq \pi(\kappa_\mu) = \lambda_\mu \geq \lambda_\xi > \kappa_\eta$. Contradiction!

QED(5)

But then $\kappa_\mu > \overline{\tau}$ and hence $\overline{\tau} = \tau_\eta, \overline{\kappa} = \kappa_\eta$. Similarly $\kappa'_\mu > \overline{\tau}'$ and $\overline{\tau}' = \tau'_\eta, \overline{\kappa}' = \kappa'_\eta$. But then:

(6) $\kappa_\mu > \rho^1_{\overline{M}}$,

since otherwise $\rho^1_{M_\eta} \geq \pi(\kappa_\mu) = \lambda_\mu > \tau_\eta$. Contradiction! by (3).

QED(6)

Hence, since $\pi : \overline{M} \longrightarrow^*_{E_{\nu_\mu}} M_\eta$, we have:

(7) $\pi : \overline{M} \longrightarrow_{E_{\nu_\mu}} : M_\eta$ is a Σ_0 ultraproduct and $\rho^1_{\overline{M}} = \rho^1_{M_\eta}$.

Recall that A is strongly $\Sigma_1(M_\eta)$ in p and A' is $\Sigma_1(M'_\eta)$ in $p' = \sigma_\eta(p)$ by the same definition. By (7) we know:

(8) $p = \pi(f)(\alpha)$ where $\alpha < \lambda_\mu, f \in \overline{M}$ and $f : \kappa_\mu \longrightarrow \overline{M}$. Hence

(9) $p' = \pi'(f')(\alpha')$ where $f' = \sigma_f(f), \alpha' = \sigma_\mu(\alpha)$.

Proof. $p' = \sigma_\eta(\pi(f)(\alpha)) = (\sigma_\eta\pi(f))(\sigma_\eta(\alpha)) = (\pi'\sigma_\zeta(f))(\sigma_\mu(\alpha))$.

QED(9)

Note. $\sigma_\mu \restriction \lambda_\mu = \sigma_\eta \restriction \lambda_\mu$ since $\mu < \eta$.

Let A be strongly $\Sigma_1(M_\eta)$ in p as witnessed by $\bigvee z B(z, X, p)$, where B is $\Sigma_0(M_\eta)$. Set:

$$B_0(u, X, p) \longleftrightarrow: \bigvee z \in u B(z, X, p).$$

Then A is strongly $\Sigma_1(M_\eta)$ in p as witnessed by $\bigvee uB_0(u,X,p)$. Note that for all u,u':

(10) $(B_0(u,X,p) \wedge u \subset u') \longrightarrow B_0(u',X,p)$.

Let B_1 be $\Sigma_0(\overline{M})$ by the same definition as B_0 over M_η. Set $\tilde{F} =: E^{M_\mu}_{\nu_\mu}, \tilde{F}' = E^{M'_\mu}_{\nu'_\mu}$. By the cofinality of the map $\overline{p} : \overline{M} \longrightarrow M_\eta$ and (10) we have:

(11)

$$\begin{aligned} AX &\longleftrightarrow \bigvee u \in \overline{M} B_0(\pi(u),X,p) \\ &\longleftrightarrow \bigvee u \in \overline{M}\{\gamma < \kappa_\mu : B_\gamma(u,X,f(\gamma))\} \in \tilde{F}_\alpha. \end{aligned}$$

But $\tilde{F}_\alpha$ is strongly $\Sigma_1(M_\mu||\nu_\mu)$ in α and $\tilde{F}'_{\alpha'}$ is $\Sigma_1(M'_\mu||\nu'_\mu)$ in α' by the same definition.

Hence by the minimality of η we conclude:

(12) There is $q \in \overline{M}$ such that the following hold:

(a) $G = \tilde{F}_\alpha$ is strongly $\Sigma_1(\overline{M})$ in q.

(b) Let G' be $\Sigma_1(\overline{M}')$ in $q' = \sigma_\gamma(q)$ by the same definition. Then $G' \subset \tilde{F}'_{\alpha'}$, where $\alpha' = \sigma_\mu(\alpha)$.

Let: $\bigvee zG_0(z,X,q)$ witness the fact that G is strongly $\Sigma_1(\overline{M})$ in q. Then:

$$\begin{aligned} AX &\longleftrightarrow \bigvee u \in \overline{M} B_0(\pi(u),X,\pi(f)(\alpha)) \\ &\longleftrightarrow \bigvee u \in \overline{M}\{\gamma < \kappa_\mu : B_1(u,X,f(\gamma))\} \in G \\ &\longleftrightarrow \bigvee v \in \overline{M} \bigvee u \in v \bigvee \in v \bigvee z \in v \\ &\quad (Y = \{\gamma < \kappa_\mu : B_1(u,X,f(\gamma))\} \wedge G_0(z,Y,q)) \end{aligned}$$

This has the form:

(13) $AX \longleftrightarrow \bigvee vB_2(v,X,r)$, where $r = \langle q,f\rangle$ and B_2 is $\Sigma_0(\overline{M})$.

For this B_2 we claim:

(14) A is strongly $\Sigma_1(\overline{M})$ in r are witnessed by $\bigvee B_2(v,X,r)$.

Proof. Let $w \subset \mathbb{P}(\overline{\kappa}) \cap \overline{M}, \overline{\overline{w}} < \overline{\kappa}$ in $\overline{M}$.

Claim. There is $v \in \overline{M}$ such that

$$\bigwedge X \in w(B_2(v,X,r) \wedge B_2(v,\overline{\kappa} \setminus X,r))$$

For the sake of simplicity we can assume without lose of generality that $X \in w \longleftrightarrow (\overline{\kappa} \setminus M) \in \omega$. Fix $u \in \overline{M}$ such that

$$\bigwedge X \in w(B_0(\pi(u), X, p) \wedge B_0(\pi(u), (\overline{\kappa} \setminus X), p))$$

For $X \in w$ set:

$$\theta(X) =: \{\gamma < \kappa_\mu : B_1(u, X, f(\gamma))\}$$

Then:

$$\bigwedge x \in w(\theta(X) \in G \vee \theta(\overline{\kappa} \setminus X) \in G)$$

By rudimentary closure, $\langle \theta(X) : X \in w\rangle \in \overline{M}$. Hence $\theta``w \in \overline{M}$ and $\mathrm{card}(\theta``w) \leq \overline{\kappa} < \kappa_\mu$ in $\overline{M}$. Thus there is $z \in \overline{M}$ such that:

$$\bigwedge X \in w(G_0(z, \theta(X), q) \vee G_0(z, \kappa_\mu \setminus \theta(X), q))$$

Claim. $\bigwedge X \in w(G_0(z, \theta(X), q) \vee G_0(z, \theta(\overline{\kappa} \setminus X), q))$.

Proof. Suppose not. Then there is $X \in w$ such that:

$$\kappa_\mu \setminus \theta(X),\ \kappa_\mu \setminus \theta(\overline{\kappa} \setminus X) \in G = \tilde{F}_\alpha.$$

Hence $\neg B_0(\pi(u), X, p)$ and $\neg B_0(\pi(u), \overline{\kappa} \setminus X, p)$. Contradiction!

QED(Claim)

Pick $V \in \overline{M}$ such that $u \in v, z \in v$ and $\theta"w \subset v$. Then:

$$\bigwedge X \in w(B_2(v, X, r) \vee B_2(v, \overline{\kappa} \setminus X), r)$$

QED(14)

(15) Let A'' be $\Sigma(\overline{M})$ in $r' = \sigma_\zeta(r)$ by the same definition. Then $A'' \subset A'$.

Proof. Let B_0' be $\Sigma_0(M')$ by the same definition as B_0 over M. Let B_1' be $\Sigma_0(\overline{M})$ by the same definition. $A''X$ says that there is $u \in \overline{M}$ with:

$$\{\gamma < \kappa_\mu' : B_1'(u, X, f'(\gamma))\} \in G'$$

where $f' = \sigma_\zeta(f)$. But $G' \subset \tilde{F}_{\alpha'}$. Hence $B_0'(\pi(u), X, \pi'(f')(\alpha'))$, where $p' = \pi'(f')(\alpha')$. Hence $A'X$.

QED(15)

Now extend $\langle I|\zeta{+}1, I'(\zeta{+}1)\rangle$ to a potential mirror pair $\langle \hat{I}, \hat{I}'\rangle$ of length $\zeta + 2$ by setting: $\nu_\zeta = \overline{\nu}, \nu_\zeta' = \overline{\nu}'$. Since $\overline{\kappa} = \kappa_\eta$, $\overline{\tau} = \tau_\eta$, we have:

$$\xi = \hat{T}(\zeta + 1), \hat{M}_\zeta^* = M_\eta^*, \hat{M}_\zeta'^* = M_\eta'^*$$

But $\zeta \leq \mu < \eta$. By the minimality of η and by (14), (15), we conclude that there is a parameter $s \in M_\eta^*$ such that:

- A is strongly $\Sigma_1(M_\eta^*)$ in s.
- If A''' has the same $\Sigma_1(M_\eta'^*)$ definition in $s'(\sigma_\xi(s))$, then $A''' \subset A''$ (hence $A''' \subset A'$).

This contradicts the fact that η was a counterexample.

QED(Lemma 3.6.40)

The argumentation used in the proof of Lemma 3.6.35, Lemma 3.6.36 and Lemma 3.6.37 actually establishes a more abstract result which is useful in other contexts:

Lemma 3.6.42. *Assume that M_i, M_i' are amenable for $i < \mu$, where μ is a limit ordinal. Assume further that:*

(a) $\pi_{i,j} : M_i \longrightarrow_{\Sigma^*} M_j$ $(i \leq j < \mu)$, *where the* $\pi_{i,j}$ *commute.*

(b) $\pi'_{i,j} : M_i' \longrightarrow_{\Sigma^*} M_j'$ $(i \leq j < \mu)$, *where the* $\pi'_{i,j}$ *commute. Moreover:*

$$\langle M_i' : i < \mu \rangle, \langle \pi'_{i,j} : i \leq j < \mu \rangle$$

has a transitivized direct limit $M', \langle \pi_i' : i < \mu \rangle$.

(c) $\sigma_i : M_i' \longrightarrow_{\Sigma^*} M_j' \min \rho^i$ $(i \leq j < \mu)$.

(d) $\pi'_{i,j}``\rho_n^i \subset \rho_n^j \leq \pi'_{i,j}(\rho_n^i)$ *for* $i \leq j < \mu, n < \omega$. *Then*

$$\langle M_i : i < \mu \rangle, \langle \pi_{i,j} : i \leq j < \mu \rangle$$

has a transitivized direct limit $M, \langle \pi_i : i < \mu \rangle$. *There is then* $\sigma : M \longrightarrow M'$ *defined by:* $\sigma\pi_i = \pi_i'\sigma_i$ $(i < \mu)$. *Moreover:*

(1) *There is a unique* ρ *such that* $\sigma : M \longrightarrow_{\Sigma^*} M' \min \rho$ *and:*

$$\pi_i'``\rho_n^i \subset \rho_n \leq \pi_i'(\rho_n^i) \text{ for } i < \mu, n < \omega.$$

(2) *There is* $i < \mu$ *such that* $\rho_n = \pi_j'(\rho_n^j)$ *for* $i \leq j < \mu$, $n < \omega$.

3.6.4 The conclusion

In this section we show that every smoothly iterable premouse is fully iterable. We first define some auxiliary concepts:

Definition 3.6.13. Let $\langle I, I'\rangle$ be a mirror pair of length η with:

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle \text{ and } I' = \langle\langle M'_i\rangle, \langle\pi'_{ij}\rangle, \langle\sigma_i\rangle, \langle\rho^i\rangle\rangle$$

Let N be a premouse such that $M'_0 = N||\mu$ for some $\mu \leq \mathsf{ON}_N$. As usual set: $\nu'_i = \sigma_i(\nu_i)$. Let:

$$I'' = \langle\langle N_i\rangle, \langle\nu''_i\rangle, \langle\pi''_{ij}\rangle, T\rangle$$

be an iteration on N of length η. (T being the same as in I). Set:

$$\mu_i = \begin{cases} \pi''_{0j}(\mu) & \text{if } \mu \in \text{dom}(\pi''_{0j}) \\ \mathsf{ON}_{N_i} & \text{if not.} \end{cases}$$

We say that the mirror pair $\langle I, I'\rangle$ is *backed* by I'' (or *M-backed by* I'') iff:

$$M'_i = N_i||\mu_i, \nu'_i = \nu''_i, \pi'_{ij} = \pi''_{ij} \restriction M'_i \text{ for } i \leq_T j < \eta.$$

Now suppose that $\langle I, I'\rangle$ is a mirror pair of length $\eta + 1$ backed by I''. Extend I to a potential iteration I^+ of length $\eta + 2$ by appointing ν_η such that $E^{M_\eta}_{\nu_\eta} \neq \varnothing$ and $\nu_\eta > \nu_i$ for $i < \eta$. This determines $\zeta = T(\eta + 1)$ and M^*_η. If we then set: $\nu'_\eta = \sigma_\eta(\nu_\eta)$, we have determined M'^*_η and turned $\langle I, I'\rangle$ into a potential mirror pair $\langle I^+, I'^+\rangle$. But ν'_η also extends I'' to a potential iteration I''^+ of length $\eta + 2$, determining N^*_η. We then say that I''^+ *potentially backs* $\langle I^+, I'^+\rangle$.

Note that if $M^*_\eta \in M_\xi$, then:

$$M'^*_\eta = \sigma_\xi(M^*_\eta) = N^*_\eta.$$

If, however, $M^*_\eta = M_\xi$, then we have $M'^*_\eta = M'_\xi$, but if is still possible that $M'^*_\eta \in N^*_\eta$ and even that $N^*_\eta \in N_\xi$. This can happen if $M'_\xi = N_\xi||\mu_\xi$ and $\mu_\xi \in N_\xi$. There might then be $\gamma > \mu_\xi$ such that τ'_η is a cardinal in $N_\xi||\gamma$. Hence $M'^*_\eta = M'_\xi \in N'_\xi||\gamma \subset N^*_\eta$. But if the largest such γ is an element of N_ξ, we then have $N^*_\eta \in N_\xi$.

Note. If I^+, I'^+, I''^+ are as above, we certainly have: $E^{M'_\eta}_{\nu'_\eta} = E^{N_\eta}_{\nu'_\eta}$.

Using Lemma 3.6.38 we can then prove:

Lemma 3.6.43. *Let I^+, I'^+, I''^+ be as above. Suppose that N^*_η is $*$-extendible by $F' = E^{N_\eta}_{\nu'_\eta}$. Then $\langle I^+, I'^+\rangle$ extends to an actual mirror pair $\langle\hat{I}, \hat{I}'\rangle$ with $\hat{\nu}_\eta = \nu_\eta$ and I''^+ extends to an iteration $\hat{I}''$ which backs $\langle\hat{I}, \hat{I}'\rangle$.*

Proof. Set $\pi'' : N^*_\eta \longrightarrow^*_{F'} N'$. Then I''^{+} extends uniquely to $\hat{I}''$ with: $N_{\eta+1} = N', \pi''_{\xi,\eta+1} = \pi''$.

Set: $\pi' =: \pi'' \upharpoonright M'^{+}_\eta$. Then:

$$\pi' : M'^{*}_\eta \longrightarrow_{\Sigma^*} M'$$

where:

$$M' = \begin{cases} \pi''(M'^{*}_\eta) & \text{if } M'^{*}_\eta \in N^*_\eta \\ M' & \text{if not} \end{cases}$$

Then $\text{crit}(\pi') = \kappa'_\nu$ and $F' = E^{M'_\eta}_{\nu'_\eta}$. Hence by Lemma 3.6.38, $\langle I, I' \rangle$ extends to a mirror $\langle \hat{I}, \hat{I}' \rangle$ of length $\eta + 2$ with: $M' = M'_{\eta+2}$. Obviously, $\hat{I}''$ backs $\langle \hat{I}, \hat{I}' \rangle$.

QED(Lemma 3.6.43)

Note. If $M'^{*}_\eta \in N^*_\eta$, then $\langle \pi', M' \rangle$ is not necessarily an ultraproduct of $\langle M'^{*}_\eta, F' \rangle$.

Using Lemma 3.6.37 we also get:

Lemma 3.6.44. *Let $\langle I, I' \rangle$ be a mirror pair of limit length η which is backed by I''. Let b be a well founded cofinal branch in I''. Then $\langle I, I' \rangle$ extend uniquely to $\langle \hat{I}, \hat{I}' \rangle$ of length $\eta + 1$ such that $b = \hat{T}``\{\eta\}$. Moreover I'' extends uniquely to $\hat{I}''$ which backs $\langle \hat{I}, \hat{I}' \rangle$.*

The proof is straightforward and is left to the reader.

But by the same lemmata we get:

Lemma 3.6.45. *Suppose that N is normally iterable. Let $M = N||\mu$. Then M is normally α-iterable.*

Proof. Fix a successful iteration strategy S for N. We must define a strategy S^* for M. Let:

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be an iteration of M of length η. We first note:

Claim. There is at most one pair $\langle I', I'' \rangle$ such that $\langle I, I' \rangle$ is a mirror pair backed by I'' and I'' is S-conforming.

Proof. By induction on $\text{lh}(I)$. We leave this to the reader.

We now define an iteration strategy S^* for M. Let I be a normal iteration of M of limit length η. If there is no pair $\langle I', I''\rangle$ satisfying the above claim, then $S^*(I)$ is undefined. If not, we set:

$$S^*(I) =: S(I'')$$

$b = S^*(I)$ is then a cofinal well founded branch is I. (Clearly, if we extend each of I, I', I'' by the branch b, we obtain $\langle \tilde{I}, \tilde{I}', \tilde{I}''\rangle$ satisfying the Claim). It is then obvious that if I is of length $\eta + 1$ and we pick $\nu > \nu_i (i < \eta)$ such that $E_\nu^{M_\eta} \neq \varnothing$, then I extends to an S^*-conforming iteration of length $\eta + 1$. Hence S^* is successful.

QED(Lemma 3.6.45)

This is fairly weak result which could have been obtained more cheaply. We now show, however, that our methods establish Theorem 3.6.1. We begin by defining the notion of a *full mirror* I' of a full iteration I.

Definition 3.6.14. Let $I = \langle I^i : i < \mu\rangle$ be a full iteration of M, inducing M_i, π_{ij} $(i \leq j < \mu)$. Let:

$$I^i = \langle\langle M_h^i\rangle, \langle \nu_h^i\rangle, \langle \pi_{hj}\rangle, T^i\rangle$$

By a *full mirror* of I we mean $I' = \langle I'^i : i < \mu\rangle$ such that

$$I'^i = \langle\langle M_h'^i\rangle, \langle \pi_{hj}'^i\rangle, \langle \sigma_h^i\rangle, \langle \rho^{i,h}\rangle\rangle$$

is a mirror of I^i for $i < \mu$, and I' induces $\langle M_i' : i < \mu\rangle, \langle \pi_{ij}' : i \leq j < \mu\rangle, \langle \sigma_i : i < \mu\rangle, \langle \rho^i : i < \mu\rangle$ such that:

(a) $\sigma_i : M_i \longrightarrow_{\Sigma^*} M_i' \min \rho^i$

(b) π_{ij}' is a partial structure preserving map from M_i' to M_j'. Moreover, they commute and $\pi_{i,i}' = \mathrm{id} \restriction M_i'$. If $\alpha < \mu$ is a limit ordinal, then $M_\alpha' = \bigcup_{i<\alpha} \mathrm{rng}(\pi_{i,\alpha}')$.

(c) $\sigma_j \pi_{ij} = \pi_{ij}' \sigma_i$ for $i \leq j < \mu$.

(d) If $i \leq j < \mu$ and $[i, j)$ has no drop point in I, then:

$$\pi_{ij}' : M_i' \longrightarrow_{\Sigma^*} M_j' \text{ and } \pi_{ij}' \text{“}\rho^i \subset \rho^i \leq \pi_{ij}'(\rho^i)$$

(e) $M_0' = M_0 = M; \sigma_0 = \mathrm{id} \restriction M$, and

$$\rho^0 = \langle \rho_M^n : n < \omega\rangle$$

(f) $M_{i+1}' = M_{l_i}'^i$ where I^i has length $l_i + 1$. Moreover, $\sigma_{i+1} = \sigma_{l_i}^i$ and $\rho^{i+1} = \rho^{i,l_i}$ and $\pi_{i,i+1} = \pi_{i,l_i}^i$.

We leave it to a reader to see that $\langle M_i : i < \mu\rangle, \langle \pi'_{ij} : i \leq j < \mu\rangle, \langle \sigma_i : i < \mu\rangle$ are uniquely characterized by (a)-(f), given the triple $\langle M, I, I'\rangle$. In particular if $\alpha < \mu$ is a limit ordinal, then:

$$M'_\alpha, \langle \pi'_{i\alpha} : i < \alpha\rangle$$

is the transitivized direct limit of

$$\langle M'_i : i < \alpha\rangle, \langle \pi'_{ij} : i \leq j < \alpha\rangle.$$

(This makes sense by (d), since I has only finitely drop points $i < \alpha$). σ_α is then defined by: $\sigma_\alpha \pi_{i\alpha} = \pi'_{i\alpha}\sigma_i$. By the method of §3.6.2 it follows that there is only one ρ^α satisfying our conditions and that, in fact, for sufficiently large $i < \alpha$ we have:

$$\rho^\alpha_n = \pi'_{i\alpha}(\rho^i_n) \text{ for } i < \omega.$$

$\langle I, I'\rangle$ is then called a *full mirror pair*.

We leave to the reader to verify:

Lemma 3.6.46. *Let $\langle I, I'\rangle$ be a full mirror pair of limit length μ. Suppose further, that, if $[i_o, \mu)$ has no drop point, then:*

$$\langle M'_i : i_0 \leq i < \mu\rangle, \langle \pi'_{ij} : i_0 \leq i \leq j < \mu\rangle$$

has a well founded limit. Then $\langle I, I'\rangle$ extends uniquely to a mirror pair of length $\mu + 1$.

We recall that a full iteration $I = \langle I^i : i < \mu\rangle$ is called *smooth* iff $M_i = M^i_0$ for all $i < \mu$. We define:

Definition 3.6.15. Let $I = \langle I^i : i < \mu\rangle$ be a full iteration of M. Let $\langle I, I'\rangle$ be a full mirror pair. Let:

$$I'' = \langle I''^i : i < \mu\rangle$$

be a smooth iteration of M inducing

$$\langle M''_i : i < \mu\rangle, \langle \pi''0_{ij} : i \leq j < \mu\rangle$$

such that $M'^i_0 \lhd M'_i \lhd M''_i$ and I''^i backs $\langle I^i, I'^i\rangle$ for $i < \mu$.

We then say that I'' *backs* $\langle M, I, I'\rangle$.

It is obvious that, if I'' backs $\langle M, I, I'\rangle$ then I'' is uniquely determined by $\langle M, I, I'\rangle$. Building on the last lemma we get:

Lemma 3.6.47. *Let $\langle I, I'\rangle$ be a full mirror pair of limit length μ. Let I'' be a smooth iteration of M of length $\mu+1$, such that $I''|\mu$ backs $\langle M, I, I'\rangle$. Then $\langle I, I'\rangle$ extends uniquely to a pair of length $\mu+1$ which is backed by I''.*

Proof. (Sketch). The extension is easily defined using Lemma 3.6.46 if we can show:

Claim. I has finitely many drop points.

We first note that if I^i has a truncation on the main branch, then so do I'^i and I''^i. Hence there are only finitely many such I^i. Now suppose that $M_0^i \neq M_i$ for infinitely many i. Let $\langle i_n : n < \omega\rangle$ be a monotone sequence of such i such that $[i_n, i_{n+1})$ has no drop. Then, letting $M_i' = M''_{i_n}||\mu_n$ for $n < \omega$, we have: $\mu_{n+1} < \pi''_{i_n, i_{n+1}}(\mu_n)$.

Hence $\pi''_{i_{n+1},\mu}(\mu_{n+1}) < \pi''_{i_n,\mu}(\mu_n)$. Contradiction!

QED(Lemma 3.6.47)

Now let S be a successful smooth iteration strategy for M. (Thus S is defined only on smooth iterations $I = \langle I^i : i \leq \eta\rangle$ such that I^η is a normal iteration of limit length. $S(I)$, if defined, is then a well founded cofinal branch b in I^η. We call S *successful* for M iff every S-conforming smooth iteration I of M can be extended in an M-conforming manner. (This is defined precisely in §3.5.2).).

Claim. Let I be a full iteration of M. There is at most one pair $\langle I', I''\rangle$ such that $\langle I, I'\rangle$ is a full mirror pair, I'' backs $\langle I, I'\rangle$ and is S-conforming.

Proof. By induction on $\mathrm{lh}(I)$ and for $\mathrm{lh}(I) = i+1$ by induction on $\mathrm{lh}(I^i)$. The details are left to the reader.

We now define a full iteration of length $i+1$ where I^i is of limit length. If there exist $\langle I', I''\rangle$ as in the above claim, we set $S^*(I) = S(I'')$. If not, then $S^*(I)$ is undefined. It follows as before that an S^*-conforming full iteration of M can be properly extended in any permissible way to an S^*-conforming iteration. More precisely:

- If I is of length $i+1$ and I^i is of limit length, then $S^*(I)$ exists.
- If I is of length $i+1$ and I^i is of successor length $j+1$ and $\nu > \nu_h^i$ for $h < j$, where $E_\nu^{M_\nu^i} \neq \varnothing$, then I extends to and S^*-conforming $\hat{I}$, $\hat{I}_i$ extends I^i and $\nu_j = \nu$ in $\hat{I}^i$.
- If I, i, j are as before and $\tilde{M} \triangleleft M_j^i$, then I extends to an S^*-conforming $\hat{I}$ of length $i+1$ such that $\tilde{M} = M_0^{i+1}$.

- If I is of limit length μ, then it extends uniquely to an S^*-conforming iteration of length $\mu + 1$.

QED(Theorem 3.6.1)

3.7 Smooth Iterability

In this section we prove Theorem 3.7.29. This will require a deep excursion into the combinatorics of normal iteration, using methods which were manly developed by John Steel and Farmer Schlutzenberg. We first answer a somewhat easier question: Let M be uniquely normally iterable and let M' be a normal iterate of M. Is M' normally iterable? Our basis tool in dealing with this is the *reiteration*: Given a normal iteration I' from M' to M'', we "reiterate" I, gradually turning it into a normal iteration I^* to an M^*. The process of reiteration mimics the iteration I'. This results in an embedding σ from M'' to M^*, thus showing that M'' is well-founded. However, σ is not necessarily Σ^*-preserving but rather Σ^*-preserving *modulo pseudoprojecta*. This means that, in order to finish the argument, we must draw on the theory of pesudoprojecta developed in §3.6. The above result is proven in §3.7.3. The path from this result to Lemma 3.7.29 is still arduous, however. It is mainly due to Schluzenberg and employs his original and surprising notion of "inflation". In order to complete the argument (in §3.7.6) we again need recourse to pseudo projecta. The remaining subsections (§3.7.1, §3.7.2, §3.7.4, §3.7.5) can be read with no knowledge of pseudoprojecta, and are of some interest in their own right.

We begin by describing a class of operations on normal iteration called *insertions*. An insertion embeds or "inserts" a normal iteration into another one.

3.7.1 Insertions

Let I be a normal iteration of M of length η. Let I' be a normal iteration of the same M having length η'. An *insertion* of I into I' is a monotone function $e : \eta \longrightarrow \eta'$ such that $E^{M_i}_{\nu_i}$ plays the same role in M_i as $E^{M'_{e(i)}}_{\nu'_{\tilde{e}(i)}}$ in $M'_{\tilde{e}(i)}$. (This is far from exact, of course, but we will shortly give a proper definition).

In one form or other, insertions have long played a role in set theory. They are implicit in the observation that iterating a single normal measure produces a sequence of indiscernibles. This situation typically arises when we have a transitive ZFC^- model M and a $\kappa \in M$ which is measurable in M with a normal ultrafilter $U \in M$. Assume that we can iterate M by U, getting:

$$M_i, \kappa_i, U_i, \pi_{i,j} : M_i \prec M_j \ \ (i \leq j < \infty),$$

where the maps $\pi_{i,j}$ are commutative and continuous at limits, $\kappa_i = \pi_{0i}(\kappa), U_i = \pi_{0i}(U)$ and:

$$\pi_{i,i+1} : M_i \longrightarrow_{U_i} M_{i+1}$$

Now let $e : \eta \longrightarrow \infty$ be any monotone function on an ordinal η. e is then an *insertion*, inducing a sequence $\langle \sigma_i : i < \eta \rangle$ of *insertion maps* such that $\sigma_i : M_i \prec M_{e(i)}$. To define there maps we first introduce an auxiliary function $\hat{e}$ defined by:

$$\hat{e}(i) =: \inf\{e(h) : h < i\}$$

Thus $\hat{e}$ is a normal function and $\hat{e}(0) = 0$.

By induction on $i < \eta$ we then define maps $\hat{\sigma}_i, \sigma_i$ as follows: We verify inductively that:

$$\hat{\sigma}_i : M_i \prec M_{\hat{e}(i)} \text{ and } \hat{\sigma}_i \bar{\pi}_{hi} = \pi_{\hat{e}(h),\hat{e}(i)} \hat{\sigma}_h$$

Since $\hat{e}(0) = 0$, we set: $\hat{\sigma}_0 = \mathrm{id} \restriction M$. If σ_i is given, we know that $\hat{e}(i) \leq e(i)$ and hence define: $\tilde{\sigma}_i = \pi_{\hat{e}(i),e(i)} \hat{\sigma}_i$. Now let $i+1 < \eta$. Then $\hat{e}(i+1) = e(i)+1$. We know that each element of M_{i+1} has the form $\pi_{i,i+1}(f)(\kappa_i)$. Hence we can define $\hat{\sigma}_{i+1}$ by:

$$\hat{\sigma}_{i+1}(\pi_{i,i+1}(f)(\kappa_i)) = \pi_{e(i),\hat{e}(i+1)}(\sigma_i(f))(\sigma_i(\kappa_i)).$$

Finally, if $\lambda < \eta$ is a limit, then $\hat{e}(\lambda) = \mathrm{lub}\{e(i) : i < \lambda\}$, and we can define $\hat{\sigma}_\lambda$ by:

$$\hat{\sigma}_\lambda \pi_{h\lambda} = \pi_{\hat{e}(h),\hat{e}(\lambda)} \hat{\sigma}_h \text{ for } h < \lambda$$

This completes the construction. The fact that $\langle u_h : h < i \rangle$ is a sequence of indiscernibles for M_i is proven by using insertions defined on finite η.

This was a simple example, but insertions continue to play a role in the far more complex theory of mouse iterations. We define the appropriate notion of insertion as follows:

Let:

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a normal iteration of M of length η. Let

$$I' = \langle \langle M'_i \rangle, \langle \nu'_i \rangle, \langle \pi'_{ij} \rangle, T' \rangle$$

be a normal iteration of the same M of length η'. Suppose that

$$e : \eta \longrightarrow \eta'$$

is monotone. Define an auxiliary function $\hat{e}$ by:

$$\hat{e}(i) =: \text{lub}\{e(h) : h < i\} \text{ for } i < \eta$$

Then $\hat{e}$ is a normal function and $\hat{e}(0) = 0$. We call e an *insertion* of I into I' iff there is a sequence $\langle \hat{\sigma}_i : i < \eta \rangle$ of *insertion maps* with the following properties:

(a) $\hat{\sigma}_i : M_i \longrightarrow_{\Sigma^*} M_{\hat{e}(i)}, \hat{\sigma}_0 = \text{id}$.

(b) $i \leq_T j \longleftrightarrow \hat{e}(i) \leq_{T'} \hat{e}(j)$. Moreover:

$$\hat{\sigma}_j \pi_{ij} = \pi'_{\hat{e}(i),\hat{e}(j)} \circ \hat{\sigma}_i, \text{ for } i \leq_T j.$$

(c) $\hat{e}(i) \leq_{T'} e(i)$ for $i < \eta'$.

Before continuing the definition, we introduce some notation. Set:

$$\pi_i = \pi'_{\hat{e}(i),e(i)},\ \sigma_i = \pi_i \hat{\sigma}_i \text{ for } i < \eta$$

We further require

(d) $\sigma_i(\nu_i) = \nu'_{e(i)}$ for $i + 1 < \eta$. More precisely, one of the following holds:

- $\nu_i \in M_i \wedge \hat{\sigma}(\nu_i) \in \text{dom}(\pi_i) \wedge \nu'_{e(i)} = \sigma_i(\nu_i)$
- $\nu_i \in M_i \wedge \text{dom}(\pi_i) = M'_{\hat{e}(i)} || \hat{\sigma}(\nu_i) \wedge \nu'_{e(i)} = \text{On} \cap M'_{\hat{e}(i)}$
- $\nu_i = \text{On} \cap M_i \wedge \text{dom}(\pi_i) = M'_{\hat{e}(i)} \wedge \nu'_{e(i)} = \text{On} \cap M'_{\hat{e}(i)}$

(e) $\hat{\sigma}_i \restriction \lambda_l = \sigma_l \restriction \lambda_l$ for $l < i < \eta$.

This completes the definition.

Note. The insertion maps $\hat{\sigma}_i, \sigma_i$ are uniquely determined by e, but we have yet to prove this fact.

Note. The map $\hat{\sigma}_i$ is total on M_i, but σ_i could be partial.

Note. We shall often write $\hat{e}_i, e_i$ for $\hat{e}(i), e(i)$.

Note. $e, \hat{e}$ are order preserving, and $\hat{e}$ takes $<_T$ to $<_{T'}$. On the other hand, $i <_T j$ does not imply $e_i <_T e_j$, although we have:

$$i <_T j \longrightarrow \hat{e}_i <_{T'} e_j \text{ and } e_i <_{T'} e_j \longrightarrow i <_T j.$$

Definition 3.7.1. The *identical insertion* is $\mathrm{id} \upharpoonright \eta$, with $\hat{\sigma}_i = \sigma_i = \mathrm{id} \upharpoonright M_i$ for $i < \eta$.

We write $\sigma_i(\nu_i)$ as an abbreviation for ν'_{e_i} when $\nu_i = \mathrm{On} \cap \mathrm{dom}(\sigma_i)$.
Note. We use the familiar abbreviation:

$$\kappa_i = \mathrm{crit}(E^{M_i}_{\nu_i}), \lambda_i = E^{M_i}_{\nu_i}(\kappa_i), \tau_i = \kappa_i^{+J^{E^{M_i}}_{\nu_i}}$$

for $i + 1 < \eta$. Similarly for $\kappa'_i, \lambda'_i, \tau'_i$ $(i + 1 < \eta')$. It follows that:

$$\sigma_i(\kappa_i) = \kappa'_{e_i}, \sigma_i(\lambda_i) = \lambda'_{e_i}, \sigma_i(\tau_i) = \tau'_{e_i}.$$

We then have:

- $\sigma_i \upharpoonright (M_i || \nu_i) \longrightarrow_{\Sigma^*} M'_{e_i} || \nu'_{e_i}$.

Note. By (e) we have:

$$n < i \longrightarrow \hat{\sigma}_i \upharpoonright J^{E^{M_i}}_{\lambda_n} = \sigma_n \upharpoonright J^{E_{M_n}}_{\lambda_n}.$$

To see this, let:

$$J^E_\lambda = J^{E_{M_n}}_{\lambda_n} = J^{E_{M_i}}_{\lambda_n} \text{ (since } n < i).$$

Similarly let:

$$J^{E'}_{\lambda'} = J^{E_{M'_{e_n}}}_{\lambda'_{e_n}} = J^{E_{M'_{\hat{e}_i}}}_{\lambda'_{e_n}} \text{ (since } e_n < \hat{e}_i).$$

Let $x \in J^E_\lambda$. Then there is a limit ordinal $\alpha < \lambda$ and a $\beta < \alpha$ such that:

$$x = \text{ the } \beta\text{-th element of } J^E_\lambda \text{ in } <^E_\alpha,$$

where $<^E_\alpha$ is the canonical well ordering of J^E_α. Let $\hat{\sigma}_i(\alpha) = \sigma_h(\alpha) = \alpha'$, $\hat{\sigma}_i(\beta) = \sigma(\beta) = \beta'$. Then:

$$\hat{\sigma}_i(x) = \sigma_h(x) = \text{ the } \beta'\text{-th element of } J^{E'}_{\alpha'} \text{ in } <^{E'}_{\alpha'}.$$

Lemma 3.7.1. *The following hold:*

(1) $\sigma_i \upharpoonright \lambda_n = \sigma_n \upharpoonright \lambda_n$ for $n \leq i < \eta$ and $n + 1 < \eta$.

Proof. This is trivial for $n = i$. Now let $n < i$. Then

$$\sigma_i \upharpoonright \lambda_n = \pi_i \hat{\sigma}_i \upharpoonright \lambda_n = \pi_i \circ (\sigma_i \upharpoonright \lambda_n).$$

Hence it suffices to prove:

Claim. $\pi_i \upharpoonright \lambda'_{e_n} = \mathrm{id}$ since $\xi < \lambda_n \longrightarrow \sigma_n(\xi) < \sigma_n(\lambda_n) = \lambda'_{e_n}$.

Proof. If $\hat{e}_i = e_i$, then $\pi_i = \mathrm{id} \upharpoonright M_{\hat{e}_i}$, where $\lambda'_{e_n} < \lambda'_{e_i} \in M_{e_i}$. Now let $\hat{e}_i < e_i$. There is a least j such that $\hat{e}_i <_{T'} (j + 1) \leq_{T'} e_i$. Let $\zeta = T'(j + 1)$. Then $\mathrm{crit}(\pi_i) = \kappa'_j \geq \lambda'_{e_n}$, since $e_n < e_i \leq j$.

QED(1)

(2) *Let* $\zeta = T(i+1)$. *Then* $\kappa'_{e_i} < \lambda'_{e_\zeta}$.

Proof. $\kappa'_{e_i} = \sigma_i(\kappa_i) = \sigma_\zeta(\kappa_i) < \sigma_\zeta(\lambda_\zeta) = \lambda'_{e_\zeta}$, since $\zeta \leq i$ and $\kappa_i < \lambda_\zeta$

QED(2)

(3) *Let* $\zeta = T(i+1), \zeta' = T'(e_i+1)$. *Then* $\hat{e}_\zeta \leq_{T'} \zeta' \leq e_\zeta$.

Proof. ζ' is by definition the least such that $\kappa'_{e_i} < \lambda'_{\zeta'}$. Hence $\zeta' < e_\zeta$ by (2). But $\hat{e}_\zeta <_{T'} \hat{e}_{i+1} = e_i + 1$. Hence $\hat{e}_\zeta \leq_{T'} \zeta'$.

QED(3)

Now we give the full determination of $T'(e_i+1)$.

(4) *Let* $j = \leq_{T'} e_\zeta$ *be least such that* $\pi'_{j,e_\zeta} \upharpoonright \kappa'_{e_i} = \mathrm{id}$. *Then* $j = T'(e_i+1)$.

Proof.

Claim 1. $\kappa'_{e_i} < \lambda'_j$.

Proof. Suppose not. Then $j < e_\zeta$ since $\kappa'_{e_i} < \lambda'_{e_\zeta}$. Set: $\kappa = \mathrm{crit}(\pi_{k,e_\zeta})$. Clearly $\kappa < \lambda'_j$, since I is a normal iteration. Thus $\kappa'_{e(i)} \leq \kappa < \lambda'_j$. Contradiction!

QED(Claim 1)

Claim 2. $\kappa'_{e_i} \geq \lambda_n$ for $n < j$.

Proof. If $j = \hat{e}_\zeta$, then $j = T(e_i+1)$ by (3) and Claim 1. The conclusion is then obvious. Now let $j > e_\zeta$. Thus $j = \mathrm{lub}\, A$ where:

$$A = \{n \mid \hat{e}_\zeta <_{T'} n+1 \leq_{T'} j\}.$$

Hence it suffices to show: **Claim 2.** $\kappa'_{e_i} \geq \lambda'_n$ for $n \in A$.

Suppose not. Let $n \in A$ be a counterexample. Let $\tau = T'(n+1)$. Then $\hat{e}_\zeta \leq_{T'} \tau$. Hence:

$$\mathrm{ran}(\pi'_{\hat{e}_\zeta,n+1}) \subset \mathrm{ran}(\pi'_{\tau,n+1}).$$

But $\kappa'_{e_i} = \sigma_i(\kappa_i) = \sigma_\zeta(\kappa_i) = \pi_i\hat{\sigma}_\zeta(\kappa_i)$. Hence:

$$\kappa'_{e_i} \in \mathrm{ran}(\pi_\zeta) \text{ where } \mathrm{crit}(\pi_{n+1,e_\zeta}) \geq \lambda'_n > \kappa'_{e_i}.$$

Hence $\kappa'_{e_i} \subset \mathrm{ran}(\pi_{\hat{e}_\zeta,n+1}) \subset \mathrm{ran}(\pi_{\tau,n+1})$, where

$$[\kappa'_n, \lambda'_n) \cap \mathrm{ran}(\pi_{\tau,n+1}) = \emptyset.$$

Hence $\kappa'_{e_i} < \kappa'_n$ and $\pi_{\tau,e_\zeta} \upharpoonright \kappa'_{e_i} = \mathrm{id}$, where $\hat{e}_\zeta \leq_{T'} \tau <_{T'} j$. Contradiction!

QED(4)

Definition 3.7.2. Let $\xi = T(i+1)$. We set:

$$e^*_i = T'(e_i+1),\ \pi^*_i = \pi'_{\hat{e}_\xi,e^*_i},\ \sigma^*_i = \pi^*_i\hat{\sigma}_\xi$$

The following are then obvious:

(5) $M'^{*}_{e_i} = M'_{e^*_i}||\mu$, where μ is maximal such that τ'_{e_i} is a cardinal in $M'_{e^*_i}||\mu$.

(6) $\sigma^*_i \upharpoonright M^*_i : M^*_i \longrightarrow_{\Sigma^*} M'^{*}_{e_i}$.

Note. If $M^*_i = M_\xi$, then τ_i is a cardinal in M_ξ. Hence $\hat{\sigma}_\xi(\tau_i)$ is a cardinal in $M'_{\hat{e}_\xi}$ and $\tau'_{e_i} = \pi^*_i \hat{\sigma}_\xi(\tau_i)$ is a cardinal in $M'_{e^*_i} = M'^{*}_{e_i}$. If $M^*_i \in M_\xi$, then $\hat{\sigma}_\xi(M^*_i) \in M'_{\hat{e}_\xi}$ and $\pi^*_i \upharpoonright \hat{\sigma}_\xi(M^*_i) : \hat{\sigma}_\xi(M^*_i) \longrightarrow_{\Sigma^*} M'^{*}_{e_i}$. (However, we cannot conclude that $M'^{*}_{e_i} \in M'_{e_i}$). Hence:

(7) Let $\xi = T(i+1)$. $\pi_{\xi,i+1}$ is a total function on M_ξ iff $\pi'_{\hat{e}_\xi, e_{i+1}}$ is total on $M'_{\hat{e}_\xi}$.

Hence, there is a drop point in $(\alpha, \beta]_T$ iff there is a drop point in $(\hat{e}_\alpha, e_\beta]_{T'}$.

(8) $\hat{\sigma}_{i+1}\pi_{\xi,i+1} = \pi'_{e^*_i, e_i+1}\sigma^*_i$, where $\xi = T(i+1)$.

Proof. $\hat{\sigma}_{i+1}\pi_{\xi,i+1} = \pi'_{\hat{e}_\xi, \hat{e}_{i+1}}\hat{\sigma}_\xi = \pi'_{e^*_i, e_{i+1}}\pi^*_i\hat{\sigma}_\xi = \pi_{e^*{}_i, e_{i+1}}\sigma^*{}_i$. QED(8)

(9) $\sigma_i(X) = \sigma^*_i(X)$ *for* $X \in \mathbb{P}(\kappa_i) \cap M^*_i$.

Proof.$\sigma_i(X) = \sigma_\xi(X)$ where $\xi = T(i+1)$, since $X \in J^{E^{M_\eta}}_{\lambda_\xi}$ and $\sigma_i \upharpoonright \lambda_\xi = \sigma_\xi \upharpoonright \lambda_\xi$ by (1). But $\sigma_\xi(X) = \pi'_{\hat{e}_\xi, e_\xi}\hat{\sigma}_\xi(X) = \pi'_{e^*_i, e_\xi}\sigma^*_i(X)$, since $\pi'_{e^*_i e_\xi} \upharpoonright \kappa_{e_i} + 1 = \mathrm{id}$.

QED(9)

Using notation from §3.2, then we have:

(10) $\langle \sigma^*_i \upharpoonright M^*_i, \sigma_i \upharpoonright \lambda_i \rangle : \langle M^*_i, F \rangle \longrightarrow \langle M'^{*}_{e_i}, F' \rangle$ where $F = E^{M_i}_{\nu_i}, F' = E^{M'_{e_i}}_{\nu_{e_i}}$.

Proof. $\alpha \in F(X) \longleftrightarrow \sigma_i(\alpha) \in \sigma_i(F(X)) = F'(\sigma^*_i(X))$ by (6) and (9).

QED(10)

But we are now, at last, in a position to prove:

(11) *The sequence* $\langle \hat{\sigma}_i : i < \eta \rangle$ *of insertion maps is uniquely determined by* e. *(Hence so is* $\langle \sigma_i : i < \eta \rangle$*, since* $\sigma_i = \pi'_{\hat{e}_i, e_i} \circ \hat{\sigma}_i$*).*

Proof. Suppose not. Let $\langle \hat{\sigma}'_i : i < \eta \rangle$ be a second such sequence. By induction on i we prove that $\hat{\sigma}_i = \sigma'_i$. For $i = 0$ this is immediate. Now let $\hat{\sigma}_i = \sigma'_i$. We must show that $\hat{\sigma}_{i+1}$ is unique. Let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n_{M_i}$. By Lemma 3.2.19 of §3.2, we know that there is at most one σ such that

$$\sigma : M_i \longrightarrow_{\Sigma^{(n)}_0} M'_{e_i},\ \sigma\pi_{\xi,i+1} = \pi'_{e^*_i \hat{e}_{i+1}}\sigma^*_i,\ \sigma \upharpoonright \lambda_i = \sigma_i \upharpoonright \lambda_i$$

Hence $\hat{\sigma}_{i+1} = \sigma'_{i+1} = \sigma$ by (8).

Now let $\mu < \eta$ be a limit ordinal. Then $\hat{\sigma}_\mu = \sigma'_\mu$ is the unique $\sigma : M_\mu \longrightarrow M'_{\hat{e}_\mu}$ defined by: $\sigma\pi_{i,\mu} = \pi'_{\hat{e}_i,\hat{e}_\mu}\hat{\sigma}_i$ for $i <_{T'} \mu$.

QED(11)

We also note:

(12) *Let* $\xi = T(i+1)$. *Then* $\pi'_{e_i^*,e_\xi} \restriction (\tau'_i + 1) = \mathrm{id}$.

(Hence $\sigma_i^* \restriction (\tau_i + 1) = \sigma_\xi \restriction (\tau_i + 1) = \sigma_i \restriction (\tau_i + 1)$.

Proof. If $e_i^* = e_\xi$, this is immediate. Now let $e_i^* < e_\xi$. Set $\pi' = \pi'_{e_i^*,e_\xi}$. Then $\kappa'_{e_i} < \tilde{\kappa} = \mathrm{crit}(\pi')$ where $\tilde{\kappa}$ is inaccessible in M'_{e_ξ}. Hence $\tau'_{e_i+1} < \tilde{\kappa}$, since $\tau'_{e_i} = (\kappa'_{e_i})^+$ in M'_{e_ξ}. QED(12)

(13) $\hat{\sigma}_{i+1}(\nu_i) = \nu'_{e_i}$.

Proof. Let $\xi = T(i+1)$. Then:

$$\begin{aligned}\hat{\sigma}_{i+1}(\nu_i) = \hat{\sigma}_{i+1}\pi_{\xi,i+1}(\tau_i) &= \pi'_{e_i^*,e_i+1}\sigma_i^*(\tau_i)\\ &= \pi'_{e_i^*,e_i+1}(\tau'_{e_i}) = \nu'_{e_i}\end{aligned}$$

since $\tau'_{e_i^*} = \sigma_i(\tau_i) = \sigma_i^*(\tau_i)$. QED(13)

Hence:

(14) $j \geq i+1 \longrightarrow \sigma_j(\nu_i) \geq \nu'_{e_i}$.

Proof. By (13) it holds for $j = i+1$. Now let $j > i+1$. Then $\kappa_i < \lambda_{i+1}$ and

$$\hat{\sigma}_j(\nu_j) = \sigma_{i+1}(\nu_i) \geq \sigma_i(\nu_i) = \nu'_{e_i}.$$

QED(14)

We also note:

(15) $e_i <_{T'} e_j \longrightarrow i \leq_T j$.

Proof. Since $e_i < \hat{e}_j$ and $\hat{e}_j \leq_T e_j$, we conclude:

$$\hat{e}_i \leq_{T'} e_i <_{T'} \hat{e}_j\text{; hence } i <_T j.$$

QED(15)

Extending insertion

Given an insertion e of I into I', when can we turn it into an e' which inserts an extension $\tilde{I}$ of I into an extension $\tilde{I}'$ of I'? Some things are obvious:

(16) *If e inserts I into I' and I'' extends I', then e inserts I into I''.*

(17) *If e inserts I of length $\nu+1$ into I' and $e(\nu) \leq_{T'} j$ in I', there is a unique e' inserting I into I' such that $e' \upharpoonright \nu = e \upharpoonright \nu$ and $e'(\nu) = j$.*

(18) *Let I be of limit length ν and let e insert I into I' of length $\nu' = \operatorname{lub} e``\nu$. Suppose that $b'$ is a cofinal well founded branch in $I'$ and $b = e^{-1}``b'$ is cofinal in I. Extend I' into $\tilde{I}$ of length $\eta+1$ by setting $T``\{\eta\} = b$. Extend $I'$ to $\hat{I}'$ of length $\eta'+1$ by: $T'``\{\eta\} = b'$. Then e extends uniquely to an insertion $\tilde{e}$ of $\tilde{I}$ into $\tilde{I}'$ with $\tilde{e}(\eta) = \eta'$.*

The proof is left to the reader.

These facts are obvious. The following lemma seems equally obvious, but its proof is rather arduous:

Lemma 3.7.2. *Let e insert I into I' where I is of length η and I' is of length $\eta'+1$, where $\eta' = e(\eta)$. Extend I to a potential iteration of length $\eta+2$ by appointing ν_η such that $\nu_\eta > \nu_i$ for $i < \eta$. Suppose $\sigma_\eta(\nu_\eta) > \nu'_j$ for all $j < \eta'$. Then we can extend I' to a potential iteration of length $\eta'+2$ by appointing: $\nu'_{\eta'} = \sigma_\eta(\nu_\eta)$. This determines $\xi = T(\eta+1)$, $e^*_\eta = T'(\eta'+1)$ and $M^*_i, M'^*_{e_i}$. If M'_{e_i} is $*$-extendible by $F = E^{M_i}_{\nu_i}$, then e extends uniquely to an $\tilde{e}$ inserting $\tilde{I}$ into $\tilde{I}'$, where $\tilde{I}'$ is an actual extension of I by ν_η and $\tilde{I}'$ is an actual extension of I' by $\nu'_{\eta'}$.*

Using Lemma 3.2.23 of §3.2 we can derive Lemma 3.7.2 from:

Lemma 3.7.3. *Let $e, I, I', \nu_\eta, \nu_{\tilde{e}_\eta}, M^*_\eta, M'^*_{\tilde{e}_i}, F, F'$ be as above. Then*

$$\langle \sigma^*_\eta, \sigma_\eta \upharpoonright \lambda_\eta \rangle : \langle M^*_\eta, F \rangle \longrightarrow^* \langle M'^*_{\tilde{e}_\eta}, F' \rangle$$

We first show that Lemma 3.7.3 implies Lemma 3.7.2. Since $M'^*_{e_\eta}$ is $*$-extendible by F' we can extend I' by setting:

$$\hat{\pi}'_{e^*_\eta, e_\eta+1} : M'^*_{\sigma e_\eta} \longrightarrow^*_{F'} M'_{e_\eta+1}$$

It follows that F is close to M^*_i; hence we can set:

$$\hat{\pi}_{\xi,\eta+1} : M^*_\eta \longrightarrow^* M_{\eta+1}$$

But by Lemma 3.2.23 there us a unique

$$\sigma : M_{\eta+1} \longrightarrow_{\Sigma^*} M_{\tilde{e}_\eta+1}$$

such that $\sigma\pi_{\xi,\eta+1} = \pi'_{e^*_\eta, \tilde{e}_\eta+1}\sigma^*_\eta$ and $\sigma \upharpoonright \lambda_\eta = \sigma_\eta \upharpoonright \lambda_\eta$. Extend e to $\tilde{e}$ by: $\tilde{e}(\eta+1) = e_\eta + 1$. The $\tilde{e}$ satisfies the insertion axioms with $\sigma_{\eta+1} = \sigma$.

QED(Lemma 3.7.2)

We derive Lemma 3.7.3 from an even stronger lemma:

Lemma 3.7.4. *Let I, I' be as above. Let $A \subset I_\eta$ be $\Sigma_1(M_\eta||\nu_\eta)$ in a parameter p and let $A' \subset \tau'_{e_\eta}$ be $\Sigma_1(M_{e_\eta}||\nu'_{e_\eta})$ in $p' = \sigma_\eta(p)$ by the same definition. Then A is $\Sigma_1(M^*_\eta)$ in a parameter q and A' is $\Sigma_1(M'^*_{e_\eta})$ in $q' = \sigma^*_\eta(q)$ by the same definition.*

We first show that this implies Lemma 3.7.3. Repeating the proof of Lemma 3.7.1(7), we have:

$$\langle \sigma^*_\eta \upharpoonright M^*_\eta, \tilde{\sigma}_\eta \upharpoonright \lambda_\eta \rangle : \langle M^*_\eta, F \rangle \longrightarrow \langle M'^*_{e_\eta}, F' \rangle$$

where $F = E^{M_\eta}_{\nu_\eta}, F' = E^{M'_{e_\eta}}_{\nu'_{e_\eta}}$.

We can code F_α by an $\tilde{F} \subset \tau_\eta$ such that F_α is rudimentary in $\tilde{F}$ and $\tilde{F}$ is $\Sigma_i(M_\eta||\nu_\eta)$ in α, τ_η. Coding $F'_{\alpha'}$ the same way by $\tilde{F}'$, we find that $\tilde{F}'$ is $\Sigma_1(M_{e_\eta}|\nu_{e_\eta})$ in α', τ'_{e_η} by the same definition, where $\sigma_\eta(\alpha) = \alpha', \sigma_\eta(\tau_\eta) = \tau'_{e_\eta}$. Hence by Lemma 3.7.4, $\tilde{F}'$ is $\Sigma_1(M'^*_\eta)$ in a q and $\tilde{F}'$ is $\Sigma_1(M'^*_{e_\eta})$ in $q' = \sigma^*_\eta(q)$ by the same definition. Hence F_α is $\Sigma_1(M'^*_\eta)$ in q and $F'_{\alpha'}$ is $\Sigma_1(M'^*_{e_\eta})$ in $q' = \sigma^*_\eta(q)$ by the same definition.

QED(Lemma 3.7.3)

Note. We are in virtually the same situation as in §3.2, where we needed to prove the extendability of the triples we called *duplications*. Lemma 3.7.2 corresponds to the earlier Lemma 3.4.17 and Lemma 3.7.4 corresponds to Lemma 3.4.20.

We now turn to the proof of Lemma 3.7.4. Its proof will be patterned on that of Lemma 3.4.20, which, in turns, we patterned on the proof of Lemma 3.4.4.

Our proof will be rather fuller than that of Lemma 3.4.20, however, since we will face some new challengers.

Suppose Lemma 3.7.4 to be false. Let I, I' be a counterexample with $\eta = \text{lh}(I)$ chosen minimally. We derive a contradiction. Let $\xi = T(\eta + 1)$.

(1) $\rho^1_{M_\eta||\nu_\eta} \leq \tau_\eta$

Proof. Suppose not. Set $\rho = \rho^1_{M_\eta||\nu_\eta}, \rho' = \rho^1_{M'_{e_\eta}||\nu'_{e_\eta}}$. Then $A \in J^{E^{M_\eta}}_\rho, A' \in J^{E^{M'_{e_\eta}}}_{\rho'}$.

Moreover, "$x = A'$" is $\Sigma^{(1)}_0(M'_\eta||\nu')$ in p, τ_η and "$x = A'$" is $\Sigma^{(1)}_0(M_\eta||\nu_\eta)$ in p', τ'_{e_η} by the same definition. Hence $\sigma_\eta(A) = A'$. Since $A \in J^{E^{M_\eta}}_{\lambda_\xi}$,

$\sigma_\eta \upharpoonright \lambda_\xi = \sigma_\xi \upharpoonright \lambda_\xi$ and $M_\xi||\lambda_\xi = M_\xi||\lambda_\xi$, we have: $\sigma_\xi(A) = \sigma_\eta(A) = A'$. But $\sigma_\eta(A) = \pi'_{e^*_\eta, e_\xi}\sigma^*_\eta(A)$ where $\pi'_{e^*_\eta, e_\eta} \upharpoonright \tau'_{e_\eta} + 1 = \text{id}$ by (10). Hence $\sigma^*_\eta(A) = A'$. Hence A is $\Sigma_1(M'^*_\eta)$ in the parameter A, and A' is $\Sigma_1(M'^*_{e_\eta})$ in the parameter $A' = \sigma^*_\eta(A)$ by the same definition. Contradiction! since η was a counterexample.

(2) $\xi < \eta$.

Proof. Suppose not. Then A is $\Sigma_1(M_\eta||\nu_\eta)$ in p and A' is $\Sigma_1(M'_{e_\eta}||\nu'_{e_\eta})$ in $p' = \sigma_\eta(p)$ by the same definition. But $\sigma_\eta = \pi'_{e^*_\eta, e_\eta}\sigma^*_\eta$, since $\xi = \eta$ and:

$$\pi'_{e^*_\eta, e_\eta} \upharpoonright \tau'_{e_\eta} + 1 = \text{id}$$

Hence A' is $\Sigma_1(M_{e^*_\eta}||\nu^*)$ in $\sigma^*_\eta(p)$ by the same definition, where $\nu^* = \sigma^*_\eta(\nu_\eta)$. But $M_\eta||\nu_\eta = M^*_\eta$ since $\rho^1_{M_\eta||\nu_\eta} \leq \tau_\eta$. But $\rho^1_{M'_{e^*_\eta}||\nu^*} \leq \tau'_{e_\eta}$, since $\sigma^*_\eta \upharpoonright M^*_\eta$ takes M^*_η in a Σ^* way to $M'_{e^*_\eta}||\nu^* \bigwedge x^1(x^1 \neq \tau_\eta)$ hold in M^*_η. But then $M'^*_{e_\eta} = M'_{e^*_\eta}||\nu^*$. Hence A is $\Sigma_1(M^*_\eta)$ in p and A' is $\Sigma_1(M'_{e_\eta})$ in $\sigma^*_\eta(p)$ by the same definition. Contradiction! QED(2)

Since $\xi < \eta$ and $\tau'_{e_\eta} = \sigma_\xi(\tau_\eta)$, we have:

$$\tau'_{e_\eta} = \sigma_\eta(\tau_\eta) = \pi_\eta\hat{\sigma}_\eta(\tau_\eta) = \pi_\eta\sigma_\xi(\tau_\eta) = \pi_\eta(\tau'_{e_\eta})$$

Hence $\text{crit}(\pi_\eta) > \tau'_{e_\eta}$ if $\hat{e}_\eta \neq e_{\eta'}$. Hence A' is $\Sigma_1(M_\eta||\nu_\eta)$ in p and A' is $\Sigma_1(M'_{\hat{e}_\eta}||\nu'_{e_\eta})$ in $\hat{\sigma}_\eta(p)$ by the same definition. But then we can set $I'' = I'|e_\eta + 1$ and define e' inserting I into I'' by:

$$e_h = \begin{cases} e_h & \text{if } h < \eta \\ \hat{e}_\eta & \text{if } h = \eta \end{cases}$$

$\langle e', \eta, I, I'' \rangle$ is obviously still a counterexample to Lemma 3.7.2. Thus we may henceforth assume:

(3) $e_\eta = \hat{e}_\eta$

(4) $\nu_\eta = \mathsf{ON}_{M_\eta}$.

Proof. $\tau_\eta < \lambda_\xi$, where λ_ξ is inaccessible in M_η. Hence, if $\nu_\eta \in M_\eta$, we would have: $\rho^1_{M_\eta||\nu_\eta} \geq \lambda_\xi > \tau_\eta$, contradicting (1). QED(4)

(5) η is not a limit ordinal.

Proof. Suppose not. Let A, A', p, p' be as above. By (2), $\xi < \eta$ where $\xi = T(\eta + 1)$. By (4) $M_\eta = M_\eta||\nu_\eta$ is an active premouse. But $\sigma_\eta : M_\eta \longrightarrow_{\Sigma^*} M'_{e_\eta}$ and $\sigma_\eta(\nu_\eta) = \nu'_{e_\eta}$. Pick $l <_T \eta$ such that:

- $\text{crit}(\pi_{l,\eta}) > \lambda_\xi$,
- $\pi_{l,\eta}$ is a total map on M_l,

- $p \in \text{rng}(\pi_{l,\eta})$.

Set $\bar{p} = \pi_{l,\eta}^{-1}(p)$. Then A is $\Sigma_1(M_l)$ in $\bar{p}$ and A is $\Sigma_1(M_\eta)$ in p by the same definition. Define a potential iteration $\bar{I}$ of length $l+2$ extending $I|l+1$ by appointing: $\bar{\nu}_l =: \pi_{l,\eta}^{-1}(\nu_\eta)$. Then $\bar{M}_l = M_l||\bar{\nu}_l$. Since $\pi_{l,\eta}(\kappa_\eta) = \kappa_\eta$ it follows that $\bar{\kappa}_l = \kappa_\eta$ and $\bar{M}_l^* = M_\eta^*$. Define $\bar{e} : l+1 \longrightarrow \eta'$ by: $\bar{e} \upharpoonright l+1 = e \upharpoonright l+1, \bar{e}_{l+1} = e_\eta + 1$ (hence $\hat{\bar{e}}_l = e_\eta$). Then $\bar{e}$ inserts $\bar{I}$ into I', giving the insertion maps:

$$\bar{\sigma}_i = \sigma_i \text{ for } i < l, \bar{\sigma}_l = \sigma_\eta \pi_{l,\eta}$$

Then $\bar{\kappa}_l = \kappa_\eta$. It follows easily that $\bar{M}_l^* = M_\eta^*$ and $\bar{\sigma}_l^* = \sigma_\eta^*$. But $l < \eta$, so by the minimality of η there is a q such that A is $\Sigma_1(M_\eta^*)$ in q and A' is $\Sigma_1(M_{e_\eta}'^*)$ in $\sigma_\eta^*(q)$ by the same definition. Contradiction! QED(5)

Now let $\eta = j+1, h = T(\eta)$. Then $e_\eta = \hat{e}_\eta = e_j + 1$. We know

$$\pi_{h,\eta} \upharpoonright M_j^* : M_j^* \longrightarrow_{\Sigma^*} M_\eta = \langle J_{\nu_\eta}^E, E_{\nu_\eta} \rangle$$

Hence M_j^* has the form:

(6) $M_j^* = \langle J_\nu^E, E_\nu \rangle$ where $E_\nu \neq \varnothing$.

(7) $\tau_\eta < \kappa_j$.

Proof. $\tau_\xi \leq \kappa_j$ since $\xi < \eta = j+1$. Hence $\tau_\eta < \lambda_\eta \leq \lambda_j$. But $\tau_\eta \in \text{rng}(\pi_{h,\eta})$, where:

$$[\kappa_j, \lambda_j) \cap \text{rng}(\pi_{h,\eta}) = \varnothing$$

QED(7)

(8) $\rho_{M_j^*}^1 \leq \tau_\eta$.

Proof. Suppose not. Then $\tau_\eta = \pi_{h,\eta}(\tau_\eta) < \pi``_{h,\eta}\rho_{M_j^*}^1 \subset \rho_{M_\eta'}^1$, contradicting (1). QED(8)

Thus:

(9) $\pi_{h,\eta} : M_j^* \longrightarrow_{E_{\nu_i}} M_\eta$ is a Σ_0 ultrapower.

(10) $\sigma_j^*(\tau_\eta) = \tau_{e_\eta}'$.

Proof. $\tau_\eta < \kappa_j < \lambda_h$ by (7). Hence:

$$\tau_{e_\eta}' = \hat{\sigma}_\eta(\tau_\eta) = \sigma_h(\tau_\eta) = \pi_{e_j^*,e_h}' \sigma_j^*(\tau_\eta) = \sigma_j^*(\tau_\eta'),$$

since $\sigma_j^*(\tau_\eta) < \sigma_j^*(\kappa_j) = \kappa_{e_j}'$ and $\pi_{e_j^*,e_h}' \upharpoonright \kappa_{e_j}' = \text{id}$.

QED(10)

(11) $\rho^1_{M^*_{e_j}} = \tau'_{e_\eta}$.

Proof. $\bigwedge x^1(x^1 \neq \tau_\eta)$ holds in M^*_j by (8). But:

$$\sigma^*_j \upharpoonright M^*_j : M_j \longrightarrow_{\Sigma^*} M'^*_{e_j}$$

Hence $\bigwedge x^1(x^1 \neq \sigma^*_j(\tau_\eta))$ holds in $M'^*_{e_j}$, where $\sigma^*_j(\tau_\eta) = \tau'_{e_j}$. QED(11)

But then:

(12) $\pi'_{e^*_j, e_\eta} : M'^*_{e_j} \longrightarrow_{E_{\nu_{e_j}}} M_{e\eta}$ is a Σ_0-ultrapower.

We can now prove:

(13) A is $\Sigma_1(M^*_j)$ in an r and A' is $\Sigma_1(M'^*_{e_j})$ in $r' = \sigma^*_j(r)$ by the same definition.

Proof. Let $p = \pi_{h,\eta}(f)(\alpha)$, where $f \in M^*_j$, $\alpha < \lambda_i$. Then $p' = \pi'_{e^*_j, e_\eta}(f')(\alpha')$, where: $f' = \sigma^*_j(f), \alpha' = \tilde{\sigma}_j(\alpha)$. Let $F =: E^{M_j}_{\nu_j}, F' = E^{M'_{e_j}}_{\nu_{e_j}}$. F_α can of course be coded by an $\tilde{F} \subset \tau_j$ which is $\Sigma_1 < (M_j || \nu_j)$ in α, τ_j and F'_α is coded by an $\tilde{F}' \subset \tau'_{e_j}$ which is $\Sigma_1(M'_{e_j})$ in α', τ'_{e_j} by the same definition. By the minimality of η we can conclude: F_α is $\Sigma_1(M^*_j)$ in a parameter a and $F'_{\alpha'}$ is $\Sigma_1(M'^*_{e_j})$ in the parameter $a' = \sigma^*_j(a)$ by the same definition. Now suppose:

$$A(\mu) \longleftrightarrow \bigvee y B(\mu, y, p) \text{ and}$$
$$A'(\mu) \longleftrightarrow \bigvee y B'(\mu, y, p')$$

where B is $\Sigma_0(M_\eta)$ and B' is $\Sigma_0(M'_{e_j})$ by the same definition. Let B^* be $\Sigma_0(M^*_j)$ and B'^* be $\Sigma_0(M'^*_{e_j})$ by the same definition. Since the map $\pi = \pi_{h,\eta}$ takes M^*_j cofinally to M_η, we have:

$$\begin{aligned} A(\mu) &\longleftrightarrow \bigvee u \in M^*_j \bigvee y \in \pi(u) B(\mu, y, \pi(f)(\alpha)) \\ &\longleftrightarrow \bigvee u \in M^*_j \{\gamma < \kappa_j : \bigvee y \in u B^*(\mu, y, f(\gamma))\} \in F_\alpha \end{aligned}$$

Hence A is $\Sigma_1(M^*_j)$ in $r = \langle a, f \rangle$. By the same argument, however, A' is $\Sigma_1(M'^*_{e_j})$ in $r' = \langle a', f' \rangle$ by the same definition. QED(13)

Now extend $I|h+1$ to a potential iteration I^+ of length $h+2$ by appointing: $\nu^+_h = \pi^{-1}_{h,\eta}(\nu_\eta)$. (Hence $M^*_j = M_h || \nu^+_h$). Set: $h' = e^*_j$. Extend $I'|h'+1$ to I'^+ of length $h'+2$ by appointing: $\nu'^+_{h'} = \pi'_{h', e_\eta}(\nu'_\eta)$. (Hence $M'^*_{e_j} = M'_{h'} || \nu'^+_{h'}$). Obviously, $\sigma^*(\nu^+_h) = \nu'^+_{h'}$. Now extend $e \upharpoonright h$ to $e^+ : h+1 \longrightarrow h'+1$ by:

$$e^+_i = \begin{cases} e_i & \text{if } i < h \\ e^*_j & \text{if } i = h \end{cases}$$

Then e^+ is easily seen to insert I^+ into I'^+, giving the insertion maps:

$$\sigma_i^+ = \begin{cases} \sigma_i & \text{for } i < h \\ \sigma_j^* = \pi'_{\hat{e}_h, h'} \circ \hat{\sigma}_j & \text{for } i = h \end{cases}$$

Then $\sigma_h^+(\nu_h^+) = \nu'^+_{h'}$. We note that $\tau_h^+ = \tau_\eta, \tau'^+_{h'} = \tau'_{e_\eta}$. It follows easily that $(M_h^+)^* = M_\eta^*, (M'^+_{h'}) = M'^*_{e_\eta}$ and $(\sigma_h^+) = \sigma_\eta^*$. By the minimality of η we conclude that A is $\Sigma_1(M_\eta^*)$ and $(\sigma_h^+)^* = \sigma_\eta^*$. By the minimality of η we conclude that A is $\Sigma_1(M_\eta^*)$ in a q and A' is $\Sigma_1(M'^*_{e_\eta})$ in $\sigma_\eta^*(q)$ by the same definition. Contradiction! QED(Lemma 3.7.4)

Composing insertions

Lemma 3.7.5. *Let e insert I into I', with insertion maps $\hat{\sigma}_i^e, \sigma_i^e$. Let f insert I' into I'' with insertion maps $\hat{\sigma}_i^f, \sigma_i^f$. Then*

(i) *fe inserts I into I''*

(ii) *$\widehat{f \circ e} = \hat{f} \circ \hat{e}$.*

(iii) *$\sigma_i^{fe} = \sigma_{e_i}^f \circ e_i^e$*

(iv) *$\hat{\sigma}_i^{fe} = \hat{\sigma}_{\hat{e}_i}^f \circ \hat{\sigma}_i^e$.*

Proof. We show that $f \circ e$ satisfies the insertion axioms (a)-(e) with $\hat{\sigma}_i^{fe} = \hat{\sigma}_{e_i}^f \circ \hat{\sigma}_i^e$. In the process we shall also verify (ii), (iii). We first note:

$$\widehat{fe}(i) = \mathrm{lub}(fe)\text{"}i = \mathrm{lub}\, f\text{"}(\mathrm{lub}\, e\text{"}i) = \hat{f}\hat{e}(i)$$

Axioms (a), (b), (c) then follow trivially. By definition we then have:

$$\begin{aligned} \sigma_i^{fe} &= \pi''_{\hat{f}\hat{e}(i), fe(i)} \hat{\sigma}_i^{ef} \\ &= \pi''_{\hat{f}e(i), fe(i)} \circ \pi''_{\hat{f}\hat{e}(i), \hat{f}e(i)} \circ \hat{\sigma}_{\hat{e}(i)}^f \circ \hat{\sigma}_i^e \\ &= (\pi''_{\hat{f}e(i), fe(i)} \circ \hat{\sigma}_{e(i)}^f) \circ (\pi'_{\hat{e}(i), e(i)} \circ \hat{\sigma}_i^e) \\ &= \sigma_{e(i)}^f \circ \sigma_i^e \end{aligned}$$

Axioms (d), (e) then follow easily. QED(Lemma 3.7.5)

We now consider "towers" of insertions. Let I^ξ be an iterate of M for $\xi < \Gamma$, where $e^{\xi,\mu}$ inserts I^ξ into I^μ for $\xi \leq \mu < \Gamma$. (We take $e^{\xi,\xi}$ as the identical insertion).

Definition 3.7.3. We call:

$$\langle\langle I^\xi : \xi < \Gamma\rangle, \langle e^{\xi,\mu} : \xi < \mu < \Gamma\rangle\rangle$$

a *commutative insertion system* iff $e^{\zeta,\mu} \circ e^{\xi,\zeta} = e^{\xi,\mu}$ for $\xi \leq \zeta \leq \mu < \Gamma$.

Now suppose that Γ is a limit ordinal. Is there a reasonable sense in which we could form the *limit* of the above system? We define:

Definition 3.7.4. $I, \langle e^\xi : \xi < \Gamma\rangle$ is a good limit of the above system iff:

- I is an iterate of M and e^ξ inserts I^ξ into I.
- $e^\mu \circ e^{\xi,\mu} = e^\xi$ for $\xi \leq \mu < \Gamma$.
- If $i < \mathrm{lh}(I)$, then $i = e^\xi(h)$ for some $\xi < \Gamma$, $h < \mathrm{lh}(I^\xi)$.

Note. Let $\eta_i = \mathrm{ht}(I^i)$ for $i < \Gamma$. It is a necessary but not sufficient condition for the existence of a good limit that:

$$\langle \eta_i : i < \Gamma\rangle, \langle e^{ij} : i \leq j < \Gamma\rangle$$

have a well founded limit.

If η, $\langle \tilde{e}^i : i < \Gamma\rangle$ is the transitivised direct limit of the above system, then any good limit must have the form $\langle I, \langle e^i : i < \Gamma\rangle\rangle$.

Fact. Let $\eta, \langle e^i : i < \Gamma\rangle$ be as above. Let $\xi < \eta$ and let $\hat{e}^i(\xi_i) = \xi$ for an $i < \Gamma$. For $i \leq j < \Gamma$ set:

$$\xi_j =: \hat{e}^{i,j}(\xi_i) = (\hat{e}^j)^{-1}(\xi)$$

Then $e^j(\xi_j) = \hat{e}^j(\xi_j) = \xi$ for sufficiently large $j < \Gamma$.

Proof. Suppose not. Then there is a monotone sequence $\langle j_n : n < \omega\rangle$ in $[i, \Gamma)$ such that $e^{j_n,j_{n+1}}(\xi_{j_n}) > \xi_{j_{n+1}}$.

Hence $e^{j_{n+1}}(\xi_{j_{n+1}}) < e^{j_n}(\xi_{j_n})$ for $n < \omega$. Contradiction! QED

We then get:

Lemma 3.7.6. *Let $\langle I^\xi\rangle, \langle e^\xi, \mu\rangle$ be a commutative system of insertions of limit length θ. Then there is at most one good limit $I, \langle e^\xi\rangle$. Moreover, if $i < \mathrm{lh}(I)$, then $|M_i| = \bigcup\{\mathrm{rng}(\tilde{\sigma}^\xi_h) : e^\xi(h) = i\}$.*

Proof. Let $\langle I\langle e^\xi\rangle\rangle, \langle I'\langle e'^\xi\rangle\rangle$ be two distinct good limits. We derive a contradiction. Set $\eta_\xi = \mathrm{lh}(I^\xi)$ for $\xi < \Gamma$. Then $\langle\eta_\xi\rangle, \langle\tilde{e}^\xi, \mu\rangle$ has a transitive direct limit $\eta, \langle f^\xi\rangle$. Moreover $\eta = \mathrm{lh}(I)$ and $e^\xi = e'^\xi = f^\xi$ for $\xi < \Gamma$. Hence $\hat{e}^\xi = \hat{e}'^\xi = \mathrm{lub}\{f^h : h < \xi\}$ for $\xi < \Gamma$. By induction on $i < \xi$ we prove:

(a) $M_i = M_i'$

(b) $\sigma_h^\xi = \sigma_h'^\xi$ for $e^\xi(h) = i$.

(c) $|M_i| = \bigcup\{\mathrm{rng}\,\sigma_h^\xi : e^\xi(h) = i\}$.

For $i = 0$ this is trivial. Now let $i = j + 1$. Then:

$$\nu_j = \nu_j' = \sigma_h^\xi(\nu_h^\xi) \text{ whenever } e^\xi(h) = j$$

This fixes $\mu =: T(j+1) = T'(j+1)$. But then we have: $M_j^* = M_j'^*$. Thus $M_i = M_i'$ and $\pi_{\mu+i} = \pi_{\mu_i}'$ are determined by:

$$\pi_{\mu+i} : M_i^* \longrightarrow_F M_i, \text{ where } F = E_{\nu_j}^{M_j} = E_{\nu_j'}^{M_j'}$$

We must still show:

Claim. If $x \in M_i$, then $x = \sigma_l^\xi(\bar{x})$ for a $\xi < \theta$ such that $e^\xi(l) = i$.

Proof. Let $n \leq \omega$ be maximal such that $\kappa_i < \rho_{M_i}^n$. Then $x = \pi_{1i}(f)(\alpha)$ for an $f \in \Gamma^n(\kappa_j, M_i^*)$. Let either $f = p \in M_i^*$ or else $f(\xi) \cong G(\xi, p)$ where $p \in M_i^*$ and G is a good $\Sigma_1^{(m)}(M_i^*)$ function for a $m < n$. Pick $\xi < \theta$ such that there are μ_ξ, j_ξ, i_ξ with:

$$e^\xi(\mu_\xi) = \mu,\ e^\xi(i_\xi) = i,\ e^\xi(j_\xi) = j$$

Assume furthermore that $\sigma_{\bar{\mu}}(\bar{p}) = p$ and $\sigma_{j_\xi}^\xi(\bar{\alpha}) = \alpha$. Since $\sigma_{j_\xi}(\nu_{j_\xi}^\xi) = \nu_j$, it follows easily that $\mu_\xi = T^\xi(i_\xi)$ and:

$$\sigma_{\bar{\mu}}^\xi \restriction M_{i_\xi}^{\xi *} : M_{i_\xi}^{\xi *} \longrightarrow_{\Sigma^*} M_i^*$$

Let $\bar{f}$ be defined from $\bar{p}$ over $M_{i_\xi}^\xi$ as f was defined from p over M_i. Let $\bar{x} = \pi_{\bar{\mu}, i_\xi}^\xi(\bar{f})(\bar{\alpha})$. Then $\sigma_{i_\xi}(\bar{x}) = x$ by Lemma 3.7.1(5). QED(Claim)

Now let $\lambda < \theta$ be a limit ordinal. We first prove:

Claim. $i <_T \lambda$ iff whenever $e(i_\xi) = i$ and $e^\xi(\lambda_\xi) = \lambda$, then $i_\xi <_{T^\xi} \lambda_\xi$.

Proof. ($\longrightarrow$) is immediate by Lemma 3.7.1(10). We prove ($\longleftarrow$). Suppose not. Let A be the set of $\xi < \theta$ such that there are i_ξ, λ_ξ with $e^\xi(i_\xi) = i$, $e^\xi(\lambda_\xi) = \lambda$. Then $i \not<_T \lambda$ but $i_\xi <_{T^\xi} \lambda_\xi$ for $\xi \in A$. Then:

$$\hat{e}^\xi(i_\xi) <_T \hat{e}^\xi(\lambda_\xi) \leq_T e^\xi(\lambda_\xi) = \lambda.$$

Set: $j = \sup\{\hat{e}^\xi(i_\xi) : \xi \in A\}$. Then $j <_T \lambda$ by the fact that $T``\{\lambda\}$ is club in λ. Hence $j < i$. Let $\xi \in A$ such that $e^\xi(j_\xi) = j$. Then $j_\xi < i_\xi$, since e^ξ is order preserving. Hence:

$$j = e^\xi(j_\xi) < \hat{e}^\xi(i_\xi) \leq j.$$

Contradiction! QED(Claim)

But then $T``\{\lambda\} = T'``\{\lambda\}$. Hence $M_\lambda = M'_\lambda, \pi_{i,\lambda} = \pi'_{i,\lambda}$ are given as the transitivized limit of:

$$\langle M_i : i <_T \lambda\rangle,\ \langle \pi_{i,j} : i \leq_T j < \lambda\rangle.$$

Finally, we show that each $x \in M_\lambda$ has the form $\sigma^\xi_{\lambda_\xi}(\bar{x})$ for an $\xi \in A$. We know that $x = \pi_{i,\lambda}(x')$ for an $i <_T \lambda$. Pick $\xi < \theta$ such that $e^\xi(i_\xi) = i$, $e^\xi(\lambda_\xi) = \lambda$ and $x' = \sigma^\xi_{i_\xi}(\bar{x}')$. Set: $\bar{x} = \pi^\xi_{i_\xi,\lambda_\xi}(\bar{x}')$. Then $\sigma^\xi_\lambda(\bar{x}) = x$ by Lemma 3.7.1(10).

QED(Lemma 3.7.6)

In the following we take a more local approach for forming a good limit and ask if and when the proven can be break down. It is of course a necessary condition that the limit be indexed in a well founded way, so we assume that.

In the following let $\mathbb{C} = \langle\langle I^\xi\rangle, \langle e^{\xi,\mu}\rangle\rangle$ be a commutative insertion system of limit length θ. Let $\eta_\xi = \text{length}(I^\xi)$ for $\xi < \theta$. Suppose that

$$\langle \eta_\xi : \xi < \theta\rangle, \langle e^{\xi,\mu} : \zeta \leq \mu < \theta\rangle$$

has the transitivized direct limit:

$$\eta,\ \langle e^\xi : \xi < \theta\rangle$$

(Thus if $\mathbb{C}$ had a good limit, it would have the form $\langle I, \langle e^\xi : \xi < \theta\rangle\rangle$).

Definition 3.7.5. Let $\mathbb{C}, \eta$, etc. be as above. Let $i < \eta$. Let I be a normal iteration of M of length $i+1$. I is a *good limit of* $\mathbb{C}$ *at* i iff whenever $\gamma < \theta$ and $e^\gamma(h) = i$, then $e^\gamma \restriction h+1$ inserts $I^\gamma | h+1$ into I.

Note. By Lemma 3.7.6 it follows that there is at most one good limit of $\mathbb{C}$ at i. To see this, let $\gamma < e$ such that $e^\gamma(h) = i$ and apply Lemma 3.7.6 to the structure:

$$\mathbb{C}' = \langle\langle \tilde{I}^\xi : \gamma \leq \xi < \theta\rangle, \dots\rangle \text{ where } \tilde{I}^\xi = I | e^{\gamma,\xi}(h) + 1.$$

Moreover, if I is a good limit of $\mathbb{C}$ at i and $h < i$, thus $I|h+1$ is the good limit of $\mathbb{C}$ at h. Thus we can unambiguously denote the good limit of $\mathbb{C}$ at i, if it exists, by: $I|i+1$. By uniqueness we then have:

$$(I|i+1)|h+1 = I|h+1 \text{ for } h < i$$

It is clear that I is the unique good limit of $\mathbb{C}$ iff $I|i+1$ exists for all $i < \eta$, and $I = \bigcup_{i<\eta} I|i+1$. We also note that $I|1 = \langle\langle M\rangle, \varnothing, \langle \text{id} \rangle, \varnothing\rangle$ is trivially the good limit at 0.

Recall that we call a premouse M *uniquely iterable* iff it is normally iterable and has the *unique branch property* -i.e. whenever I is a normal iteration of M of limit length, then it has at most one cofinal well founded branch. (Similarly for uniquely α-iterable). In the later subsection of §3.7 we shall always assume unique iterability of M and make use of the following two lemmas:

Lemma 3.7.7. *Let $\mathbb{C}, \eta$ be as above and let M be uniquely η-iterable. Let $i+1 < \eta$. If $I|i+1$ exists, then so does $I|i+2$.*

Proof. Let $I = I|i+1$. Pick $\mu < \theta$ such that $e^\mu(i_\mu) = i$ and $e^\mu(i_\mu+1) = i+1$. Set: $\nu_i = \sigma^\mu_{i_\mu}(\nu^\mu_{i_\mu})$. For $\mu \leq \delta < \theta$, we have $\nu_\delta = \sigma^\delta_{i_\delta}(\nu^\delta_{i_\delta})$ and $\nu^\delta_{i_\delta} \geq \nu^\delta_j$ for $j < i_\delta$.

It follows easily that $\nu_i > \nu_j$ in I whenever $j < i$. Thus ν_i determines a potential extension of $I|i+1$, giving: $\xi = T'(i+1), M^*_i$. Let $F = E^{M_i}_{\nu_i}$ in I.

Set:

$$\pi'_{\eta,i+1} : M^*_i \longrightarrow^*_F M'_{i+1}$$

This gives us an iteration I' of length $i+2$ extending I, it follows by Lemma 3.7.2 that $e^\mu|i_\mu+2$ inserts $I^\mu|i_\mu+2$ into I'. But this holds for sufficiently large $\mu < \theta$. Now let $\overline{\mu} < \theta$ with $e^{\overline{\kappa}} = i+1$. Let $\mu \geq \overline{\mu}$ be as above. Then $e^{\overline{\mu},\mu}(h) = i_\mu + 1$, and $e^{\overline{\mu},\mu} \upharpoonright h+1$ inserts $I^{\overline{\mu}}|h+1$ into $I^\mu|i_\mu+2$. Hence $e^{\overline{\mu}} = e^\mu \circ e^{\overline{\mu},\mu}$ inserts $I^{\overline{\mu}}|h+1$ into I'.

QED(Lemma 3.7.7)

Now let $\delta < \eta$ be a limit ordinal and let $I|i+1$ be defined for all $i < \delta$. If $I|\delta+1$ defined? Not necessarily. Set: $I = \bigcup_{i<\delta} I|i+1$. Then I is a normal iteration of length δ. Hence it has a unique cofinal well founded branch b. We can then extend I to I' of length $\delta+1$, taking $T'``\{\delta\} = b$. However I' will only be a good limit of $\mathbb{C}$ at δ if a certain condition on b is fulfilled:

Lemma 3.7.8. *Let $\mathbb{C}, I, b, I'$, etc. be as above. Assume that there are arbitrarily large $\gamma < \theta$ such that:*

(*) *$e^\gamma(\overline{\delta}) = \delta$ for some $\overline{\delta}$. Moreover, either $\hat{e}^\gamma(\overline{\delta}) \in b$ or $\hat{e}^\gamma(\overline{\delta}) = \delta$ and $\hat{e}^\gamma(i) \in b$ whenever $i <_{T^\gamma} \overline{\delta}$.*

Then I' is a good limit of $\mathbb{C}$ at δ.

Proof. Let γ, $\overline{\delta}$ as in (*). We show that $e^\gamma \restriction \overline{\delta}+1$ inserts $I^\gamma|\overline{\delta}+1$ into $I'|\delta+1$. We consider two cases:

Case 1: $\hat{e}^\gamma(\overline{\delta}) \in b$.

Let $\xi = \hat{e}^\gamma(\overline{\delta})$. Then $\xi \leq_{T'} \delta$. It is easily verified that $e^\gamma \restriction \overline{\delta}+1$ inserts $I^\gamma|\overline{\delta}+1$ into I' with $\hat{\sigma} = \hat{\sigma}^\gamma_{\overline{\delta}}$, $\sigma = \sigma^\gamma_{\overline{\gamma}}$ defined as follows:

By the above Fact there is $\gamma' > \gamma$ such that $e^{\gamma'}(\delta') = \xi$, where $\delta' = \hat{e}^{\gamma,\gamma'}(\overline{\delta})$. Thus $e^{\gamma'} \restriction \delta'+1$ inserts $I_{\delta'}|\delta+1$ into $I|\xi+1$. Set:

$$\hat{\sigma} =: \hat{\sigma}^\gamma_{\delta'} \circ \hat{\sigma}^{\gamma,\gamma'}_{\overline{\delta}}, \sigma =: \pi'_{\xi,\delta} \circ \hat{\sigma}$$

QED(Case 1)

Case 2: $e^\gamma(\overline{\delta}) = \delta$.

Then e^γ takes $\overline{\delta}$ cofinally to δ. Thus $e^\gamma \restriction \overline{\delta}+1$ inserts $I^\gamma|\overline{\delta}+1$ into $I|\delta+1$, where $\sigma = \sigma^\gamma_{\overline{\delta}} = \hat{\sigma}^\gamma_{\overline{\delta}}$ is defined by:

$$\sigma\pi^\gamma_{i,\overline{\delta}} = \pi_{e^\gamma_{\overline{\delta}}(i),\delta} \circ \hat{\sigma}^\gamma_i$$

The verification is again straightforward.

QED(Case 2)

Now let $\mu < \theta$ be arbitrary such that $e^\mu(\delta') = \delta$. Let $\gamma > \mu$ satisfy (*) with $e^\gamma(\overline{\delta}) = \delta$. Then $e^{\mu,\gamma}$ inserts $I^\mu|\delta'+1$ into $I^\gamma|\overline{\delta}+1$ and e^γ inserts $I^\gamma|\overline{\delta}+1$ into $I'|\delta+1$. Hence $e^\mu = e^\gamma \cdot e^{\mu,\gamma}$ inserts $I^\mu|\delta'+1$ into $I'|\delta+1$.

QED(Lemma 3.7.8)

Remark. It follows that every $\gamma < \theta$ such that $\delta \in \text{rng}(e^\gamma)$ satisfies (*).

Building on what we have just proven, we show that we can disperse with the iterability assumption if the length of the commutative system has cofinality greater than ω.

Lemma 3.7.9. *Let $\mathbb{C}$ be a commutative insertion system of length θ. If $\text{cf}(\theta) > \omega$, then $\mathbb{C}$ has a good limit.*

Proof.

Claim. $\langle \eta_i : i < \theta \rangle, \langle e^{\xi,\mu} : \xi \leq \mu < \theta \rangle$ has a transitivized direct limit:

$$\eta, \langle e^{\xi} : \xi < \theta \rangle$$

Proof. Suppose not. Let $\langle u, <^* \rangle, \langle e^{\xi} : \xi < \theta \rangle$ be a direct limit, where $<^*$ is a linear ordering of u. Then there are x_n $(n < \omega)$ such that $x_{n+1} <^* x_n$ for $n < \omega$. Since $\mathrm{cf}(\theta) > \omega$, there must be $\gamma < \theta$ such that $x_n \in \mathrm{rng}(e^{\gamma})$ for $n < \omega$. Let $e^{\gamma}(\alpha_n) = x_n$ $(n < \omega)$. Then $\alpha_{n+1} < \alpha_n$ in η_{δ} for $n < \omega$. Contradiction!

QED(Claim)

We now prove by induction on $i < \eta$ that $\mathbb{C}$ has a good limit $I|i$ at i.

Case 1. $i = 0$. The 1-step iteration of M: $\langle \langle M \rangle, \varnothing, \langle \mathrm{id} \rangle, \varnothing \rangle$ is the good limit at 0 (with $e_0^0 = \hat{e}_0^0 = \mathrm{id} \restriction \{0\}$).

Case 2. $i = h + 1$.

Let $\nu_i, \xi = T'(i+1), M_i^*, F = E_{\nu_i}^{M_i}$ be as in the proof of Lemma 3.7.7. The proof of Lemma 3.7.7 goes through exactly as before if we can show:

Claim. M_i^* is extendible by F.

Proof. Suppose not. Then there are $f_n \in \Gamma^*(\kappa_i, M_i^*), \alpha_n \in \lambda_i$ $(n < \omega)$ such that

$$\{\langle \mu, \tau \rangle : f_{n+1}(\mu) \in f_n(\tau)\} \in F_{\langle \alpha_{n+1}, \alpha_n \rangle} \quad \text{for } n < \omega$$

Let $p_n \in M_i^*$ such that either $p_n = f_n$ or f_n is defined by: $f_n(\beta) \cong G(p_n, \beta)$, where G is good over M_{ξ}^*. Since $\mathrm{cf}(\theta) > \omega$, we can pick $\gamma < \theta$ such that

- $e^{\gamma}(i_{\gamma}) = i, e^{\gamma}(\xi_{\gamma}) = \xi$
- $\sigma_{\xi_{\gamma}}^{\gamma}(\overline{p}_n) = p_n$ $(n < \omega)$
- $\sigma_{i_{\gamma}}^{\gamma}(\overline{\alpha}_n) = \alpha_n$ $(n < \omega)$
- $[\overline{e}^{\gamma}(\xi_{\gamma}), e^{\gamma}(\xi_{\gamma})]_T$ has no drop point in I. (Hence $\sigma_{\xi_{\gamma}}^{\gamma *}, M_{\xi_{\gamma}}^{\gamma} \longrightarrow_{\Sigma^*} M_{\xi}$, since $\sigma_{\xi_{\gamma}}^{\gamma} = \pi_{\xi_{\gamma}} \hat{\sigma}_{\xi_{\gamma}}^{\gamma}$).

We note that $\xi_{\gamma} = T^{\gamma}(i_{\gamma} + 1)$. (Suppose not. Let $t = T^{\gamma}(i_{\gamma} + 1)$. Then $\xi \in [\hat{e}^{\gamma}(t), e^{\gamma}(t)]$ by Lemma 3.7.1 (3). But thus $t < \xi$ and $\xi < t$ are both impossible. Contradiction!) It follows that:

$$\sigma_\xi^\gamma \upharpoonright M_{i_\gamma}^{\gamma *} \longrightarrow_{\Sigma^*} M_i^*$$

If $\overline{f}_n$ is defined from $\overline{p}_n$ as f_n was defined from p_n, we then have:

$$\{\langle \mu, \tau \rangle : \overline{f}_{n+1}(\mu) \in \overline{f}_n(\tau)\} \in \overline{F}_{\langle \overline{\alpha}_{n+1}, \overline{\alpha}_n \rangle}$$

where $\overline{F} = E_{\nu_{i_\gamma}}^{M_{i_\gamma}^\gamma}$. But:

$$\pi_{\xi_\gamma, i_\gamma}^\gamma : M_{i_\gamma}^{\gamma *} \longrightarrow_{\overline{F}}^* M_{i_\gamma + 1}^\gamma$$

Hence $M_{i_\gamma+1}^\gamma$ would be ill founded. Contradiction!

QED(Case 2)

Case 3: $i = \mu$ is a limit ordinal.

Let b' be the set of $j < \mu$ such that for some $\gamma < \theta$ and $\overline{\mu} < \eta_\gamma$ we have $e^\gamma(\overline{\mu}) = \mu$ and $j = \hat{e}^\gamma(i)$ for an $i \leq_{T^\gamma} \overline{\mu}$. Let b be the closure of b' under limit points below μ. Then b is a cofinal branch in I. Moreover, b satisfies $(*)$.

τ_{i_n} is not a cardinal in Lemma 3.7.8. Hence we can simply repeat the proof of Lemma 3.7.8 if we can show:

Claim. b is a well founded branch in I.

Proof. We must first show:

Subclaim. b has at most finitely many drop points.

Proof. Suppose not. Let $\langle i_n : n < \omega \rangle$ be monotone such that $i_n + 1$ is a drop point in b. Since $i_n + 1$ is not a limit point in b, we have $i_n + 1 \in b'$. Hence for each n there is a $\gamma < \theta$ and a $\overline{\mu}$ such that $e^\gamma(\overline{\mu}) = \mu, \hat{e}^\gamma(h_n + 1) = i_n + 1, h_n + 1 <_{T^\gamma} \overline{\mu}$. If γ has this property, so will every larger $\gamma' < \theta$. Since $\mathrm{cf}(\theta) > \omega$, we know that sufficiently large $\gamma < \theta$ will have the property for all n. We can also suppose without lose of generality that $e^\gamma(t_n) = t_n$, where $t_n = T(i_n + 1)$ in I. Just as in Case 2 we then have $I_n = T^\gamma(h_n + 1)$. As in Case 2 we can assume γ chosen big enough that $[\hat{e}^\gamma(\overline{t}_n), e^\gamma(\overline{t}_n))_T$ has no drop point in I. (Hence the map $\sigma_{\overline{t}_n}^\gamma$ is Σ^*-preserving). Then τ_{i_n} is not a cardinal in M_{t_n} and $\tau_{i_n} = \sigma_{h_n}^\gamma(\tau_{h_n}) = \sigma_{\overline{t}_n}^\gamma(\tau_{h_n})$. Hence τ_{h_n} is not a cardinal in $M_{h_n}^\gamma$. Hence $h_n + 1$ is a drop point in I^γ. Hence T^γ“$\{\overline{\mu}\}$ has infinitely many drop points. Contradiction!

QED(Subclaim)

We now prove the claim. Suppose not, Let $b'' =: b' \setminus \beta$, where $\beta < \bar{\mu}$ is big enough that no $i \in b''$ is a drop point. Then there is a monotone sequence $\langle i_n : n < \omega \rangle$ such that $i_n \in b'', x_n \in M_{i_n}$ and

$$x_{n+1} \in \pi_{i_n, i_{n+1}}(x_n) \text{ for } n < \omega$$

Pick $\gamma < \theta$ big enough that $e^\gamma(\bar{\mu}) = \mu$ and $\hat{e}^\gamma(h_n) = i_n$, where $h_n <_{T^\gamma} \bar{\mu}$. We can also pick it big enough that $x_n = \hat{\sigma}_{i_n}(\bar{x}_n)$ for $n < \omega$. Hence

$$\bar{x}_{n+1} \in \pi^\gamma_{h_n, h_{n+1}}(\bar{x}_n) \text{ for } n < \omega$$

Hence $M^\gamma_{\bar{\mu}}$ is ill founded. Contradiction!

QED(Lemma 3.7.9)

3.7.2 Reiterations

From now on assume that M is a uniquely normally iterable mouse (i.e. every normal iteration of limit length has exactly one cofinal well founded branch). (Our results will go through mutatis mutandis if we assume unique normal α-iterability for a regular cardinal $\alpha > \omega$).

Interpolating extenders

Let $I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$ be a normal iteration of M of length $\eta + 1$. A "reiteration" of I occurs when we "interpolate" new extender which were not on the sequence $\langle \nu_i : i < \eta \rangle$. This rounds very vague, or course, but we can make it more explicit by considering the case of a single extender $F = E^{M_\eta}_\nu$ which we had neglected to place on the sequence. Set: $\tau = \tau^{+M_\eta || \nu}, \kappa = \text{crit}(F), \lambda = \lambda(F) =: F(u)$. For the moment let us assumer that τ is a cardinal in M_η. The interpolation gives rise to a new iteration I'. I' coincides with I up to the point at which F should have been applied. At that point we apply F and thereafter simply copy what we did in I. The point s at which F should have been applied is defined as follows:

$$s = \text{ the least point such that } s = \eta \text{ or } s < \eta \text{ and } \nu < \nu_s$$

We want $I|s+1 = I'|s+1$, but at stage s we apply F instead of $E^{M_s}_{\nu_s}$. Thus we set: $\nu_s = \nu$. This determines $t = T'(s+1)$ and M'^*_s. We then form:

$$\pi'_{t,s+1} : M'^*_r \longrightarrow^*_F M'_{s+1}$$

There is then an obvious insertion f of $I|t+1$ into $I'|s+2$ defined by:

$$f \upharpoonright t = \text{id}, \ f(t) = s+1$$

f induces the new insertion embeddings:

$$\hat{\sigma}_t = \mathrm{id} \restriction M_t,\ \pi_t = \pi'_{t,s+1},\ \sigma_t = \pi_t \hat{\sigma}_t$$

If $t = \eta$ (hence $s = \eta$), then $I' = I'|s+2$ is fully defined. Now let $t < \eta$.

Then $M'^*_s = M_t||\mu$, where $\mu \leq \mathsf{ON}_{M_t}$ is maximal with: τ is a cardinal in $M_t||\mu$. But then $\tau \in J^{E^{M_\eta}}_{\nu_t} \subset J^{E^{M_\eta}}_{\nu}$, so τ is a cardinal in $J^{E^{M_\eta}}_{\nu_t}$. Hence $\mu \geq \nu_t$ and $\sigma_t(\nu_t)$ is defined. Set: $\nu'_{s+1} = \sigma_t(\nu_t)$. This defines a potential extension of $I'|s+2$, since

$$\nu'_s = \pi_t(\tau) < \pi_t(\nu_t) = \nu'_{s+1}$$

where $\pi_t = \pi'_{t,s+1}$.

Now define e on η by:

$$e \restriction t = \mathrm{id}, e(t+i) = s+1+i \text{ for } t+i \leq \eta$$

Then $e \restriction t+1 = f$. It is easily seen that $\hat{e}(t) = t$ and $e(t) = s+1$. But for $i \neq t$ we have $\hat{e}(i) = e(i)$. We prove:

Claim. e inserts I into a unique I' of length $e(\eta)+1$.

To show this we prove the following subclaim by induction on i:

Subclaim. If $t+1+i \leq \eta$, then $e \restriction (t+1+i+1)$ inserts $I|(t+1+i+1)$ into a unique $I'' = I'|(s+2+i+1)$ of length $s+2+i+1$.

Proof. Case 1: $i = 0$.

We have seen that $\sigma_t(\nu_t)$ exists and that $\sigma_t(\nu_t) > \nu'_t$. Hence we can appoint $\nu'_{t+1} = \sigma(\nu_t)$, which determines $\xi = T'(s+2)$ and M'^*_{s+1}. M'^*_{s+1} is $*$-extendible by $F = E^{M'_{s+1}}_{\nu'_{s+1}}$ by the fact that M is uniquely iterable. By Lemma 3.7.2 we conclude that $e|t+2$ inserts $I|t+2$ into a unique $I'|s+3$ extending $I'|s+2$.

QED(Case 1)

Case 2: $i = j+1$.

Then $I'|s+2+i$ is given. Set: $h = t+1+j$. Then $e(h) = \hat{e}(h) = s+2+j$. We are given: $\sigma_h(\nu_h) = \hat{\sigma}_h(\nu_h)$. Set $\nu'_{e(h)} =: \sigma_h(\nu_h)$. This determines a potential extension of $I'|e(h)+1$, since:

$$\nu'_{e(h)} > \sigma_h(\nu_l) \geq \nu'_{e(l)} \text{ for } t \leq l < h$$

But M'^*_h is $*$-extendible by $E^{M'_{e(h)}}_{\nu_{e(h)}}$ by unique iterability. Hence by Lemma 3.7.2, $e|h+2$ inserts $I|h+2$ into a unique $I'|e(h)+2$ extends $I'|e(h)+1$ by Lemma 3.7.2.

QED(Case 2)

Case 3: $i = \lambda$ is a limit ordinal.

We first observe that the componentwise union $I' = \bigcup_{i<\lambda} I'|e(i)$ is the unique iteration of length $e(\lambda)$ into which $e|\lambda$ inserts $I|\lambda$. Now let b' be the unique cofinal well founded branch in $I'|e(\lambda)$. Then $b = \{i : e(i) \in b'\}$is the unique cofinal well founded branch in $I|\lambda$. Hence $b = T``\{\lambda\}$. By Lemma 3.7.1 (18), $e|\lambda + 1$ inserts $I|\lambda + 1$ into a unique $I'|e(\lambda) + 1$ extending $I'|e(\lambda)$.

QED(Case 3)

QED(Claim)

We must still consider the case that τ is not a cardinal in M_η. If $t < \eta$, then τ is not a cardinal in $J_{\lambda_t}^{E^{M_t}}$ since $J_{\lambda_t}^{E^{M_t}} = J_{\lambda_t}^{E^{M_\eta}}$ and λ_t is a cardinal in M_η. $M_s'^*$ thus has the form: $M_t||\mu = M_\eta||\mu$. (Hence we truncate to the same place that we would if we applied F directly to M_η). Clearly $\mu < \lambda_t < \nu_t$ if $t < \eta$. Hence the "copying" process we performed in the previous case is impossible. (Note, too, that $t = s$, since if $t < s$, then λ_t would be inaccessible in $J_\nu^{E^{M_s}}$ and $\tau < \lambda_t$ would be a cardinal in $J_{\lambda_t}^{E^{M_s}} = J_{\lambda_t}^{E^{M_t}}$. Contradiction!). We set:

$$I^\nu = I|t + 1$$

We can extend I^* to I' by setting $\nu_t' = \nu$. Set $e \restriction t = \mathrm{id}, e(t) = s + 1 = t + 1$. Then e inserts I^* into I'.

The I' which we have described above is called a *simple reiteration* of I. If I' is obtained by a chain of simple reiterations, we also call it a simple reiteration. However, we must still show that an infinite chain of simple reiterations has a well founded limit. This will require considerable effort. Before doing that we develop the notion of *normal reiteration*, which is easier to deal with.

Now let $\langle I^i : i < \omega \rangle$ be a chain of simple reiterations with

$$I^0 = \langle \langle M_h^i \rangle, \langle \nu_h^i \rangle, \langle \pi_h^i \rangle, T^i \rangle \text{ of length } \eta_i.$$

Let I^{i+1} be obtained from I^i by interpolating $F_i = E_{\nu_i}^{M_{\xi_i}^i}$ into I^i, giving rise to the insertion e^i of I^{i*} into I^{i+1}. In an effort to tame the complexity of these structures, we could impose the normality condition: $\nu_i < \nu_j$ for $i < j < \omega$. It turns out that we can impose a far more powerful normality condition by requiring that F_i be interpolated in the *earliest possible* I^h with $h \leq i$, rather than necessarily into I_i itself. This gives the concept of *normal reiteration*, which is clearly analogous to that of normal iteration. First, however, we must redo our definitions in order to make this notion precise. To say that I^h is a *possible* candidate for interpolation of F_i means simply that $h \leq i$ and $I^h|t+1 = I^i|t+1$, where t is defined from as before from ν_i, I^i. In a normal reiteration it will then turn out that either $t = \eta_h$ or $\nu_t^i \leq \nu_t^h$ (ν_t^i will exits if $h < i$). In a normal reiteration we will then have: $I^j|t+1 = I^i|j+1$ for $h \leq j \leq i$.

We now give a precise definition of the operation we perform when we apply F_i to I^h.

Definition 3.7.6. Let $I = \langle\langle M_h^i\rangle, \langle\nu_h^i\rangle, \langle\pi_h^i\rangle, T\rangle$ be a normal iteration of M of length η. Let

$$I' = \langle\langle M_h'^i\rangle, \langle\nu_h'^i\rangle, \langle\pi_h'^i\rangle, T'\rangle$$

be a normal iteration of M of length η'. Let $F = E_\nu^{M'_\eta} \neq \varnothing$. Set:

$$\kappa =: \mathrm{crit}(F), \lambda = \lambda(F) =: F(\kappa), \tau = \kappa^{+M||\nu}.$$

Let s be least such that

$$s = \eta' \ \vee \ (s < \eta' \wedge \nu' < \nu_s)$$

Let t be least such that:

$$t = \eta^i \ \vee \ (t = \eta^i \wedge \kappa' < \lambda'_t)$$

(Hence $t \leq s$).

Assume that $I|t+1 = I'|t+1$ and $\nu'_t \leq \nu_t$. We define an operation:

$$W(I, I', \nu) = \langle I^*, I'', e\rangle$$

by cases as follows:

Case 1: $t = \eta$ and τ is a cardinal in M_η.

Extend I to I'' by appointing $\nu''_\eta = \nu$. Then $\pi''_{\eta,\eta+1} : M \longrightarrow_F^* M_{\eta+1}$. e is then the insertion of I into I'' defined by $e \upharpoonright \eta = \mathrm{id}, e(\eta) = \eta + 1$. (Hence $\pi_\eta = \pi'_{\eta,\eta+1}$ and $\sigma_\eta = \mathrm{id} \upharpoonright M_\eta, \tilde{\sigma}_\eta = \tilde{\pi}_\eta \sigma_\eta$). We set: $I^* = I$.

Case 2: $t < \eta$ and τ is a cardinal in M_η. We set $I''|s+1 = I'|s+1$. We then appoint $\nu''_s = \nu$. Thus $t = T''(s+1)$ and $M_s''^* = M_t||\mu$, where $\mu \leq \mathsf{ON}_{M_t}$ is maximal such that τ is a cardinal in $M_t||\mu$. But τ is a cardinal in $J_{\nu_t}^{E^{M_t}} = J_{\nu_t}^{E^{M_\eta}}$. Hence $\mu \geq \nu_t$. Let f be the insertion of $I|t+1$ into $I''|s+2$ defined by

$$f \upharpoonright t = \mathrm{id}, f(t) = s + 1.$$

Then:

$$\hat{\sigma}_t = \mathrm{id} \upharpoonright M_t, \pi_t = \pi_{t,s+1}, \sigma_t = \pi_t \circ \sigma_t$$

(Hence $\sigma_t(\nu_t) > \nu_t''$ as before).

Now define e on $\eta + 1$ by

$$e \upharpoonright t = \text{id}, e(t+i) = s+1+i.$$

Set $\eta'' =: e(\eta)$. I'' is then the unique iteration of length $\eta'' + 1$ extending $I'|s+2$ such that e inserts I into I''. We set: $I^* =: I$.

The existence and uniqueness proofs are exactly as before.

Case 3: τ is not a cardinal in M_η. If $t < \eta$, then τ is not a cardinal in $J_{\nu_t}^{E^{M_t}}$. Hence $M_s''^* = M_t||\mu$, where $\mu < \nu_t$. Set: $I^* =: I|t+1$. Set: $\nu_s'' =: \nu$. This gives:

$$\pi''_{t,s+1} : M_s''^* \longrightarrow_F^* M''_{s+1}$$

which defines $I'' = I''|s+2$. e is thus the insertion of I^* into I'' defined by: $e \upharpoonright t = \text{id}, e(t) = s+1$.

Note that $e \upharpoonright t = \text{id}$ (hence $\hat{e} \upharpoonright t+1 = \text{id}$ in all three cases.)

This completes the definition. We are now in a position to define the notion of *normal reiteration*. First, however, we prove a particularly useful lemma:

Lemma 3.7.10. *If $j \in (t,s]$ and $s < \mu$, then $j \not<_{T''} \mu$.*

Proof. We proceed by induction on μ.

Case 1: $\mu = s+1$. Then $t = T''(\mu)$ and $j \not<_{T''} t$, since $t < j$. Hence $j \not<_{T''} \mu$.

Case 2: $\mu > s+1$ is a successor. Let $\mu = \gamma + 1$. Then $\gamma \geq s+1$ and $\gamma = e(\overline{\gamma})$ where $\bar{\gamma} \geq t$. Let $\xi = T''(\gamma+1)$. Let $j \in (t,s]$ such that $j <_{T''} \mu$, then $j \leq_{T''} \xi$. We derive a contradiction. Let $\overline{\xi} = T(\overline{\gamma}+1)$. Then:

$$\hat{e}(\overline{\xi}) \leq_{T''} \xi \leq_{T''} e(\overline{\xi}).$$

If $\overline{\xi} = t$, then $t \leq_{T'} \xi \leq_{T''} s+1$. Hence $\xi \notin (t,s]$ by Case 1. Hence either $\xi = t < j$ or $\xi = s+1 >_{T'} j$, contradicting the induction hypothesis. If $\overline{\xi} < t$ then $\xi = \hat{e}(\overline{\xi}) = e(\overline{\xi}) = \overline{\xi} < j$. Contradiction! If $\overline{\xi} > t$, then $\xi = \hat{e}(\overline{\xi}) = e(\xi) \geq s+1$. Hence $j <_{T'} \xi < \mu$, contradicting the induction hypothesis.

QED(Case 2)

Case 3: μ is a limit ordinal.

Pick $i <_{T''} \mu$ such that $i > s$. Then $j \not<_{T''} i$ by the induction hypothesis. Hence $j \not<_{T''} \mu$.

QED(Lemma 3.7.10)

As we have seen, if e is an insertion of I to I' and $h = T(i+1)$, then the determination of $e^*(i) = T'(e(i)+1)$ is important. In the case of the e defined above, this determination is as follows:

Lemma 3.7.11. *Let $h = T(i+1)$. If $\kappa_i < \kappa$, then $\hat{e}(h) = h = T''(e(i)+1)$. If $\kappa_i \geq \kappa$, then $e(h) = T''(e(i)+1)$, where $e(h) > s+1$.*

Proof. Let $h' = T''(e(i)+1)$. We know:

$$\hat{e}(h) \leq_{T'} h' \leq_{T'} e(h).$$

The cases: $h < t$ and $h > t$ are straightforward. Now let $h = t$. As in Case 2 of the above proof we conclude: $h' = t$ or $h' = s+1$. But $\kappa''_{e(i)} = \pi(\kappa_i)$, where $\pi = \pi''_{t,s+1}$. Hence, if $\kappa_i < \kappa = \text{crit}(\pi)$ we have: $\pi(\kappa_i) = \kappa_i < \lambda_t$. Hence $h' = t$. If $\kappa \leq \kappa_i$, then: $\pi(\kappa_i) \geq \pi(\kappa) = \lambda \geq \lambda_i$. Hence $h' = s+1$.

QED(Lemma 3.7.11)

We now turn to the definition of a *normal reiteration.*

$R = \langle\langle I^i : i < \eta\rangle, \langle\nu_i : i+1 < \eta\rangle, \langle e^{i,j} : i \leq_T j\rangle, T\rangle$ is a *normal reiteration on M* iff the following hold:

(a) $\eta \geq 1$ and each $I^i = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_h\rangle, \tau^i\rangle$ is a normal iteration of M of length $\eta_i + 1$.

(b) T is a tree on η such that $iTj \longrightarrow i < j$.

(c) $F_i =: E^{M^i_{\eta_i}}_{\nu_i} \neq \varnothing$. Moreover, $\nu_i < \nu_j$ for $i < j$.

Set: $\kappa_i =: \text{crit}(F_i), \lambda_i = \lambda(F_i) =: F_i(\kappa_i)$, $\tau_i = \tau(F_i) =: \kappa^{+J^E_{\nu_i}}$, where $E = E^{M^i_{\eta_i}}$.

(d) $e^{i,j}$ inserts a segment $I^i|\mu$ into I^j. Moreover, $e^{h,i} = e^{ij} \circ e^{hi}$ for $h \leq_T i \leq_T j$. e^{ii} is the identical insertion on I^i.

(e) Set: $s = s_i =:$ the least s such that $s = \eta_i$ or $s < \eta_i$ and $\nu_i < \nu^i_s$. Then: $I^i|s+1 = I^j|s+1$ and $\nu^j_s = \nu_i$ for $i < j \leq \eta$.

(f) Let $i+1 < \eta$. Let h be least such that $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then h is the immediate predecessor of $i+1$ in T. (In symbols: $h = T(i+1)$). Before continuing with the definition, we note some consequences:

Set:

$$t = t_i =: \text{ the least } t \text{ such that } t = \eta_i \text{ or } t < \eta_i \wedge \kappa < \lambda^i_t$$

(Hence $t_i \leq s_i$). In the following assume: $h = T(i+1), t = t_i$. Then:

(1) $I^i|t+1 = I^h|t+1$. Moreover $\nu^h_t \geq \nu^i_t$ if $t < \eta_h$.
Proof. If $h = i$ this is trivial. Now let $h < i$. Then

$$\kappa < \lambda_h = \lambda^i_{s_h} \text{ by (e).}$$

Hence $t \leq s_h$. Clearly by (e) we have:

$$I^h|s_h + 1 = I^i|s_h + 1 \text{ and } \nu^i_{s_h} = \nu_h \tag{*}$$

Hence $I^h|t+1 = I^i|t+1$. If $t = s_h$, we then have: $\nu^h_t > \nu_h = \nu^i_t$ if $t < \eta_h$. If $t < s_h$, then: $\nu^h_t = \nu^i_t$ by (*).

QED(1)

(2) h is least such that $I^i|t = I^h|t$.
Proof. Let $l < t$. Then $\lambda^i_{s_l} = \lambda l \leq \kappa < \lambda^i_t$. Hence $s_l < t$. But $\nu^h_{s_l} = \nu_l < \nu^l_{s_l}$ if $s_l < \eta_l$. Hence $I^l|t \neq I^h|t$.

QED (2)

By (1), the conditions for forming $W(I^h, I^i, \nu_i)$ are given. Our next axiom reads:

(g) Let $h = T(i+1)$. Then $e^{h,i+1}$ inserts I^i_* into I^{i+1} where:

$$\langle I^i_*, I^{i+1}, e^{h,i+1}\rangle = W(I^h, I^i, \nu_i)$$

We define:

Definition 3.7.7. $i+1$ is a *drop point* (or *truncation point*) in R iff τ_i is not a cardinal in $M^h_{\eta_h}$ where $h = T(i+1)$. (This is the only case in which $I^i_* \neq I^h$ is possible).

Our final axioms read:

(h) If $\lambda < \eta$ is a limit ordinal, then $T``\{\lambda\}$ is club in $\lambda$. Moreover, $T``\{\lambda\}$ contain at most finitely many drop points.

(i) If λ is as above and $(h, \lambda)_T$ has no drop points, then $e^{i,\lambda}$ inserts I^h into I^λ and:

$$I^\lambda, \langle e^{i,\lambda} : h \leq_T i \leq_T \lambda\rangle$$

is the good limit of:

$$\langle I^i : h \leq_T i <_T \lambda\rangle, \langle e^{i,j} : h \leq_T i \leq_T j < \lambda\rangle$$

Note. As usual, we will then refer to $I^\lambda, \langle e^{i,\lambda} : i <_T \lambda\rangle$ as the direct limit of:

$$\langle I^i : i \leq_T \lambda\rangle, \langle e^{i,j} : i \leq_T j < \lambda\rangle,$$

since the missing points are supplied by: $e^{l,\lambda} = e^{h,\lambda} \circ e^{l,h}$ for $l \leq h$.

Definition 3.7.8. If $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is a reiteration of length η and $o < \mu \leq \eta$, we let $R|\mu$ denote:

$$\langle\langle I^i : i < \mu\rangle, \langle\nu_i : i + 1 < \mu\rangle, \langle e^{i,j} : i \leq_T j < \mu\rangle, T \cap \mu^2\rangle$$

Lemma 3.7.12. *If R is a reiteration and $0 < i \leq \mathrm{lh}(R)$. Then $R|i$ is a reiteration.*

Lemma 3.7.13. *Let $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{ij}\rangle, T\rangle$ be reiteration of length $\gamma + 1$, where I^i have length $\eta_i + 1$ for $i \leq \gamma$. Let $E_\nu^{M^\gamma_{\eta_\gamma}} \neq \varnothing$, where $\nu > \nu_i$ for $i < \gamma$. Then there is a unique extension of B to a reiteration R' of length $\gamma + 2$ such that $R'|\gamma + 1 = R$ and $\nu'_\gamma = \nu$.*

Proof. Let $i = T'(\gamma + 1)$. Then $W(I^i, I^\gamma, \nu)$ is defined.

A much deeper result is:

Lemma 3.7.14. *Let R be a reiteration of limit length η. There is a unique extension R' such that $R'|\eta = R$ and $\mathrm{lh}(R') = \eta + 1$.*

The proof of this theorem will be the main task of this subsection. It will require a long train of lemmas.

For now on let:

$$R = \langle\langle I^\xi\rangle, \langle\nu_\xi\rangle, \langle e^{\xi,\mu}\rangle, T\rangle$$

be a reiteration of limit length η. Let:

$$I^\xi = \langle\langle M^\xi_i\rangle, \langle\nu^\xi_i\rangle, \langle\pi^\xi_{ij}\rangle, T^\xi\rangle$$

be of length $\eta_\xi + 1$ for $\xi < \eta$.

Lemma 3.7.15. *Let $\xi < \mu < \eta$. Then:*

(a) $s_\xi < s_\mu$

(b) $\nu_\xi = \nu^\mu_{s_\xi}$

Proof. (b) holds by (e) in Definition **??**. We prove (a). Suppose not. $\eta_\mu > s_\xi$ since $\nu^\mu_{s_\xi}$ exists. Hence $s_\mu < \eta_\mu$. Hence $\nu_\mu < \nu^\mu_{s_\mu} \leq \nu^\mu_{s_\xi} = \nu_\xi$. Contradiction!

QED(Lemma 3.7.15)

Lemma 3.7.16. *Let $\xi + 1 \leq_T \mu$. Then $e^{\xi+1,\mu} \restriction s_\xi + 1 = \mathrm{id}$.*

We proved by induction on μ. For $\mu = \xi+1$ it is trivial. Now let $\xi+1 <_T \mu+1$ and let it hold at $\gamma = T(\mu+1)$. Then $\xi < \gamma$ and hence: $\kappa_\mu \geq \lambda_\xi = \lambda^\mu_{s_\xi}$. Hence $t_\mu \geq s_\xi + 1$ and:

$$e^{\gamma,\mu+1} \restriction t_\mu = \mathrm{id}$$

by (g). Hence:

$$e^{\xi+1,\mu+1}(\alpha) = e^{\gamma,\mu+1} e^{\xi+1,\gamma}(\alpha) = \alpha \text{ for } \alpha \leq s_\eta.$$

Now let μ be a limit ordinal and let the induction hypothesis hold at γ for all γ with: $\xi + 1 \leq_T \gamma <_T \mu$. For $i \leq_T j <_T \mu$ we then have: $e^{i\mu}(\alpha) = e^{j\mu} e^{ij}(\alpha) = e^{j\mu}(\alpha)$.

Let $\alpha \leq s_\xi$ be least such that $\alpha < e^{\xi+1,\mu}(\alpha)$. Let $\xi + 1 \leq_T \delta <_T \mu$ such that $e^{\delta,\mu}(\overline{\alpha}) = \alpha$. Then $\overline{\alpha} < \alpha = e^{\xi+1,\delta}(\alpha)$. Hence $e^{\delta,\mu}(\overline{\alpha}) = \overline{\alpha} < \alpha$. Contradiction!

QED(Lemma 3.7.16)

Definition 3.7.9. $\hat{s}_\gamma =: \mathrm{lub}\{s_\xi : \xi < \gamma\}$.

Lemma 3.7.17. *Let $\gamma = T(\xi + 1)$. Then $\hat{s}_\gamma \leq t_\xi \leq s_\gamma$.*

Proof.

(1) $\hat{s}_\gamma \leq t_\eta$, since if $i < \gamma$, then $\lambda_i = \lambda^\gamma_{s_i} \leq \kappa_\xi$.

(2) $t_\xi \leq s_\gamma$.

This is trivial for $\gamma = \xi$. Now let $\gamma < \xi$. Then $\kappa_\eta < \lambda_\gamma = \lambda^\xi_{s_\gamma}$. Hence $t_\xi \leq s_\gamma$.

QED(Lemma 3.7.17)

Definition 3.7.10. X is *in limbo* at μ iff $X \subset \hat{s}_\mu$ and there is no pair $\langle i, j\rangle$, such that $i \in X$, $j \geq \hat{s}_\mu$ and $i <_{T^\mu} j$.

Lemma 3.7.18. *If $\xi + 1 \leq_T \mu$, then $(t_\xi, s_\xi]$ is in limbo at μ.*

Proof. By induction on μ.

Case 1: $\mu = \xi + 1$ by Lemma 3.7.10.

Case 2: $\mu = \delta + 1 >_T \xi + 1$.

Let $\gamma = T(\delta+1)$. Then it holds at γ. Moreover, $\hat{s}_\gamma \leq t_\gamma \leq s_\gamma$. Let $i \in (t_\xi, s_\xi]$ and $i <_{T^\mu} j$, where $j \geq \hat{s}_\mu = s_\delta + 1$. We derive a contradiction.

$j \geq \hat{s}_\mu = s_\delta + 1$. Hence $j = s_\delta + 1 + l$. Hence $e^{\gamma,\mu}(k) = j$, where $k = t_\delta + l$. Since $e^{\gamma,\mu}(i) = i$, we conclude: $i <_{T^\gamma} k$, where $\hat{s}_\gamma \leq t_\delta \leq k$. Contradiction!

QED(Case 2)

Case 3: μ is a limit ordinal.

Suppose $i \in (t_\xi, s_\xi]$ with $i \leq_{T^\mu} h$, $h \geq \hat{s}_\mu$. Then $h = e^{\gamma+1,\mu}(\overline{h})$ for a γ such that

$$\xi + 1 <_{T^\mu} \gamma + 1 <_{T^\mu} \mu$$

But $e^{\gamma+1,\mu} \upharpoonright s_\gamma + 1 = \mathrm{id}$ by Lemma 3.7.16. Hence $\overline{h} > s_\gamma$. Hence $\overline{h} \geq \hat{s}_\gamma = s_\gamma + 1$. Hence $i \not<_{T^{\gamma+1}} \overline{h}$ by the induction hypothesis. Hence $i \not<_{T^\mu} h$.

QED(Lemma 3.7.18)

By Lemma 3.7.16, $I^\xi | s_\xi + 1 = I^\gamma | s_\xi + 1$ for $\xi \leq \gamma < \eta$. The componentwise union:

$$\tilde{I} = \bigcup_{\xi<\eta} I^\xi | s_\xi$$

is then a normal iteration of length

$$\tilde{\eta} = \mathrm{lub}\{s_\xi : \xi < \eta\}$$

For $\xi < \tilde{\eta}$ set:

Definition 3.7.11. $\gamma(i) =:$ the least γ such that $i \leq s_\gamma$.

(Hence $\hat{s}_\gamma \leq i \leq s_\gamma$). The following lemma establishes an important connection between the normal iteration $\tilde{I}$ and the reiteration R.

Lemma 3.7.19. *Let $i \leq_{\tilde{T}} j$. Then $\gamma(i) \leq_T \gamma(i)$.*

Proof. Suppose not. Let i, j be a counterexample. Then $\gamma(i) \not\leq_T \gamma(j)$. Hence $i < j$ and $\gamma(i) < \gamma(j)$. Set: $\gamma = \gamma(j)$. There is $\mu + 1 \leq_T \gamma$ such that $T(\mu + 1) < \gamma(i) < \mu + 1$. Set $\tau = T(\mu + 1)$. Then $s_\tau < i$, since $\tau < \gamma(i)$. Hence $t_\mu \leq s_\tau < i$ by Lemma 3.7.17. But $i \leq s_{\gamma(i)} \leq s_\mu$, since $\gamma(i) \leq \mu$. Hence $i \not<_{T^\gamma} j$ by Lemma 3.7.18, since $j \geq \hat{s}_\gamma$. Hence $i \not<_{\tilde{T}} j$, since $I^\gamma | s_\gamma + 1 = \tilde{I} | s_\gamma + 1$. Contradiction!

QED(Lemma 3.7.19)

Lemma 3.7.20. *Let $\tau = T(\xi + 1) \leq_T \mu$. Then:*

$$\mathrm{crit}(e^{\tau,\mu}) = t_\xi \text{ and } e^{\tau,\mu}(t_\xi) \leq \hat{s}_\mu$$

Proof. By induction on μ.

Case 1. $\mu = \xi + 1$. $e^{\tau,\xi+1}(t_\xi) = s_\xi + 1 = \hat{s}_{\xi+1} > t_\eta$, but

$$e^{t,\xi+1}(i) = \hat{e}^{\tau,\xi+1}(i) = i \text{ for } i < t_\xi$$

Case 2. $\mu = \delta + 1$ is a successor.

Let $\gamma = T(\delta + 1)$. Then:

$$\begin{aligned} e^{\tau,\mu}(t_\xi) &= e^{\gamma,\mu} \circ e^{\tau,\mu}(\hat{s}_\gamma) \\ &\leq e^{\gamma,\mu}(t_\delta) = s_\delta + 1 = \hat{s}_\mu \end{aligned}$$

By the induction hypothesis we have:

$$e^{\tau,\mu}(t_\xi) = e^{\gamma,\mu} \circ e^{\tau,\gamma}(e_\xi) \geq e^{\tau,\gamma}(t_\xi) > t_\eta$$

For $i < t_\xi$ we have:

$$e^{\tau,\mu}(i) = e^{\gamma,\mu}e^{\tau,\gamma}(i) = e^{\gamma,\mu}(i) = i$$

(since $i < t_\gamma$).

QED(Case 2)

Case 3. μ is a limit cardinal. Then $e^{\tau,\mu} \restriction t_\xi = \text{id}$, since $e^{\tau,\gamma} \restriction t_\xi = \text{id}$ for $t \leq_T \gamma <_T \mu$ (cf. the proof of Lemma 3.7.16). Moreover $e^{\tau\mu}(t_\xi) \geq e^{\tau\gamma}(t_\xi) > t_\xi$.

Claim. $e^{\tau,\mu}(t_\xi) \leq \hat{s}_\mu$.

Proof. Let $h < e^{\tau,\mu}(t_\xi)$. Then $h = e^{\gamma,\tau}(\overline{h})$ where $\xi \leq_T \gamma <_T \mu$. Assume w.l.o.g. that $\gamma = T(\delta + 1)$, where $\delta + 1 <_T \mu$. Then:

$$\overline{h} < e^{\tau,\gamma}(t_\xi) \leq \hat{s}_\gamma \leq t_\delta.$$

But $e^{\gamma,\mu} \restriction t_\delta = \text{id}$ by the induction hypothesis.

Hence:

$$h = e^{\gamma,\mu}(\overline{h}) = \overline{h} < \hat{s}_\gamma \leq \hat{s}_\mu$$

QED(Lemma 3.7.20)

In order to prove Theorem 3.7.14 we must find a cofinal branch b in T such that

$$\langle I^i : i \in b \rangle, \langle e^{i,j} : i < j \text{ in } b \rangle$$

has a good limit. An obvious necessary condition is that

$$\langle \eta_i : i \in b \rangle, \langle e^{i,j} : i < j \text{ in } b \rangle$$

have a transitivized direct limit:

$$\eta, \langle e^i : i \in b \rangle.$$

Note. This does not say that e^i inserts I^i into a good limit I. It simply gives us a system of indices which, with luck, might be used to construct a good limit.

We obtain a rather surprising result:

Lemma 3.7.21. *Let b be any cofinal branch in T. Then the commutative system:*

$$\langle \eta_i : i \in b \rangle, \langle e^{i,j} : i \leq j \textit{ in } b \rangle$$

has a well founded limit.

Note. This is surprising since, as we shall see, there is only one branch which yields a good limit, whereas these could be many cofinal branches.

We now turn to the proof of Lemma 3.7.21. Let $i_0 \in b$ such that there is no drop point in $b \setminus i_0$. Hence $e^{i,j}(\eta_i) = \eta_i$ for $i \leq j$, $i, j \in b$. Let $\hat{\eta} + 1$, $\langle e^i : i \in b \setminus i_0 \rangle$ be the direct limit of

$$\langle \eta_i + 1 : i \in b \setminus i_0 \rangle, \langle e^{i,j} : i \leq j \text{ in } b \setminus i_0 \rangle$$

We claim that $\hat{\eta}$ is well founded.

Set: $\tilde{\kappa}_\tau =: t_\xi$ for $\tau, \xi + 1 \in b \setminus i_0, \tau = T(\xi + 1)$. Using Lemma 3.7.20 it is straightforward to see that:

(a) $\hat{e}^{\tau,\mu} \upharpoonright \tilde{\kappa}_\tau = \mathrm{id}$ for $\tau \leq \mu$ in $b \setminus i_0$.

(b) $\tilde{\kappa}_\tau < e^{\tau,\xi+1}(\tilde{\kappa}_\tau) \leq \tilde{\kappa}_{\xi+1}$.

(c) $e^{\tau,\xi+1}(\tilde{\kappa}_\tau + j) = e^{\tau,\xi+1}(\tilde{\kappa}_\tau) + j$.

(d) If τ is a limit ordinal, then:

$$\eta_\tau = \bigcup \{ \operatorname{rng} e^{i,\tau} : i_0 < i < \tau \text{ in } b \}.$$

Given this, the conclusion follows from a sublemma, which -in an effort to simplify notation- we formulate abstractly:

Sublemma. *Let η be a limit ordinal. Let $\langle\delta_i : i < \eta\rangle$ be a sequence of ordinals and $e_{ij} : \delta_i \longrightarrow \delta_j$ $(i \leq j < \eta)$ be a commutative system of order preserving maps. Let*

$$\Delta,\ \langle e_i : i < \eta\rangle$$

be the direct limit of

$$\langle\delta_i : i < \eta\rangle,\ \langle e_{i,j} : i \leq j < \eta\rangle$$

Let $<_\Delta$ be the induced order on Δ. Assume that $\kappa_i < \delta_i$ for $i < \eta$ such that the following hold:

(a) $e_{i,j} \restriction \kappa_i = \mathrm{id}$

(b) $\kappa_i < e_{i,i+1}(\kappa_i) \leq \kappa_{i+1}$

(c) $e_{i,i+1}(\kappa_i + j) = e_{i,i+1}(\kappa_i) + j$

(d) $\delta_\lambda = \bigcup_{i<\lambda} \mathrm{rng}(e_{i,\lambda})$ *for limit* $\lambda < \eta$.

Then $<_\Delta$ is well founded.

Proof. Set $\tilde{\Delta} = \mathrm{wfc}(\langle\Delta, <_\Delta\rangle)$. Assume w.l.o.g. that $\tilde{\Delta}$ is transitive and $<_\Delta \cap \tilde{\Delta}^2 = \in \cap \tilde{\Delta}^2$. Thus, our assertion amounts to: $\tilde{\Delta} = \Delta$.

(1) $\kappa_j \geq \kappa_i$ for $j > i$.

Proof. Otherwise $e_{i,j+1}(\kappa_i) > \kappa_j$ where $\kappa_j < \kappa_i$, contradicting (a).

(2) $\kappa_j > \kappa_i$ for $j > i$.

Proof. $\kappa_j \geq \kappa_{j-1} > \kappa_i$ by (b).

(3) Let $e_i(h) \in \tilde{\Delta}$. Let $\mu \leq \delta_i$ and:

$$e_{i,j}(h + l) = e_{i,j}(h) + l \text{ for } i \geq j \text{ and } h + l < \mu.$$

Then $e_i(h + l) = e_i(h) + l$ for $h + l \leq \mu$.

Proof. Suppose not. Let l be the least counterexample. Then $l > 0$. Let $e_j(\alpha) = e_i(h) + l$ for a $j \geq i$. Then $e_{ij}(h) < \alpha < e_{ij}(h) + l$, since

$$e_j e_{ij}(h) < e_j(k) < e_j(e_{ij}(h) + l)$$

Hence $\alpha = e_{ij}(h) + k$ for a $k < l$. Hence:

$$e_j(\alpha) = e_j(e_{ij}(h) + k) = e_i(h) + k < e_j(h) + l = e_j(\alpha).$$

Contradiction!

QED(3)

Taking $h = 0$, we have $e_{ij}(l) = i$ for $l < \kappa_i$. Hence:

(4) $\kappa_i \subset \tilde{\Delta}$ and $e_i \upharpoonright \kappa_i = \mathrm{id}$.

(5) Let $e_{ij}(h) \geq \kappa_j$. Then $e_{ij}(h+l) = e_{ij}(h) + l$ for all $h + l < \delta_i$.

Proof. By induction on $j \geq i$. The case $i = j$ is trivial. Now let $j = k+1$, where it holds at k. Then $e_{i,k}(h) \geq \kappa_k$, since otherwise:

$$e_{ij}(h) = e_{k,k+1}e_{ik}(h) = e_{i,k}(h) < \kappa_h < \kappa_j.$$

Hence:

$$\begin{aligned} e_{i,k}(h+l) &= e_{kj}e_{ik}(h+l) = e_{kj}(e_{ik}(h)+l) \\ &= e_{kj}(h) + l \end{aligned}$$

since if $e_{ik}(h) = \kappa_k + a$, then:

$$\begin{aligned} e_{k,k+1}(h+l) &= e_{k,k+1}(\kappa_k + a + l) = e_{k,k+1}(\kappa_k) + a + l \\ &= e_{h,k+1}(\kappa_k + a) + l = e_{k,k+1}(h) + l \end{aligned}$$

Now let j be a limit ordinal. Then:

$$\delta_j, \langle e_{ij} : i < j \rangle$$

is the limit of

$$\langle \delta_i : i < j \rangle,\ \langle e_{h,i} : h \leq i < j \rangle$$

and we apply (3).

QED(5)

We now prove $\Delta \subset \tilde{\Delta}$ by cases as follows:

Case 1: For all $i < \eta, h < \delta_i$ there is $j > i$ such that $e_{ij}(h) < \kappa_j$.

Then $e_i(h) = e_j e_{i,j}(h) \subset \kappa_j$, since $e_j \upharpoonright \kappa_j = \mathrm{id}$. Thus $\Delta = \bigcup_i \mathrm{rng}(e_i) \subset \bigcup_i \kappa_i \subset \tilde{\Delta}$.

Case 2: Case 1 fails.

Then there is i such that for some $h < \delta_{i_0}$, we have: $e_{ij}(h) \geq \kappa_i$ for all $j \geq i$. Since $e_{jk}e_{ik}(h) \geq e_{ik}(h) \geq \kappa_k$ for $i_0 \leq j \leq k$, there is for each $j \geq i_0$ a least h_j such that $e_{jl}(h_j) \geq \kappa_l$ for all $l \geq j$.

Claim. $e_{ij}(h_j) = h_j$ for $i_0 \leq i \leq j$.

Proof. Suppose not. Let j be the least counterexample. Using (3) it follows that $j = l+1$ is a successor. Then $h_j < e_{l,j}(h_l)$. But $h_j \geq \kappa_j \geq e_{lj}(\kappa_l)$.

Hence $h_j = e_{lj}(\kappa_l) + a = e_l(\kappa_l + a)$, where $\kappa_l + a < h_l$. But for $j' \geq j$ we have:

$$h_{l,j}(\kappa_l + a) = h_{j,j'}(e_{l,j}(\kappa_l) + a) \geq \kappa_{j'}.$$

Hence $h_l \leq \kappa_l + a < h_l$. Contradiction!

QED(Claim)

But then $e_i(h_i) = e_j(h_j)$ for $i_0 \leq i \leq j < \eta$. Now let $\tilde{h} = e_i(h_i)$ for $i_0 \leq i < \eta$. Then:

Claim. $\tilde{h} = \bigcup\{h_i : i_0 \leq i < \eta\}$.

Proof. $\tilde{h} = \bigcup_i e_i``h_i$. But if $a < h_i$, then $e_{ij}(a) < \kappa_j$ for some $j \geq i$ by the minimality of h_i. Hence $e_i(a) = e_j(e_{i,j}(a)) = e_{i,j}(a) < h_j$, since $e_j \restriction \kappa_j = \mathrm{id}$.

QED(Claim)

Hence $\tilde{h} \in \tilde{\Delta}$ and:

$$e_j(h_j + l) = \tilde{h} + l \text{ for } h_j + l < \delta_j,$$

by (3), (5). Hence $\mathrm{rng}(e_j) \subset \tilde{\Delta}$ and $\Delta = \tilde{\Delta}$. This proves the sublemma and with it Lemma 3.7.21.

QED(Lemma 3.7.21)

Note that $\eta_0 \geq \tilde{\kappa}_i$ for $i \in b \setminus i_0$ where $e^i(\eta_i) = \hat{\eta}$. Hence as a corollare of the proof we have:

Corollary 3.7.22. *Set $\tilde{\eta}_i =$ the least h such that $e^{i,j}(h) \geq \tilde{\kappa}_j$ for all $j \geq i$. Then $\tilde{\eta}_i$ is defined for sufficiently large i and $e^i(\tilde{\eta}_i) = \tilde{\eta}$. Moreover $\tilde{\eta} =$* $\mathrm{lub}\{\tilde{\eta}_i : i < \eta\}$.

However, in order to prove Theorem 3.7.14 we must find the "right" cofinal branch in T. Lemma 3.7.19 suggests an obvious strategy: Let $\tilde{b}$ be the unique well founded cofinal branch in $\tilde{I}$. Set:

$$\hat{b} = \{\gamma(i) : i \in \tilde{b}\}, b = \{\tau : \bigvee \gamma \in \hat{b},\ \tau \leq_T \gamma\}$$

Then b is a cofinal branch in T. We show that this branch works, thus establishing the existence assertion of Theorem 3.7.14.

By Lemma 3.7.21, the commutative system

$$\langle \eta_i + 1 : i \in b \rangle, \langle e^{i,j} : i \leq j \text{ in } b \rangle$$

has a transitivized direct limit:

$$\hat{\eta}+1, \langle e^i : i \in b \rangle$$

This gives us a system of indices with which to work.

We must show that the commutative insertion system:

$$\langle I^h : h \in b \rangle, \langle e^{h,j} : h \leq j \text{ in } b \rangle$$

has a good limit I. By induction on $i < \hat{\eta}$ we, in fact, show:

Lemma 3.7.23. *Let $i < \hat{\eta}$. Then the above commutative system has a good limit $I|i+1$ with respect to i in the sense of Definition 3.7.5 at the end of §3.7.1. In other words, $I|i+1$ has length $i+1$ and $e^\xi \restriction h+1$ inserts $I^\xi|h+1_i$ into $I|i+1$ whenever $e^\xi(h) = i$.*

Remark on notation. In §3.7.1 we showed that there can be at most one good limit below i. We denote this, if it exists, by $I|i+1$. But then $(I|i+1)|h+1 = I|h+1$ by uniqueness.

We recall that we defined: $\tilde{\kappa}_\tau = t_\xi$ where $\tau = T(\xi+1), \xi+1 \in b$, and that $\tilde{\kappa}_\tau = \text{crit}(e^{\tau,j}) = \text{crit}(e^\tau)$ for $\tau < j$ in b.

But then $\tilde{I} = \bigcup_{\tau \in b} I^\tau|\tilde{\kappa}_\tau$, since if $\tau = T(\xi+1), \xi+1 \in b$, then:

$$I^\tau|\tilde{\kappa}_\tau = (I^\xi|s_{\eta+1})|\tilde{\kappa}_\tau = \tilde{I}|\tilde{\kappa}_\tau.$$

But $\bigcup_{\tau \in b} \tilde{\kappa}_\tau = \bigcup_{i<\eta} s_i + 1,$ since if $\tau = \delta+1$, then:

$$\hat{s}_\tau = s_\delta + 1 \leq t_\xi = \tilde{\kappa}_\tau.$$

We prove Lemma 3.7.23 by induction on $i \leq \hat{\eta}$.

Case 1. $i < \tilde{\eta} = \text{lh}(\tilde{I})$.

Let $e^\xi(h) = i$. Let $\xi <_T \tau \in b$, where $i+1 < \tilde{\kappa}_\tau$. Then $e^\xi|h+1 = (e^\tau|i+1)(e^{\xi,\tau}|h+1)$ where $e^\tau|i+1 = \text{id}$. Hence:

$$e^\xi|h+1 = e^{\xi,\tau}|h+1 \text{ inserts } I^\xi|h+1 \text{ into } I^\tau|h+1 = I|h+1$$

QED(Case 1)

Case 2. $i = \tilde{\eta}$.

Let $\tilde{b}$ be the unique cofinal well founded branch in $\tilde{I}$. Let $M_{\tilde{\eta}}, \langle\tilde{\pi}_{i,\tilde{\eta}} : i \in \tilde{b}\rangle$ be the transitivized direct limit of: $\langle M_i : i \in b\rangle$, $\langle\tilde{\pi}_{ij} : i \leq_T j \in \tilde{b}\rangle$. This gives us $I|\tilde{\eta}+1$. We must prove that whenever $e^{\xi}(\overline{\eta}) = \tilde{\eta}, \xi \in b$, then e^{ξ} inserts $I^{\xi}|\overline{\eta}+1$ into $I|\tilde{\eta}+1$. By Lemma 3.7.8 it suffices to show that for arbitrarily large $\xi \in b$:

(∗) $e^{\xi}(\overline{\eta}) = \tilde{\eta}$, where either $\hat{e}^{\xi}(\overline{\eta}) \in \tilde{b}$ or else $\hat{e}^{\xi}(\overline{\eta}) = \tilde{\eta}$ and $\hat{e}^{\xi}(i) \in \tilde{b}$ for all $i <_{T^{\xi}} \overline{\eta}$.

We know: $\tilde{\kappa}_\tau = \mathrm{crit}(e^{\tau,\xi+1}) = t_\xi$ for $\tau = T(\xi+1)$, $\xi+1 \in b$. Set:

$$\tilde{\lambda}_\tau =: e^{\tau,\xi+1}(\tilde{\kappa}_\tau) = s_\xi + 1 \text{ for } \tau = t(\xi+1),\ \xi+1 \in b.$$

Then:

(1) $\tilde{b} \cap \bigcup_{\tau\in b}(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) = \varnothing$.

Proof. Suppose not. Let $i \in \tilde{b} \cap (\tilde{\kappa}_\tau, \tilde{\lambda}_\tau)$ where $\tau \in b$. Let $\mu > \tau$ such that:

$$\mu \in \hat{b} = \{\gamma(i) : i \in \tilde{b}\}.$$

Let $\mu = \gamma(j), j \in \tilde{b}$. Then $\hat{s}_\mu \leq j \leq s_\mu$. Then $i < j$ in $\tilde{b}$, since:

$$i \leq s_\xi < \hat{s}_\mu \leq j, \text{ where } \tau = T(\xi+1),\ \xi+1 \in b.$$

But $\tilde{T}|s = T^{\mu}|s_\mu + 1$. Hence $i <_{T^{\mu}} j$ in I^{μ}. But:

$$(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) = (t_\xi, s_\xi].$$

Hence $(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau)$ is in limbo at μ, since $\xi+1 \leq_T \mu$. Hence $i \not<_{T^{\mu}} j$. Contradiction!

QED(1)

Set:

$$A = \{\tau \in b : \hat{s}_\tau < \tilde{\kappa}_\tau\}.$$

The set A strongly determines what happens at $\tilde{\eta}$. We first consider the case:

Case 2.1. A is cofinal in b.

There is then a $\tau_0 \in b$ such that $\hat{s}_\tau = \tilde{\kappa}_\tau$ for all $\tau \in b \setminus \tau_0$. (Recall that, if $T = T(\xi+1)$ and $\xi+1 \in b$, then $\tilde{\kappa}_\tau = t_\xi$ and $\hat{s}_\tau \leq t_\xi \leq s_\tau < \tilde{\lambda}_\tau$ by Lemma 3.7.17.) By (1) we have:

$$\tilde{b} \setminus \tau_0 \subset B =: \{\hat{s}_i : \tau_0 \leq i \text{ in } b\} = \{\tilde{\kappa}_i : \tau_0 \leq i \text{ in } b\}.$$

(2) $\tilde{b}\setminus\tau_0 = B$.

Proof. Suppose not. Let $i \in B\setminus\tilde{b}_0$ be the least counterexample. Then $i > \tau_0$. Moreover, i is not a limit ordinal, since otherwise $i = \mathrm{lub}\{\hat{s}_j : j \in B \cap i\}$, where $B \cap i \subset \tilde{b}$ and $\tilde{b}$ is closed in $\tilde{\eta}$. Hence:

$$i = \hat{s}_{\xi+1} = s_\xi + 1, \text{ where } \xi + 1 \in b\setminus(\tau_0 + 1).$$

Let $\tau = T(\xi + 1)$. Then $\tau \geq \tau_0$ in b and

$$\hat{s}_\tau = \tilde{\kappa}_\tau = t_\xi, \ s_\xi + 1 = \tilde{\lambda}_\xi.$$

Hence $\hat{s}_\tau = \tilde{T}(s_\xi + 1)$, where $\hat{s}_\tau \in B$. Clearly $\hat{s}_\tau \in \tilde{b}$, by the minimality of i. Now let $j+1 \in \tilde{b}$ such that $\hat{s}_\tau = \tilde{T}(j+1)$. Then $j+1 \geq \tilde{\lambda}_\tau = s_\xi+1$, since $j + 1 > \tilde{\kappa}_\tau$ and $(\tilde{\kappa}_\tau, \tilde{\lambda}_\tau) \cap \tilde{b} = \varnothing$. Let $\gamma = \gamma(j + 1)$. Then $j + 1 = \hat{s}_\gamma = \tilde{\kappa}_\gamma$ is a successor ordinal. Hence $\hat{s}_\gamma = s_\delta + 1$, where $\gamma = \delta + 1$. Let $\mu = T(\delta + 1)$. Then $\hat{s}_\mu = \hat{\kappa}_\mu = \hat{T}(s_\delta + 1)$. Hence $\hat{s}_\mu = \hat{s}_t$. Hence $\mu = \tau, \delta = \xi$ and $i = \hat{s}_\xi + 1 = j + 1 \in \tilde{b}$. Contradiction!

QED(2)

But then every $\tau \in b\setminus\tau_0$ satisfies (*), since:

(3) Let $\tau_0 \leq \tau \in b$. Then $e^\tau(\tilde{\kappa}_\tau) = \tilde{\eta}$ and $e^\tau \restriction \tilde{\kappa}_\tau = \mathrm{id}$. (Hence $\hat{e}^\xi(\tilde{\kappa}_\tau) = \tilde{\kappa}_\tau \in \tilde{b}$).

Proof. We know that if $\tau = T(\xi + 1), \xi + 1 \in b$, then:

$$e^{\tau,\xi+1} \restriction \tilde{\kappa}_\tau = \mathrm{id}, \ e^{\tau,\xi+1}(\tilde{\kappa}_\tau) = \tilde{\lambda}_\tau = s_\xi + 1 = \tilde{\kappa}_{\xi+1}$$

Using this we prove by induction on $\xi \in b\setminus\tau_0$ that if $\tau_0 \leq \tau < \xi$, $\tau \in b$, then:

$$e^{\tau,\xi} \restriction \tilde{\kappa}_\tau = \mathrm{id}, e^{\tau,\xi}(\tilde{\kappa}_\tau) = \tilde{\kappa}_\xi.$$

At limit ξ we use the fact that:

$$e^{\tau,\xi}(i) = \bigcup_{\tau \leq \tau' \in b} e^{\tau',\xi}{}''e^{\tau,\tau'}(i).$$

But then the same proof shows:

$$e^\tau \restriction \tilde{\kappa}_\tau = \mathrm{id}, e^\tau(\tilde{\kappa}_\tau) = \tilde{\eta},$$

since:

$$\tilde{\eta} = \sup_{\tau \in b\setminus\tau_0} \tilde{\kappa}_\tau = \sup_{\tau \in b\setminus\tau_0} \hat{s}_\tau = \sup_{\xi+1 \in b\setminus\tau_0} s_\xi + 1.$$

QED(Case 2.1)

Case 2.2. A is cofinal in b.

We shall make use of the following general lemma on normal reiteration:

Lemma 3.7.24. *Let $\xi \leq_T \mu, i \leq \eta_\xi$ such that $\hat{s}_\mu \leq j < e^{\xi,\mu}(i)$. Then $j \in \mathrm{rng}(e^{\sigma,\mu})$.*

Proof. Suppose not. Let μ be the least counterexample. Then $\mu > \xi$.

Case 1. μ is a limit ordinal.

Let ζ such that $\xi \leq \zeta < \mu$ and $j = e^{\zeta,\mu}(j')$. Then $j' \geq \tilde{\kappa}_\zeta$, since otherwise:

$$j = j' < \kappa_\zeta < \tilde{\lambda}_\zeta < \hat{s}_\mu.$$

Contradiction! Thus $\hat{s}_\zeta \leq j' \leq e^{\zeta,\mu}(i)$. By the minimality of μ we conclude:

$$j' \in \mathrm{rng}(e^{\zeta,\mu});$$

hence $j = e^{\zeta,\mu}(j') \in \mathrm{rng}(e^{\zeta,\mu})$. Contradiction!

Case 2. $\mu = \zeta + 1$ is a successor.

Let $\tau = T(\zeta + 1)$. Then $j \geq \hat{s}_\mu = s_\zeta + 1 = \tilde{\lambda}_\tau$. Moreover:

$$e^{\tau,\mu}(\tilde{\kappa}_\tau + h) = \tilde{\lambda}_\tau + h \text{ for } h \leq \eta_\tau.$$

Let $j = \tilde{\lambda}_\tau + h, e^{\xi,\mu}(i) = \tilde{\lambda}_\tau + k$. Hence $h < k$. Set $j' = \tilde{\kappa}_\tau + h$. Then $e^{\tau,\mu}(j') = j$, where $\hat{s}_\tau \leq \tilde{\kappa}_\tau \leq j' < e^{\xi,\mu}(i)$. By the minimality of μ we conclude: $j' \in \mathrm{rng}(e^{\xi,\tau})$. Hence $j = e^{\tau,\mu}(j') \in \mathrm{rng}(e^{\xi,\mu})$. Contradiction!

QED(Lemma 3.7.24)

Let $\tau_0 \in b$ such that $\tilde{\eta} \in \mathrm{rng}(\tilde{e}^{\tau_0})$. Then $\tilde{\eta} \in \mathrm{rng}(e^\tau)$ for all $\tau \in b \setminus \tau_0$. Set:

$$\tilde{\eta}_\tau = (e^\tau)^{-1}(\tilde{\eta}) \text{ for } \tau \in b \setminus \tau_0.$$

Then:

(4) $e^\tau(\tilde{\kappa}_\tau) < \tilde{\eta}$ for $\tau \in b \setminus \tau_0$.

Proof. Let $\tau < \gamma \in A$. Then $e^{\tau,\gamma}(\tilde{\kappa}_i) \leq \hat{s}_\gamma < \tilde{\kappa}_\gamma$ by Lemma 3.7.20. Hence:

$$e^\tau(\tilde{\kappa}_\tau) = e^\gamma \cdot e^{\tau,\gamma}(\tilde{\kappa}_\tau) = e^{\tau,\gamma}(\tilde{\kappa}_\tau) < \tilde{\kappa}_\gamma < \tilde{\eta}.$$

QED(4)

Now set:

$$B =: \bigcup_{\tau \in b \setminus \tau_0} [\tilde{s}_\tau, \tilde{\kappa}_\tau).$$

Note. $[\hat{s}_\tau, \tilde{\kappa}_\tau) = \varnothing$ if $\tau \notin A$.

(5) Let $\tau_0 \leq \tau \in b$. Then $B \subset \text{rng}(e^\tau)$.

Proof. Let $\tau \leq \gamma \in A$. Let $j \in [\hat{s}_\gamma, \tilde{\kappa}_\gamma)$. Then

$$\hat{s}_\gamma \leq j \leq \tilde{\eta}_\gamma = e^{\tau,\gamma}(\tilde{\eta}_\gamma).$$

But then by Lemma 3.7.24 we have:

$$[\hat{s}_\gamma, \tilde{\kappa}_\gamma) \subset \text{rng}(e^{\tau,\gamma}).$$

But $e^\gamma \upharpoonright [\hat{s}_\gamma, \tilde{\kappa}_\gamma) = \text{id}$. Hence:

$$[\hat{s}_\gamma, \tilde{\kappa}_\gamma) \subset \text{rng}(e^\tau) = \text{rng}(\tilde{e}^\gamma \tilde{e}^\tau).$$

QED(5)

Since B is cofinal in $\tilde{\eta}$, we conclude:

(6) $e^\tau``\tilde{\eta}_\tau$ is cofinal in $\tilde{\eta}$ for $\tau \in b \setminus \tau_0$. Using this we then get:

(7) Let $\tau \in b \setminus \tau_0$. Then:

$$\tilde{b} \cap (\text{rng}(e^\tau) \cup \text{rng}(\hat{e}^\tau))$$

is cofinal in $\tilde{\eta}$.

Proof. Suppose not. Then there is a $i_0 < \tilde{\eta}$, such that

$$\tilde{b} \cap (\text{rng}(e^\tau) \cup \text{rng}(\hat{e}^\tau)) \subset i_0.$$

Note that if $\gamma \in A$, then $[\hat{s}_\gamma, \tilde{\kappa}_\gamma) \subset \text{rng}(e^\tau)$. Hence $(\hat{s}_\gamma, \tilde{\kappa}_\gamma] \subset \text{rng}(\hat{e}^\tau)$. We shall derive a contradiction by showing that A is not cofinal in b. In particular, we show:

Claim. Let $i_0 < j \in \tilde{b}$. Let $\gamma_0 = \gamma(j)$. Assume that $\gamma \leq \delta \in b$. Then $\hat{s}_\delta = \tilde{\kappa}_\delta \in \tilde{b}$.

Proof. We proceed by induction on δ. There are three cases:

Case 2.2.1. $\delta = \gamma_0$.

It suffices to show: $\gamma_0 \notin A$, since then $\hat{s}_{\gamma_0} \leq j < \tilde{\lambda}_\gamma, j \notin (\tilde{\kappa}_{\gamma_0}, \tilde{\lambda}_{\gamma_0})$, where $\hat{s}_{\gamma_0} = \tilde{\lambda}_{\gamma_0}$. Hence $j = \hat{s}_\gamma = \hat{\kappa}_\gamma \in \tilde{b}$. Suppose not. $j \in [\hat{s}_\gamma, \tilde{\kappa}_\gamma]$ since $(\tilde{\kappa}_\gamma, \tilde{\lambda}_\gamma) \cap \tilde{b} = \varnothing$. But $[\hat{s}_\gamma, \tilde{\kappa}_\gamma] \subset \text{rng}(e^\tau) \cup \text{rng}(\hat{e}^\tau)$. Contradiction!, since $j < i_0$.

QED(Case 2.2.1)

Case 2.2.2. $\delta = \xi + 1 > \gamma_0$ is a successor.

Let $\mu = T(\xi + 1)$. Hence, $\gamma_0 \leq \mu \in b$. Then $s_\mu = \tilde{\kappa}_\mu \in \tilde{b}$. Let $j + 1$ be the immediate successor of s_μ in $\tilde{b}$. Then $\tilde{\kappa}_\mu < j + 1$. Hence

$j+1 \geq \tilde{\lambda}_\mu = s_\xi + 1$, since $(\tilde{\kappa}_\mu, \tilde{\lambda}_\mu) \cap \tilde{b} = \varnothing$. Let $\gamma = \gamma(j+1)$. Then $j+1 \in [\hat{s}_\gamma, \tilde{\kappa}_\gamma]$. Hence, as in Case 2.2.1, $\tilde{\kappa}_\gamma = \hat{s}_\gamma$, since otherwise:

$$[\hat{s}_\gamma, \tilde{\kappa}_\gamma] \subset \mathrm{rng}(e^\tau) \cup \mathrm{rng}(\hat{e}^\tau).$$

Then $j+1 = \hat{s}_\gamma = \tilde{\kappa}_\gamma$ and $\hat{s}_\gamma = s_\xi + 1$, where $\gamma = \zeta+1$. Let $\xi = T(\zeta+1)$. Then $\tilde{\kappa}_\eta = \tilde{T}(j+1)$, where $j = s_\zeta$. Hence $\tilde{\kappa}_\eta = \hat{s}_\mu = \tilde{T}(j+1)$. Hence $\eta = \mu$, since otherwise $\eta > \mu$ and $\hat{s}_\mu < \hat{s}_\eta = \tilde{\kappa}_\eta$. Hence $\xi = \zeta$, since $\xi + 1 = \zeta + 1 =$ the immediate successor of μ in b. Hence $\hat{s}_\delta = \tilde{\kappa}_\delta \in \tilde{b}$.

QED(Case 2.2.2)

Case 2.2.3. $\delta > \gamma_0$ is a limit ordinal.

Then $\hat{s}_\delta = \sup_{i<\delta} \hat{s}_i \in \tilde{b}$, since $\tilde{b}$ is closed in $\tilde{\eta}$. But then $\hat{s}_\delta = \tilde{\kappa}_\delta$, since otherwise:

$$[\hat{s}_\delta, \tilde{\kappa}_\delta) \subset \mathrm{rng}(e^\tau), \text{ where } \hat{s}_\delta > i_0.$$

QED(Case 2.2.3)

This proves (7).

We now show that (*) holds for all $\tau \in b \setminus \tau_0$.

(8) Let $\tau \in b \setminus \tau_0$. If $i <_{T^\tau} \tilde{\eta}_\tau$, then $\hat{e}^\tau(i) \in \tilde{b}$.

Proof. Set: $\overline{b} = (\hat{e}^\tau)^{-1}{}''\tilde{b}$.

Claim 1. $\overline{b}$ is cofinal in $\tilde{\eta}_\tau$.

Proof. Let $i < \tilde{\eta}_\tau$. Set $i' = e^\tau(i)$. By (7) there is $j' \in \tilde{b}$ such that

$$j' > i' \text{ and } j' \in \mathrm{rng}(e^\tau) \cup \mathrm{rng}(\hat{e}^\tau).$$

If $e^\tau(j) = j'$, then $j > i$ and $\hat{e}^\tau(j) \leq_{\tilde{T}} j' \in \tilde{b}$. Hence $\hat{e}^\tau(j) \in \tilde{b}$ and $j \in \overline{b}$. If $\hat{e}^\tau(j) = j'$, then $\hat{e}^\tau(i) < j' \in \tilde{b}$. Hence $j > i$ and $j \in \overline{b}$.

QED(Claim 1)

Claim 2. $\overline{b}$ is a branch in T^τ.

Proof. Let $i <_{T^\tau} j \in \overline{b}$. Then $\hat{e}^\tau(i) \leq_{\tilde{T}} \hat{e}^\tau(j) \in \tilde{b}$. Hence $\hat{e}^\tau(i) \in \tilde{b}$ and $i \in \overline{b}$.

QED(Claim 2)

Claim 3. $\overline{b}$ is well founded.

This follows by standard methods, given that $\tilde{b}$ is well founded. But then $\overline{b} = T^{\tau}{}''\{\tilde{\eta}_\tau\}$ by uniqueness.

QED(Case 2)

Case 3. $i > \tilde{\eta}$.

Then $e^\tau(\tilde{\eta}_\tau + i) = \tilde{\eta} + i$ by Lemma 3.7.24. Using this, it follows easily by Lemma 3.7.8 and Lemma 3.7.7 that $I|i+1$ exists. We leave the details to the reader.

QED(Lemma 3.7.23)

This proves the existence part of Theorem 3.7.24. We must still prove uniqueness.

Definition 3.7.12. Let b be a cofinal branch in:

$$R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle,$$

where R is a reiteration of limit length η. b is *good for* R iff R extends to R' of length $\eta+1$ with $b = T``\{\eta\}$.

We have proven the existence of a good branch b. Now we must show that it is the only one. Suppose not. Let b^* be a second good branch, inducing R^* of length $\eta+1$ with: $b^* = T^*``\{\eta\}$. Since b, b^* are distinct cofinal branches in T, there is $\tau_0 < \eta$ such that:

$$(b\setminus\tau_0) \cap (b^*\setminus\tau_0) = \varnothing.$$

$I' = (I^\eta)^{R'}$ has length $\hat{\eta}$ and $I^* = (I^\eta)^{R^*}$ has length η^*. However:

$$\tilde{\eta} = \bigcup_{i<\eta} s_i + 1, \; \tilde{I} = \bigcup_{i<\eta} I|s_i + 1$$

remain unchanged. Moreover $I = I'|\tilde{\eta} = I^*|\tilde{\eta}$. Since $\hat{b}$ is the unique cofinal well founded branch in $\tilde{I}$, we must have:

$$\tilde{b} = T'``\{\tilde{\eta}\} = T^*``\{\tilde{\eta}\}.$$

Now let $\gamma > \tau_i$ such that:

$$\gamma = \gamma(i) \in \hat{b} = \{\gamma(i) : i \in \tilde{b}\}$$

Then $\gamma \in b\setminus\tau_0$. Let $\gamma = \gamma(i)$ where $i \in \tilde{b}$. Then $\hat{s}_\gamma \leq i \leq s_\gamma$.

Let δ be least such that $\delta \in b^*$ and $\delta > \gamma_0$. Then $\delta = \xi + 1$ and $\tau =:$ $T^*(\xi+1) < \gamma$. Then $t_\xi \leq s_\tau$. But

$$s_\tau < \hat{s}_\gamma \leq i \leq s_\gamma, \text{ where } s_\gamma + 1 = \hat{s}_{\gamma+1} \leq \hat{s}_\xi = s_\xi + 1.$$

Hence $i \in (t_\xi, s_\xi]$. But then:

$$i < s_\xi + 1 = \tilde{\lambda}^*_\tau \leq \tilde{\kappa}^*_\delta = \mathrm{crit}(e^{*\delta})$$

Hence $e^{*\delta}(i) = i \in b^*$. But $i <_{T^*} \tilde{\eta}$, since $i \in \tilde{b}$. Hence, letting $e^{*\delta}(\tilde{\eta}^*_\delta) = \tilde{\eta}$, we have:

$$i <_T \eta^*_\delta, \text{ where } \tilde{\eta}^*_\delta \geq \hat{s}_0 = s_\xi + 1.$$

But this is impossible, since $(t_\xi, s_\xi]$ is in limbo at δ. Contradiction!

QED(Theorem 3.7.14)

We have shown that, if M is uniquely normally iterable, then it is uniquely normally iterable in the sense that every normal reiteration of limit length has exactly one good branch. As we stated at the outset, the result can be relativized to a regular $\theta > \omega$. In this case we restrict ourselves to θ-reiterations.

Definition 3.7.13. Let $\theta > \omega$ be regular. A normal reiteration $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is called a *θ-reiteration* iff $\mathrm{lh}(R) < \theta$ and $\mathrm{lh}(I^i) < \theta$ for all i. M is *uniquely normally θ-reiterable* iff every θ-reiteration of limit length $< \theta$ has one good branch.

We have shown that, if M is uniquely normally θ-iterable, then it is uniquely normally θ-reiterable. But what if M is, in fact, $\theta + 1$ iterable? Can we strengthen the the conclusion correspondingly? We define:

Definition 3.7.14. Let θ, R be as above. R is a θ+1-reiteration iff $\mathrm{lh}(R) \leq \theta$ and $\mathrm{lh}(I^i) < \theta$ for all i. M is *uniquelly normally $\theta + 1$ reiterable* iff every θ-reiteration of length $\leq \theta$ has a unique good branch.

Now suppose M be normally $\theta + 1$-iterable. Let R be a $\theta + 1$ reiteration of length θ. Define $\tilde{I}, \tilde{b}, \hat{b}, b$ exactly as before. Then b is a cofinal branch in T. (It is also the unique such branch, since if b' were another such, then $b \cap b'$ s club in θ. Hence $b = b'$). b has at most finitely many drop points, since otherwise some proper segment of b would have infinitely many drop points. Suppose that $\gamma \in b$ and $b \setminus \gamma$ has no drop points. Then:

$$\langle\langle I^i : i \in b\setminus\gamma\rangle, \langle e^{i,j} : i < j \in b\setminus\gamma\rangle\rangle$$

has a unique good limit:

$$\langle I, \langle e^i : i \in b\setminus\gamma\rangle\rangle$$

by Lemma 3.7.9. Hence b is a good branch. Thus we have:

Lemma 3.7.25. *If M is uniquely normally iterable, then it is uniquelly normally reiterable. Moreover if $\theta > \omega$ is regular, then:*

(a) *If M is uniquely normally θ-iterable, then it is uniquely normally θ-reiterable.*

(b) *If M is uniquely normally $\theta + 1$-iterable, then it is uniquely normally $\theta + 1$-reiterable.*

Remark. The assumption that M is uniquely normally iterable can be weakened somewhat. We define:

Definition 3.7.15. Let S be a normal iteration strategy for M. S is *insertion stable* iff whenever I is an S-conforming iteration of M and e inserts $\overline{I}$ into I, then $\overline{I}$ is an S-conforming iteration.

Now suppose that M is iterable by an insertion stable strategy S. We can define the notion of a *normal reiteration on* $\langle M, S\rangle$ exactly as before, except that we require each of the component normal iterations I^i to be S-conforming. (We could also call this an *S-conforming normal reiteration on M*). All of the assertions we have proven in this subsection go through for reiterations on $\langle M, S\rangle$, with nominal changes in formulation and proofs. For instance, if we alter the definition of *good branch* mutatis mutandis, our proofs give:

> $\langle M, S\rangle$ is *uniquely reiterable* in the sense that every reiteration of limit length has exactly one good branch.

We close this section with two technical lemmas which will be of use later. Both assume the unique iterability (or θ-iterability) of M.

Lemma 3.7.26. *Let I, I' be normal iterations of M. There is at most one pair $\langle R, \xi\rangle$ such that*

$$R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle,$$

is a reiteration of M, $\mathrm{lh}(R) = \xi + 1, I = I^0, I' = I^\xi$.

Proof. Assume such R, ξ to exist. Ww show that R, ξ are defined by a recursion:

$$R|i + 1 \cong F(R|i)$$

where ξ is least such that $F(R|\xi + 1)$ is undefined. F will be defined solely by reference to I, I'. We have:

$$R|1 = \langle\langle I\rangle, \varnothing, \langle\,\mathrm{id}\restriction\mathrm{lh}(I)\rangle, \varnothing\rangle.$$

At limit $\lambda, R\restriction\lambda + 1 = F(R|\lambda)$ is given by the unique good branch in $R|\lambda$. Now let $R|i+1$ be given. If $I^i = I'$, then $F(R|i+1)$ is undefined. If not, let $s = s_i$. Then $I^i|s+1 = I'|s+1$, since $\nu_i = \nu_s^{i+1} = \nu'_s$. If $s+1 < \mathrm{lh}(I^i)$, then $\nu_i = \nu'_s < \nu_s^i$. Hence $I^i|s+2 \neq I'|s+2$. We have shown:

$$\begin{aligned} s = {} & \text{the maximal } s \text{ such that } s+1 \leq \mathrm{lh}(I^i) \\ & \text{and } I^i|s+1 = I'|s+1. \end{aligned}$$

But then $R|i+2$ is uniquely defined from $R|i+1$ and $\nu_i = \nu'_s$.

QED(Lemma 3.7.26)

For later reference we state a further lemma about reiterations:

Lemma 3.7.27. *Let $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ be a reiteration of length $\mu+1$. Let I^i be of length η_i for $i \leq \mu$. Set:*

$$A_j = A_j^R =: \{i : i <_T j \text{ and } (i,j]_T \text{ has no drop point in } R\}$$

for $j \leq \mu$. Set:

$$\sigma_{i,j} = \sigma_{\eta_i}^{i,j} \text{ for } i \in A_j \text{ or } i = j$$

. Then:

(a) $e^{i,\mu}(\eta_i) = \eta_\mu$ *for* $i \in A_\mu$.

(b) $\sigma_{i,\mu} : M_{\eta_i} \longrightarrow_{\Sigma^*} M_{\eta_\mu}$ *for* $i \in A_\mu$.

(c) *If μ is a limit ordinal, then*

$$M_\eta = \bigcup_{i \in A_\mu} \text{rng}(\sigma_{i,\mu}).$$

Proof. We prove it by induction on μ.

Case 1. $\mu = 0$. Then $A_\mu = \varnothing$ and there is nothing to prove.

Case 2. $\mu = j+1$ is a successor. If μ is a drop point, then $A_\mu = \varnothing$ and there is nothing to prove. Assume that it is not a drop point. Then $h = T(\mu)$ is the maximal element of A_μ. (c) holds vacuously. We now prove (a), (b) for $i = h$. By our construction, $e^{h,\mu}(\eta_h) = \eta_h$ could only fail if μ is a drop point, so (a) holds. We now prove (b) for $i = h$. If $t_j < \eta_h$, then $\hat{e}^{h,\mu} = e^{h,\mu}$ and:

$$\sigma_{h,\mu} = \hat{\sigma}_{\eta_h}^{h,\mu} = \sigma_{\eta_h}^{h,\mu}.$$

Hence (b) holds. Now let $t_j = \eta_h$. Then $\eta_\mu = s_j + 1$ and:

$$\sigma_{\eta_h}^{h,\mu} : M_{\eta_h}^h \longrightarrow_F^* M_{\eta_\mu}^\mu,$$

where $F = E_{\nu_j}^{M_j}$. Hence (b) holds.

Now let $i < h$. Then $i \in A_h^{R|h+1}$. This gives us $\sigma_{ih} = \sigma_{\eta_i}^{i,h}$. Then (a)-(c) holds for $R|h+1$ by the induction hypothesis.

By Lemma 3.7.5 we then easily get:

$$\sigma_{h,\mu}\sigma_{i,h} = \sigma_{i,\mu}.$$

It follows easily that (a), (b) hold at i.

QED(Case 2)

Case 3. μ is a limit ordinal. Then $A_\mu = [i_0, \mu)_T$ for a $i_0 <_T \mu$. We know that:

$$\eta_\mu, \langle e^{i,\mu} : i \in A_\mu \rangle$$

is the transitivized direct limit of:

$$\langle \nu_i : i \in A_\mu \rangle, \langle e^{i,j} : i \leq j \text{ in } A_\mu \rangle$$

Hence (a) holds at μ. But:

$$I^\mu, \langle e^{i,\mu} : i \in A_\mu \rangle$$

is the good limit of:

$$\langle I^i : i \in A_\mu \rangle, \langle e^{i,j} : i \leq j \text{ in } A_\mu \rangle$$

(where $e^{j\mu} e^{ij} = e^{i,\mu}$). But then (c) holds by Lemma 3.7.7. Hence (b) holds, since (b) holds for $R|i+1$ whenever $i \in A_\mu$ (hence $A_i = A_\mu \cap i$).

QED(Lemma 3.7.27)

3.7.3 A first conclusion

In this section we prove:

Theorem 3.7.28. *Let M' be a normal iterate of M. Then M' is normally iterable.*

We prove it in the slightly stronger form:

Lemma 3.7.29. *Let $\tilde{I} = \langle \langle \tilde{M}_i \rangle, \langle \tilde{\nu}_i \rangle, \langle \tilde{\pi}_{i,j} \rangle, \tilde{T} \rangle$ be a normal iteration of M of length $\tilde{\eta} + 1$. Let $\tilde{\sigma} : N \longrightarrow_{\Sigma^*} \tilde{M}_{\tilde{\eta}} \min \tilde{\rho}$. Then N is normally iterable.*

First, however, we prove a technical lemma. Recalling the Definition 3.7.6 of the function $W(I, I', \nu)$, we prove:

Lemma 3.7.30. *Let $W(I, I', \nu) = \langle I^*, I'', e \rangle$, where $F, \nu, \kappa, \tau, \lambda, s, t$ are as in 3.7.6. Let I, I^*, I', I'' be of length $\eta + 1, \eta^* + 1, \eta' + 1, \eta'' + 1$ respectively. Let $\sigma = \tilde{\sigma}_{\eta^*}$ be induced by e. Set:*

$$M_* = M_\eta || \mu \text{ whose } \mu \text{ is maximal such that } \tau \text{ is a cardinal } M_\eta || \mu.$$

(Hence $\mathbb{P}(\kappa) \cap M_* = \mathbb{P}(\kappa) \cap J_{\nu'}^{E^{M'}_{\eta'}}$). Then:

(a) $\sigma : M_* \longrightarrow_{\Sigma^*} M''_{\eta''}$

(b) $\sigma(X) = F(X)$ *for* $X \in \mathbb{P}(\kappa) \cap M^*$ *(hence* $\kappa = \mathrm{crit}(\sigma)$*).*

Proof. Case 1. $t = \eta$ and τ is a cardinal in M_η.

Then $\eta^* = \eta, M_* = M, \eta'' = \eta + 1$ and:

$$\sigma_\eta = \pi_\eta = \pi''_{\eta,\eta+1} : M_\eta \longrightarrow^*_F M''_{\eta+1}$$

QED(Case 1)

Case 2. $t < \eta$ and τ is a cardinal in M_η. Then $\eta^* = \eta, M_* = M_\eta$. Moreover, $\hat{\sigma}_\eta = \sigma_\eta$; hence (a) holds. Set:

$M''_* = M_t||\mu$ where μ is maximal such that τ is a cardinal in $M_t||\mu$.

Then $M''_* = M''^*_s$ and:

$$\sigma_t = \pi_t = \pi''_{t,s+1} : M''_* \longrightarrow^*_F M''_{\eta+1}.$$

Note that $\mu \geq \lambda_t$, since λ_t in inaccessible in M_η and $\tau < \lambda_t$ is a cardinal in M_η. Then $\sigma_\eta \upharpoonright \lambda_t = \sigma_t \upharpoonright \lambda_t$ and $J^{E^{M_t}}_{\lambda_t} = J^{E^{M_\eta}}_{\lambda_t}$. Hence $\sigma_\eta \upharpoonright J^{E^{M_\eta}}_{\lambda_t} = \sigma_t \upharpoonright J^{E^{M_t}}_{\lambda_t}$. Hence:

$$\sigma_\eta(X) = \sigma_t(X) = F(X) \text{ for } X \in \mathbb{P}(\kappa) \cap M.$$

QED(Case 2)

Case 3. τ is not a cardinal in M_η. Then $\eta^* = t, \eta'' = s + 1$, and:

$$\sigma_t = \pi_t : M_* \longrightarrow^*_F M''_{s+1}$$

QED(Lemma 3.7.30)

Corollary 3.7.31. *Let:*

$$R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle,$$

be a reiteration where:

$$I^i = \langle\langle M^i_k\rangle, \langle\nu^i_k\rangle, \langle\pi^i_{k,l}\rangle, T^i\rangle \text{ is of length } \eta_i + 1.$$

Let $\xi = T(i+1)$. *Let* I^i_* *have length* $\eta^* + 1$. *Set:* $M^i_* = M^\xi_{\eta^*}||\mu$, *where* μ *is maximal such that* τ_i *is a cardinal in* $M^\xi_{\eta^*}$. *Then:*

$$\sigma^{\xi,i+1}_{\eta^*} : M^i_* \longrightarrow_{\Sigma^*} M^{i+1}_{\eta_{i+1}} \text{ and:}$$

$$\sigma^{\xi,i+1}_{\eta^*}(X) = E^i_{\nu_i}(X) \text{ for } X \in \mathbb{P}(\kappa_i) \cap M^i_*.$$

Note. $\mathbb{P}(\kappa_i) \cap M^i_* = \mathbb{P}(\kappa_i) \cap J^{E^{M^i}_{\eta_i}}_{\nu_i}$.

Note. This does not say that $M^{i+1}_{\eta_{i+1}}$ is a $*$-ultrapower of M^i_* by $E^{M'_{\eta_i}}_{\nu_i}$.

We now make use of the notion of *mirror* defined in §3.6.

This suggests the following definition:

Definition 3.7.16. Let $I^* = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a normal iteration of length η.

By a *reiteration mirror* (RM) of I^* we mean a pair $\langle R, I'\rangle$ such that

(a) $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is a reiteration of M of length η, where

$$I^i = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_{hj}\rangle, T^i\rangle \text{ is of length } \eta_i.$$

(b) $I' = \langle\langle M'_i\rangle, \langle\pi'_{ih}\rangle, \langle\sigma^i\rangle, \langle\rho^i\rangle\rangle$ is a mirror of I^*. (Hence $\sigma_i(\nu^*_i) = \nu_i$).

(c) $M'_i = M^i_{\eta_i}$.

(d) If $h = T(i+1)$, then

$M'^*_i = M^h_{\eta_h}||\mu$, where μ is maximal such that τ_i is a cardinal in $M^h_{\eta_h}$ and $\pi'_{h,i+1} = \sigma^{h,i+1}_{\eta^*_h}$, where $\eta^*_h + 1 = \mathrm{lh}(I^i_*)$.

Definition 3.7.17. $\langle I^*, R, I'\rangle$ is called an *RM-triple* if $\langle R, I'\rangle$ is an RM of I^*.

We obviously have:

Lemma 3.7.32. *$i+1$ is a drop point in I^* iff it is a drop point in R.*

Moreover:

Lemma 3.7.33. *If $(i,j]_T$ has no drop point, then $\pi'_{ij} = \sigma^{ij}_{\eta_i}$.*

Proof. By induction on j, using Lemma 3.7.27. We leave this to the reader.

Lemma 3.7.34. *Let $\langle I, R, I'\rangle$ be an RM-triple of length $\eta+1$. Let $E^{N_\eta}_\nu \neq \varnothing$, where $\nu > \nu_i$ for $i < \eta$. Then $\langle I, R, I'\rangle$ extends to a triple of length $\eta+2$, with $\nu = \nu_\eta$ (hence $\nu'_\eta = \sigma_\eta(\nu)$).*

Proof. By Lemma 3.7.25, R is uniquely reiterable. Hence R extends to $\dot{R}$ of length $\eta + 2$ with $\dot{\nu}_\eta = \sigma_\eta(\nu)$. Set: $M'_{\eta+1} =:$ the final model of $\dot{I}^{\xi+1}, \xi =: \dot{T}(\eta+1), \pi' =: \sigma^{\xi,\eta+1}_{\eta^*}$, where $\eta^* = \ln(I^\eta_*)$. The choice of ν_η determines $\dot{M}^*_\eta = M^\xi_\eta || \mu$. Then:

$$\pi^{-1} : \dot{M}^*_\eta \longrightarrow_{\Sigma^*} M_{\eta+1}, \pi(X) = E^{M'_\eta}_\nu(X) \text{ for } X \in \mathbb{P}(\kappa) \cap \dot{M}^*_\eta.$$

The conclusion then follows by Lemma 3.6.38.

QED(Lemma 3.7.34)

By Lemma 3.7.25 and Lemma 3.6.37 we then have:

Lemma 3.7.35. *Let $\langle I, R, I' \rangle$ be an RM-triple of limit length η. Let b be the unique good branch in R. Then there is a unique extension to an RM-triple of length $\eta + 1$. Moreover, $b = T``\{\eta\}$ in the extension.*

Proof. R extends uniquely to $\dot{R}$ of length $\eta + 1$. We now extend I' to $\dot{I}'$ by taking $\dot{M}'$ as the final model of $\dot{I}'^\eta$. Pick $i < \eta$ such that $b \setminus i$ has no drop point in R. For $j \in b \setminus i$ set:

$$\dot{\pi}'_{j,\eta} = \dot{\sigma}^{i,\eta}_{\eta_j} \text{ (where } \eta_j + 1 = \mathrm{lh}(I^j) \text{ in } R).$$

By Lemma 3.7.33, we know:

$$\dot{\pi}'_{j,\eta} \pi'_{h,j} = \dot{\pi}'_{h,\eta} \text{ for } h \leq j \text{ in } b \setminus i.$$

By Lemma 3.7.27 it follows that:

$$\dot{M}, \langle \dot{\pi}'_{j,\eta} : j \in b \setminus i \rangle$$

is the direct limit of:

$$\langle M'_h : h \in b \setminus i \rangle,\ \langle \pi'_{h,j} : h \leq j \text{ in } b \setminus i \rangle.$$

(For $h \in b \cap i$, we then set: $\dot{\pi}'_{h,\eta} = \pi'_{i,\eta} \pi'_{h,i}$.)

The conclusion is immediate by Lemma 3.6.37.

(Lemma 3.7.35)

Now let $N, \tilde{I}$ be as in the premise of Lemma 3.7.2. In particular, $\tilde{I}$ is a normal iteration of M of length $\tilde{\eta} + 1$ and:

$$\tilde{\sigma} : N \longrightarrow_{\Sigma^*} \tilde{M}_{\tilde{\eta}} \min \tilde{\rho}.$$

Using the last two lemmas, we define a successful strategy for N. We first fix a function G such that whenever $\Gamma = \langle I, R, I'\rangle$ is an RM triple of length $\mu + 1$ and $E_\nu^{M_\mu} \neq \varnothing$ with $\mu > \nu_j$ for $j < \mu$, then $G(\Gamma, \nu)$ is an extension of Γ to an RM triple of length $\mu + 1$ with $\nu_\mu = \nu$. In all other cases $G(\Gamma, \nu)$ is undefined. Now let I be any normal iteration of N. There can obviously be only one RM triple $\Gamma = \langle I, T, I'\rangle$ with the properties:

(a) $I^0 = \tilde{I}, \sigma_0 = \tilde{\sigma}, \rho^0 = \tilde{\rho}$.

(b) If $i + 1 < \mathrm{lh}(I)$, then:

$$\Gamma|i + 2 = G(\Gamma|i + 1, \nu_i),$$

since $\Gamma|\lambda+1$ is uniquely determined at limit stages λ by Lemma 3.7.35.

Denote this Γ by $\Gamma(I)$ if it exists. We define the strategy S as follows:

Let I of limit length. If $\Gamma(I)$ is undefined, then so is $S(I)$. Now let $\Gamma(I) = \langle I, R, I'\rangle$ be defined. Set:

$$S(I) = \text{ the unique cofinal, well founded branch in } R.$$

(This exists by Lemma 3.7.35). We then get:

Lemma 3.7.36. *Let I be a normal iteration of N. If I is S-conforming, then $\Gamma(I)$ is defined.*

Proof. By induction on $\mathrm{lh}(I)$, using Lemma 3.7.34 and Lemma 3.7.35.

QED(Lemma 3.7.36)

In particular, if I is of limit length, it follows by Lemma 3.7.35 that $S(I)$ is defined and is a cofinal, well founded branch in I. This proves Theorem 3.7.28.

Theorem 3.7.28 is stated under the assumption that M is uniquely normally iterable in V. As usual, we can relativize this to a regular cardinal $\theta > \omega$. We call M' a θ-iterate of M is it is obtained by a normal iteration of length $< \theta$. Modifying our proof slightly we get:

Lemma 3.7.37. *Let $\theta > \omega$ be regular.*

(a) *If M is uniquely normally θ-iterable and M' is a θ-iterate of M then M' is normally θ-iterable.*

(b) *If M is uniquely normally $\theta+1$-iterable and M' is a θ-iterate of M, then M' is normally $\theta+1$-iterable.*

Note. In proving (b) we must restate Lemma 3.7.29 as:

Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a normal iteration of length $\eta+1<\theta$. Let $\sigma : N \longrightarrow_{\Sigma^*} M_\eta \min\rho$. Then M is normally $\theta+1$-iterable.

Note. In proving Lemma 3.7.37, we restrict ourselves to θ-reiterations $R = \langle\langle I^i\rangle, \dots\rangle$ meaning that $\mathrm{lh}(I^i) < \theta$ for $i<\theta$. Thus we restrict to *θ-reiteration mirror* $\langle R, I'\rangle$, meaning that R is a θ-reiteration. Lemma 3.7.34 is then stated for RM-triples of length $\eta+1<\theta$. Lemma 3.7.35 is stated for RM-triples of length $\eta\le\theta$. All steps fo through as before.

Note. An easy modification of the proof shows that, if M is normally iterable by a insertion stable strategy, then every S-conforming iterate of M is normally iterable.

This is a relatively weak result, and could, in fact, have been obtained without use of the pseudo projecta. (However, we would not know how to do it without the use of reiteration). What we really want to prove is that M is smoothly iterable. The above proof indicates a possible strategy for doing so, however: If M is "smoothly reiterable", and:

$$\sigma : N \longrightarrow_{\Sigma^*} M \min\rho$$

we could use the same procedure to define a successful smooth iteration strategy for N. In §3.7.4 we shall define "smooth reiterability" and show that if holds for M.

3.7.4 Reiteration and Inflation

By a *smooth reiteration* of M we mean the result of doing (finitely or infinitely many) successive normal reiterations. We define:

Definition 3.7.18. A *smooth reiteration* of M is a sequence $S = \langle\langle I_i : i < \mu\rangle, \langle e_{i,j} : i \le j < \mu\rangle\rangle$ such that $\mu \ge 1$ and the following hold:

(a) I_i is a normal iteration of M of successor length η_i+1.

(b) $e_{i,j}$ inserts an $I_i|\alpha$ into I_j, where $\alpha \le \eta_i+1$.

(c) $e_{h,j} = e_{i,j} \circ e_{h,i}$.

(d) If $i+1<\mu$, there is a normal reiteration:

$$R_i = \langle\langle I_i^l\rangle, \langle\nu_i^l\rangle, \langle e_i^{k,l}\rangle, T_i\rangle$$

of length $\eta_i + 1$ such that $I_i = I_i^0, I_{i+1} = I_i^{\eta_i}$ and $e_{i,i+1} = e_i^{0,\eta_i}$.

Note. R_i is unique by Lemma 3.7.21. Hence so is $\langle e_{i,j} : i \leq j < \mu\rangle$, which we call the *induced sequence*.

Call i a *drop point* in S iff R_i has a truncation on the main branch.

(e) If $\lambda < \mu$ is a limit ordinal, then there are at most finitely many drop points $i < \lambda$. Moreover, if $h < \lambda$ and (h, λ) is free of drop points, then:

$$I_\lambda, \langle e_{i,\lambda} : h \leq i < \lambda\rangle$$

is the good limit of:

$$\langle I_i : h \leq i < \lambda\rangle, \langle e_{i,j} : h \leq i \leq j < \lambda\rangle$$

This completes the definition. We call μ the *length* of S.

Note. Since $e_{l,\lambda} = e_{h,\lambda}e_{l,h}$ for $l < h < \lambda$, we follow our usual convention, calling:

$$I^\lambda, \langle e_{i,\lambda} : i < \lambda\rangle$$

the *good limit* of:

$$\langle I^i : i < \lambda\rangle, \langle e_{i,j} : i \leq j < \lambda\rangle$$

We call M *smoothly reiterable* if every smooth reiteration of M can be properly extended in any legitimate way. We note:

Fact 1. If I is a normal iteration of M, then $\langle\langle I\rangle, \varnothing, \langle \mathrm{id} \restriction I\rangle, \varnothing\rangle$ is a smooth reiteration of M of length 1.

Fact 2. If $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$ is a smooth reiteration of M of length $i+1$, and $R = \langle I^i\rangle, \langle\nu^i\rangle\rangle$ is a normal reiteration of length $\eta+1$ with $I^0 = I_i$, then S extends to S' of length $i+2$ with $I'_{i+1} = I^\eta$ and $e'_{i,i+1} = e^{0,\eta}$ (hence $R = R_i^{S'}$).

Fact 3. Let $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$ be a smooth reiteration of M of limit length λ. Assume:

(a) S has finitely many drop points.

(b) S has a good limit: $I, \langle e_i : i < \lambda\rangle$.

Then S extends uniquely to S' of length $\lambda+1$ with $I'_\lambda = I, e'_{i,\lambda} = e_i$.

Clearly, then, saying that M is smoothly reiterable is the same as saying that, whenever S is as in Fact 3, then (a), (b) are true. In the next subsection (§3.7.5) we shall prove the smooth iterability of M. The proof is, in all essentials, due to Farmer Schlutzenberg, and is based on his remarkable theory of *inflations*. This subsection is devoted an exposition of that theory.

Before proceeding to the precise definition of *inflation*, however, we give an introduction to Schlutzenberg's methods. Let $R = \langle\langle I^i\rangle, \langle\nu_i\rangle, \langle e^{i,j}\rangle, \tilde{T}\rangle$ be a reiteration of M. Schultzenberg calls I' an "inflation" of I^0, since it was obtained by introducing new extenders into the original sequence. He makes the key observation that the pair $\langle I^0, I^i\rangle$ determines a unique record of the changes made in passing from I^0 to I^i. We shall call that record the *history* of I^i and denote it by $\mathrm{hist}(I^0, I^i)$.

Definition 3.7.19. Let $\eta_i + 1 = \mathrm{lh}(I^i)$ for $i < \mathrm{lh}(R)$. For $\alpha \leq \eta_i$, set:

$$l(\alpha) = l^i(\alpha) =: \text{ the least } i \text{ such that } I^i|\alpha+1 = I^l|\alpha+1.$$

Let $s_i, t_i, \hat{s}_i = \mathrm{lub}_{h<i}\, s_h$ be defined as in §3.7.2. Then:

Lemma 3.7.38. (a) *$l(\alpha) =$ that $l \leq i$ such that $\hat{s}_l \leq \alpha$ and either $l = i$ or $l < i$ and $\alpha \leq s_l$.*

(b) *$I^j|\alpha+1 = I^l|\alpha+1$ for $l \leq j \leq i$.*

Proof.

(a) $\hat{s}_l \leq \alpha$, since otherwise $s_j+1 > \alpha$ for a $j < l$. Hence $I^j|s_j+1 = I^i|s_j+1$ where $\alpha+1 \leq s_j+1$. Hence $j \geq l$. Contradiction!

Suppose $l \neq i$. Then $\alpha \leq s_l$, since otherwise $s_l+1 \leq \alpha$ and $I^i|\alpha+1 \neq I^l|\alpha+1$, since $\nu^i_{s_l} < \nu^l_{s_l}$.

QED(a)

(b) Suppose not. Then $i \neq l, \alpha \leq s_l$ and $I^l|s_l+1 = I^j|s_l+1$ for $l \leq j < \mathrm{lh}(R)$. Contradiction! QED(Lemma 3.7.38)

Hence $\hat{s}_i \leq \alpha \longrightarrow l^i(\alpha) = i$.

Lemma 3.7.39. *If $h \leq i$ and $I^h|\alpha+1 = I^i|\alpha+1$ then $\nu^i_\alpha \leq \nu^h_\alpha$ if $\alpha < \eta_h$.*

Proof. By induction on i.

Case 1. $i = 0$ (trivial).

Case 2. $i = h + 1$.

Then $I^i|s_h + 1 = I^h|s_h + 1$ and $\nu^i_{s_h} \leq \nu^h_{s_h}$. Thus it holds for $\alpha \leq s_h$ by the induction hypotheses. But $l(\alpha) = i$ for $\alpha > s_h$.

Case 3. i is a limit.

Then $I^i|s_j + 1 = I^j|s_j + 1$ for $j < i$. Hence it holds for $\alpha < \hat{s}_i = \mathrm{lub}_{j<i}\, s_j$ by the induction hypothesis. But $l(\alpha) = i$ for $\alpha \geq \hat{s}_i$.

QED(Lemma 3.7.39)

The next lemma is crucial to developing the theory of inflations:

Lemma 3.7.40. *Let $\alpha \leq \eta_i, l = l(\alpha)$. Set:*

$$a = \{\gamma \leq \eta_0 : e^{0,l}(\gamma) < \alpha\}.$$

There is a unique e inserting $I^0|a + 1$ into $I^i|\alpha + 1$ such that $e \upharpoonright a = e^{0l} \upharpoonright a$ and $e(a) = \alpha$.

Proof. By induction on i.

Case 1. $i = 0$. Set $a = \alpha, e = \mathrm{id} \upharpoonright \alpha + 1$.

Case 2. $i = h + 1$.

If $\alpha \leq s_h$, then $I^i|\alpha + 1 = I^h|\alpha + 1$. Hence $l = l^h(\alpha)$ and the result holds by the induction hypothesis.

If $\alpha > s_h$, then $l(\alpha) = i$, since $I^i|s_h + 1 \neq I^h|s_h + 1$. Then $\alpha = s_h + 1 + j$. Let $\mu = \tilde{T}(h+1)$. Then $e^{\mu,i}(\overline{\alpha}) = \alpha$, where $\overline{\alpha} = t_h + j$. But $\hat{s}_\mu \leq t_h \leq s_\mu$ by Lemma 3.7.17. Hence $l^\mu(t_h) = l^\mu(\overline{\alpha}) = \mu$. Clearly:

$$a = \{\gamma \leq \eta_0 : e^{0,\mu}(\gamma) < \overline{\alpha}\}$$

Since $\mu \leq h$, the induction hypothesis gives a unique f inserting $I^0|a + 1$ into $I^\mu|\overline{\alpha} + 1$ such that $f \upharpoonright a = e^{0,\mu} \upharpoonright a$ and $f(a) = \overline{\alpha}$. Thus $e = e^{\mu,l} f$ has the desired properties.

QED(Case 2)

Case 3. i is a limit ordinal.

Then $I^i|s_j + 1 = I^j|s_j + 1$ for $j < i$. Hence the assertion holds for $\alpha < \hat{s}_i = \mathrm{lub}_{j<i}\, s_j$ by the induction hypothesis. But $l(\alpha) = i$ for $\hat{s}_i \leq \alpha$. Then

there is $j <_T i$ such that $\alpha = e^{j,i}(\overline{\alpha})$. Let $j = T(\xi + 1)$ where $\xi + 1 <_T i$. Then $\overline{\alpha} \geq \mathrm{crit}(e^{j,i}) = t_\xi$. But $\hat{s}_j \leq t_\xi \leq s_j$. Hence $l^j(\overline{\alpha}) = l^j(t_\xi) = j$. Since $e^{0,i} = e^{j,i} \circ e^{0,j}$, we conclude as in Case 2 that:

$$a = \{\gamma < \eta : e^{0,j}(\gamma) < \overline{\alpha}\}$$

By the induction hypothesis there is f inserting $I^0|a+1$ into $I^j|\overline{\alpha}+1$ such that $\hat{f} \restriction a = e^{0,j} \restriction a$ and $f(a) = \overline{\alpha}$. Hence $e = e^{f,i} \circ f$ has the desired properties.

QED(Lemma 3.7.40)

Definition 3.7.20. For $i < \mathrm{lh}(R), \alpha \leq \eta_0$ set:

$$\begin{aligned} a^i_\alpha &=: \mathrm{lub}\{\xi < \eta_0 : e^{0l}(\xi) < \alpha\} \text{ where } l = l^i(\alpha) \\ e^i_\alpha &=: \text{ the unique } e \text{ inserting } I^0|a^j_\alpha + 1 \text{ into } I^i|\alpha + 1 \text{ such} \\ &\qquad \text{that } e \restriction a^i_j = e^{0,l} \restriction a^i_j \text{ and } e(a^i_\alpha) = \alpha \end{aligned}$$

It follows easily that:

Lemma 3.7.41. (a) *If $l = l^i(\alpha)$, then $\alpha \leq \eta_l$ and $l = l^l(\alpha), a^i_\alpha = a^l_\alpha$ and $e^i_\alpha = e^j_\alpha$.*

(Hence $e^i_\alpha = e^h_\alpha$ and $a^i_\alpha = a^h_\alpha$ whenever $I^i|\alpha + 1 = I^h|\alpha + 1$).

(b) *If $e^{\mu,i}(\overline{\alpha}) = \alpha, \hat{s}_\mu \leq \overline{\alpha}, \hat{s}_i \leq \alpha$, then:*

$$l^\mu(\overline{\alpha}) = \mu, l^i(\alpha) = i, a^\mu_{\overline{\alpha}} = a^i_\alpha, \text{ and } e^{\mu,i} e^\mu_{\overline{\alpha}} = e^i_\alpha.$$

(c) $e^i_{\eta_i} \restriction a^i_{\eta_i} = e^{i,\eta_i} \restriction a^i_{\eta_i}$; $e^i_{\eta_i}(a^i_{\eta_i}) = \eta_i$ *($l^{\eta_i} = \eta_i$, since $\eta_i \geq \hat{s}_i$).*

(d) *If there is no truncation on the main branch of $R|i+1$, then $e^{0,i} = e^i{}_{\eta_i}$ and $a_{\eta_i} = \eta_0$ (since $e^{0,i}(\eta_0) = \eta_i$).*

The proof is left to the reader.

We now fix an $i < \mathrm{lh}(R)$ and set:

$$\begin{aligned} I &= \langle\langle M_\alpha\rangle, \langle\nu_\alpha\rangle, \langle\pi_{\alpha,\beta}\rangle, T\rangle =: I^0 \\ I' &= \langle\langle M'_\alpha\rangle, \langle\nu'_\alpha\rangle, \langle\pi'_{\alpha,\beta}\rangle, T'\rangle =: I^i \\ a &= \langle a^i_\alpha : \alpha \leq \eta_i\rangle, e_\alpha = e^i_\alpha \text{ for } \alpha \leq \eta_i. \end{aligned}$$

$\langle a, \langle e_\alpha : \alpha \leq \eta'\rangle\rangle$ is then called the *history* of I' from I. We shall show that it is completely determined by the pair $\langle I, I'\rangle$. a_α is called the *ancestor* of α in this history.

We prove:

Theorem 3.7.42. *Let $I, I', a, \langle e_\alpha : \alpha \leq \eta_i\rangle$ be as above. Then:*

(1) $a : \mathrm{lh}(I') \longrightarrow \mathrm{lh}(I)$ *and e_α inserts $I|a_\alpha + 1$ into $I'|\alpha + 1$ for $\alpha < \mathrm{lh}(I')$. Moreover, $e_\alpha(a_\alpha) = \alpha$.*

(2) *Let $a_\alpha < \eta$. If $\tilde{\nu}_\alpha = \sigma^{e_\alpha}_{a_\alpha}(\nu_{a_\alpha})$ exists and $\alpha + 1 < \mathrm{lh}(I')$, then $\nu'_\alpha \leq \tilde{\nu}_\alpha$.*

(3) *Let $a_\alpha < \eta, \alpha + 1 < \mathrm{lh}(I'), \nu'_\alpha = \tilde{\nu}_\alpha$. Then:*

$$a_{\alpha+1} = a_\alpha + 1,\ e_{\alpha+1} \restriction a_\alpha + 1 = e_\alpha.$$

For $\alpha + 1 < \mathrm{lh}(I^i)$, define the index *of α ($\mathrm{in}(\alpha) = \mathrm{in}^i(\alpha)$) as:*

$$\mathrm{in}(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is as in } (3) \\ 1 & \text{if not} \end{cases}$$

(4) *If $\mathrm{in}(\alpha) = 1, \gamma = T'(\alpha + 1)$, then $a_{\alpha+1} = a_\gamma$.*

(5) *If $\beta \leq_{T'} \alpha$, then $e^{-1}_\alpha \restriction \beta = e^{-1}_\beta \restriction \beta$.*

Note. *Ignoring our formal definition of $\langle a, e\rangle$ and using only (1), (5), we get:*

- $e_\alpha \restriction a_\beta = e_\beta \restriction a_\beta$.
- $a_\beta \leq_T a_\alpha$ *since:*

$$\hat{e}_\alpha(a_\beta) = \hat{e}_\beta(a_\beta) \leq_{T'} e_\beta(\beta) = \beta \leq_{T'} \alpha = e_\alpha(a_\alpha).$$

- *If α is a limit ordinal, then:*

$$a_\alpha = \bigcup_{\beta <_{T'} \alpha} a_\beta \text{ and } e_\alpha \restriction a_\alpha = \bigcup_{\beta <_{T'} \alpha} e_\beta \restriction a_\beta,$$

since $e^{-1}_\alpha \restriction \alpha = \bigcup_{\beta <_{T'} \alpha} e^{-1}_\beta \restriction \beta$.

Note. *By (1), (4) and (5) we get:*

- *If in$(\alpha) = 1$, $\gamma = T'(\alpha + 1)$, then $e_{\alpha+1} \restriction a_{\alpha+1} = e_\gamma \restriction a_\gamma$.*

Note. *Since e_α, e_β are monotone and $a_b e = e^{-1}_\beta``\beta$, the statement:*

$$e^{-1}_\alpha \restriction \beta = e^{-1}_\beta \restriction \beta$$

is equivalent to:

$$e_\beta \restriction a_\beta = e_\alpha \restriction a_\beta \text{ and } e_\alpha(a_\beta) \geq \beta.$$

(6) *If $R|i+1$ has a truncation on the main branch, then there is $\alpha \in (\hat{e}_{\eta_i}(a_{\eta_i}), \eta_i]_{T'}$ which is a drop point in I'.*

Note. By Lemma 3.7.41 (a) we have:

$$\hat{e}_{\eta_i}(a_{\eta_i}) = \text{lub}\, e_{\eta_i} {}``a_{\eta_i} = \text{lub}\, e^{0,i_0} a_{\eta_0} = \hat{e}^{0,i}(a_{\eta_i}).$$

We prove Theorem 3.7.42 by induction on i:

Case 1. $i = 0$.

Trivial, since $a_\alpha = \alpha, e_\alpha = \text{id} \upharpoonright \alpha + 1$.

Case 2. $i = h + 1$.

(1) is given.

(2) If $\alpha \leq s_h$, then $I^i|\alpha + 1 = I^h|\alpha + 1$, hence $l^i(\alpha) = l^h(\alpha), e^i_\alpha = e^h_\alpha, \tilde{\nu}^i_\alpha = \tilde{\nu}^h_\alpha$. By the induction hypothesis $\nu^h_\alpha = \tilde{\nu}^h_\alpha$. But $\nu^i_\alpha < \nu^h_\alpha$. Now let $\alpha > s_h$. Then $l(\alpha) = i$ and $\alpha = s_h + 1 + j$ for some j. Let $\mu = \tilde{T}(h+1)$. Then $e^{\mu,i}(\overline{\alpha}) = \alpha$ where $\overline{\alpha} = t_h + 1$. Just as in the proof of Lemma 3.7.40 (Case 2), we have: $\mu = l^\mu(t_h) = l^\mu(\overline{\alpha})$ and $e^{\mu,i} \circ e^\mu_{\overline{\alpha}} = e_\alpha$. Hence:

$$\tilde{\nu}^i_\alpha = \sigma^{e^i_\alpha}_a(\nu^0_a) = \sigma^{\mu,\alpha}_{\overline{\alpha}} \sigma^{e^\mu_{\overline{\alpha}}}(\nu^0_a) = \sigma^{\mu,\alpha}_{\overline{\alpha}}(\tilde{\nu}^\mu_{\overline{\alpha}})$$

(Since if $e = e_1 \circ e_0$, then $\sigma^e_\beta = e^{e_1}_{e_0(\beta)} \circ e^{e_0}_\beta$). By the induction hypothesis: $\nu^\mu_{\overline{\alpha}} \leq \tilde{\nu}^\mu_{\overline{\alpha}}$. Hence:

$$\nu^i_\alpha = \sigma^{\mu,\alpha}_{\overline{\alpha}}(\nu^\mu_{\overline{\alpha}}) \leq \sigma^{\mu,\alpha}_{\overline{\alpha}}(\tilde{\nu}^\mu_{\overline{\alpha}}) = \tilde{\nu}'_\alpha.$$

QED(2)

(3) If $\alpha < s_h$, then $\nu^i_\alpha = \nu^h_\alpha, \tilde{\nu}^h_\alpha = \tilde{\nu}^i_\alpha$, since $I^i|s_h + 1 = I^h|s_h + 1$. Hence $\nu^h_\alpha = \tilde{\nu}^h_\alpha$.

Hence $a^h_{\alpha+1} = a^h_\alpha + 1, e^h_{\alpha+1} \upharpoonright a^h_{\alpha+1} = e^h_\alpha$ by the induction hypothesis. But $l^i(\alpha + 1) = l^h(\alpha + 1)$. Hence: $a^n_{\alpha+1} = a^i_{\alpha+1}, a^h_\alpha = a^i_\alpha, e^h_{\alpha+1} = e^i_{\alpha+1}, e^h_\alpha = e^i_\alpha$. The conclusion is immediate. Now let $\alpha = s_h$. We still have $e^h_\alpha = e^i_\alpha$; hence $\tilde{\nu}^h_\alpha = \tilde{\nu}^i_\alpha$. But $\nu^i_\alpha < \nu^h_\alpha \leq \tilde{\nu}^h_\alpha$. Contradiction! Now let $\alpha > s_h$. We again have: $\alpha = s_h + 1 + j, \alpha = e^{\mu,i}(\overline{\alpha})$, where $\mu = T(h+1)$ and $\overline{\alpha} = t_h + j$. As before, we have $l^i(\alpha) = i, l^\mu(\overline{\alpha}) = \mu$. Moreover $\tilde{\nu}^i_\alpha = \sigma^{\mu,i}_{\overline{\alpha}}(\tilde{\nu}^\mu_{\overline{\alpha}})$ and $\nu^i_\alpha = \sigma^{\mu,i}_{\overline{\alpha}}(\nu^\mu_{\overline{\alpha}})$. Hence $\nu^\mu_{\overline{\alpha}} = \tilde{\nu}^\mu_{\overline{\alpha}}$. Hence:

$$a^\mu_{\overline{\alpha}+1} = a^\mu_{\overline{\alpha}} + 1, e^\mu_{\overline{\alpha}+1} \upharpoonright \overline{\alpha} + 1 = e^\mu_{\overline{\alpha}}.$$

But $i = l'(\alpha) = l^i(\alpha+1), \mu = l^\mu(\overline{\alpha}) = l^\mu(\overline{\alpha}+1)$, and $e^{\mu,i}(\overline{\alpha} + 1) = \alpha+1$. Hence:

$$a = a^i_\alpha = a^\mu_{\overline{\alpha}} \text{ and } a_{\alpha+1} = a^i_{\alpha+1} = a^\mu_{\overline{\alpha}+1} = a + 1.$$

Moreover, we have:

$$e^i_{\alpha+1} \upharpoonright a+1 = e^{\mu,i} e^{\mu}_{\overline{\alpha}+1} \upharpoonright a+1 = e^{\mu,i} e^{\mu}_{\overline{\alpha}} = e_\alpha.$$

QED(3)

(4) If $\alpha < s_n$ the result follows by the induction hypothesis, since $I^i|\alpha+2 = I^h|h+2$. Now let $\alpha = s_h$. Then $\text{in}(\alpha) = 1$ as shown above. Let $\mu = \tilde{T}(h+1), \gamma = t_h$. Then $e^{\mu,i}(\gamma) = \alpha+1$. Hence $a^{\mu}_{\gamma} = a^i_{\alpha+1}$. But $I^i|\gamma+1 = I^{\mu}|\gamma+1$. Hence $l^{\mu}(\gamma) = l^i(\gamma)$ and $a^i_{\gamma} = a^{\mu}_{\gamma} = a^i_{\alpha+1}$. Now let $\alpha > s_h$. Then $i = h+1$ is not a drop point in R, since otherwise $\eta_i = s_h + 1 = \alpha$. Hence $\alpha + 1 \not< \text{lh}(I^i) = \eta_i + 1$. Contradiction! Then $\alpha = s_h + 1 + j$ and $\alpha = e^{\mu,i}(\overline{\alpha})$ where $\overline{\alpha} = t_h + j$ and $\mu = \tilde{T}(h+1)$. Note that $e^{\mu,i}(\xi) = \hat{e}^{\mu,i}(\xi) = \text{lub}\, e^{\mu,i}``\xi$ for $\xi > t_h$. Clearly $\alpha+1 = e^{\mu,i}(\overline{\alpha}+1)$. As in the foregoing proofs we have:

$$\sigma^{\mu,i}(\nu^{\mu}_{\overline{\alpha}}) = \nu^i_{\alpha};\ \sigma^{\mu,i}(\tilde{\nu}^{\mu}_{\overline{\alpha}}) = \tilde{\nu}^i_{\alpha}.$$

Hence $\nu^{\mu}_{\overline{\alpha}} < \tilde{\nu}^{\mu}_{\overline{\alpha}}$ and $\text{in}(\overline{\alpha}) = 1$. By the induction hypothesis we conclude: $a^{\mu}_{\overline{\gamma}+1} = a^{\mu}_{\overline{\gamma}}$, where $\overline{\gamma} = T^{\mu}(\overline{\alpha}+1)$. But, as before, $a^{\mu}_{\overline{\alpha}+1} = a^i_{\alpha+1}$, since $e^{\mu,i}(\overline{\alpha}+1) = \alpha+1, l^{\mu}(\overline{\alpha}+1) = \mu$, $l^i(\alpha+1) = i$. Thus it suffices to show:

Claim. $a^{\mu}_{\overline{\gamma}} = a^i_{\gamma}$, where $\gamma = T^i(\alpha+1)$.

We consider two cases:

Case A. $\kappa^{\mu}_{\overline{\alpha}} > \kappa_i$. Then $e^{\mu,i}(\overline{\gamma}) = \gamma$ by Lemma 3.7.10 (1). As before $l^{\mu}(\overline{\gamma}) = \mu, l^i(\gamma) = i$ and $a^{\mu}_{\overline{\gamma}} = a^i_{\gamma}$.

Case B. $\kappa^{\mu}_{\overline{\alpha}} < \kappa_i$. Then $\gamma = \overline{\gamma}$ by Lemma 3.7.10(1). Then $\overline{\gamma} \leq t_h$, where $I^i|t_h+1 = I^{\mu}|t_h+1$. Hence $a^i_{\overline{\gamma}} = a^{\mu}_{\overline{\gamma}}$.

QED (4)

(5) If $\alpha \leq s_h$, then $I^h|\alpha+1 = I^i|\alpha+1$ and $a^h_{\gamma} = a^i_{\gamma}, e^h_{\gamma} = e^i_{\gamma}$ for $\gamma \leq \alpha$. Hence the conclusion follows by the induction hypothesis. Now let $\alpha > s_h$. Then $\alpha = s_h + 1 + j$ for some j. Let $\mu = \tilde{T}(h+1)$. Then $e^{\mu,i}(\overline{\alpha}) = \alpha$ where $\overline{\alpha} = t_h + 1$. But $\overline{\alpha} \geq \text{crit}(e^{\mu,i}) = t_h \geq \hat{s}_{\mu}$. Hence:

$$l^{\mu}(\overline{\alpha}) = \mu, a^{\mu}_{\overline{\alpha}} = a^i_{\alpha}, e^i_{\alpha} = e^{\mu,i} \cdot e_{\overline{\alpha}}.$$

Let $\beta <_{T^i} \alpha$. We consider two cases:

Case A. $\beta > s_h$.

Then $\beta = s_h + 1 + r$ for an $r < j$. Hence, letting $\overline{\beta} = t_h + r$, we have $e^{\mu,i}(\overline{\beta}) = \beta$ and:

$$l^{\mu}(\overline{\beta}) = \mu, a^{\mu}_{\overline{\beta}} = a^i_{\beta}, e^i_{\beta} = e^{\mu,i} \cdot e_{\overline{\beta}}.$$

It follows easily that $\overline{\beta} <_{T^\mu} \overline{\alpha}$. Hence by the induction hypothesis:

$$(e^\mu_{\overline{\beta}})^{-1} \upharpoonright \overline{\beta} = (e^\mu_{\overline{\alpha}})^{-1} \upharpoonright \overline{\alpha}.$$

Hence:

$$\begin{aligned}(e^i_\beta)^{-1} \upharpoonright \beta &= (e^\mu_{\overline{\beta}})^{-1} \cdot (e^{\mu,i})^{-1} \upharpoonright \beta \\ &= (e^\mu_{\overline{\alpha}})^{-1} \cdot (e^{\mu,i})^{-1} \upharpoonright \beta \\ &= (e^i_\alpha)^{-1} \upharpoonright \beta.\end{aligned}$$

QED(Case A)

Case B. $\beta \leq s_h$.

Then $\beta \leq t_h$, since $(t_h, s_h]$ is in limbo at $\hat{s}_i = s_h+1$. Hence $e^{\mu,i} \upharpoonright \beta = \mathrm{id}$, since $t_h = \mathrm{crit}(e^{\mu,i})$. But then:

$$\beta = \hat{e}^{\mu,i}(\beta) \leq_{T^\mu} \alpha = e^{\mu,i}(\overline{\alpha}).$$

Hence $\beta \leq_{T^\mu} \overline{\alpha}$. Moreover $I^i|\beta + 1 = I^\mu|\beta + 1$, since $\hat{e}^{\mu,i} \upharpoonright \beta + 1 = \mathrm{id}$. Hence $a^\mu_\beta = a^i_\beta$ and $e^\mu_\beta = e^i_\beta$. But:

$$(e^\mu_{\overline{\alpha}})^{-1} \upharpoonright \beta = (e^\mu_\beta)^{-1} \upharpoonright \beta$$

since $\beta \leq_{T^\mu} \overline{\alpha}$. Hence:

$$(e^i_\alpha)^{-1} \upharpoonright \beta = (e^\mu_{\overline{\alpha}})^{-1}(e^{\mu i})^{-1} \upharpoonright \beta \quad = (e^\mu_\beta)^{-1}(e^{\mu i})^{-1} \upharpoonright \beta = (e^i_\beta)^{-1} \upharpoonright \beta$$

QED(Case B)

This proves (5).

(6) If $i = h + 1$ is a drop point on $R|i + 1$, then $M'^*_{s_h} \neq M_{t_i}$, where $\eta^i = s_h + 1, t_i = T^i(s_h + 1)$. Hence η_i is a drop point in I^i. Now suppose that $h + 1$ does not drop in $R|i + 1$. Let $\mu = \tilde{T}(h + 1)$. Then there must be a drop point on the main branch of $R|\mu + 1$. Hence I^μ has a drop point in $(\varepsilon, \eta_\mu]_{T^\mu}$ where $\varepsilon = \hat{e}^\mu_{\eta_\mu}(a^\mu_{\eta_\mu})$. Since $e^{\mu,i}(\eta_\mu) = \eta_i$, it follows easily from Lemma 3.7.10(7) that there is a drop point on I^i in $(\hat{e}^{\mu,i}(\varepsilon), t_i]_{T^i}$. Since $\hat{s}_\mu \leq \eta_\mu, \hat{s}_i \leq \eta_i$, we have:

$$\mu = l^\mu =: l^\mu(\eta_\mu),\ i = l^i = l^i(\eta_i).$$

Hence $a^\mu_{\eta_\mu} = a^i_{\eta_i}$. Clearly:

$$\hat{e}^{\mu,i}(\varepsilon) = \mathrm{lub}\, e^{\mu,i} “\varepsilon.$$

Since $e^\mu_{\eta_\mu} \upharpoonright a^\mu_{\eta_\mu} = e^{0,\mu} \upharpoonright a^\mu_{\eta_\mu}$, we have: $\varepsilon = \mathrm{lub}\, e^{0,\mu} “a^\mu_{\eta_\mu}$. Hence:

$$\hat{e}^{\mu,i}(\varepsilon) = \mathrm{lub}\, e^{0,i} “a^i_{\eta_i} = \hat{e}^i_{\eta_i}(a^i_{\eta_i}).$$

Hence I^i has a drop in $(\hat{e}^i_{\eta_i}(a^i_{\eta_i}), \eta_i]_{T^i}$.

QED(6)

This completes Case 2.

Case 3. $i = \lambda$ is a limit ordinal.

(1) is given.

(2) Set $\hat{s} = \hat{s}_\lambda = \text{lub}_{i<\lambda} s_i$. Then $I^\lambda | s_i + 1 = I^i | s_i + 1$ for $i < \lambda$. Thus (2) holds by the induction hypothesis for $\alpha < \hat{s}$. Now let $\alpha \geq \hat{s}$ then $l^\lambda(\alpha) = \lambda$. Pick $\mu < \lambda$ such that $\alpha \in \text{rng}(e^{\mu,\lambda})$ and there is no drop in $(\mu, \lambda)_{T^\lambda}$. Let $i = h{+}1$, where $\mu = T(h{+}1), h{+}1 <_{T^\lambda} \lambda$. If $e^{\mu,\lambda}(\hat{\alpha}) = \alpha$, then $\hat{\alpha} \geq t_h$, since $e^{\mu,\lambda} \upharpoonright t_h = \text{id}$. Hence $\overline{\alpha} \geq s_h + 1 = \hat{s}_i$, where $e^{i,\lambda}(\overline{\alpha}) = \alpha$. Hence $l^i =: l^i(\overline{\alpha}) = i$. Hence $a^i_{\overline{\alpha}} = a^\lambda_\alpha$ and $e^\lambda_\alpha = e^{i,\lambda} e^i_{\overline{\alpha}}$. We are assuming that:

$$\tilde{\nu}^\lambda_\alpha = \sigma^{e^\lambda_\alpha}_{a^\lambda_\alpha}(\nu^0_{a^\lambda_\alpha}) \text{ exists.}$$

But then:

$$\tilde{\nu}^i_{\overline{\alpha}} = \sigma^{e^i_{\overline{\alpha}}}_{a^i_{\overline{\alpha}}}(\nu^0_{a^i_{\overline{\alpha}}}) \text{ exists and } \sigma^{i,\lambda}_{\overline{\alpha}}(\tilde{\nu}^i_{\overline{\alpha}}) = \tilde{\nu}^\lambda_\alpha.$$

Clearly: $\nu^\lambda_\alpha = \sigma^{i,\lambda}_{\overline{\alpha}}(\nu^i_{\overline{\alpha}})$. But $\nu^i_{\overline{\alpha}} \leq \tilde{\nu}^i_{\overline{\alpha}}$ by the induction hypothesis. Hence $\nu^\lambda_\alpha \leq \tilde{\nu}^\lambda_\alpha$.

QED(2)

(3) For $\alpha < \hat{s}_\lambda$ it holds by the induction hypothesis, so let $\alpha \geq \hat{s}_\lambda$. Let $\mu, h, i, \overline{\alpha}$ be as in (2). Then $l^\lambda(\alpha) = \lambda, l^i(\alpha) = i$. We assume $\text{in}^\lambda(\alpha) = 0$, i.e.:

$$\alpha < \eta_\lambda \text{ and } \nu^\lambda_\alpha = \tilde{\nu}^\lambda_\alpha.$$

But then:

$$\overline{\alpha} < \eta_i \text{ and } \nu^i_{\overline{\alpha}} = \tilde{\nu}^i_{\overline{\alpha}} \text{ hence } \text{in}^i(\overline{\alpha}) = 0$$

Hence $a^i_{\overline{\alpha}+1} = a^i_{\overline{\alpha}} + 1$ and $e^i_{\overline{\alpha}+1} \upharpoonright a^i_{\overline{\alpha}} + 1 = e^i_{\overline{\alpha}}$. But $l^i(\overline{\alpha} + 1) = i, l^\lambda(\overline{\alpha} + 1) = \lambda$. Hence

$$a^\lambda_{\alpha+1} = a^i_{\overline{\alpha}+1} = a^i_{\overline{\alpha}} + 1$$

and

$$\begin{aligned} e^\lambda_{\alpha+1} \upharpoonright a^\lambda_{\alpha+1} &= e^{i\lambda} e^i_{\overline{\alpha}+1} \upharpoonright a^i_{\overline{\alpha}} + 1 \\ &= e^{i\lambda} e^i_{\overline{\alpha}} = e^\lambda_\alpha \end{aligned}$$

QED(3)

(4) For $\alpha < \hat{s}_\lambda$ it holds by the induction hypothesis, so let $\alpha \geq \hat{s}_\lambda$. Let $\mu, h, i, \overline{\alpha}$ be as in (2) with the additional stipulation that $\gamma \in \text{rng}(e^{\mu,\lambda})$ where $\gamma = T^\lambda(\alpha + 1)$. Let $e^{i,\lambda}(\overline{\gamma}) = \gamma$. Then either $\gamma \geq \hat{s}_\lambda$ and $\overline{\gamma} \geq \hat{s}_i = s_h + 1$, or $\gamma < \hat{s}_\lambda$ and $\bar{\gamma} = \gamma$. It follows easily that $\overline{\gamma} = T^i(\overline{\alpha} + 1)$. Moreover $\text{in}^i(\overline{\alpha}) = 1$, since $\text{in}^\lambda(\alpha) = 1$. But then $a^i_{\overline{\alpha}} = a^i_{\overline{\gamma}}$.

But $a^i_{\overline{\alpha}} = a^\lambda_\alpha$. Moreover $a^i_{\overline{\gamma}} = a^\lambda_\gamma$. (If $\gamma \geq \hat{s}_\lambda$, this is because $l^i(\overline{\gamma}) = i$. If $\gamma < \hat{s}_\lambda$, it is because $I^i|\gamma + 1 = I^i|\overline{\gamma} + 1$).

QED(4)

(5) If $\alpha < \hat{s}$, it follows by the induction hypothesis, since $I^\lambda|\alpha{+}1 = I^i|\alpha{+}1$ for ß $< \lambda, \alpha \leq s_i$. Now let $\alpha \geq \hat{s}$. Fix $\beta <_{T^\lambda} \alpha$. Let $\mu, i, h, \overline{\alpha}$ be as before with μ chosen big enough that $\beta \in \mathrm{rng}(e^{\mu,\lambda})$ and $\beta < t_h = \mathrm{crit}(e^{\mu,\lambda})$ if $\beta < \hat{s}$. Let $\alpha = e^{i,\lambda}(\overline{\alpha}), \beta = e^{i,\lambda}(\overline{\beta})$. Since:

$$e^{i,\lambda}(\overline{\beta}) = \beta <_{T^\lambda} \alpha = e^{i\lambda}(\overline{\alpha}),$$

we conclude: $\overline{\beta} <_{T^i} \overline{\alpha}$. Hence:

$$(e^i_{\overline{\alpha}})^{-1} \upharpoonright \beta = (e^i_{\overline{\beta}})^{-1}\overline{\beta}$$

by the induction hypothesis. Since $\hat{s}_i \leq \overline{\alpha}$, we again have:

$$a^i_{\overline{\alpha}} = a^\lambda_\alpha, e^\lambda_\alpha = e^{i,\lambda} e^i_{\overline{\alpha}}.$$

If $\beta \geq \hat{s}$, then $\hat{s}_i \leq \overline{\beta}$ and we have :

$$a^i_{\overline{\beta}} = a^\lambda_\beta, e^\lambda_\beta = e^{i,\lambda} e^i_{\overline{\beta}}.$$

Hence:

$$\begin{aligned}(e^\lambda_{\overline{\alpha}})^{-1} \upharpoonright \beta &= (e^i_{\overline{\alpha}})^{-1}(e^{i\lambda})^{-1} \upharpoonright \beta \\ &= (e^i_{\overline{\beta}})^{-1}(e^{i\lambda})^{-1} \upharpoonright \beta \\ &= (e_\beta)^{-1} \upharpoonright \beta.\end{aligned}$$

Now suppose that $\beta < i$. Then $\beta = \overline{\beta} < \mathrm{crit}(e^{i\lambda})$. Hence $I^i|\beta + 1 = I^\lambda|\beta + 1$ and:

$$a^i_\beta = a^\lambda_\beta, e^i_\beta = e^\lambda_\beta \text{ where } e^{i\lambda} \upharpoonright \beta + 1 = \mathrm{id}\,.$$

Hence we again have:

$$a^i_{\overline{\beta}} = a^\lambda_\beta, e^\lambda_\beta = e^{i\lambda} e^i_{\overline{\beta}},$$

and we argue exactly as before.

QED(5)

(6) Suppose $R|\lambda + 1$ has a truncation on the main branch. Clearly $\eta_\lambda \geq \hat{s}_\lambda$, so $l^\lambda(\eta_\lambda) = \lambda$. Let $\mu, i, h, \overline{\alpha}$ be as in (2) with $\alpha = \eta_\lambda$. Then $[i, \lambda]_{T^\lambda}$ is free of drops. Hence $e^{i,\lambda}(\eta_i) = \eta_\lambda$. But $R|i + 1$ then has a drop on the main branch. Hence there is a drop in $(\hat{e}^i_{\eta_i}(a^i_{\eta_i}), \eta_i]_{T^{i+1}}$. By Lemma 3.7.1 (7) it follows that there is a drop in $(\hat{e}^{i,\lambda}(\varepsilon), \eta_\lambda]_{T^\lambda}$,

where $\varepsilon = e_{\eta 0}(a^i_{\eta 0})$. But $l^i(\eta_i) = i$, since $\eta_i \geq \hat{s}_i$. Hence $a^i_{\eta_i} = a^\lambda_{\eta\lambda}$ and $\varepsilon = \hat{e}_{\eta_i}(a^i_{\eta_i}) = \operatorname{lub} e^{0,i}``a^i_{\eta_i}$. Moreover $e^{i,\lambda}(\varepsilon) = \operatorname{lub} e^{i,\lambda}``\varepsilon$. Hence $\hat{e}^{i,\lambda}(\varepsilon) = \operatorname{lub} e^{o,\lambda}``a^\lambda_{\eta_\lambda} = \hat{e}^\lambda_{\eta_\lambda}(a^\lambda_{\eta_\lambda})$.

QED(6)

This completes the proof of Lemma 3.7.42.

Inflations

Following Farmer Schlutzenberg we now define:

Definition 3.7.21. Let I be a normal iteration of M of successor length $\eta+1$. Let I' be a normal iteration of M. I' is an *inflation* of I iff there exist a pair $\langle a, e\rangle$ satisfying (1)-(5) in Theorem 3.7.42 (with $e = \langle e_\alpha : \alpha < \mathrm{lh}(I')\rangle$). We call any such pair a *history* of I' from I.

By the remark accompanying the statement of Theorem 3.7.42 we have:

Lemma 3.7.43. *Let I' be an inflation of I with history $\langle a, e\rangle$. Then:*

(a) *If $\beta \leq_{T'} \alpha$, then $a_\beta \leq_T a_\alpha$ and $e_\alpha \restriction a_\beta = e_\beta \restriction a_\beta$.*

(b) *If $\alpha \leq \mathrm{lh}(I')$ is a limit ordinal, then:*

$$a_\alpha = \bigcup_{\beta <_{T'} \alpha} a_\beta \text{ and } e_\alpha \restriction a_\alpha = \bigcup_{\beta_{T'} \alpha} e_\beta \restriction a_\beta.$$

(c) *If $\alpha + 1 < \mathrm{lh}(I'), \mathrm{in}(\alpha) = 1, \gamma = T'(\alpha+1)$, then:*

$$a_{\alpha+1} = a_\gamma \text{ and } e_{\alpha+1} \restriction a_{\alpha+1} = e_\gamma \restriction a_g a.$$

Lemma 3.7.44. *Let I, I' be as above. Then there is at most one history of I' from I.*

Proof. Let $\langle a, e\rangle$ be a history. By the conditions (1)-(5), this history satisfies a recursion of the form:

$$\langle a_\alpha, e_\alpha\rangle = F(\langle\langle a, e\rangle : \xi < \alpha\rangle),$$

where F is defined by reference to the pair $\langle I, I'\rangle$ alone. To see this we note:

(a) $a_0 = \varnothing$, $e_0(\varnothing) = \varnothing$ by (1).

(b) Let a_α, e_α be given. Then:

- $a_{\alpha+1} = \begin{cases} a_\alpha + 1 & \text{if } \mathrm{in}(\alpha) = 0 \\ a_\beta & \text{where } \beta = T'(\alpha+1) \text{ if } \mathrm{in}(\alpha) = 1 \end{cases}$
- $e_{\alpha+1}(a_\alpha + 1) = \alpha + 1$
- $e_{\alpha+1} \restriction a_{\alpha+1} = \begin{cases} e_\alpha & \text{if } \mathrm{in}(\alpha) = 0 \\ e_\beta \restriction a_{\alpha+1} & \text{if } \beta = T'(\alpha+1) \text{ and } \mathrm{in}(\alpha) = 1 \end{cases}$

In order to determine $\mathrm{in}(\alpha)$, however, we need only to know $a_\alpha, e_\alpha, I, I'$.

(c) If λ is a limit ordinal, then:

$$a_\lambda = \bigcup_{\alpha <_{T'} \lambda} a_\alpha;\ e_\lambda \restriction a_\lambda = \bigcup_{\alpha <_{T'} \lambda} e_\alpha \restriction a_\alpha;\ e_\lambda(a_\lambda) = \lambda.$$

QED(Lemma 3.7.44)

Definition 3.7.22. Let I' be an inflation of I. We denote the unique history of I' from I by: $\mathrm{hist}(I, I')$.

Note. Schlutzenberg's original definition replaced (5) in Definition 3.7.21 by the following statement, which we now prove as a lemma:

Lemma 3.7.45. *Let $\mu \leq a_\alpha$ such that $\hat{e}_\alpha(\mu) \leq_{T'} \beta \leq_{T'} e_\alpha(\mu)$. Then $a_\beta = \mu$. Moreover $e_\beta \restriction \mu = e_\alpha \restriction \mu$. (Hence $e_\mu(\mu) = \beta, \hat{e}_\beta(\mu) = \hat{e}_\alpha(\mu) = \sup e_\alpha``\mu$).*

Proof. Suppose not. Let α be the least counterexample. Let $\mu \leq a_\alpha, \hat{e}_\alpha(\mu) \leq_{T'} \beta \leq_{T'} e_\alpha(\mu)$. We derive a contradiction by showing:

$$a_\beta = \mu, e_\beta \restriction a_\beta = e_\alpha \restriction a_\beta.$$

Case 1. $\mu = a_\alpha$.

Then $a_\beta \leq_T a_\alpha$ and $e_\beta \restriction a_\beta = e_\alpha \restriction a_\alpha$. But $a_\beta = a_\alpha = \mu$, since otherwise $e_\alpha(a_\beta) < \hat{e}_\alpha(a_\alpha) \leq \beta$. Hence $a_\beta \in e_\alpha^{-1}``\beta$ but $a_\beta = e_\beta^{-1}``\beta$. Hence $e_\alpha^{-1} \neq e_\beta^{-1} \restriction \beta$. Contradiction!

Case 2. $\mu < a_\alpha$.

Then there is $\gamma < \alpha$ such that:

$$\mu \leq a_\gamma, e_\alpha \restriction a_\gamma = e_\gamma \restriction a_\gamma.$$

(Clearly $\alpha > 0$. This holds by (3) or (4) if α is a successor and by Lemma 3.7.43 if α is a limit.) Hence:

$$\hat{e}_\gamma(\mu) \leq_{T'} \beta \leq_{T'} e_\gamma(\mu).$$

Hence:

$$a_\beta = \mu, e_\beta \restriction a_\beta = a_\gamma \restriction a_\beta = a_\alpha \restriction a_\beta$$

by the minimality of α.

QED(Lemma 3.7.45)

Remark. (5) can be equivalently replaced by Lemma 3.7.45 in the definition of "inflation". It can also be equivalently replaced by the conjunction of (a) and (b) in Lemma 3.7.43.

Extending inflations

By Definition 3.7.21 it follows easily that:

Lemma 3.7.46. *Let I' be an inflation of I with history $\langle a, e\rangle$. Let $1 \leq \mu \leq \mathrm{lh}(I')$. Then $I'|\mu$ is an inflation of I with history $\langle a \restriction \mu, e \restriction \mu\rangle$.*

Proof. (1)-(5) continue to hold.

Taking $\mu = 1$ it becomes evident that an inflation might say very little about the original iteration I. Hence it is useful to have lemmas which enable us to extend a given inflation I' to an I'' of greater length, thus "capturing" more of I. We prove two such lemmas:

Lemma 3.7.47. *Let I be a normal iteration of M of length $\eta' + 1$. Let I' be an inflation of I of length $\eta' + 1$ with history $\langle a, e\rangle$, where $a_{\eta'} < \eta$. Let $\tilde{\nu} = \sigma^{e_{\eta'}}_{a_{\eta'}}(\nu'_{a_{\eta'}})$ be defined with: $\tilde{\nu} > \nu'_i$ for $i < \eta$. Extend I' to I'' of length $\eta' + 2$ by appointing $\nu'_{\eta'} = \tilde{\nu}$. Then I'' is an inflation of I with history $\langle a', e'\rangle$ where:*

- *$a' \restriction \eta' + 1 = a$, $e'_\eta = e_\eta$ for $\eta \leq \eta'$,*
- *$a'_{\eta'+1} = a_{\eta'} + 1, e'_{\eta'+1} \restriction a_{\eta'} + 1 = e_{\eta'}$,*
- *$e'_{\eta'+1}(a_{\eta'} + 1) = \eta' + 1$.*

Proof. We must show that (1)-(5) are satisfied. The only problematical case is (5). We must show that if $\gamma <_{T''} \eta' + 1$, then

$$e_\gamma^{-1} \restriction \gamma = e'^{-1}_{\eta'+1} \restriction \gamma.$$

It suffices to prove it for $\gamma = T''(\eta' + 1)$. Let $\overline{\gamma} = T(a_{\eta'} + 1)$. Then

$$\hat{e}_{\eta'}(\overline{\gamma}) \leq_{T'} \gamma \leq_{T'} e_{\eta'}(\gamma)$$

by Lemma 3.7.1 (3). Hence

$$a_\gamma = \overline{\gamma} \text{ and } e_\gamma \restriction a_\gamma = e_{\eta'} \restriction a_\gamma$$

by Lemma 3.7.46. But then

$$e_\gamma^{-1} \restriction \gamma = e_{\eta'}^{-1} \restriction \gamma = (e'_{\eta'+1})^{-1} \restriction \gamma$$

since $e_{\eta'}(\overline{\gamma}) = e'_{\eta'+1}(\overline{\gamma}) \geq \gamma$.

QED(Lemma 3.7.47)

Lemma 3.7.48. *Let I' be an inflation of I of limit length η'. Let b be the unique cofinal well founded branch in I'. Extend I' to I'' of length $\eta'+1$ by appointing: $\{\xi : \xi <_{T''} \eta'\} = b$. Then I'' is an inflation of I with history $\langle a', e'\rangle$, where:*

$$a' \restriction \eta' = a,\ a'_{\eta'} = \sup_{\beta \in b} a'_\beta,\ e' \restriction \eta' = e \restriction \eta',$$
$$e'_\eta \restriction a'_{\eta'} = \bigcup_{\beta \in b} e_\beta \restriction a_\beta,\ e'_{\eta'}(a'_\eta) = \eta'.$$

Proof. (1)-(5) are satisfied.

Composing Inflations

We now show that if I' in an inflation of I and I'' is an inflation of I', then I'' is an inflation of I.

Theorem 3.7.49. *Let I, I', I'' be normal iteration of M with:* $\mathrm{lh}(I) = \eta + 1, \mathrm{lh}(I') = \eta' + 1$. *Let I' be an inflation of I with:*

$$\mathrm{hist}(I, I') = \langle a, e\rangle.$$

Let I'' be an inflation of I' with:

$$\mathrm{hist}(I', I'') = \langle a', e'\rangle.$$

Then I'' is an inflation of I with:

$$\mathrm{hist}(I, I'') = \langle a'', e''\rangle,$$

where: $a''_\alpha = a_{a'_\alpha}$, $e''_\alpha = e'_\alpha e_{a'_\alpha}$.

Proof. We verify (1)-(5).

(1) $a'' = a \cdot a'$ clearly maps $\mathrm{lh}(I'')$ into $\mathrm{lh}(I)$. Since e'_α inserts $I'|a'_\alpha + 1$ into $I''|\alpha + 1$ and $e_{a'_\alpha}$ inserts $I|a''_\alpha + 1$ into $I'|a'_\alpha + 1$, then $e'_\alpha \cdot e_{a'_\alpha}$ inserts $I|a''_\alpha + 1$ into $I''|\alpha + 1$.

QED(1)

Now let:

$$\begin{aligned}
I &= \langle\langle M_\alpha\rangle, \langle\nu_\alpha\rangle, \langle\pi_{\alpha,\beta}\rangle, T\rangle \\
I' &= \langle\langle M'_\alpha\rangle, \langle\nu'_\alpha\rangle, \langle\pi'_{\alpha,\beta}\rangle, T'\rangle \\
I'' &= \langle\langle M''_\alpha\rangle, \langle\nu''_\alpha\rangle, \langle\pi''_{\alpha,\beta}\rangle, T''\rangle
\end{aligned}$$

We recall by Lemma 3.7.5 that if e inserts I into I' and e' inserts I' into I'' then $e'e$ inserts I into I''. Moreover:

$$\sigma_\xi^{e' \cdot e} = \sigma_{e'(\xi)}^{e'} \cdot \sigma_\xi^e.$$

Thus, in particular:

$$\sigma_\xi^{e''_\alpha} = \sigma_\xi^{e'_\alpha \cdot e_{a'_\alpha}} = \sigma_{e'_\alpha(\xi)}^{e'_\alpha} \cdot \sigma_\xi^{e_{a'_\alpha}} \text{ for } \xi < a''_\alpha.$$

(2) If $\tilde{\nu}''_\alpha = \sigma_{a''_\alpha}^{e_{a''_\alpha}}(\nu_{a''_\alpha})$ exists and $\alpha < \mathrm{lh}(I'')$, then:

$$\tilde{\nu}''_\alpha = \sigma_\alpha^{e_{a'_\alpha}} \cdot \sigma_{a'_\alpha}^{e_{a'_\alpha}}(\nu_{a_{a'_\alpha}}) = \sigma_\alpha^{e_{a'_\alpha}}(\tilde{\nu}'_{a'_\alpha}).$$

But then $\nu'_{a'_\alpha} \leq \tilde{\nu}'_{a'_\alpha}$ and:

$$\nu''_\alpha \leq \sigma_\alpha^{e'_\alpha}(\nu'_{a'_\alpha}) \leq \tilde{\nu}''_\alpha.$$

QED(2)

Now let:

$\mathrm{in}(\alpha) =$ the index of α with respect to I, I',

$\mathrm{in}'(\alpha) =$ the index of α with respect to I', I'',

$\mathrm{in}''(\alpha) =$ the index of α with respect to I, I''.

(3) It is easily seen that if $\mathrm{in}''(\alpha) = 0$, then $\mathrm{in}(a'_\alpha) = \mathrm{in}'(\alpha) = 0$. Hence:

$$a'_{\alpha+1} = a'_\alpha + 1, a''_{\alpha+1} = a_{a'_{\alpha+1}} = a_{(a'_\alpha+1)} = a''_\alpha + 1.$$

Moreover:

$$\begin{aligned} e''_{\alpha+1} \upharpoonright a''_\alpha + 1 &= e'_{\alpha+1} e_{a'_\alpha+1} \upharpoonright a_{a'_\alpha} + 1 \\ &= e'_{\alpha+1} \cdot e_{a'_\alpha} \\ &= e'_{\alpha+1} \upharpoonright (a'_\alpha + 1) \cdot e_{a'_\alpha} \\ &= e'_\alpha \cdot e_{a'_\alpha} = e''_\alpha. \end{aligned}$$

QED(3)

(4) Assume $\mathrm{in}''(\alpha) = 1$. Then either $\mathrm{in}'(\alpha) = 1$ or $\mathrm{in}(a'_\alpha) = 1$.

Case 1. $\mathrm{in}'(\alpha) = 1$.

Let $\gamma = T''(\alpha + 1)$. Thus $a'_\gamma = a'_{\alpha+1}$. Hence

$$a''_\gamma = a_{a'_\gamma} = a_{a'_{\alpha+1}} = a''_{\alpha+1}.$$

Case 2. $\mathrm{in}(a'_\alpha) = 1$ but $\mathrm{in}'(\alpha) = 0$.

Let $\gamma = T'(a'_\alpha + 1)$. Then:

$$a_\gamma = a_{(a'_\alpha+1)} = a_{a'_{\alpha+1}} = a''_{\alpha+1}.$$

Let $\beta = T''(\alpha + 1)$. Then:

$$\hat{e}_\alpha(\gamma) \leq_{T''} \beta \leq_{T''} e_\alpha(\gamma).$$

Hence by Lemma 3.7.45:

$$\gamma = a'_\beta,\ a''_{\alpha+1} = a_\gamma = a_{a'_\beta} = a''_\beta.$$

QED(4)

(5) Let $\beta <_{T''} \alpha$. Then $a'_\beta \leq_{T''} a'_\alpha$ and hence:

$$a''_\beta = a_{a'_\beta} \leq_T a_{a'_\alpha} = a''_\alpha.$$

But then $(e'_\alpha)^{-1} \upharpoonright \beta = (e'_\alpha)^{-1} \upharpoonright \beta$ and

$$(e_{a'_\beta})^{-1} \upharpoonright a'_\beta = (e_{a'_\alpha})^{-1} \upharpoonright a'_\beta.$$

Hence:

$$\begin{aligned} [(e''_{be})^{-1} \upharpoonright \beta &= (e'_{a'_\beta})^{-1} (e'_\beta)^{-1} \upharpoonright \beta \\ &= (e_{a'_\beta})^{-1} (e'_\alpha)^{-1} \upharpoonright \beta \\ &= (e_{a'_\alpha})^{-1} (e'_\alpha)^{-1} \upharpoonright \beta \\ &= (e''_\alpha)^{-1} \upharpoonright \beta. \end{aligned}$$

QED(5)

This proves Theorem 3.7.49.

3.7.5 Smooth Reiterability

In §3.7.2 we proved that if M is uniquely normally iterable, then it is normally reiterable. In this section we prove the fact announced in §3.7.4. that if M is uniquely normally iterable, then it is smoothly reiterable. Just as before, it will also be of interest to know whether this theorem can be relativized to a regular cardinal $\kappa > \omega$. We called a normal reiteration $R = \langle\langle I^i\rangle, \dots\rangle$ a κ-iteration iff each of its component normal iteration I^i has length less than κ. If we are given a smooth κ-reiteration $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$, we call it a *smooth κ-reiteration* iff each of its induced reiteration R_i $(i + 1 < \mathrm{lh}(S))$ is a κ-reiteration of length less than κ. We proved previously that, if M is uniquely normally κ-iterable, then it is normally κ-reiterable. In the present case the proofs are more subtle, and the best we can get is:

Theorem 3.7.50. *Let $\kappa > \omega$ be regular. Let M be uniquely normally $\kappa + 1$-iterable. Then it is smoothly $\kappa + 1$-reiterable. (Hence if M is uniquely normally iterable, it is uniquely smoothly reiterable).*

We don't see any way to weaken the hypothesis of this theorem. Thus, for instance, if we only know that M is uniquely normally ω_1-iterable, we have no proof that it is smoothly ω_1-iterable.

We prove Theorem 3.7.50. From now on we take "reiteration" as meaning "κ-reiteration" and "smooth reiteration" as meaning "smooth κ-reiteration". We assume M to be uniquely normally κ+1-iterable. The desired conclusion then is given by:

Lemma 3.7.51. *Let $S = \langle\langle I_i\rangle, \langle e_{i,j}\rangle\rangle$ be a smooth reiteration of M of limit length $\mu \leq \kappa$. Then:*

(a) *S has at most finitely many drop points.*

(b) *S has a good limit $I, \langle e_i : i < \mu\rangle$.*

Proof. Case 1. $\mu = \kappa$.

(a) is immediate by $\mathrm{cf}(\kappa) > \omega$, since if S had infinitely many drop points, then so would $S|\gamma + 1$ for some $\gamma < \kappa$.

To prove (b), let (i, κ) be free of drop points, where $i < \kappa$. We must show that $\langle\langle I_j : i \leq j < \kappa\rangle, \langle e^{hj} : i \leq h \leq j < \kappa\rangle\rangle$ has a good limit:

$$I, \langle e^j : i \leq j < \kappa\rangle.$$

(We then set: $e^h = e^i \cdot e^{h,i}$ for $h < i$). But this is immediate by Lemma 3.7.9.

QED(Case 1)

The hard case is:

Case 2. $\mu < \kappa$.

By induction on μ we prove (a), (b) and:

(c) If $i < \mu$, then I is an inflation of I_i with history $\langle a^i, \langle e^i_\alpha : \alpha \leq \eta_i \rangle\rangle$, where $\eta_i + 1 = \mathrm{lh}(I_i)$.

(d) If $i < \mu$ and (i, μ) has no drop point in S, then $a^i_\mu = \eta_i$ and $e^i_\mu = e_i$.

Assume that this holds at every limit ordinal $\lambda < \mu$. Then:

Claim 1. Let $i \leq j < \mu$. Then

(i) I_j is an inflation of I_i with history $\langle a^{ij}, \langle e^{i,j}_\alpha : \alpha \leq \eta_j \rangle\rangle$.

(ii) If the interval (i, j) has no drop point in S, then $a^{i,j}_{\eta_j} = \eta_i$ and $e_{i,j} = e^{i,j}_{\eta_j}$.

Proof. Suppose not. Let j be the least counterexample. Then $i < j$ since (i), (ii) hold trivially for $i = j$. But j is not a limit ordinal since otherwise (i), (ii) hold by the induction hypothesis. Hence $j = h + 1$. We first show that it holds for $i = h$.

(i) is immediate by Theorem 3.7.42. We now prove (ii) for $i = h$. Let R, ξ be the unique objects such that:

$$R = \langle \langle I^l \rangle, \langle \nu^l \rangle, \langle e^{k,l} \rangle, T \rangle$$

is a normal reiteration of length $\xi + 1$ and $I_h = I^0, I_j = I^\xi$. Then $e_{h,j} = e^{0,\xi}$. Since R has no truncation on its main branch, $e_{h,j}$ inserts I_h into I_j and $e_{h,j}(\eta_h) = \eta_j$. But $a^{h,j}_\alpha = \{a < \eta_h : e_{h,j}(\alpha) < \eta_j\}$. Hence $a^{h,j}_{\eta_j} = \eta_h$. But:

$$e_{h,j} \restriction \eta_h = e^{h_j}_{\eta_j} \restriction \eta_h \text{ and } e_{h,j}(\eta_h) = e^{h,j}_{\eta_j}(\eta_h) = \eta_j$$

Hence $e_{i,h} = e^{h,j}_{\eta_j}$.

But then $i < h$. We know that (i), (ii) hold at h and that

$$a^{i,j}_\alpha = a^{i,h}_{a^{h,j}_\alpha};\ e^{i,j}_\alpha = e^{h,j}_\alpha \cdot e^{i,h}_{a^{h,j}_\alpha},$$

where $a^{h,j}_{\eta_i} = \eta_h, a^{i,h}_{\eta_h} = \eta_i, e_{i,h} = e^{i,h}_{\eta_h}, e_{hj} = e^{h,j}_{\eta_j}$. Thus:

$$\begin{aligned} a^{i,j}_{\eta_i} &= a^{i,h}_{\eta_h} = \eta_i \text{ and} \\ e_{i,j} &= e_{h,j} \cdot e_{i,h} = e^{h,j}_{\eta_j} \cdot e^{i,h}_{\eta_h} = e^{i,j}_{\eta_j} \end{aligned}$$

Contradiction!

QED(Claim 1)

We now attempt to prove (a)-(d), taking an indirect approach. Call I a *simultaneous inflation* if it is an inflation of I_i for each $i < \mu$. Our job is to find a simultaneous inflation which also satisfies the conditions (a), (b) and (d). There is no shortage of simultaneous inflations. For instance the normal iteration of length 1:

$$\langle\langle M\rangle, \varnothing, \langle \mathrm{id} \restriction M\rangle, \varnothing\rangle$$

is a simultaneous inflation. Starting with this, we attempt to form a tower of simultaneous inflations $I^{(i)}$, where $I^{(\xi)}$ is an iteration of length $\xi + 1$ extending $I^{(i)}$ for $i < \xi$. The attempt will have only limited success. If we have constructed $I^{(\xi)}$ for ξ below a limit ordinal λ, we shall, indeed, be able to construct $I^{(\lambda)}$. In attempting to go for $I^{(\xi)}$ to $I^{(\xi+1)}$, however, we may encounter a "bad case", which blocks us from going further. Using the $\kappa+1$-normal iterability of M we can, however, show that, if the bad case does not occur, we reach $I^{(\kappa)}$. But this turns out to be a contradiction. Hence the bad case must have occurred below κ. A close examination of this "bad case" then reveals it to be a very good case, since it gives $I = I^{(\xi)}$ satisfying (a)-(d).

In the following let:

$$I_i = \langle\langle M^i_\alpha\rangle, \langle\nu^i_\alpha\rangle, \langle\pi^i_{\alpha,\beta}\rangle, T^i\rangle \text{ be of length } \eta_i + 1.$$

We attempt to construct:

$$I = \langle\langle M_\alpha\rangle, \langle\nu_\alpha\rangle, \langle\pi_{\alpha,\beta}\rangle, T\rangle \text{ of length } \eta + 1$$

satisfying (a)-(d).

We successively construct:

$$I^{(\xi)} = \langle\langle M^{(\xi)}_\alpha\rangle, \langle\nu^{(\xi)}_\alpha\rangle, \langle\pi^{(\xi)}_{\alpha,\beta}\rangle, T^{(\xi)}\rangle \text{ of length } \eta + 1.$$

The intention is that $I^{(\xi)} = I|\xi + 1$ will be defined up to an $\eta < \theta$ and that $I = I^{(\eta)}$ will have the desired properties (a)-(d). The proof that there is such an η is highly indirect and non constructive. We shall require:

(A) $I^{(\xi)}$ is an inflation of I_i with history

$$\langle a^{(\xi),i}, e^{(\xi),i}\rangle \text{ for } i < \mu.$$

(B) $i < \xi \longrightarrow I^{(i)} = I^{(\xi)}|i+1$.

Note. By (B) we can write $M_\alpha, \nu_\alpha, \pi_{\alpha,\beta}, T, I$ instead of $M_\alpha^{(\xi)}$, etc. without reference to ξ. Similarly we can write a^i, e^i instead of $a^{(\xi),i}, e^{(\xi),i}$. Thus, for $\alpha \leq \xi$ we have:

$$a^i_\alpha \leq \eta_i \text{ and } e^i_\alpha \text{ inserts } I^i|a^i_\alpha + 1 \text{ into } I|\alpha+1.$$

(C) Let $\alpha \leq \xi$. Then $\alpha = \bigcup_{i<\mu} e^i_\alpha``a^i_\alpha$.

By (C) we have:

(1) $\alpha = \sup\{\hat{e}^i_\alpha(a^i_\alpha) : i < \mu\}$, since $\hat{e}^i_\alpha(a^i_\alpha) = \text{lub}\, e^i_\alpha``a^i_\alpha$.

Set: $e^{i,j}_{(\alpha)} = e^{i,j}_{a^i_\alpha}$. Hence by (C) we have:

(2) $I|\alpha+1, \langle e^i_\alpha : i < \mu\rangle$ is the good limit of

$$\langle I^i|a^i_\alpha + 1 : i < \mu\rangle, \langle e^{i,j}_{(\alpha)} : i \leq j < \mu\rangle$$

Now set: $\sigma^i_{(\alpha)} = \sigma^{e^i_\alpha}_{a^i_\alpha}, \sigma^{i,j}_{(\alpha)} = \sigma^{e^{i,j}_{(\alpha)}}_{a^i_\alpha}$. Then: $\sigma^h_{(\alpha)} e^{h,i}_{(\alpha)} = e^h_{(\alpha)}$. We can define $\hat{\sigma}^i_{(\alpha)}, \hat{\sigma}^{(i)}_{(\alpha)}$, similarly. Note, however, that $\sigma^i_{(\alpha)}$ might be a partial function on $M^i_{a^i_\alpha}$, whereas $\hat{\sigma}^i_{(\alpha)}$ is a total function. Nonetheless we do have:

(3) $\sigma^i_{(\alpha)} : M^i_{a^i_\alpha} \longrightarrow_{\Sigma^*} M_\alpha$ for sufficiently large $i < \kappa$.

Proof. $\sigma^i_{(\alpha)} = \pi_{\hat{e}^i_{(\alpha)}(a^i_\alpha),\alpha} \cdot \hat{\sigma}^i_{(\alpha)}$, where:

$$\hat{\sigma}^i_{(\alpha)} : M^i_{a^i_\alpha} \longrightarrow_{\Sigma^*} M_{e^i_{(\alpha)}(a^i_\alpha)}.$$

By (1) we can pick i big enough that there is no truncation in $(e^i_\alpha(a^i_\alpha), \alpha]_T$. Hence $\pi_{e^i_{(\alpha)}(a^i_\alpha),\alpha}$ is Σ^*-preserving.

QED(3)

We construct $I^{(\xi)} = I|\xi+1$ by recursion on ξ as follows:

Case 1. $\xi = 0$.

$I^{(0)} = \langle\langle M\rangle, \varnothing, \langle \text{id} \restriction M\rangle, \varnothing\rangle$ is the 1-step iteration of M. (A)-(C)hold trivially.

Case 2. $\xi = \theta + 1$ and $a^i_\theta < \eta_i$ for arbitrarily large $i < \mu$. Let D be the set of i such that:

$$a^i_\theta < \eta_i \text{ and } \sigma^i_{(\theta)} : M^i_{a^i_\theta} \longrightarrow_{\Sigma^*} M_\theta$$

Then D is unbounded in μ by (3). Clearly:

$$\sigma^{i,j}_{(\theta)} : M^i_{a^i_\theta} \longrightarrow_{\Sigma^*} M^i_{a^i_\theta} \text{ for } i \in D, j \in D \setminus i.$$

Hence:

$$\sigma^{i,j}_{(\theta)}(\nu^i_{a^i_\theta}) \geq \nu^i_{a^i_\theta} \text{ for } i \in D, j \in D \setminus i.$$

But then for sufficiently large $i \in D$ we have:

$$\sigma^{i,j}_{(\theta)}(\nu^i_{a^i_\theta}) = \nu^i_{a^i_\theta} \text{ for } j \in D \setminus i.$$

(To see this, suppose not. Then there is a monotone sequence $\langle i_n : n < \omega \rangle$ such that $i_n \in D$ and

$$\sigma^{i_n, i_{n+1}}_{(\theta)}(\nu^{i_n}_{a^{i_n}_\theta}) > \nu^{i_{n+1}}_{a^{i_{n+1}}_\theta}.$$

Set $\gamma_n = \sigma^{i_n}_{(\theta)}(\nu^{i_n}_{a^{i_n}_\theta})$. Then: $\gamma_n > \gamma_{n+1}$. Hence M_θ is ill founded. Contradiction!)

Let D' be the set of such $i \in D$. Then there is $\nu \in M_\theta$ such that $\nu = \sigma^i_{(\theta)}(\nu^i_{a^i_\theta})$ for $i \in D$.

Claim. $\nu > \nu_\delta$ for $\delta < \theta$.

Proof. Pick an $i \in D$ large enough that $\delta \in e^i_\theta$"a^i_θ. Let $e^i_\theta(\bar{\delta}) = \delta$. Then $\nu^i < \nu^i_{a^i_\theta}$. Hence

$$\nu_\delta = \nu = \sigma^i_{(\theta)}(\nu^i_{\bar{\delta}}) < \sigma^i_{(\theta)}(\nu^i_{a^i_\theta}) = \nu$$

QED(Claim)

We are now in a position to apply the extension lemma Lemma 3.7.47. Extend $I^{(\theta)}$ to $I^{(\theta+1)}$ by setting $\nu_\theta = \nu$. For each $i \in D', I' = I^{(\theta+1)}$ is an inflation of I_i with history $\langle a^{i'}, e^{i'} \rangle$, where:

$$a^{i'} \upharpoonright \theta + 1 = a^i, a^{i'}_{e+1} = a^i_e + 1, e^{i'} \upharpoonright a^i_\theta = e^i \upharpoonright a^i_\theta \text{ and } e^{i'}_{\theta+1}(a^{i'}_{\theta+1}) = \theta + 1.$$

But D' is cofinal in μ. It follows easily that I' is an inflation of each I_i $(i < \mu)$. Thus (A) holds for $\xi = \theta + 1$. (B) follows trivially. (C) holds trivially for $\alpha \leq \theta$. But then (c) holds for $\alpha = \xi = \theta + 1$, since $\sigma^i_\theta(a^i_\theta) = \theta$ for $i < \mu$ and $\theta = \bigcup_{\delta < \mu} e^i_\theta$"$a^i_\theta$.

QED(Case 2)

Case 3. $\xi = \theta + 1$ and Case 2 fails.

Then $a^i_\theta = \eta^i$ for sufficiently large i. This is the "bad case" in which $I^{(\theta+1)}$ is undefined.

Case 4. $\xi = \lambda$ is a limit ordinal.

Let $\tilde{I} = I|\lambda$ be the componentwise union: $\tilde{I} = \bigcup_{\gamma<\lambda} I^{(\gamma)}$. $\tilde{I}$ is then an inflation of I_i ($i < \mu$) with history:

$$a^i \upharpoonright \lambda =: \bigcup_{\gamma<\lambda} a^i \upharpoonright \gamma,\ e' \upharpoonright \lambda = \bigcup_{\gamma<\lambda} e^i|\gamma.$$

Let b be the unique well founded cofinal branch in $\tilde{I}$. Extend $\tilde{I}$ to $I' = I^{(\lambda)}$ of length $\lambda + 1$ by setting: $T``\{\lambda\} = b$. By Lemma 3.7.48, I' is then an inflation of each I_i with history $\langle a'^i, e'^i \rangle$ such that:

$$a'^i \upharpoonright \lambda = a^i \upharpoonright \lambda, e'^i \upharpoonright \lambda = e^i \upharpoonright \lambda,\ a'^i_\lambda = \bigcup_{\beta \in b} a^i_\beta,\ \tilde{e}^i_\lambda(a^i_\lambda) = \lambda.$$

(A), (B) are then trivially satisfied. But then so is (C) since

$$\bigcup_{i\in\mu} e^i_i ``a^i_\lambda = \bigcup_{i\in\mu} \bigcup_{\beta\in b} e^i_\beta ``a^i_\beta = \bigcup_{\beta\in b} \bigcup_{i<\mu} e^i_\beta ``a^i_\beta = \bigcup b = \lambda.$$

QED(Case 4)

We note that the construction in Case 4 goes through for $\lambda = \kappa$, since M is $\kappa + 1$-normally iterable. Hence $I^{(\kappa)}$ would exist if the bad case did not occur. This is impossible, however, since:

(4) If λ is a limit ordinal and $I^{(\lambda)}$ exists, then $\mathrm{cf}(\lambda) \le \mu$ or $\mathrm{cf}(\lambda) \le \eta_i$ for some $i < \mu$.

Proof. Suppose first that $\lambda > \hat{e}^i_\lambda(a^i_\lambda)$ for all $i < \mu$. Since $\lambda = \mathrm{lub}_{i<\mu}\, \hat{e}^i_\lambda(a^i_\lambda)$ by (1), we conclude that $\mathrm{cf}(\lambda) \le \mu$. Otherwise $\lambda = \hat{e}^i_\lambda(a^i_\lambda) = \mathrm{lub}\, e^i_\lambda ``a^i_\lambda$. Hence a^i_λ is a limit ordinal. Hence $\mathrm{cf}\,\lambda \le a^i_\lambda \le \eta_i$.

QED(4)

Hence the "bad case" occurs at $\xi = \delta + 1$, where $\delta < \kappa$. $I = I^{(\delta)}$ is the final element of our tower. For sufficiently large $i < \mu$ we have: $a^i_\delta = \eta_i$. Thus if $i \le j < \mu$ we have:

$$a^{i,j}_{\eta_j} = a^{i,j}_{a^j_\delta} = a^i_\delta = \eta_i,\ e^{i,j}_{\eta_i} = e^{i,j}_{(\delta)}.$$

We now show:

(5) There are only finitely many drop points $h + 1 < \mu$ in S.

Proof. Suppose not. Since the assertion is true for all $\mu' < \mu$, we conclude that here are cofinally many truncation points $h + 1 < \mu$ in

S. By (1), we can then pick such an $h+1>i$, where i is chosen such that $(\hat{e}^i_\delta(a^i_\delta),\delta)_T$ has no truncation point in I. But we can also choose i large enough that $a^i=\eta_i$. By Theorem 3.7.42(6) there is a drop point:

$$\alpha\in(\hat{e}^{i,i+1}_{\eta_i}(a^i_{\eta_i}),\eta_{i+1}]_{T^{i+1}}.$$

By Lemma 3.7.1(7) we then conclude that there is a drop point in $(\hat{e}^i_{\eta_i}(a^i_{\eta_i}),\delta)_T$. Contradiction!

QED(5)

Now suppose i_0 is chosen large enough that there is no drop point in (i,δ) in S, and that $a^i_\theta=\eta_j$ for $i_0\le j<\theta$. By Claim (1)(ii), we have

$$a^{i,j}_{\eta_i}=\eta_i \text{ and } e_{i,j}=e^{i,j}_{\eta_i}=e^{ij}_{(\theta)}$$

for $i_0\le i\le j<\theta$. By (2) we have:

$$I,\langle e^i_\theta : i_0\le i<\mu\rangle$$

is the good limit of

$$\langle I^i|\eta_i+1 : i_0\le i<\mu\rangle,\ \langle e_{i,j} : i_0\le j<\mu\rangle$$

We have thus proven (a), (b) in Lemma 3.7.51. (c) and (d) are immediate by the construction.

This proves Lemma 3.7.51 and, with it, Theorem 3.7.50.

Note. By the same method we get:

> *Let S be an insertion stable strategy for M and assume that $\langle M,S\rangle$ is $\kappa+1$-normally-iterable. Then $\langle M,S\rangle$ is κ-smoothly-iterable.*

The proofs require only cosmetic changes.

We note the following consequence of Lemma 3.7.51:

Lemma 3.7.52. *Let $S=\langle\langle I_i\rangle,\langle e_{i,j}\rangle\rangle$ be a smooth reiteration of M of length μ, where each I_i is of length η_i+1. For $j<\mu$ set:*

$$A_j=\{i<j : (i,j] \text{ has no drop points in } S\},\ A^*_j=A_j\cup\{j\}.$$

(Hence $i\in A_j\longrightarrow A_i=i\cap A_j$). For $i\in A^*_j$ set: $\pi_{i,j}=\sigma^{e_{i,j}}_{\eta_i}$. Then:

(a) $\pi_{i,j} \cdot \pi_{h,i} = \pi_{h,j}$ *for* $h \leq i \leq j$ *in* A_j^*.

(b) $\pi_{i,j} : M_{\eta_i} \longrightarrow_{\Sigma^*} M_{\eta_i}$.

(c) *If* $j = \lambda$ *is a limit ordinal, then:*

$$M_{\eta_\lambda},\ \langle \pi_{i,\lambda} : i \in A_\lambda \rangle$$

is the direct limit of:

$$\langle M_{\eta_\lambda} : i \in A_\lambda \rangle,\ \langle \pi_{i,j} : i < j \text{ in } A_\lambda \rangle$$

Proof.

(a) Since $e_{h,i}(\eta_h) = \eta_i$ and $e_{i,j}(\eta_i) = \eta_j$, we have: $\sigma_{(\eta_h)}^{e_{h,j}} = \sigma_{(\eta_i)}^{e_{i,j}} \cdot \sigma_{(\eta_h)}^{e_{h,i}}$.

We prove (b), (c) by induction on j as follows:

Case 1. $j = 0$. Then $A_j = \varnothing$ and there is nothing to prove.

Case 2. $j = i + 1$. We must prove (b). If $i + 1$ is a drop point, then $A_j = \varnothing$ and there is nothing to prove. If not, it suffices to prove it for $h = i$, by (a) and the induction hypothesis. Then the main branch of R_i has no drop point in R_i, where R_i is the unique reiteration from I^i to I^{i+1}. Then $\pi_{i,i+1} = (\sigma_{\eta_i}^{0,\gamma})^{R_i}$, where $\gamma + 1 = \mathrm{lh}(R_i)$. But:

$$\sigma_{\eta_i}^{0,\gamma} : M_{\eta_i} \longrightarrow_{\Sigma^*} M_{\eta_{h+1}} \text{ in } R_i.$$

QED(Case 2)

Case 3. $j = \lambda$ is a limit ordinal.

It suffices to prove (c), since (b) then follows by the induction hypothesis. In S we have:

$$I_\lambda,\ \langle e_{i,\lambda} : l \in A_\lambda \rangle$$

is the good limit of

$$\langle I_i : i \in A_\lambda \rangle,\ \langle \pi_{i,j} : i \leq j \text{ in } A_\lambda \rangle$$

But then $M_\eta = \bigcup_{i \in A_\lambda} \mathrm{rng}(\sigma_{\eta_i}^{i,\lambda})$. This implies (c).

QED(Lemma 3.7.52)

3.7.6 The final conclusion

We now apply the method of §3.7.3 to show that M is smoothly iterable. In §3.5.2 we defined a smooth iteration of N to be a sequence $I = \langle I_i : i < \mu \rangle$ of normal iterations, inducing sequences $\langle N_i : i < \mu \rangle$, $\langle \pi_{i,j} : i \leq j < \mu \rangle$ with the following properties:

- N_i is the initial model of I_i. Moreover $N_0 = N$.
- Let $i + 1 < \mu$. Then I_i is of successor length. N_{i+1} is the final model of I_i and $\pi_{i,i+1}$ is the partial embedding of N_i into N_{i+1} determined by I_i.
- $\pi_{i,j}\pi_{h,i} = \pi_{h,i}$.
- Call $i + 1 < \mu$ a *drop point* in I iff I_i has a truncation on its main branch. If the interval $(i, j]$ has no drop point, then:

$$\pi_{i,j} : N_i \longrightarrow_{\Sigma^*} N_j.$$

- If $\lambda < \mu$ is a limit ordinal, $i_0 < \lambda$ and (i, λ) has no drop point, then:

$$N_\lambda, \langle \pi_{i,\lambda} : i_0 \leq i < \mu \rangle$$

  is the direct limit of

$$\langle N_i : i_0 \leq i < \mu \rangle, \ \langle \pi_{i,j} : i \leq j < \mu \rangle.$$

$\langle \langle N_i \rangle, \ \langle \pi_{i,j} \rangle \rangle$ is called the *induced* sequence.

Call a smooth iteration I *critical* if it has successor length $\eta + 1$ and I_η is of limit length. By a *strategy for* N we mean a partial function S defined on critical smooth iterations such that $S(I)$, if defined, is a well founded cofinal branch in I_η, where $\mathrm{lh}(I) = \eta + 1$.

A smooth iteration $I = \langle I_i : i < \mu \rangle$ is S-conforming iff whenever $i < \mu$ and $\lambda < \mathrm{lh}(I_i)$ is a limit ordinal, and $I^* = I \upharpoonright i \cup \{\langle I_i \upharpoonright \lambda, i \rangle\}$, then:

$$T^{i\prime\prime}\{\lambda\} = S(I^*) \text{ if } S(I^*) \text{ is defined.}$$

S is a *successful strategy* for N iff every S-conforming smooth iteration I of N can be properly extended in any legitimate S-conforming way. In other words:

(A) Let I have length $\eta+1$ and let I_η have length $i+1$. Let $Q = N_i^\eta$ be the final model of I_η. Let $E_\nu^Q \neq \varnothing$, where ν is greater than all the indices ν_j^η $(j < i)$ employed in I_η. Then Q is $*$-extendible by E_ν^Q.

(B) If I is critical, then $S(I)$ is defined.

(C) Let I have limit length μ. Then there are only finitely many drop points in I. Moreover, if $l_0 < \mu$ and (i_0, μ) is free of drops, then:

$$\langle N_i : i_0 \leq i < \mu \rangle,\ \langle \pi_{i,j} : i \leq j < \mu \rangle$$

has a well founded direct limit:

$$N_\mu, \langle \pi_{i,\mu} : i_0 \leq i < \mu \rangle$$

We say that N is *smoothly iterable* iff it has a successful smooth iteration strategy.

These concepts can, of course, be relativized to an ordinal α. To this end we define the *total length* of $I = \langle I_i : i < \mu \rangle$ to be:

$$\mathrm{tl}(I) = \sum_{i<\mu} \mathrm{lh}(I_i).$$

The notion of *α-successful smooth iteration* strategy is then defined as before, except that we restrict ourselves to iteration of total length less than α.

Note that if $\kappa > \omega$ is regular, then there are only two ways that a smooth iteration $I = \langle I_i : i < \mu \rangle$ can have total length κ. Either $\mu = \kappa$ and $\mathrm{lh}(I_i) < \kappa$ for $i < \kappa$, or else $\mu = \eta + 1 < \kappa$, $\mathrm{lh}(I_\eta) = \kappa$ and $\mathrm{lh}(I_i) < \kappa$ for $i < \eta$.

In this section we shall prove:

Theorem 3.7.53. *Let M be uniquely normally iterable. Then it is smoothly iterable.*

Note. There is of course, considerable interest in relativizing this theorem to $\alpha < \infty$. We shall later show that, if $\kappa > \omega$ is regular, then the theorem can be relativized to $\kappa + 1$. That will require fairly modest changes in the proof we give now.

Until further notice, assume M to be uniquely normally iterable. We prove our Theorem 3.7.53 in the slightly stronger form:

Lemma 3.7.54. *Let I be a normal iteration of M of length $\eta+1$. Let:*

$$\sigma : N \longrightarrow_{\Sigma^*} M_\eta \min \rho$$

Then N is smoothly iterable.

In §3.7.3 we used the premiss of Lemma 3.7.54 to derive the normal iterability of N. We first briefly review that proof, since our new proof will build upon it. Our main tool was the *reiteration mirror* (RM). Given a normal iteration of N:

$$I = \langle\langle N_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle \text{ of length } \eta,$$

we define a reiteration mirror of I to be a pair $\langle R, I'\rangle$ such that:

(a) $R = \langle\langle I^i\rangle, \langle\nu'_i\rangle, \langle e^{i,j}\rangle, T\rangle$ is a reiteration of M of length η, where:

$$I^i = \langle\langle M^i_h\rangle, \langle\nu^i_h\rangle, \langle\pi^i_{h,j}\rangle, T^i\rangle \text{ is of length } \eta_i + 1$$

(b) $I' = \langle\langle M'_i\rangle, \langle\pi'_{i,h}\rangle, \langle\sigma_i\rangle, \langle\rho^i\rangle\rangle$ is a mirror of I with $\sigma_i(\nu_i) = \nu'_i$.

(c) $M'_i = M^i_{\eta_i}$.

(d) If $h = T(i+1)$, then:

$M'^{*}_i = M'_h||\mu$ where μ is maximal such that τ'_i is a cardinal in M'_h.

Moreover:

$$\pi'_{h,i+1} = \sigma^{h,i+1}_{\eta^*_h}, \text{ where } \eta^*_h = \mathrm{lh}(I^i_*).$$

$\langle I, R, I'\rangle$ is called an RM triple of length η if and only if $\langle R, I'\rangle$ is an RM of I.

We observed that:

Lemma 3.7.34 *Let $\Gamma = \langle I, R, I'\rangle$ be an RM triple of length $\eta + 1$. Let $E^{M_\eta}_\nu \neq \varnothing$, where $\nu > \nu_i$ for all $i < \eta$. Then Γ extends to an RM triple $\dot{\Gamma} = \langle\dot{I}, \dot{R}, \dot{I}'\rangle$ of length $\eta + 2$ with $\dot{\nu} = \nu$.*

We fixed a function G such that whenever (Γ, ν) is such a pair, then $G(\Gamma, \nu) = \langle\dot{I}, \dot{R}, \dot{I}'\rangle$ is such an extension.

We also observed that:

Lemma 3.7.35. *Let $\Gamma = \langle I, R, I'\rangle$ be an RM-triple of limit length η. Let b be the unique good branch in R. Then there is a unique extension to an RM-triple $\dot{\Gamma}$ of length $\eta + 1$. Moreover, $b = \dot{T}$“$\{\eta\}$ in this extension.*

We also noted that:

Lemma 3.7.32. *$i + 1$ is a drop point in I iff it is a drop point in R.*

Lemma 3.7.33. *If $(i, j]_T$ has no drop point in I, then $\pi'_{i,j} = \sigma^{i,j}_{\eta_i}$.*

Clearly, if $\Gamma = \langle I, R, I' \rangle$ is an RM-triple of length η and $1 \leq i < \eta$, then $\Gamma | i = \langle I|i, R|i, I'|i \rangle$ is a RM triple of length i. Now let:

$$\sigma : N \longrightarrow_{\Sigma^*} \tilde{M}_{\tilde{\eta}} \min \tilde{\rho},$$

where $\tilde{I} = \langle \langle \tilde{M}_i \rangle, \langle \tilde{\nu}_i \rangle, \langle \tilde{\pi}_{i,j} \rangle, \tilde{T} \rangle$ is a normal iteration of M of length $\tilde{\eta} + 1$. We define:

Definition 3.7.23. Let I be a normal iteration of N of length μ. By a *good triple for* I we mean an RM triple $\Gamma(I) = \langle I, R, I' \rangle$ such that:

(a) $R = \langle \langle I^i \rangle, \langle \nu'_i \rangle, \langle e^{ij} \rangle, T \rangle$, $I' = \langle \langle M'_i \rangle, \langle \pi'_{i,j} \rangle, \langle \sigma_i \rangle, \langle \rho^i \rangle \rangle$ with $I^0 = \tilde{I}, \sigma_i = \tilde{\sigma}, \rho^0 = \tilde{\rho}$.

(b) If $i + 1 < \mu$, then $\Gamma | i + 2 = G(\Gamma | i + 1, \nu'_i)$.

By the fact that M is uniquely normally iterable and Γ is an RM-triple, it follows that, if $\eta < \mu$ is a limit ordinal then $\Gamma | \eta + 1$ is obtained from $\Gamma | \eta$ as in Lemma 3.7.35. It follows easily that I can have at most one good triple, which we denote by $\Gamma(I)$, if it exists, we then define a strategy S for N as follows:

Let I be a normal iteration N of limit length. If $\Gamma(I)$ is undefined, then so is $S(I)$. If not, then we let:

$$b = \text{ the unique good branch in } R,$$

where $\Gamma(I) = \langle I, R, I' \rangle$. We set: $S(I) = b$, We then noted:

Lemma 3.7.36. *If I is an S-conforming iteration, then $\Gamma(I)$ is defined.*

But this means that I can be extended one step further, using Lemma 3.7.34 and 3.7.35. Hence S is a successful normal iteration strategy.

Building upon this, we now try to define a successful smooth iteration strategy for N. Note that, given the function G, the operation $\Gamma(I)$ is uniquely characterized by $\tilde{\sigma}, \tilde{I}, \tilde{\rho}$. Thus we can write: $\Gamma_{\tilde{\sigma}, \tilde{I}', \tilde{\rho}}(I)$. We now try to define $\Gamma(I)$ for *smooth iterations* I of N.

Definition 3.7.24. Let $I = \langle I_i : i < \mu \rangle$ be a smooth iteration of N inducing $\langle N_i : i < \mu \rangle, \langle \pi_{i,j} : i \leq j < \mu \rangle$. Let

$$I_i = \langle \langle N^i_h \rangle, \langle \nu^i_h \rangle, \langle \pi^i_{h,j} \rangle, T^i \rangle \text{ be of length } \eta_i.$$

By a Γ-sequence for I, we mean any sequence $\Gamma = \langle \Gamma_i : i < \mu \rangle$ such that:

(a) $\Gamma_i = \Gamma_{\dot{I}_i,\sigma_i,\rho_i}(I_i) = \langle I_i, R_i, I_i' \rangle$ is an RM triple where:

$$\sigma_i : N_i \longrightarrow_{\Sigma^*} \dot{M}_i \min \rho^i$$

and $\dot{I}_i$ is the first iteration in R_i and $\dot{M}_i$ is the final model in $\dot{I}_i$.

We set:

$$R_i = \langle \langle I_i^h \rangle, \langle \nu_i^h \rangle, \langle e_i^{h,j} \rangle, T^i \rangle$$

$$I_i' = \langle \langle M_h^{\prime(i)} \rangle, \langle \pi_{h,j}^{\prime i} \rangle, \langle \sigma_h^i \rangle, \langle \rho^{i,h} \rangle \rangle$$

(Hence $\dot{I}_i = I_i^0, \dot{M}_i = M_0^{\prime i}$.)

(b) $\dot{I} = \langle \dot{I}_i : i < \mu \rangle$ is a smooth reiteration of M such that $R_i =$ the unique reiteration from $\dot{I}_i$ to $\dot{I}_{i+1}$ for $i + 1 < \mu$.

$\dot{I}$ then induces partial insertions $\dot{e}_{i,j}$ with:

$$\dot{I}_{i+1} = I_i^{\eta_i}, \dot{e}_{i,i+1} = e_i^{0,\eta_i} \text{ for } i + 1 < \mu$$

and

$$\begin{aligned} &\dot{I}_\lambda, \langle \dot{e}_{i,\lambda} : i < \lambda \rangle \text{ is the good limit of} \\ &\langle \dot{I}_i : i < \lambda \rangle, \langle \dot{e}_{i,j} : i \leq j < \lambda \rangle \text{ for limit } \lambda < \mu. \end{aligned}$$

(c) There is a commutative system $\langle \dot{\pi}_{i,j} : i \leq j < \mu \rangle$ such that $\dot{\pi}_{i,j}$ is a partial map from $\dot{M}_i$ to $\dot{M}_j$ and:

$$\dot{\pi}_{i,i+1} = \pi_{0,\eta_i}^{\prime i} \text{ for } i + 1 < \mu.$$

Moreover:

$$\begin{aligned} &\dot{M}_\lambda, \langle \dot{\pi}_{i,\lambda} : i < \lambda \rangle \text{ is the limit of} \\ &\langle \dot{M}_i : i < \lambda \rangle, \langle \dot{\pi}_{i,j} : i \leq j < \lambda \rangle \text{ for limit } \lambda < \mu. \end{aligned}$$

(d) $\dot{\sigma}_{i+1} = \sigma_{\eta_i}^i, \rho^{i+1} = \rho^{i,\eta_i}$ for $i + 1 < \mu$.

(e) $\tilde{I} = \dot{I}_0, \tilde{\sigma} = \dot{\sigma}_0, \tilde{\rho} = \dot{\rho}^0$.

(f) Suppose that I has no drop point in $[i, j]$. Then:

 (i) $\dot{\pi}_{i,j} : \dot{M}_i \longrightarrow_{\Sigma^*} \dot{M}_j$

 (ii) $\dot{\pi}_{i,j} \cdot \sigma_i = \dot{\sigma}_j \pi_{i,j}$

 (iii) $\dot{\pi}_{i,j}``\dot{\rho}_n^i \subset \rho_n^j \leq \dot{\pi}_{i,j}(\dot{\rho}_n^i)$ for $n < \omega$.

This completes the definition.

Recall that $h+1$ is a drop point in R_i iff it is a drop point in I_i. We call $i+1$ a drop point in $\dot{I}$ iff r_i has a drop point on its main branch. Similarly, $i+1$ is a drop point in I iff I_i has a drop point on its main branch. Hence $i+1$ is a drop in $\dot{I}$ iff it is a drop point in I.

Lemma 3.7.55. *There is at most one Γ-sequence for I.*

Proof. By induction on $i < \mu$ we show that the sets:

$$\Gamma_i, \dot{I}_i, \langle \dot{e}_{h,i} : h < i \rangle, \dot{M}_i, \langle \dot{\pi}_{h,i} : h < i \rangle, \sigma_i, \rho^i$$

are uniquely determined by $\Gamma | i = \langle \Gamma_h : h < i \rangle$.

Case 1. $i = 0$.

$\dot{I}_0, \sigma_0, \rho^0$ are explicitly given by (e). Hence so are:

$$\dot{M}_0 = \text{ the final model of } \dot{I}_0, \Gamma_{\dot{I}_0, \dot{\sigma}_0, \rho^0}(I_0)$$

Case 2. $i = h+1$. Then

- $\dot{I}_i = I_h^{\eta_h}, \dot{e}_{j,i} \cdot \dot{e}_{j,h}$ for $h < i$.
- $\dot{M}_i$ is defined from $\dot{I}_{i,j}$ and $\dot{\pi}_{j,i} = \pi'^h_{0,\eta_h} \dot{\pi}_{j,h}$ for $h < i$.
- $\sigma_i = \sigma^h_{\eta_h}, \rho^i = \rho^{h,\eta_h}$.
- $\Gamma_i = \Gamma_{\dot{I}_i, \sigma_i, \rho^i}(I_i)$

Case 3. $i = \lambda$ is a limit ordinal.

- $\dot{I}_\lambda, \langle \dot{e}_{h,\lambda} : h < \lambda \rangle$ are given by (b).
- $\dot{M}_\lambda, \langle \dot{\pi}_{h,\lambda} : h < \lambda \rangle$ are given by (c).
- σ_λ is defined by: $\sigma_\lambda \pi_{h,\lambda} = \dot{\pi}_{h,\lambda} \sigma_h$ for $[h, \lambda)$ drop free in I (by (f)).
- By Lemma 3.6.42, ρ^λ is the unique ρ such that

$$\sigma_\lambda : N_\lambda \longrightarrow_{\Sigma^*} \dot{M}_\lambda \min \rho \text{ and}$$

$$\dot{\pi}_{i,\lambda} "\rho^i \subset \rho \leq \dot{\pi}_{i,\lambda}(\rho^i) \text{ if } (i, \lambda) \text{ is drop free.}$$

- $\Gamma_\lambda = \Gamma_{\dot{I}_\lambda, \sigma_\lambda, \rho^\lambda}(I_\lambda)$.

QED(Lemma 3.7.55)

We denote the unique Γ-sequence for I by $\Gamma(I)$, if it exits. Writing $\dot\sigma^{i,j}_l$ for $\sigma^{\dot e_{i,j}}_l$ and $\dot\eta_i$ for $\mathrm{lh}(\dot I_i)$ we have:

Lemma 3.7.56. *Let $\Gamma = \Gamma(I)$. If $(i, j]$ has no drop point in I, then $\dot\pi_{i,j} = \dot\sigma^{i,j}_{\dot\eta_i}$.*

Proof. We recall that if $i+1$ is not a drop point, then

$$\dot\pi_{i,i+1} = \pi''_{0,\eta_i} = \sigma^{e_i^{0,\eta_i}}_{\dot\eta_i} = \dot\sigma^{i,i+1}.$$

(Here $\eta_i + 1 = \mathrm{lh}(R_i), \dot\eta_i + 1 = \mathrm{lh}(I_i^0)$). Using this and Lemma 3.7.52, we prove the assertion by induction on j.

QED(Lemma 3.7.56)

Lemma 3.7.57. *Let $I = \langle I_i : i < \mu\rangle$ be of limit length μ. Assume that $\Gamma = \Gamma(I)$ exits. Then there are unique: $N_\mu, \langle\pi_{i,\mu}\rangle, \dot I_\mu, \langle\dot e_{i,\mu}\rangle, \dot M_\mu, \langle\dot\pi_{i,\mu}\rangle, \sigma_\mu, \rho^\mu$ such that:*

(a) *$N_\mu, \langle\pi_{i,\mu} : i < \mu\rangle$ is the direct limit of:*

$$\langle N_i : i < \mu\rangle, \langle\pi_{i,j} : i \le j < \mu\rangle.$$

(b) *$\dot I_\mu, \langle\dot e_{i,\mu} : i < \mu\rangle$ is the good limit of*

$$\langle\dot I_i : i < \mu\rangle, \langle\dot e_{i,j} : i \le j < \mu\rangle$$

(c) *$\dot M_\mu$ is the final model of $\dot I_\mu$.*

(d) *$\dot M_\mu, \langle\dot\pi_{i,\mu} : i < \mu\rangle$ is the direct limit of:*

$$\langle\dot M_i : i < \mu\rangle, \langle\dot\pi_{i,j} : i \le j < \mu\rangle.$$

(e) $\sigma_\mu : N_\mu \longrightarrow_{\Sigma^*} \dot M_\mu \min \rho^\mu$.

(f) *For sufficient $i < \mu$ we have:*

$$\sigma_\mu\pi_{i,\mu} = \dot\pi_{i,\mu}\sigma_i; \dot\pi_{i,\mu}``\rho^i \subset \rho^\mu \le \dot\pi_{i,\mu}(\rho^i)$$

Proof. (b) is immediate by Theorem 3.7.50. We let $\dot M_\mu$ be defined as in (c). Let $i < \mu$ such that (i, μ) has no drop points in I, Then (i, μ) has no drop points in $\dot I = \langle\dot I_i : i < \mu\rangle$. By Lemma 3.7.56 we know that $\dot\pi_{h,j} = \dot\sigma^{h,j}_{\dot\eta_h}$ for $i \le h \le j < \mu$. Set: $\dot\pi_{h,\mu} = \dot\sigma^{h,\mu}_{\dot\eta_h}$ for $h \in [i, \mu)$. Then (d) follows by Lemma

3.7.52. We know that $\sigma_j \pi_{hj} = \dot{\pi}_{hj} \sigma_h$ for $i \leq h \leq j < \mu$. Hence we can define σ_μ as in (f). σ_μ is obviously unique. But then there is a unique ρ^μ satisfying (e), (f) by Lemma 3.6.42. QED(Lemma 3.7.57)

We now define the strategy S. Let I be a critical smooth iteration. Then I has length $\eta + 1$ and I_η is of limit length. If $\Gamma(I)$ is undefined, the so is $S(I)$. If not, then:

$$\sigma_\eta : N_\eta \longrightarrow_{\Sigma^*} \dot{M}_\eta \min \rho^\eta$$

where $\dot{I}, \dot{M}_\eta, \sigma_\eta, \rho^\eta$ are as in the definition of "Γ-sequence". Moreover, $\Gamma_\eta = \Gamma_{\dot{I}_\eta, \sigma_\eta, \rho^\eta}(I_\eta)$. We then set:

$$S(I) =: S_\eta(\dot{I}_\eta) = \text{ the unique cofinal, well founded branch in } \dot{I}_\eta.$$

But then:

Lemma 3.7.58. *Let $I = \langle I_i : i < \mu \rangle$ be any S-conforming smooth iteration. Then $\Gamma(I)$ exists.*

Proof. Let $I = \langle I_i : i < \mu \rangle$. Define a partical function on μ by:

$$\Gamma_i =: \text{ the unique } x \text{ such that } \Gamma(I|i + 1) = \langle \Gamma_h : h < i \rangle \cup \{\langle x, i \rangle\}.$$

By induction on i we show:

Claim. Γ_i exists.

Case 1. $i = 0$.

Clearly $\Gamma_i = \Gamma_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}(I_0)$. But this holds for any I_0 which is a normal iteration of N. Hence by induction on $\mathrm{lh}(I_0)$, we have: I_0 is $S_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}$-conforming, where $S_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}$ is the normal iteration strategy for N defined from the function $\Gamma_{\tilde{I}, \tilde{\sigma}, \tilde{\rho}}$.

QED(Case 1)

Case 2. $i = h + 1$.

Set $\dot{I}_i = I_h^{\eta_h}, \sigma_i = \sigma_{\eta_h}^h, \rho^i = \rho^{h, \eta_h}$. Clearly, then:

$$\Gamma_i = \Gamma_{\dot{I}_i, \sigma_i, \rho^i}(I_i)$$

where $\dot{I}_i$ is a normal iterate of M and:

$$\sigma : N_i \longrightarrow_{\Sigma^*} \dot{M}_i \min \rho^i,$$

$\dot{M}_i$ being the final model of $\dot{I}_i$. Since this holds for any normal iterate I_i of N_i, we conclude by induction on $\mathrm{lh}(I_i)$ that I_i is $S_{\dot{I}_i, \sigma_i, \rho^i}$-conforming. Hence $\Gamma_i = \Gamma_{\dot{I}_i, \sigma_i, \rho^i}$ exists.

QED(2)

Case 3. $i = \lambda$ is a limit.

It is easily seen that $\langle \Gamma_h : h < \lambda \rangle = \Gamma(I \upharpoonright \lambda)$. Let $\dot{I}_\lambda, \dot{M}_\lambda, \sigma_\lambda, \rho^\lambda$ be as in Lemma 3.7.57. Clearly we have: $\Gamma_\lambda = \Gamma_{\dot{I}_\lambda, \dot{M}_\lambda, \sigma_\lambda, \rho^\lambda}(I_\lambda)$. Exactly as before, we conclude that I_λ is $S_{\dot{I}_\lambda, \dot{M}_\lambda, \sigma_\lambda, \rho^\lambda}$-conforming, hence that Γ_λ exists.

QED(Claim)

But then it is easily seen that $\langle \Gamma_i : i < \mu \rangle = \Gamma(I)$.

QED(Lemma 3.7.58)

But then S is successful, since, if I is S-conforming, then I can be extended un any S-conforming way -i.e. (A)-(C)hold. (A) follows by Lemma 3.7.34. (B) follows by Lemma 3.7.35. (C) follows by Lemma 3.6.47.

This proves Lemma 3.7.54 and with it Theorem 3.7.53. We now show how to relativize this to a regular cardinal $\kappa > \omega$. We assume that M is uniquely $\kappa + 1$-normally iterable. By a κ-reiteration of M we mean a reiteration of length $\leq \kappa$ in which each component normal iteration is of length $< \kappa$. If we understand "reiteration" as meaning a κ-reiteration of length $< \kappa$, and "smooth iteration" as meaning a smooth iteration of total length $< \kappa$, then a literal repetition of the above proof shows:

Lemma 3.7.59. *Let M be uniquely normally $\kappa + 1$-iterable. Let $\tilde{I}$ be a normal iteration of M of length $\eta + 1 < \kappa$. Let*

$$\sigma : N \longrightarrow_{\Sigma^*} \tilde{M}_\eta \min \rho$$

Then N is smoothly κ-iterable.

The following strength of κ+1-iterability is needed for this, however, in order to justify the use of Theorem 3.7.50. We now show that, under the premises of Lemma 3.7.59, N is in fact, smoothly $\kappa + 1$-iterable. Let $I = \langle I_i : i < \mu \rangle$ be a smooth iteration of N of total length κ. As mentioned earlier, one of two cases hold, which we consider separately:

Case 1. $\mu = \eta + 1 < \kappa$ and I_η is of length κ.

We assume I to be S-conforming. Then $I|\eta$ is S-conforming. Then $I|\eta$ is S-conforming and I_η is $S_{\dot{I}_\eta, \sigma_\eta, \rho^\eta}$-conforming. Hence:

$$\Gamma_{\dot{I}_\eta, \sigma_\eta, \rho^\eta}(I_\eta) = \langle I_\eta, R, I' \rangle \text{ exists,}$$

where R is a reiteration of M of length κ. But then R has a well founded cofinal branch b. Hence b is cofinal in I_η. b has only finitely many drop points in I_η, since otherwise, by the fact that $\kappa > \omega$ is regular, there would be $\lambda \in b$ such that $h \cap \lambda = T^\eta``\{\lambda\}$ has infinitely many drop points. Contradiction! Let $i \in b$ such that $b \setminus i$ has no drop points. Using the fact that $\kappa > \omega$ is regular, it follows easily that

$$\langle M_h : h \in b \setminus i \rangle, \langle \pi_{h,j} : h \leq j \text{ in } b \setminus i \rangle$$

has a well founded limit. (If $x_{n+1} \in x_n$ is the limit, these would be a $\xi \in b \setminus i$ such that $x_n = \overline{N}_\xi(\overline{x}_n)$ for $n < \omega$. Hence $\overline{x}_{n+1} \in \overline{x}_n$ in N_ξ. Contradiction!)

QED(Case 1)

Case 2. $\mu = \kappa$.

I has only finitely many drop points, since otherwise these would be $\xi < \kappa$ such that $I|\xi$ has infinitely many drop points. Contradiction! Let the interval (i, κ) be drop free. Since $\kappa > \omega$ is regular, it again follows that:

$$\langle M_h : i \leq h < \kappa \rangle, \langle \pi_{h,j} : i \leq h \leq j < \kappa \rangle$$

has a well founded limit.

QED(Case 2)

This proves Theorem 3.6.2.

3.8 Unique Iterability

3.8.1 One small mice

Although we have thus far developed the theory of mice in considerable generality, most of this book will deal with a subclass of mice called *one small*. These mice were discovered and named by John Steel. It turns out that a great part of many one small mice are uniquely normally iterable. Using the notion of Woodin cardinal defined in the preliminaries we define:

Definition 3.8.1 (1-small)**.** A premouse M is one small iff whenever $E^M_\nu \neq \varnothing$, then

$$\text{no } \mu < \kappa = \text{crit}(E^M_\nu) \text{ is Woodin in } J^{E^M}_\kappa$$

Note. Since J^E_κ is a ZFC model, we can employ the definition of "Woodin cardinal" given in the preliminaries. An examination of the definition shows that the statement "μ is Woodin" is, in fact, first order over H_τ where $\tau = \mu^+$. Thus the statement "μ is Woodin in M" makes sense for any transitive ZFC^- model M. It means that $\mu \in M$ and "μ is Woodin" hold in H^M_τ where $\tau = \mu^{+M}$ (taking $\tau = \text{card}\, M$ if no $\xi > \mu$ is a cardinal in M). We then have:

Lemma 3.8.1. *Let M be a premouse such that $E_\nu^M \neq \varnothing$ and let us set:*

$$\kappa = \mathrm{crit}(E_\nu^M), \lambda = \lambda(E_\nu^M) =: E_\gamma^M(\kappa), \tau = \tau(E_\gamma^M) =: \kappa^{+E^M}.$$

The following are equivalent:

(a) *No $\mu < \kappa$ is Woodin in J_κ^E*

(b) *No $\mu \leq \kappa$ is Woodin in J_τ^E*

(c) *No $\mu < \lambda$ is Woodin in J_λ^E*

(d) *No $\mu \leq \lambda$ is Woodin in J_γ^E.*

Proof: (d)→(c)→(b)→(a) is clear. We now show (a)→(d). Assume (a). Since $J_\kappa^E \prec J_\lambda^E$ we have (c). But then (b) holds. Since $\pi : J_\tau^E \longrightarrow J_\nu^E$ cofinally, we conclude that π is elementary on J_τ^E. Hence (d) holds. QED (Lemma 3.8.1).

Recalling the typology developed in §3.3, we have:

Lemma 3.8.2. *Every active one-small premouse is of type 1.*

Proof: Suppose not. Let $M = \langle J_\nu^E, F\rangle$ be a counterexample. We derive a contradiction by proving:

Claim. κ is Woodin in M, where $\kappa = \mathrm{crit}(F)$.

Proof: Let $A \subset \kappa$, $A \in M$. We show that some $\tau < \kappa$ is A-strong on J_κ^E. It is easily seen that $\langle J_\kappa^E, B\rangle \prec \langle J_\lambda^E, F(B)\rangle$ whenever $B \subset \kappa$, $B \in M$. Hence it suffices to find a $\tau < \lambda$ such that τ is $F(A)$-strong in J_λ^E.

Claim. κ is $F(A)$-strong in J_λ^E.

Proof: Suppose not. Then there is $\xi < \lambda$ such that whenever $G \in J_\lambda^E$ is an extender at κ on J_λ^E, then $F(A) \cap \xi \neq G(A) \cap \xi$ (where $A = F(A) \cap \kappa$). Let ξ be the least such. Since M is not of type 1, there is $\bar{\lambda} < \lambda$ such that $\bar{F} = F \restriction \lambda$ is a full extender at κ in M. Hence $\bar{F} \in J_\lambda^E$. But:

$$\langle J_{\bar{\lambda}}^E, \bar{F}(A)\rangle \prec \langle J_\lambda^E, F(A)\rangle$$

Since for $\alpha_1, \ldots, \alpha_n < \bar{\lambda}$ we have:

$$\begin{aligned}\langle J_{\bar{\lambda}}^E, \bar{F}(A)\rangle \models \varphi[\vec{\alpha}] &\longleftrightarrow \langle J_\lambda^E, F(A)\rangle \models \varphi[\vec{\alpha}] \\ &\longleftrightarrow \langle \vec{\alpha}\rangle \in F(e)\end{aligned}$$

where $e = \{\langle\vec{\xi}\rangle < \kappa : \langle J_\kappa^E, A\rangle \models \varphi[\vec{\xi}]\}$. Hence $\xi < \bar{\lambda}$ by minimality. Hence $\bar{F} \in J_{\bar{\lambda}}^E$ and $F(A) \cap \xi = \bar{F}(A) \cap \xi$. Contradiction! QED (Lemma 3.8.2).

We leave it to the reader to show:

- If M is one small and $\mu \in M$, then $M||\mu$ is one small (for limit μ).
- Let $\langle M_i : i < \lambda \rangle$ be a sequence of one small premice. Let $\pi_{ij} : M_i \longrightarrow_{\Sigma^*} M_j$ for $i \leq j < \lambda$, where the π_{ij} commute. Let $M_\lambda, \langle \pi_{i\lambda} : i < \lambda \rangle$ be the direct limit of $\langle M_i : i < \lambda \rangle, \langle \pi_{ij} : i \leq j < \lambda \rangle$. Then M_λ is one small.

It then follows easily that:

Lemma 3.8.3. *Any full iterate of a small mouse is one small.*

In particular, any normal iterate of a one small mouse is one small.

In §3.8.2 we shall show that there is a large class of one small premice, all of which have the normal uniqueness property. That will be our main result in this section.

3.8.2 Woodiness and non unique branches

In the preliminaries we defined the notion of A-strong. We now adapt this notion to certain admissible structures in place of V.

Definition 3.8.2. $N = J_\alpha^E$ is a *limit structure* iff N is acceptable and there are arbitrarily large $\tau \in N$ such that $N \models \tau$ is a cardinal.

Definition 3.8.3. Let $N = J_\alpha^E$ is a limit structure. $\kappa \in N$ is *strong in N* iff for arbitrarily large $\xi \in N$ there is $F \in N$ such that:

- F is an extender at κ on N of length $\geq \xi$.
- N is extendible by F.
- Let $\pi : N \longrightarrow N' = J_{\alpha'}^{E'}$. Then $J_\xi^{E'} = J_\xi^E$.

Hence, if ξ is a cardinal in N, it follows that $H_\xi^N = H_\xi^{N'}$.

Definition 3.8.4. Let $A \subset N$, where $N = J_\alpha^E$ is as above, $\kappa \in N$ is A-strong in N iff $\langle N, A \rangle$ is amenable and for arbitrarily large $\xi \in N$ there is $F \in N$ such that

- F is an extender at κ of length $\geq \xi$
- N is extendible by F (hence so is $\langle N, A \rangle$)

- Let $\pi : \langle N, A\rangle \longrightarrow \langle N', A'\rangle = \langle J_\alpha^{A'}, A'\rangle$. Then $J_\xi^E = J_\xi^{E'}$ and $A \cap J_\xi^E = A' \cap J_\xi^E$.

Definition 3.8.5. N is Woodin for $A \subset N$ iff there are arbitrarily large $\kappa \in N$ which are A-strong in N.

Hence if $N = J_\xi^{E^M}, \xi \in M$, then $M \models$ "ξ is Woodin" if and only if ξ is Woodin for all $A \in M$ such that $A \subset N$.

In this subsection we shall prove:

Theorem 3.8.4. *Let M be a premouse. Let*

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$$

be a normal iteration of M of limit length η. Set:

$$\tilde{\eta} = \sup_{i<\eta} \kappa_i = \sup_{i<\eta} \lambda_i;\ N = J_{\tilde{\eta}}^E =: \bigcup_{i<\eta} M_i|\nu_i$$

Assume that b_0, b_1 are distinct cofinal well founded branches in T (hence $\tilde{\eta} = \sup b_h$ for $h = 0, 1$). Then N is Woodin with respect to every $A \subset N$ such that $A \in M_{b_0}, M_{b_1}$.

The proof will require many steps. We first prepare the ground by reformulating the definition of "strong" and "A-strong".

Note that if $A \subset \mathsf{ON}$, then $A \cap J_\xi^E = A \cap \xi$ for $\xi \in N$. Thus, if $F \in N$ verifies A-strongness, then so does $F|\xi$. In the following we shall make frequent use of this fact. Since, in the book, we have generally worked with full extenders, we pause now to remind ourselves what it means to say:

$$F \text{ is an extender at } \kappa \text{ on } M \text{ of length } \xi$$

We take M as being acceptable. The above statement then means that the following hold:

(a) $\xi > \kappa$ is Gödel closed (i.e. closed under Gödel pairs $\prec\ ,\ \succ$).

(b) $\kappa \in M$ and $\mathbb{P}(\kappa) \cap M \in M$

(c) $F : \mathbb{P}(\kappa) \cap M \longrightarrow \mathbb{P}(\xi)$

(d) F has an *extension* $\tilde{\pi}$ characterized by:

- $\tilde{\pi} : H_\kappa^M \longrightarrow_{\Sigma_0} H$ cofinally, where H is transitive

- $F(X) = \tilde{\pi}(X) \cap \xi$ for $X \in \mathbb{P}(\kappa) \cap M$
- Each $x \in H$ has the form $\tilde{\pi}(f)(\bar{\xi})$, where $\bar{\xi} < \xi$ and $f \in H^M_\kappa$ is a function on κ.

Then $\tilde{\pi}$ is uniquely characterized by F. Moreover, $\tilde{\pi}$ is definable from F by an "ultrapower" construction which is absolute in ZFC^- models. Thus $\tilde{\pi} \in M$ if $F \in M$ and $M \models \mathsf{ZFC}^-$. But then $\tilde{\pi} \in M$ if $F \in M$ and M is a limit structure in the above sense, since then M is a union of transitive ZFC^- models.

$\pi : M \longrightarrow_F M'$ here means that $\langle M', T \rangle$ is the Σ_0 lift-up of $M, \tilde{\pi}$. We say that M is *extendable by* F if $\langle M', \pi \rangle$ exists.

Definition 3.8.6. Let $M = \langle J^E_\alpha, B \rangle$ be acceptable. Let F be an extender on M at $\kappa \in M$ of length $\xi \leq \alpha$. Let $\tilde{\pi}$ be the extension of F and let $\tilde{\pi}(J^E_\kappa) = J^{E'}_\lambda$. F is *strong* with respect to M iff $J^E_\xi = J^{E'}_\xi$. If F is strong, we define a function $\tilde{F}$ on $\mathbb{P}(J^E_\kappa) \cap M$ by $\tilde{F}(a) =: \tilde{\pi}(a) \cap J^E_\xi$.

Note that $\tilde{F}(a) = F(a)$ for $a \subset \kappa$.

Note. If M is a premouse, $E_\nu \neq \varnothing$ and τ_ν is a cardinal in M, then E_ν is a strong extender on M at κ of length λ_ν. If $\nu \in M$, then $E_\nu \in M$, but the case $\nu = \alpha$ can give us trouble.

Definition 3.8.7. Let M, F, κ, ξ be as above. Let $A \subset M$. F is A-strong in M iff

- $\langle M, A \rangle$ is amenable
- F is strong in M
- $\tilde{F}(A \cap J^E_\kappa) \cap J^E_\xi = A \cap J^E_\xi$.

We note:

Fact. Let F be an extender on M at $\kappa \in M$ of length η. Let $\kappa < \mu < \xi$, where μ is Gödel closed. Define $F' = F|\mu$ by:

$$F'(X) = F(X) \cap \mu \text{ for } X \in \mathbb{P}(\kappa) \cap M.$$

Then:

(a) F' is an extender on M at κ of length μ

(b) If F is strong in M, so is F'

(c) If F is A-strong in M and $\langle J^E_\mu, A \cap J^E_\mu \rangle$ is amenable, so is F'

(d) If M is extendible by F, then it is extendible by F'.

We sketch the proof of (b). Let π be the extension of F with:

$$\pi : J^E_\tau \longrightarrow_{\Sigma_0} H \text{ cofinally, where } \tau = \kappa^{+M}.$$

Similarly for π', F'. Let:

$$\pi' : J^E_\tau \longrightarrow_{\Sigma_0} H' \text{ cofinally}$$

Define:

$$k : H' \longrightarrow_{\Sigma_0} H \text{ cofinally}$$

by $k(\pi'(f)(\xi)) = \pi(f)(\xi)$ where $\xi < \mu$ and $f \in J_\kappa$ is a function on κ. Then $k \upharpoonright \mu = \mathrm{id}$, since:

$$k(\xi) = k(\pi'(\mathrm{id} \upharpoonright \tau)(\xi)) = \pi(\mathrm{id} \upharpoonright \tau)(\xi) = \xi$$

But then $\bar{k} = k \upharpoonright J^{E'}_\mu$ maps $J^{E'}_\mu$ cofinally to J^E_μ, since $k(J^{E'}_\xi) = J^E_\xi$ for limit $\xi < \mu$. Now let h', h be the Σ_1 Skolem function of $J^{E'}_{\mu'}, J^E_\mu$ respectively. Then

$$\bar{k}(h'(i, \langle \vec{\xi} \rangle)) = h(i, \langle \vec{\xi} \rangle)$$

for $i < \omega$, $\xi_1, \ldots \xi_n < \mu$. It follows easily that $\bar{k}$ is an isomorphism of $J^{E'}_\mu$ onto J^E_μ. Hence $\bar{k} = \mathrm{id}$, $J^{E'}_\mu = J^E_\mu$. QED (part (b)).

We shall sometimes make use of the following:

$$F' = F|\mu, \ D = F' \circ G$$

Then:

Lemma 3.8.5. *Let M be acceptable. Let F be strong on M at κ of length μ. Let $G \in M$ be strong on M at $\bar{\kappa} < \kappa$ of length κ. Assume that M is extendable by F. Set: $D = F \cdot G$. Then:*

(a) $D \in M$

(b) *D is strong on M at $\bar{\kappa}$ of length μ.*

Proof: Let $\pi \colon M \longrightarrow_F M'$. The statement: G is strong over M at $\bar{\kappa}$ of length κ is a first order statement:

$$M \models \varphi(G, \bar{\kappa}, \kappa).$$

Hence:

$$M' \models \varphi(\pi(G), \bar{\kappa}, \pi(\kappa)).$$

Set $D = \pi(G)|\mu$. Then $D \in M'$ is strong on M' at $\bar{\kappa}$ of length μ. But for $X \in \mathbb{P}(\bar{\kappa}) \cap M$ we have:

$$D(X) = \pi(G)(X) \cap \mu = F \cdot G(X).$$

But $F \cdot G \in J_\tau^E = H_\tau^M$ where $\tau = \kappa^{+M}$. Hence $D = F \cdot G \in M$. QED (Lemma 3.8.5)

Note We did not assume: $F \in M$. If we dropped the assumption $G \in M$, we would still get (b), though we have not proven this.

Lemma 3.8.6. *Let $N = J_\alpha^E$ be a limit structure. Let $F \in N$ be a strong extender at κ on N of length η, where η is regular in N. Then N is extendible by F.*

Proof: Suppose not. Let

$$D = \{\langle f, \alpha\rangle \in N : \alpha < \xi \text{ and } f \text{ is a function on } \kappa = \text{crit}(F)\}$$

Let $e \subset D^2$ be defined by:

$$\langle f, \alpha\rangle \, e \, \langle g, \beta\rangle \longleftrightarrow \langle \alpha, \beta\rangle \in F(\{\langle \xi, \zeta\rangle : f(\xi) \in g(\zeta)\})$$

Our assumption says that e is ill-founded. Hence there is a sequence $\langle f_i, \alpha_i\rangle_{i<\omega}$ such that

$$\langle f_{i+1}, \alpha_{i+1}\rangle \, e \, \langle f_i, \alpha_i\rangle, \text{ for } i < \omega$$

Let $\langle f_0, \alpha_0\rangle \in J_\gamma^E$ where $\gamma > \xi$ is regular in N. We can assume without lose of generality that $\langle f_i, \alpha_i\rangle \in J_\gamma^E$. If not, replace f_i by f_i' where

$$f_i'(\xi) = \begin{cases} f_i(\xi) & \text{if } f_i(\xi) \in J_\gamma^E \\ 0 & \text{otherwise} \end{cases}$$

But then $e' = e \cap J_\gamma^E$ is ill-founded, where $e' \in N$. Since N is a union of transitive ZFC^- models, it follows by absoluteness that:

$$N \models e' \text{ is ill-founded.}$$

But then there is $\langle\langle f_i, \alpha_i\rangle : i < \omega\rangle \in N$ such that

$$\langle f_{i+1}, \alpha_{i+1}\rangle \, e' \, \langle f_i, \alpha_i\rangle \text{ for } i < \omega$$

Let $\tilde{\pi} \in N$ be the extension of F. Then:

$$\tilde{\pi} : J_\tau^E \longrightarrow_{\Sigma_0} H \text{ cofinally.}$$

Set: $X_i = \{\langle\xi,\zeta\rangle : f_{i+1}(\xi) \in f_i(\xi) \in f_i(\zeta)\}$. Let $\tau = \kappa^{+N}$, we have $\langle X_i : i < \omega\rangle \in J^E_\tau$. Set

$$\langle\tilde{X}_i : i < \omega\rangle = \tilde{\pi}(\langle X_i : i < \omega\rangle)$$

Then $\tilde{X}_i \cap \eta = F(X_i)$ for $i < \omega$. Since η is regular in N and F is strong, we have:

$$\langle\alpha_i : i < \omega\rangle \in J^E_\xi \subset H$$

But $\langle\alpha_{i+1}, \alpha_i\rangle \in F(X_i) \subset \tilde{X}_i$ for $i < \omega$. Hence H satisfies the statement:

There is $g : \omega \longrightarrow \tilde{\pi}(\kappa)$ such that $\langle g(i+1), g(i)\rangle \in \tilde{X}_i$ for $i < \omega$

But then J^E_τ satisfies:

There is $g : \omega \longrightarrow \kappa$ such that $\langle g(i+1), g(i)\rangle \in X_i$ for $i < \omega$

Hence $f_{i+1}(g(i+1)) \in f_i(g(i))$ for $i < \omega$. Contradiction! QED (Lemma 3.8.6)

But then by Fact 1, it follows easily that:

Lemma 3.8.7. *Let N be a limit structure, $\kappa \in N$. Then κ is strong in N iff for arbitrarily large $\eta \in N$ there is $F \in N$ which is strong for N at κ of length η.*

Lemma 3.8.8. *Let N, κ be as above. Let $A \subset N$. Then κ is A-strong in N iff for arbitrarily large $\xi \in N$ there is $F \in N$ which is A-strong for N at κ of length ξ.*

The proofs are left to the reader.

Before embarking on the proof of Theorem 3.8.4 we digress in order to prove a lemma which will be important later chapter.

Lemma 3.8.9. *Let $M = \langle J^E_\nu, F\rangle$ be an active premouse. Let $\rho^1_M = \lambda$. Then $M \models$ "κ is Woodin". (Hence M is not 1-small.)*

Proof. We must show that if $A \in M$, $A \subset J^E_\kappa$, then there is $\kappa' < \kappa$ which is A-strong for J^E_κ at κ'. Since we can canonically code A as a subset of κ, we shall assume: $A \subset \kappa$. Let $\pi\colon J^E_\tau \longrightarrow J^E_\nu$ be the extension of F. Since $\pi \restriction J^E_\kappa \colon \langle J^E_\kappa, A\rangle \longrightarrow \langle J^E_\lambda, F(A)\rangle$, it suffices to show that the above statement holds of $\langle J^E_\lambda, A'\rangle$, where $A' = F(A)$.

By §3.3 we know: $h_M(\lambda) = M$. Hence $\emptyset \in R^1_M$, since $\rho^1_M = \lambda$. We shall, in fact, show:

Claim. Let $\tau < \eta < \lambda$ such that η is regular in J^E_λ. Then there is an extender $G \in J^E_\lambda$ at κ which is A'-strong.

Set: $N = \langle J^E_\lambda, T\rangle = M^1 = M^{1,\emptyset}$. Then N is amenable. Since η is regular in N, it follows by acceptability that $\bar{N} = \langle J^E_\eta, T \cap J^E_\eta\rangle$ is amenable. But $\bar{N} \prec_{\Sigma_0} N$. By the downward extension lemma, there is a unique $\bar{M}$ such that $\bar{N} = \bar{M}^{1,\emptyset}$ and $\emptyset \in P^1_{\bar{M}}$. Moreover, there is a unique σ such that

$$\sigma\colon \bar{M} \longrightarrow_{\Sigma_0^{(1)}} M \text{ and } \emptyset \in P^1_{\bar{M}}.$$

$\bar{M}$ is recoverable from $\bar{N}$ in any transitive ZFC^- model containing $\bar{N}$. Hence $\bar{M} \in J^E_\lambda$. But $\bar{M} = \langle J^{\bar{E}}_{\bar{\nu}}, \bar{F}\rangle$. It follows easily that $\bar{F}$ is an injective function and that $\mathrm{dom}(\bar{F}) = \mathrm{dom}(F) = \mathbb{P}(\kappa) \cap M = \mathbb{P}(\kappa) \cap \bar{M}$. Moreover $\bar{F}(X) \subset \bar{\lambda}$, where $\bar{\lambda} = \bar{F}(\kappa)$ is the largest cardinal in $\bar{M}$. But for each $\xi < \bar{M}$ there is $X \in \mathbb{P}(\kappa) \cap M$ such that $\bar{F}(X) \notin J^{EM}_\xi$. It follows easily that $\bar{F}$ is an extender at κ on J^E_τ with base $|J^E_\tau|$ and extension $\bar{F}\colon J^E_\tau \longrightarrow J^{\bar{E}}_{\bar{\nu}}$. Now let $G = \bar{F} \restriction \eta$. Then $G \in M$ is an extender at κ on M. Let $\tilde{\pi}\colon |J^E_\tau| \longrightarrow H$ be the extension of G. Then $H = J^{\tilde{E}}_{\tilde{\nu}}$ and $\tilde{\pi}\colon J^E_\tau \longrightarrow J^{\tilde{E}}_{\tilde{\nu}}$ cofinally. There is a cofinal map $\tilde{\sigma}\colon J^{\tilde{E}}_{\tilde{\nu}} \longrightarrow_{\Sigma_0} J^{\bar{E}}_{\bar{\nu}}$, defined by:

$$\tilde{\sigma}(\tilde{\pi}(f)(\alpha)) = \bar{\pi}(f)(\alpha)$$

for $\alpha < \eta$, $f \in J^E_\tau$, $f\colon \kappa \longrightarrow J^E_\tau$. Clearly $\tilde{\sigma} \restriction \eta = \mathrm{id}$. Hence $\sigma\tilde{\sigma} \restriction \eta = \mathrm{id}$. Hence $J^{\tilde{E}}_\eta = J^E_\eta$ and G is strong. Moreover, $G(A) \cap \eta = A' \cap \eta = A \cap \eta$ and G is A'-strong.

□

We are now ready to embark upon the proof of Theorem 3.8.4.

The proof will have many steps. We shall in fact, first prove it under a simplifying assumption, in order to display the method more clearly.

Since b_0, b_1 are distinct and T is a tree, there is an $\alpha < \eta$ such that $(b_0 \setminus \alpha) \cap (b_1 \setminus \alpha) = \varnothing$. Define a sequence $\langle \delta_i : i < \omega\rangle$ by:

$$\begin{aligned}
\delta_0 = &\ \text{the least } \xi \in b_i \setminus (\alpha + 1)\\
\delta_{2i+1} = &\ \text{the least } \xi \in b_1 \text{ such that } \xi > \delta_{2i}\\
\delta_{2i+2} = &\ \text{the least } \xi \in b_0 \text{ such that } \xi > \delta_{2i+1}
\end{aligned}$$

By minimality, each δ_i is a successor ordinal. Note that

$$T(\delta_{2i+1}) < \delta_{2i} < \delta_{2i+1}$$

since otherwise, setting $\xi = T(\delta_{2i+1})$, we would have $\xi \geq \delta_{2i}, \xi \in b_1$; hence $\xi > \delta_{2i}$. But then $\delta_{2i+1} \leq \xi < \delta_{2i+1}$. Contradiction! A similar argument shows:

$$T(\delta_{2i+2}) < \delta_{2i+1} < \delta_{2i+2}$$

Hence:

(1) $T(\delta_{i+1}) < \delta_i < \delta_{i+1}$ for $i < \omega$.

Set

(2) $\gamma_i =: \delta_i - 1$, $\gamma_i^* = T(\delta_i)$.

By (1) we then have

(3) $\kappa_{\gamma_{i+1}} < \lambda_{\gamma_{i+1}^*} \leq \lambda_{\gamma_i} \leq \kappa_{\gamma_{i+2}}$.

We have $\lambda_{\gamma_i} \leq \kappa_{\gamma_{i+2}}$ since $(\gamma_i+1)T(\gamma_{i+2}+1)$. Now note that for $n < \omega$ we have:

(4) If n is even, then $\langle \delta_{n+i} : i < \omega$ has the same definition as $\langle \delta_i : i < \omega \rangle$ with δ_n in place of α. Similarly for n odd, with b_0, b_1 reversed.

Hence we may without lose of generality assume α chosen large enough that:

(5) No $\xi \in (b_h \setminus \alpha)$ is a drop point $(h = 0, 1)$. Thus $M_{\gamma_i^*} = M_{\gamma_i}^*$ and we have:

(6) $\pi_{\gamma_i^*, \delta_i} : M_{\gamma_i^*} \longrightarrow^*_{E_{\nu_{\gamma_i}}} M_{\delta_i}$.

Clearly

(7) $\sup_{i<\omega} \gamma_i = \sup_{i<\omega} \delta_i = \nu$, since otherwise $\sup_{i<\omega} \gamma_i \in (b_0 \setminus \alpha) \cap (b_1 \setminus \alpha)$.

By (6) we conclude:

(8) τ_{γ_i} is a cardinal in M_ξ for $\xi \geq \gamma_i^*$.

Set:

(9) $N = J_{\tilde{\xi}}^E =: \bigcup_i J_{\kappa_{\gamma_i}}^{E^{M_{\gamma_i}}} = \bigcup_i J_{\nu_{\gamma_i}}^{E^{M_{\gamma_i}}}$.

Until further notice we make the following simplifying assumption:

$$\text{(SA)} \quad E_{\nu_{\gamma_i}}^{M_{\gamma_i}} | \kappa_{\gamma_{i+1}} \in M_{\gamma_i} \ (i < \omega)$$

This would be true e.g. if M were passive and no truncation occurred in the iteration, since then $E_{\nu_{\gamma_i}}^{M_{\gamma_i}} \in M_{\gamma_i}$.

Using this assumption we get:

(10) $N \models$ there are arbitrarily large strong cardinals.

Proof. Since we can choose α (and hence κ_{γ_0}) arbitrarily large, it suffices by (4) to show:

Claim. κ_{γ_0} is strong in N.

Proof. Set: $F_n = E^{M_{\gamma_n}}_{\nu_{\gamma_n}} | \kappa_{\gamma_{n+1}}$. Since $\kappa_{\gamma_{n+1}} < \lambda_{\gamma_n}$ by (3) and $M_{\gamma_n} || \lambda_{\gamma_n} = N || \lambda_{\gamma_n}$, we conclude that F_n is strong at κ_{γ_n} on N of length $\kappa_{\gamma_{n+1}}$. But $\kappa_{\gamma_{n+1}}$ is regular in N. Hence N is extendable by F_n. But we also have $F_n \in M_{\gamma^*_{n+1}}$, since either $\gamma^*_{n+1} = \gamma_n$ or $\gamma^*_{n+1} < \gamma_n$ and $\lambda_{\gamma^*_{n+1}}$ is a cardinal in M_{γ_n}. But $\tau_{\gamma_{n+1}}$ is a cardinal on $M_{\gamma^* n+1}$. Hence:

$$F_n \in M_{\gamma^*_{n+1}} || \tau_{\gamma_{n+1}} = M_{\gamma_{n+1}} || \tau_{\gamma_{n+1}} \subset N.$$

Set $G_0 = F_0$. Then $G_0 \in N$ is strong on N at κ_{γ_0} of length κ_{γ_1}. If we then set: $G_{n+1} = F_{n+1} \cdot G_n$ for $n < \omega$, we get

$$G_{n+1} \in N \text{ is strong on } \kappa_{\gamma_0} \text{ on } N \text{ of length } \kappa_{\gamma_{n+1}}$$

by successive application of Lemma 3.8.5. QED (10)

(11) Let $A \subset \mathrm{On}_N$, $A \in M_{b_0} \cap M_{b_1}$. Then N is Woodin for A.

Proof. Assume α is so chosen that $A \in \mathrm{rng}(\pi_{\gamma^*_0, b_0}) \cap \mathrm{rng}(\pi_{\gamma^*_1, b_1})$. It follows easily that:

$$F_n(A \cap \kappa_{\gamma_n}) = A \cap \kappa_{\gamma_{n+1}}.$$

Hence $G_n(A \cap \kappa_{\gamma_0}) = A \cap \kappa_{\gamma_{n+1}}$. Then $G_n \in N$ is A-strong for N at κ_{γ_0} of length $\kappa_{\gamma_{n+1}}$. QED (11)

We now face the task of redoing this without the special assumption (SA). We first choose α large enough that we can avoid a certain undesirable situation:

Definition 3.8.8. If $M = \langle |M|, F \rangle$ is an active premouse, we call F the *top extender* of M.

Definition 3.8.9. $n \in \omega$ is *undesirable* if and only if M_{δ_n} has a top extender F with $\mathrm{crit}(F) \in [\kappa_{\gamma_n}, \kappa_{\gamma_{n+1}})$.

(12) If α is chosen sufficiently large, then no $n < \omega$ is undesirable.

Proof: Suppose not. Then there are infinitely many undesirable n. But then these are undesirable n, m such that $n < m$ and n, m are both add or both even. Then $\delta_{n+1} <_T \delta_{m+1}$. Let F be a top extender of $M_{\delta_{n+1}}$, $\bar{\kappa} = \mathrm{crit}(F)$. Then:

$$\bar{\kappa} < \kappa_{\delta_{n+1}} = \mathrm{crit}(\pi_{\delta_{n+1}, \delta_{m+1}}) \text{ by undesirablity.}$$

Hence $\bar{\kappa} = \mathrm{crit}(F')$, where:

$$\pi_{\delta_{n+1}, \delta_{m+1}} : \langle |M_{\delta_{n+1}}|, F \rangle \longrightarrow \langle |M_{\delta_{m+1}}|, F' \rangle$$

and F' is therefore a top extender of $M_{\delta_{m+1}}$. But $\bar{\kappa} < \kappa_{\gamma_{n+1}} \leq \kappa_{\gamma_m}$ by (3). Hence m is not undesirable. Contradiction! QED(12)

From now on let α be chosen as in (12). We wish to prove Theorem 3.8.4. Since α (and with it κ_{γ_0}) can be chosen as large as we wish, it will suffice to show:

(13) There is $\bar{\kappa}$ such that

- $\kappa_{\gamma_0} \leq \bar{\kappa}$ and $\bar{\kappa}$ is strong in N
- If $A \subset \mathrm{On} \cap N$, $A \in M_{b_0} \cap M_{b_1}$ such that $A \in \mathrm{rng}(\pi_{\gamma_0^*, b_0}) \cap \mathrm{rng}(\pi_{\gamma_1^*, b_1})$,

then $\bar{\kappa}$ is A-strong for N.

Our main tool in proving this will be:

Lemma 3.8.10. *Let $a \in \mathbb{P}(\kappa_{\gamma_1}) \cap N$ such that $F(a \cap \kappa_{\gamma_1}) = a$. There are G, F such that $\kappa_{\gamma_0} \leq \bar{\kappa} < \kappa_{\gamma_1}$ and:*

- *G is strong on N at $\bar{\kappa}$ of length κ_{γ_1}*
- *$G(a \cap \bar{\kappa}) = a$*
- *$G \in N$.*

Proof. We assume the lemma to be false and derive a contradiction. Knowing that we must fail, we nonetheless make ω many successive attempts to produce such a G. But this sequence of attempts give a descending sequence $\langle \beta_i \mid i < \omega \rangle$ of ordinals with: $\beta_{i+1} < \beta_i$ for $i < \omega$.

Assume α chosen large enough that $\lambda_0 < \kappa_{\gamma_0}$. We successively construct

$$\langle \beta_n, \bar{G}_n, \bar{\kappa}_n \rangle (n < \omega) \text{ such that}$$

- $\kappa_{\gamma_0} \leq \bar{\kappa}_n < \kappa_{\gamma_1}$
- $\bar{G}_n$ is strong on N at $\bar{\kappa}_n$ of length κ_{γ_1}
- $\bar{G}_n(a \cap \bar{\kappa}_n) = a$
- $\bar{G}_n = F|\kappa_{\gamma_1}$, where $F = E^{M_{\beta_n}}_{\nu_{\beta_n}}$ is a top extender of M_{β_n}.

We set $\beta_0 = \gamma_0$, $\bar{G}_0 = F_0$, $\bar{\kappa}_0 = \kappa_{\gamma_0}$. Since $\bar{G}_0 \notin N$, we have seen that $F = E^{M_{\beta_0}}_{\nu_{\beta_0}}$ must be the top extender of M_{β_0}. Hence all conditions are fulfilled at $n = 0$. Now let $\langle \beta_n, \bar{G}_n, \bar{\kappa}_n \rangle$ be given. γ_1^* is the least ordinal η such that $\kappa_{\gamma_1} < \lambda_\eta$. Hence $\gamma_1^* \leq \beta_n$. But $\gamma_1^* < \beta_n$, since otherwise:

$$\pi_{\gamma_1^*, \delta_1} : \langle |M_{\beta_n}|, F \rangle \longrightarrow \langle M_{\delta_1}, F' \rangle$$

where $F = E^{M_{\beta_n}}_{\nu_{\beta_n}}$. Moreover:

$$\mathrm{crit}(\pi_{\gamma_1^*, \delta_1}) = \kappa_{\gamma_1} > \bar{\kappa}_n.$$

Hence $\bar{\kappa}_n = \mathrm{crit}(F') \in [\kappa_{\gamma_0}, \kappa_{\gamma_1})_T$ where F' is a top extender of M_{δ_1}. Hence 1 is undesirable. Contradiction! by (12). Since $\gamma_1^* < \beta_n$, there must be a least β such that $\beta+1 \leq_T \beta_n$, $\kappa_{\gamma_1} < \lambda_\beta$, and $(\beta+1, \beta_n]_T$ has no trancation. Set:

$$\beta_{n+1} =: \beta, \bar{\kappa} = \bar{\kappa}_{n+1} =: \mathrm{crit}(E^{M_\beta}_{\nu_\beta}), \bar{G}_{n+1} = \bar{G} =: E^{M_\beta}_{\nu_\beta}|\kappa_{\gamma_1}.$$

Let $h = T(\beta+1)$, $\pi = \pi_{h,\beta_n}$. Then $\pi\colon M^*_\beta \longrightarrow M_{\beta_n}$ where M_{β_n} has a top extender $F = E^{M_{\beta_n}}_{\nu_{\beta_n}}$. Thus M^*_β has a top extender F' and $\pi(\mathrm{crit}(F')) = \mathrm{crit}(F) = \bar{\kappa}_n$. Hence $\mathrm{crit}(F') = \bar{\kappa}_n < \bar{\kappa}$, since otherwise:

$$\bar{\kappa_n} = \mathrm{crit}(F) \geq \pi(\bar{\kappa}) \leq \lambda_\beta > \kappa_{\gamma_1} > \bar{\kappa_n}.$$

Contradiction! We have shown:

(1) $\bar{\kappa}_n < \bar{\kappa}$

We now show:

(2) $\bar{\kappa} \neq \kappa_{\gamma_n}$

Suppose not. Then: $\gamma^*_1 = h$, $M^*_\beta = M_h$ and $\pi_{h,\delta_1}\colon M^*_\beta \longrightarrow M_{\delta_1}$. Hence M_{δ_1} has a top extender $\tilde{F}$ with $\mathrm{crit}(\tilde{F}) = \mathrm{crit}(F') = \bar{\kappa}_n \in [\kappa_{\gamma_0}, \kappa_{\gamma_1})$. Hence 1 is undesirable. Contradiction! QED (2)

(2) $\bar{\kappa} < \kappa_{\gamma_n}$

Suppose not. Then $\kappa_{\gamma_n} < \bar{\kappa}$: Hence either $\gamma^*_1 = h$ or $\gamma^*_1 < h$ and $\lambda_{\gamma^*_1}$ is a cardinal in M_h. In either case $J^{M^*_{gamma_1}}_{\tau_{\gamma_1}} = J^{M_h}_{\tau_{\gamma_1}}$ and $\tau_{\gamma_1} < \bar{\kappa}$ is a cardinal in M_h. But then $M^*_\beta = M_h$, since otherwise $F' \in M_h$; $F'|\bar{\kappa} = F|\bar{\kappa}$, since $\pi \restriction \bar{\kappa} = \mathrm{id}$. Hence:

$$\bar{G}_n = F'|\kappa_{\gamma_1} \in J^{E^{M_{\gamma_1}}}_{\tau_{\gamma_1}} \subset N.$$

Contradiction! But then $\beta+1$ is not a drop point. We have seen, however, that $\gamma^*_1 < \beta_n$. Hence β is not the least $\beta+1 \leq_T \beta_n$ such that $\kappa_{\gamma_1} < \lambda_\beta$ and $(\beta+1, \beta_n]_T$ has no drop point. Contradiction! QED (2) Hence:

(3) $\bar{G} = E^{M_\beta}_{\gamma_\beta}|\kappa_{\gamma_1}$ is strong on N at $\bar{\kappa}_0$ of length κ_{γ_1}.

Proof. $N||\lambda_\beta = M_\beta|\lambda_\beta$ and hence $E^{M_\beta}_{\nu_\beta}$ is strong on N at $\bar{\kappa}$ of length $\lambda_\beta > \kappa_{\gamma_1}$.

(4) $\bar{G}(a \cap \bar{\kappa}) = a$.

Proof. Let $G^* = E^{M_\beta}_{\nu_\beta}$, $\bar{a} = a \cap \bar{\kappa}_n$, $a' = F'(\bar{a})$, $\tilde{a} = F(\bar{a})$. Then $\tilde{a} \cap \kappa_{\gamma_1} = \bar{G}_n(\bar{a}) = a$. Since:

$$\bar{\kappa} = \mathrm{crit}(G^*) = \mathrm{crit}(\pi), \tilde{a} = \pi(a'),$$

we have: $a' \cap \bar{\kappa} = \tilde{a} \cap \bar{\kappa} = a \cap \bar{\kappa}$. But $G^*(a \cap \bar{\kappa}) = G^*(a') \cap \lambda_{\nu_\beta} = \tilde{a} \cap \lambda_{\nu_\beta}$, since $\pi(a') = \pi_{\beta+12,\beta_n} G^*(a')$ and $\mathrm{crit}(\pi_{beta+1,\beta_n}) \geq \lambda_{\nu_\beta}$. Hence

$$\bar{G}(a \cap \bar{\kappa}) = \tilde{a} \cap \kappa_{\gamma_1} = a.$$

QED (4)

By our assumption we conclude: $\bar{G} \notin N$. But then:

(5) $G^* = E^{M_\beta}_{\nu_\beta}$ is a top extender on M_β.

Proof. Suppose not. Then $G^* \in M_\beta$. But $J^{M_{\gamma_1^*}}_{\tau_{\gamma_1}} = J^{E^{M_\beta}}_{\tau_{\gamma_1}}$ and τ_{γ_1} is a cardinal in M_β, since either $\gamma_1 = \beta$ or $\gamma_1 < \beta$ and λ_{γ_1} is a cardinal in M_β. Hence:

$$\bar{G} = G^* | \kappa_\beta \in J^{E_{\gamma_1^*}}_{\tau_{\gamma_1}} \subset J^{E_{\gamma_1^*}}_{\lambda_{\gamma_1^*}} \subset N$$

. QED (5)

This completes the construction. It is evident that $\bar{\beta}_{n+1} < \bar{\beta}_n$ for $n < \omega$. Contradiction! QED(Lemma 3.8.12)

We can now prove (13): Let G be as in Lemma 3.8.12. Set $G_n = F_{n+1} \cdot G$. Since $G \in N$ is strong on N at $\bar{\kappa}$ of length κ_{γ_1} and we set

$$G_n = F_{n+1} \cdot G \, (n < \omega)$$

it follows by successive application of Lemma 3.8.5 that:

$$G_n \in N \text{ is strong on N at } \bar{\kappa} \text{ of length } \kappa_{\gamma_{n+1}}.$$

Moreover, if $A \subset \mathrm{On} \cap N$ such that

$$A \in \mathrm{rng}(\pi_{\gamma_0^*, b_[0}) \cap \mathrm{rng}(\pi_{\gamma_1^*, b_1}).$$

Then:

$$F_n(A \cap \kappa_{\gamma_n}) = A \cap \kappa_{\gamma_{n+1}} \text{ for } n < \omega.$$

Hence $F_0(A \cap \kappa_{\gamma_n}) = A \cap \kappa_{\gamma_{n+1}}$ and:

$$G_n(A \cap \bar{\kappa}) = A \cap \kappa_{\gamma_{n+1}} \, (n \in \omega).$$

Hence $\bar{\kappa}$ is A-strong in N. QED (13)

This proves Theorem 3.8.4.

Note Strictly speaking, we have only proven that if $A \subset \mathrm{On} \cap N$ and $A \in M_{b_0} \cap M_{b_1}$, then N is Woodin for A.

We now show that this implies the full result. We use the fact that any $A \subset N$ can be coded by a set $\tilde{A} \subset \tilde{\eta}$. Let $N = J^E_{\tilde{\eta}}$ and suppose that $\alpha \leq \tilde{\eta}$ is Gödel-closed. By Corollary 2.4.12 we know $M = h_M``(\omega \times \alpha)$, where $M = J^E_\alpha$. Let k_α be the canonical $\Sigma_1(M)$ uniformization of

$$\{\langle \nu, x \rangle : x = h_M((\nu)_0, (\nu)_1)\}$$

Then k_α injects M into α and is uniformly $\Sigma_1(M)$. Set $k = k_{\tilde{\eta}}$. Then:

(a) $k_\alpha = k \upharpoonright \alpha$ if $\alpha < \tilde{\xi}$ is Gödel-closed.

(b) $k_\mu^{-1} = k^{-1} \upharpoonright \mu$ if $\mu < \tilde{\eta}$ is a cardinal in N (since J_μ^E is Σ_1-elementary submodel of N).

(c) $k_\alpha \in N$ for Gödel-closed $\alpha < \tilde{\eta}$.

(d) Let $A \subset N$ and set $\tilde{A} = k``A$. If $\mu < \tilde{\eta}$ is a cardinal in $N$, then $\tilde{A} \cap \mu = k``_\mu(A \cap J_\mu^E)$ (hence $\langle N, \tilde{A}\rangle$ is amenable if $\langle N, A\rangle$ is amenable.

Theorem 3.8.4 then follows from

(14) Let $A \subset N$ such that $\langle N, \tilde{A}\rangle$ is amenable and N is Woodin with respect to $\tilde{A}$. Then N is Woodin with respect to A.

Proof: Let $G \in N$ be $\tilde{A}$-strong in N at κ of length μ, where $\mu > \omega$ is regular in N.

Claim. G is A-strong in N (i.e. $\tilde{G}(A \cap J_\kappa^E) = A \cap J_\mu^E$).

Proof: N is extendable by G. Set:

$$\pi : N \longrightarrow_G N' = J_{xi}^{E'}$$

Let k', k'_α be defined over N like k, k_α over N. Since G is strong in N we have: $J_\mu^E = J_\mu^{E'}$ and $k_\mu = k'_\mu$. Let $\nu = \pi(\kappa)$. Then $k'_\nu = k' \upharpoonright J_\nu^{E'}$. Hence for $y \in J_\mu^E$ we have:

$$\begin{aligned} y \in \tilde{G}(A \cap J_\kappa^E) &\longleftrightarrow k_\mu(y) \in k'_\nu{}''\tilde{G}(A \cap J_\kappa^E) \\ &\longleftrightarrow k_\mu(y) \in k'_\nu{}''\pi(A \cap J_\kappa^E) \\ &\longleftrightarrow k_\mu(y) \in \pi(k'_\nu{}''(A \cap J_\kappa^E)) \\ &\longleftrightarrow k_\mu(y) \in G(\tilde{A} \cap \kappa) \\ &\longleftrightarrow k_\mu(y) \in \tilde{A} \cap \mu = k_\mu{}''(A \cap J_\mu^E) \\ &\longleftrightarrow y \in A \cap J_\kappa^E \end{aligned}$$

This proves (14) and with it Theorem 3.8.4.

Note. The notion of *premouse* which we develop in this book is based on the notion developed by Mitchell and Steel in [MS]. However, they employ a different indexing of the extenders than we do. Their indexing makes it much easier to prove Theorem 3.8.4, since our special assumption (SA), when reformulated for their premice, turns out to the outright.

We note a further consequence of our theorem:

Lemma 3.8.11. *Let $N = J_{\tilde{\eta}}^E$ be as in Theorem 3.8.4. There are arbitrarily large $\nu \in N$ such that $E_\nu \neq \varnothing$.*

Proof: Suppose not. Let $\alpha < \eta$ be a strict upper bound of the set of such ν. Then N is a constructible extension of J_α^E (in the sense of Definition of E in §2.5). By Theorem 3.8.4 some $\kappa > \alpha$ is strong in N. In particular, there is $F \in N$ which is an extender at κ on N and N is extendible by F. Let $\pi : N \longrightarrow_F N'$. Then $\langle N', \pi \rangle$ is the extension of $\langle N, \bar{\pi} \rangle$ where $\bar{\pi} : J_\tau^E \longrightarrow J_\nu^E$ is the extension of F (with $\tau = \kappa^{+N}$). Then $\bar{\pi} \in N$. Hence ν is not regular in N since $\tau < \nu$ and $\nu = \sup \bar{\pi}"\tau$. Clearly, however, $N' = J_{\eta'}^{E'}$ is a constructible extension of $J_{\alpha'}^E$, where $\alpha' \geq \alpha$. Hence $N \subset N'$. ν is regular in N', since $\nu = \pi(\tau)$. But then ν is regular in N. Contradiction! QED(Lemma 3.8.11)

We have actually proven a stronger result than we have stated. Theorem 3.8.4 does, in fact, *not* require that the cofinal branches b_0, b_1 be well founded. Let b be any cofinal branch in I, Let i_0 be such that $i_0 \in b$ and no $i \in b \setminus i_0$ is a truncation point. Let:

$$M_b, \langle \pi_{i,b} \mid i \in b \rangle$$

be defined by taking

$$M_b, \langle \pi_{i,b} \mid i \in b \setminus i_0 \rangle$$

as the direct limit of

$$\langle M_i \mid i \in b \setminus i_0 \rangle, \langle \pi_{i,j} \mid i_0 \leq i \leq j \text{ in } b \rangle$$

and then setting:

$$\pi_{j,b} =: \pi_{i_0,b} \cdot \pi_{j,i_0} \text{ for } j \in b \cap i_0$$

M_b may not be well founded, but we assume it to be *grounded* in the sense that its well founded core $\text{wfc}(M_b)$ is transitive and:

$$E \cap \text{wfc}(M_b) = E_{M_b} \cap \text{wfc}(M_b).$$

(M_b is thus defined up to isomorphism and $\text{wfc}(M_b)$ is defined uniquely.) If we define $\tilde{eta}$, N as in Theorem 3.8.4 it follows easily that $\tilde{\eta}, N \subset \text{wfc}(M_b)$ (since $\pi_{i,b} \restriction \kappa_i = \text{id}$ for $i \in b$). We then obtain the following stronger result of Lemma 3.8.4:

Theorem 3.8.12. *Let M, I, $\tilde{\eta}$, N be as in Theorem 3.8.4. Let b_0, b_1 be distinct cofinal branches in I. Let $A = A_0 \cap N = A_1 \cap N$, where $A_h \in M_{b_h}$ for $h = 0, 1$. Then N is Woodin with respect to A.*

As before, the proof is by showing that there are arbitrarily large $\kappa < \tilde{\eta}$ which are A-strong in N. The steps are virtually the same, requiring only cosmetic changes. (Basically, this is because our proofs only talked about $\langle N, A \rangle$ rather than M_{b_0} and M_{b_1}.) Theorem 3.8.12 will play an important role in Chapter 5. It was first noticed by Woodin.

3.8.3 One smallness and unique branches

We now apply the method of the previous subsection to one small mice. We let $M, b_0, b_1, \alpha, \gamma_n (n < \omega)$, etc. be as before, but also assume that M is one small. It is easily seen that every normal iterate of M must be one small. Hence M_{b_0}, M_{b_1} are one small. Letting $\eta, \tilde{\eta}, N$ be as before, we set:

Definition 3.8.10. $Q =: J_\beta^{E^N}$, where $\beta = \min(\mathrm{On}_{M_{b_0}}, \mathrm{On}_{M_{b_1}})$.

By Theorem 3.8.4 we obviously have:

Lemma 3.8.13. *$\tilde{\eta}$ is Woodin in Q.*

From now on, assume w.l.o.g. that $\mathrm{On}_{M_{b_0}} \leq \mathrm{On}_{M_{b_1}}$ (i.e. $\mathrm{On}_{M_{b_0}} = \beta$). Then:

Lemma 3.8.14. *$M_{b_0} = Q$.*

Proof: Suppose not. Then there is $\nu \geq \tilde{\eta}$ such that $E_\nu^{M_{b_0}} \neq \varnothing$. But then $\nu > \tilde{\eta}$, since $\tilde{\eta}$ is a limit of cardinals in M_{b_0} and ν is not. Taking ν as minimal, we then have $J_\nu^{E^{M_{b_0}}} = J_\nu^{E^N} \models \tilde{\eta}$ is Woodin. Hence M_{b_0} is not one small. Contradiction! QED (Lemma 3.8.14)

But then we can essentially repeat our earlier argument to show:

Lemma 3.8.15. *Let $A \subset N$ be $\Sigma^*(Q)$ such that $\langle N, A\rangle$ is amenable. Then N is Woodin for A.*

Proof: As before, we can assume w.l.o.g. that $A \subset \mathrm{On}_Q$. Let A be $\Sigma^*(Q)$ in a parameter p by Σ^* definition φ. We assume α to be chosen as before, but now large enough that for $h = 0, 1$:

- $p \in \mathrm{rng}(\pi_{\gamma_h^*}, b_h)$
- If $N \neq Q$, then $N \in \mathrm{rng}(\pi_{\gamma_h^*, b_h})$
- If $\mathrm{On}_{M_{b_h}} > \mathrm{On}_Q$ (hence $h = 1$), then $Q \in \mathrm{rng}(\pi_{\gamma_1^*}, b_1)$.

Since $M_{b_0} = Q$ we have

$$\pi_{\gamma_{2i}^*, b_0} : M_{\gamma_{2i}}^* \longrightarrow_{\Sigma^*} Q \text{ with critical point } \kappa_{2i}.$$

Let A_{2i} be defined over $M_{\gamma_{2i}}^*$ in $p_{2i} = \pi_{\gamma_{2i}^*, b_0}^{-1}(p)$ by φ. Set:

$$N_{2i} = \begin{cases} \pi_{\gamma_{2i}^*, b_i}^{-1}(N) & \text{if } N \in Q \\ M_{\gamma_{2i}^*} & \text{if not} \end{cases}$$

Then $\langle N_{2i}, A_{2i}\rangle$ is amenable and:

$$(\pi_{\gamma^*_{2i},b_0} \upharpoonright N_{2i}) : \langle N_{2i}, A_{2i}\rangle \longrightarrow_{\Sigma_0} \langle N, A\rangle$$

It follows easily that $A_{2i} \cap \kappa_{\gamma_{2i}} = A \cap \kappa_{\gamma_{2i}}$ and

$$E^{M_{\gamma_{2i}}}_{\nu_{\gamma_{2i}}}(A \cap \kappa_{\gamma_{2i}}) = \pi_{\gamma^*_{2i},\gamma_{2i}+1}(A \cap \kappa_{\gamma_{2i}}) = A \cap \lambda_{\gamma_{2i}}.$$

If $\mathrm{On} \cap M_{b_1} = \mathrm{On} \cap Q$, it follows by symmetry that $M_{b_1} = Q$. Hence:

$$\pi_{\gamma^*_{2i+1},b_1} : M^*_{2i+1} \longrightarrow_{\Sigma^*} Q \text{ with critical point } \kappa_{\gamma_{2i+1}}.$$

If we then define A_{2i+1}, N_{2i+1}, p_{2i+1} as before. We get:

$$E^{M_{\gamma_i}}_{\nu_{\gamma_i}}(A \cap \kappa_{\gamma_i}) = \pi_{\gamma^*_i,\gamma_i+1}(A \cap \kappa_{\gamma_i}) = A \cap \lambda_{\gamma_i}$$

for $i \in \omega$. If $M_{b_1} \neq Q$ we set:

$$A_{2i+1} = \pi^{-1}_{\gamma^*_{2i+1},b_1}(A), N_{2i+1} = \pi^{-1}_{\gamma^*_{2i+1},b_1}(N)$$

and get the same results. As before we define $F_i = E^{M_{\gamma_i}}_{\nu_{\gamma_i}} | \kappa_{\gamma_{i+1}}$. Then :

$$F_i(A \cap \kappa_{\gamma_i}) = A \cap \kappa_{\gamma_{i+1}} \text{ for } i \in \omega.$$

In particular, F is A-strong on N at κ_{γ_i} of length κ_{γ_i+1}. Now let $a = A \cap \kappa_{\gamma_1}$. By Lemma 3.8.12 there are G, $\bar{\kappa}$ such that $\kappa_{\gamma_0} < \bar{\kappa} < \kappa_{\gamma_1}$ and :

- $G \in N$ is strong on N at $\bar{\kappa}$ of length κ_{γ_1}
- $G(a \cap \bar{\kappa}) = a$.

Successively, define G_n $(n \in \omega)$ by:

$$G_0 = G, G_{n+1} = F_{n+1} \cdot G_n.$$

Just as before we get: $G_n(A \cap \bar{\kappa}) = A \cap \kappa_{\gamma_{n+1}}$ and:

$$G_n \text{ is } A\text{-strong on } N \text{ at } \bar{\kappa} \text{ of length } \kappa_{\gamma_{n+1}}.$$

But this holds for arbitrarily large $\bar{\kappa}$, since, by making α large enough, we can make κ_{γ_0} as large as we want. QED (Lemma 3.8.15)

Note that, by lemma 3.8.15, we san conclude that if $\rho^\omega_Q \geq \tilde{\eta}$ and $A \in \Sigma^*(Q)$ such that $A \subset N$, then N is Woodin with respect to A. We now prove:

Lemma 3.8.16. $\rho^\omega_Q \geq \tilde{\eta}$.

Proof: Suppose not. We consider two cases:

Case 1 $\rho^n_Q \geq \tilde{\eta}$, $\rho^{n+1}_Q < \tilde{\eta}$ for any $n < \omega$.

(This includes the case $N = Q$.) Then there is a $\underline{\Sigma}^{(n)}_1(Q)$ set $B \subset \tilde{\eta}$ such that $\langle N, B\rangle$ is not amenable. Let:

$$B(\xi) \longleftrightarrow \bigvee z^n A(z, \xi),$$

where A is a $\Sigma^{(n)}_0$ in a parameter p. Define $B' \subset \tilde{\eta}$ by:

$$B'(\prec \xi, \zeta \succ) \longleftrightarrow \bigvee z \in J^{E^N}_\zeta A(z, \xi) \text{ for } \xi, \zeta < \tilde{\eta}.$$

Claim 1 $\langle N, B'\rangle$ is amenable.

Proof. If $\tau \in N$ is regular in N, then $B' \cap \tau \in N$, since:

$$\prec \xi, \zeta \succ \in B' \cap \tau \longleftrightarrow \bigvee z \in J^{E^N}_\tau A(z, \xi).$$

By Claim 1 there are arbitrarily large $\kappa < \tilde{\eta}$ which are Woodin with respect to B'. Choose such a κ large enough that $B \cap \kappa \notin N$.

Claim 2 There is $\xi_0 \in B \cap \kappa$ such that $\neg B'(\prec \xi_0, \zeta \succ)$ for all $\zeta < \kappa$.

Proof. If not: $B \cap \kappa = \{\xi \mid \bigvee \zeta < \kappa B'(\prec \xi, \zeta \succ)\}$. Hence $B \cap \kappa \in N$. Contradiction. QED(Claim 2)

Let $F \in N$ be B'-strong in N at κ of length μ, where $\bigvee \zeta < \mu B'(\prec \xi_0, \zeta \succ)$. Set: $B'' = \{\zeta \mid B'(\prec \xi_0, \zeta \succ)\}$. Then:

$$\emptyset = F(\emptyset) = F(B'' \cap \kappa) = B'' \cap \mu \neq \emptyset.$$

Contradiction! QED(Case 1)

Case 2 $\rho^n_Q > \eta > \rho^{n+1}_Q$ for an $n < \omega$.

Let $Q^* = Q^{n, p^n_Q}$. Then each element of Q^* has the form: $h(i, \prec p, \xi, \tilde{\eta} \succ)$ where $i < \omega$, $\xi < \tilde{\eta}$ and $h = h_{Q^*}$ is the Σ_1 Skolem function for Q^*. Set:

$$f(\prec i, \xi \succ) \simeq h(i, \prec p, \xi, \tilde{\eta} \succ) \text{ if } i < \omega, \xi < \tilde{\eta} \tag{3.1}$$
$$f(\alpha) \text{ undefined otherwise.} \tag{3.2}$$

Then $|Q^*| = f\text{"}\tilde{\eta}$. Set:

$$\bar{f}(\zeta) \simeq \begin{cases} f(\zeta) & f(\zeta) < \tilde{\eta} \\ \text{otherwise undefined.} \end{cases} \tag{3.3}$$

Then $\tilde{\eta} = \bar{f}"\tilde{\eta}$. We consider two subcases:

Case 2.1 There is $\delta < \tilde{\eta}$ such that $\text{lub}\,\bar{f}"\delta = \tilde{\eta}$.

Let:

$$\zeta = \bar{f}(\xi) \longleftrightarrow \bigvee z \in Q^* H(z, \xi, \zeta)$$

where H is $\underline{\Sigma}_0^{(n)}(Q)$. Let $\eta^* = \text{ht}(Q^*)$. For $\gamma < \eta$ set:

$$\zeta = \bar{f}_\gamma(\xi) \longleftrightarrow: \bigvee z \in S_\gamma^{E^{Q^*}} H(z, \xi, \zeta).$$

Then $\bar{f}_\gamma \in Q$. Hence $\text{lub}\,\bar{f}_\gamma"\delta < \tilde{\eta}$, since $\tilde{\eta} \in Q$ is Woodin, hence regular in Q. But:

$$\bigcup_{\gamma < \eta^*} \bar{f}_\gamma"\delta = \bar{f}"\delta \text{ is unbounded in } \tilde{\eta}.$$

Set:

$$g(\mu) = \text{lub}\{\gamma < \eta^* \mid \bar{f}_\gamma"\delta < \mu\}$$

Then:

$$g(\mu) < \eta^* \text{ for } \mu < \tilde{\eta} \text{ but } \text{lub}_{\mu < \tilde{\eta}}\, g(\mu) = \eta^*.$$

We are now in a position to imitate the proof in Claim 1. Assume $B \in \underline{\Sigma}_1^{(n)}(Q)$ where $B \subset \tilde{\eta}$ and $\langle N, B\rangle$ is not amenable. We can suppose δ to be chosen large enough that $B \cap \delta \notin N$. Let:

$$B(\xi) \longleftrightarrow \bigvee z^n A(z, \xi) \text{ where } A \text{ is } \underline{\Sigma}_0^{(n)}(Q).$$

Set:

$$B'(\prec \xi, \zeta \succ) \longleftrightarrow \bigvee \gamma < g(\zeta) \bigvee z \in S_\gamma^{E^{Q^*}} A(z, \xi)$$

for $\xi, \zeta < \tilde{\eta}$. Then

$$B(\xi) \longleftrightarrow \bigvee \zeta < \tilde{\eta} B'(\prec \xi, \zeta \succ).$$

Claim 1 $\langle N, B'\rangle$ is amenable.

Proof. If $\tau \in N$ is regular in N, then $B' \cap \tau \in N$, since: $g(\zeta) \leq g(\tau)$ for $\zeta < \tau$ and $\langle S_\gamma^{E^{Q^*}} \mid \gamma < g(\tau)\rangle \in Q$. Thus $B' \cap \tau \in Q$. Hence $B' \cap \tau \in N = H_{\tilde{\eta}}^Q$. QED(Claim 1)

But then there are arbitrarily large $\kappa \in N$ which are Woodin for B' in N. Choose such a κ such that $\kappa \geq \delta$. Exactly as before we get: **Claim 2** There is $\xi_0 \in B \cap \kappa$ such that $\neg B'(\prec \xi_0, \zeta \succ)$ for all $\zeta < \kappa$.

Now let $F \in N$ be B'-strong in N at κ of length μ such that $\mu > \zeta$ for all ζ such that $B'(\xi_0, \zeta)$. Set:

$$B''(\zeta) \longleftrightarrow: B'(\xi_0, \zeta).$$

Then:

$$\emptyset = F(\emptyset) = F(B'' \cap \kappa) = B'' \cap \mu \neq \emptyset.$$

Contradiction! QED (Case 2.1)

Case 2.2 Case 2.1 fails.

Then $\operatorname{lub} \bar{f}"\delta < \tilde{\eta}$ for all $\delta < \tilde{\eta}$. We again derive a contradiction. Let $B \subset \tilde{\eta}$ be $\Sigma_1(Q^*)$ in $q \in Q^*$ such that $\langle N, B\rangle$ is not amenable. Note that $f"\gamma \prec_{\Sigma_1} Q^*$ whenever $\gamma < \tilde{\eta}$ is Gödel closed. Moreover, $Q^* = f"\tilde{\eta}$. Let $B \cap \delta \notin N$, where $\delta < \tilde{\eta}$ such that δ is Gödel closed and $q \in f"\delta$. Define a sequence δ_n $(n < \omega)$ by: $\delta_0 = \delta$, $\delta_{n+1} =$ the least $\delta \subset \bar{f}"\delta_n$ such that δ is regular in N. Set: $\tilde{\delta} = \operatorname{lub}_{n<\omega} \delta_n$. We consider two cases:

Case 2.2.1 $\tilde{\delta} < \tilde{\eta}$.

Let $X = f"\tilde{\delta}$. Then $X \prec_{\Sigma_1} Q^*$ such that $q, \tilde{\eta} \in X$. Let $\sigma \colon \bar{Q}^* \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Then $\sigma \colon \bar{Q}^* \longrightarrow_{\Sigma_1} Q^*$. It is easily seen that $X \cap \tilde{\eta} = \tilde{\delta}$. Since $\tilde{\eta} \in X$ we have:

$$\tilde{\delta} = \operatorname{crit}(\sigma), \sigma(\tilde{\delta}) = \tilde{\eta}.$$

Let $\sigma(\bar{q}) = q$. Then $\bar{B} = B \cap \tilde{\delta}$ is $\Sigma_1(\bar{Q}^*)$ in $\bar{q}$. By the extension of embeddings lemma there are $\bar{Q}$, $\bar{p}$, σ' such that $\bar{Q}^* = \bar{Q}^{n,\bar{p}}$ and $\sigma' \subset \sigma$ such that

$$\sigma' \colon \bar{Q} \longrightarrow_{\Sigma_1^{(n)}} Q \text{ and } \sigma'(\bar{p}) = p.$$

Since $Q = J^E_\alpha$, where $E \in N$ and $N = J^E_{\tilde{\eta}}$, we conclude that $\bar{Q} = J^{\bar{E}}_{\bar{\alpha}}$ where $\bar{E} \subset \bar{N}$, $\bar{N} = J^{\bar{E}}_{\tilde{\delta}}$. Since $\sigma(\tilde{\delta}) = \tilde{\eta}$, $\sigma \restriction \tilde{\delta} = \operatorname{id}$, we conclude $\bar{E} = E \cap \bar{N}$. WE now show:

Claim $\bar{\alpha} < \tilde{\eta}$.

Proof. Suppose not. Since $\tilde{\eta}$ is Woodin in Q, we know that $E_\nu \neq \emptyset$ for arbitrarily large $\nu < \tilde{\eta}$. Let ν be least such that $\tilde{\delta} \leq \nu$ and $E_\nu \neq \emptyset$. Then $\tilde{\delta} < \nu$, since $\tilde{\delta}$ is a limit cardinal in N. Then $E_\nu \neq \emptyset$ and $\tilde{\delta}$ is Woodin in $J^E_\nu = J^{\bar{E}}_\nu$. Hence N is not 1-small. Contradiction! QED(Claim)

But then $\bar{Q} = J^{\bar{E}}_{\bar{\alpha}} \in N$, since $\bar{E} = E \cap \bar{N}$ and $N = J^E_{\tilde{\delta}}$. Hence $\bar{Q}^* = \bar{Q}^{n,\bar{p}} \in \bar{N}$. Hence $B \cap \tilde{\delta} \in N$ since $B \cap \tilde{\delta}$ is $\underline{\Sigma}_1(Q^*)$. Hence $B \cap \delta \in N$. Contradiction! QED(Case 2.2.1)

All that remains is:

Case 2.2.2 $\tilde{\delta} = \tilde{\eta}$.

Let $C = \{\delta_n \mid n < \omega\}$. Then C is Q-definable in parameters and $\langle N, C\rangle$ is amenable, since $u \cap C$ is finite for $u \in N$. But then there is $\kappa \in N$ which is

Woodin with respect to C. Let $\mu < \kappa$ such that $C \cap (\kappa \setminus \mu) = \emptyset$. Let F be C-strong at κ in N of length τ such that $C \cap (\tau \setminus \kappa) \neq \emptyset$. Then:

$$\emptyset = F(\emptyset) = F(C \cap (\kappa \setminus \kappa)) = C \cap (\tau \setminus \kappa) \neq \emptyset.$$

Contradiction! QED(Lemma 3.8.16)

Making use of this we prove:

Lemma 3.8.17. *There is no truncation on the branch b_0.*

Proof: Suppose not. Let $\mu + 1$ be the least truncation point. Let $\mu^* = T(\mu+1)$ (hence $\mu + 1 \leq_T \gamma_0 + 1$ and $\mu^* \leq_T \gamma_0^*$). Then $\rho^\omega_{M^*_\mu} \leq \kappa_\mu$. Hence $\rho^\omega_{M_{b_0}} \leq \kappa_\mu < \tilde{\eta}$, since $\mathrm{crit}(\pi_{\mu^*,b}) = \kappa_\mu$. Contradiction! QED (Lemma 3.8.17)

Hence $\pi_{0,b_0} : M \longrightarrow_{\Sigma^*} Q$. We shall use this fact to garner information about M. We know:

(a) $Q = J^E_\beta$ is a constructible extension of $N = J^E_{\tilde{\eta}}$.

(b) $\tilde{\eta} = \mathrm{lub}\{\nu : E_\nu \neq \varnothing\}$

(c) $\rho^\omega_Q \geq \tilde{\eta}$ (hence Q is sound).

(d) If $A \subset N = J^E_{\tilde{\eta}}$, $A \in \underline{\Sigma}(Q)$, then N is Woodin for A.

Note. By soundness we have: $\underline{\Sigma}^*(Q) = \underline{\Sigma}_\omega(Q)$.

We shall prove:

Lemma 3.8.18. *Let* $\eta_0 = \mathrm{lub}\{\nu : E^M_\nu \neq \varnothing\}$. *Then:*

(a) $\eta_0 \leq \mathsf{ON}_M$ *is a limit ordinal. Hence M is a constructible extension of* $N_0 = J^{E^M}_{\nu_0}$.

(b) $\rho^\omega_M \geq \eta_0$. *Hence M is sound.*

(c) *Let $A \in \underline{\Sigma}_\omega(M)$ such that $A \subset N$. Then N_0 is Woodin for A.*

Proof: Set $\pi = \pi_{0,r_0}$. For $i \in b_0$ set: $\pi_i = \pi_{i,b_0}$. Then $\pi_i : M_i \longrightarrow_{\Sigma^i} Q$. We find prove (a). Suppose not $\eta_0 \neq 0$, since otherwise the iteration would be impossible. Hence there is a maximal ν, such that $E^M_\nu \neq \varnothing$. The statement $E^M_\nu \neq \varnothing$ is $\Sigma_r(M)$ in ν and the statement "ν is maximal" is $\Pi_1(M)$. Hence these statement hold in Q of $\pi(\nu)$. But $\pi(\nu) < \tilde{\eta}$ is not maximal. Contradiction! QED(a)

We now prove (b). If not, then $\rho^\omega_M \leq \nu$ where $E^M_\nu \neq \varnothing$. But $\rho^\omega_{M||\nu} \leq \lambda$, where $\kappa = \mathrm{crit}(E^M_\nu)$ and $\lambda = \lambda(E^M_\nu) =: E^M_\nu(\kappa)$. Hence $\rho^\omega_M \leq \lambda < \nu$. Hence

$$\rho^\omega_Q \leq \pi(\rho^\omega_M) \leq \pi(\lambda) < \pi(\nu) < \tilde{\eta}$$

Contradiction! QED(b)

We now prove (c). Let $A \subset N_0$ be $\Sigma_\omega(\ ,\ )$. Since M is sound, A is $\underline{\Sigma}^*(M)$ by Corollary 2.6.30. Let A be $\Sigma^*(M)$ in q and let A' be $\Sigma^*(Q)$ in $q' = \pi(q)$ by the same definition. Pick $n < \omega$ such that $\rho^n_M = \eta_0$ and $\rho^n_Q = \tilde{\eta}$. Clearly, every $\Sigma_\omega(H^n_M, A)$ statement translates uniformly into a statement which is $\Sigma^*(M)$ in q. Similarly for Q, A', q'. Hence:

$$\pi \upharpoonright N_0 | \langle N_0, A\rangle \prec \langle N, A'\rangle$$

But the statement "N is Woodin for A'" is elementary in $\langle N, A'\rangle$. Hence N_0 is Woodin for A. QED(Lemma 3.8.18)

We now define:

Definition 3.8.11. A premouse M is *restrained* iff it is one small and does not satisfy the condition (a)-(c) in Lemma 3.8.18.

We have proven:

Theorem 3.8.19. *Every restrained premouse has the normal uniqueness property.*

By theorem 3.6.1 and theorem 3.6.2 we conclude:

Corollary 3.8.20. *Let $n > \omega$ be regular. Let M be a restrained premouse which is normally $\kappa + 1$-iterable. Then M is fully $\kappa + 1$-iterable.*

Hence, if $\alpha > \omega$ is a limit cardinal and M is normally α-iterable, then M is fully α-iterable. This holds of course for $\alpha = \infty$ as well.

We also note the following fact:

Lemma 3.8.21. *Let M be restrained. Then every normal iterate of M is restrained.*

Proof: Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_i\rangle, T\rangle$ be the iteration of M to $M' = M_\mu$.

Case 1: There is a truncation on the main brach $b = \{i : i \leq_T \mu\}$. Let $i + 1$ be the last truncation point. Then $\kappa_i < \lambda_h$ where $h = T(i + 1)$. Hence

$\rho^{\omega}_{M^*_h} \leq \lambda_h < \nu_h$. Hence $\rho^{\omega}_{M} \leq \pi_{h,\nu}(\rho^{\omega}_{M^*_h}) < \pi_{h,\mu}(\nu_h)$, where $E^{M'}_{\pi_{h,\mu}(\nu_h)} \neq \varnothing$. Hence M' is restrained.

Case 2: Case 1 fails. Then $\pi_{0,1} : M \longrightarrow_{\Sigma^*} M'$.

Case 2.1: $\rho^{\omega}_{M} < \nu$ for a ν such that $E^{M}_{\nu} \neq \varnothing$. This is exactly like Case 1. There remains the case:

Case 2.2: Case 2.1 fails. Then $\eta = \text{lub}\{\nu : E^{M}_{\nu} \neq \varnothing\}$ is a limit ordinal and M is a constructible extension of $J^{E^M}_{\nu}$. But then there is $A \subset J^{E}_{\nu}$ such that $A \in \underline{\Sigma}_{\omega}(M)$ and $J^{E^M}_{\nu}$ is not Woodin for A. Repeating the proof of Lemma 3.8.18, it follows that $\pi_{0,n}$ is an elementary embedding of M into M'. If A is $\Sigma_{\omega}(M)$ in p and A' is $\Sigma_{\omega}(M')$ is $\pi(p)$, it follows that $N' = J^{E^{M'}}_{\nu'}$ is not Woodin for A', where

$$\nu' = \text{lub}\{\nu : E^{M'}_{\nu} \neq \varnothing\} = \pi_{0,\mu}(\eta)$$

Hence M' is restrained. QED(Lemma 3.8.21)

Note. We could also show that every smooth iterate of a restrained premouse is restrained. This does not hold for full iterates, however, since there can be a restrained M such that $M||\mu$ is not restrained for some $\mu \in M$.

3.8.4 The Bicephalus

In this section we verify some technical lemmas which will be needed in Chapter 5. There are we'll need to consider "two headed mice", also known as *bicephali*.

Definition 3.8.12. By a *bicephalus* we mean a structure $M = \langle |M|, F^0, F^1 \rangle$ s.t,

- $|M| = J^{E}_{\nu}$ is a passive premouse,
- $\langle |M|, F^n \rangle$ is an active premouse for $n = 0, 1$.

The possibility that $F^0 \neq F^1$ is not excluded. (Ultimately, however, we will aim to show that in all interesting cases, we have $F^0 = F^1$. Using this we shall show that the inner model K^c constructed in Chapter 5 is uniquely determined.) By Theorem 3.3.24 we have;

Lemma 3.8.22. *Let $M = \langle |M|, F^0, F^1 \rangle$ be a bicephalus. Let G be an extender at $\kappa \in M$ on M. Let;*

$$\pi \colon M \longrightarrow_G M' = \langle |M'|, F'^0, F'^1 \rangle.$$

Then M' is a bicephalus.

Note. Here we are using Σ_0 ultrapowers. This makes sense if we consider that M' is obtained by first applying G to the ZFC^- model $|M|$ and then recovering $F^{0\prime}$, $F^{1\prime}$ by:

$$F^{h\prime} = \bigcup_{u \in M} \pi(u \cap F^h) \text{ for } h = 0, 1$$

When we normally iterate bicephali, we shall apply the Σ_0 ultrapowers on non-truncating branches.

By Theorem 3.3.25 we have:

Theorem 3.8.23. *Let $M_0 = \langle |M_0|, F^0, F^1 \rangle$ be a bicephalus. Let $\pi_{i,j} \colon M_i \longrightarrow M_j$ $(i \leq j \leq \eta)$ be a system of commuting maps such that*

- $\pi_{i,i+1} \colon M_i \longrightarrow_{G_i} M_{i+1}$*, where Gi is an extender in M_i,*
- *M_i is transitive and the $\pi_{i,j}$ commutes,*
- *If $\lambda \leq \eta$ is a limit ordinal, then*

  $$M_\lambda, \langle \pi_{i,\lambda} \mid i < \lambda \rangle$$

  is the transitivased direct limit of:

  $$\langle M_i \mid i < \lambda \rangle, \langle \pi_{i,j} \mid i \leq j < \lambda \rangle.$$

  Then each M_i is a bicephalus.

Definition 3.8.13. By a *precephalus* we mean either a premouse or a pre-bicephalus. If M is a precephalus, $\nu \subset M$ is a limit ordinal, and E_ν^M is uniquely determined, we set: $M||\nu = \langle |M|, E_\nu^M \rangle$. If, however, $\nu = \mathrm{ht}(M)$ and $M = \langle |M|, F^0, F^1 \rangle$ is a bicephalus, we set $M||\nu =: M$. $\mathbb{F}_\nu^M$ is then defined to be :

$$\{E_\nu^M\} \text{ if uniquely defined, } \{F^0, F^1\} \text{ if not.}$$

Using this we can define the notion of a *normal iteration*:

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle F_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$$

of a precephalus M. This is defined exactly as before in §3.4 except that:

- If $h = T(i+1)$, we apply $F_i \in \mathbb{F}_{\nu_i}$ to M_i^*
- If $i+1$ is not a drop point (i.e. τ_i is a cardinal in M_h) and M_h is a bicephalus, then M_{i+1} is the Σ_0-ultrapower of M_h:

  $$\pi_{h,i+1} \colon M_h \longrightarrow_{F_i} M_{i+1}$$

- In all other cases, set:

$$\pi_{h,i+1} \colon M_i^* \longrightarrow_{F_i}^n M_{i+1},$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho_{M_i^*}^n$.

As usual we set:

$$\kappa_i =: \mathrm{crit}(F_i), \tau_i =: \kappa_i^{+M_i||\nu_i}.$$

and:

$$\lambda_i =: F_i(\nu_i) = \text{the largest cardinal in } M_i||\nu_i.$$

(Thm κ_i, τ_i are dependent on the choice of F_i, whereas λ_i depends only on ν_i.) We again have:

$$T(i+1) =: \text{the least } h \text{ such that } \kappa_i < \lambda_h \text{ or } i = h.$$

This, of course, means that in the definition of "normal iteration" given in §3.4.2, we must make appropriate changes in (b), (c), and (f). If I is the iteration of a bicephalus M, it follows easily by induction on i that

$$M_i \text{ is a bicephalus if and only if } [0, i)_T \text{ has no drop.}$$

We leave this to the reader. If M is not a bicephalus, then I is a normal iteration in the new sense if and only if in the old sense, Lemma 3.4.1 and Lemma 3.4.10 still hold.

Note. It may seem strange that, if $h = T(i+1)$ and $M_h = M_i^* = \langle |M_h|, F_h^0, F_h^1 \rangle$ is a bicephalus, we take the Σ_0 ultraproduct of M_h rather that the $*$-ultraproduct. But $|M_h|$ is then a ZFC^- model and we are -in effect- applying $E_{\nu_i}^{M_i}$ to $|M|$. For this the Σ_0-ultrapower is appropriate. We then recover F_{i+1}^0, F_{i+1}^1 by:

$$F_{i+1}^l = \bigcup_{u \in |M_h|} \pi_{h,i+1}(u \cap F_h^l).$$

We can turn an iteration:

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle F_i \rangle, \langle \tau_{i,j} \rangle, T \rangle$$

of length to $i+1$ into a *potential* iteration of length $i+2$ by appointing a pair of indices $\langle \nu_i, F_i \rangle$ such that $\nu_i > \nu_j$ for $j < i$ and $F_i \in \mathbb{F}_{\nu_i}^{M_i}$. We leave it to the reader to amend the definition in §3.3.2 appropriately. Given the choice of ν_i, F_i we can then define $h = T(i+1)$, $M_h^* = M_h||\beta$ (for appropriate β) as usual. We do not know, however, whether M_i^* is extendable by F_i. In place of Theorem 3.4.4 we then have:

Theorem 3.8.24. *Let I be a normal iteration of M of length $i+1$. Extend it to a potential normal iteration of length $i+2$ by appointing appropriate ν_i, F_i, then F_i is close to M_i^*.*

This means that whenever M_i^* is not a bicephalus, we shall have:

$$\pi_{h,i+1} \colon M_i^* \longrightarrow_{F_i}^* M_{i+1},$$

whereas we take the Σ_0-ultraproduct otherwise.

The proof of Theorem 3.8.24 is a simple variant of the earlier proof.

Our main result here is that Theorem 3.8.4 holds for bicephali as well as for premice. In fact, we can almost literally repeat the proof. This seems problematic at first glance, since our proof makes frequent use of the notation $E_{\nu_i}^{M_i}$ in describing a normal iteration of a precephalus M, although $M_i = \langle |M_i|, F^0, F^1 \rangle$ might be a bicephalus. If then $\nu_i = \mathrm{ht}(M_i)$, we let $E_{\nu_i}^{M_i}$ denotes that $F \in \{F^0, F^1\}$ which we chose to apply to M_i^* at stage i. Let M be a precephalus and let

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle F_i \rangle, \langle \pi_{i,j} \rangle, T \rangle$$

be a normal iteration of M of limit length η. Let b_0, b_1 be distinct cofinal well fouded branches in I. Pick $\alpha < \eta$ such that $(b_0 \setminus \alpha) \cap (b_1 \setminus \alpha) = \emptyset$ and define δ_i, γ_i, γ_i^* exactly as before. If we make the special assumption:

$$\text{(SA) } E_{\nu_{\gamma_i}} | \kappa_{\gamma_{i+1}} \in M_{\gamma_i},$$

We can literally repeat the steps (1)-(11).

We now attempt to redo the proof without (SA). The situation is complicated by the fact that a bicephalus M can have two distinct top extenders. Nontheless we define the notion *undesirable* able exactly as before. (Note that the definition speaks of "a top extender" rather than "the top extender".) We again prove:

(12) If α is sufficiently large, then no n is undesirable.

Proof. Assign to each undesirable n an integer $\langle i_n, j_n \rangle$ as follows:

- $i_n = \begin{cases} 0 & n \text{ is even} \\ 1 & n \text{ is odd.} \end{cases}$
- $j_n = 0$ if M_{δ_n} is a premouse or $M_{\delta_n} = \langle |M|, F^0, F^1 \rangle$ is a prebicephalus with $\mathrm{crit}(F^0) \in [\kappa_{\gamma_n}, \kappa_{\gamma_{n+1}})$.

- $j_n = 1$ if not.
  (Hence $\mathrm{crit}(F^1) \in [\kappa_{\gamma_n}, \kappa_{\gamma_{n+1}})$.)

If (12) fails, there are infinitely many undesirable n. In particular, there are undesirable n, m such that

$$n < m \text{ and } \langle i_n, j_n\rangle = \langle i_m, j_m\rangle.$$

This gives a contradiction exactly as before. (We leave this to the reader.)

If we have chosen α large enough that (12) holds, we can then literally repeats the proof of Lemma 3.8.12 and the definition of (13). QED(Theorem 3.8.4)

We call a bicephalus $\langle |M|, F^0, F^1\rangle$ 1-*small* if and only if $\langle |M|, F^0\rangle$, $\langle |M|, F^1\rangle$ are 1-small premice. (Since $|M|$ is a ZFC^- model, this is equivalent to: $|M| \models$ There is no Woodin cardinal.) The proofs in §3.8.3 then go through literally as before for 1-small precephali. In particular, Lemma 3.8.18 goes through (although we must change the definition of η_0 to:

$$\eta_0 = \mathrm{lub}\{\nu \mid \mathbb{F}_\nu \neq \emptyset\}).$$

If M has top extenders, then η_0 is obviously a successor ordinal. Hence M is restrained. In particular, every prebicephalus is restrained. Hence:

Lemma 3.8.25. *Let I be a normal iteration of a prebicephalus M of limit length. Then I has at most one cofinal well founded branch.*

Chapter 4

Properties of Mice

4.1 Solidity

In §2.5.3 we introduced the notion of *soundness*. Given a sound M, we were then able to define the n-th projectum $\rho_M^n (n < \omega)$. We then defined the n-th *reduct* $M^{n,a}$ with respect to a parameter a (consisting of a finite set of ordinals). We then defined the n-th set P_M^n of *good parameters* and the set R_M^n of *very good* parameters. (Soundness was, in fact, equivalent to the statement: $P^n = R^n$ for $n < \omega$). We then defined the n-th *standard parameter* $p_M^n \in R_M^n$ for $n < \omega$. This gave us the classical fine structure theory, which was used to analyze the constructible hierarchy and prove such theorems as $\square$ in L. Mice, however, are not always sound. We therefore took a different approach in §2.6, which enabled us to define $\rho_M^n, M^{n,a}, P_M^n, R_M^n$ for all acceptable M. (In the absence of soundness we could, of course, have: $R_M^n \neq P_M^n$). In fact R_M^n could be empty, although P_M^n never is. P_M^n was defined in §2.6.

P_M^n is a subset of $[\mathrm{On}_M]^{<\omega}$ for acceptable $M = \langle J_\alpha^A, B\rangle$. Moreover, the reduct $M^{n,a}$ is defined for any $n < \omega$ and $a \in [\mathrm{On}_M]^{<\omega}$. The definition of P_M^n, M^n are recapitulated in §3.2.5, together with some of their consequences. R_M^n is defined exactly as before, taking $= R_M^n = \varnothing$ if n is not weakly sound. At the end of §2.6 we then proved a very strong downward extension lemma, which we restate here:

Lemma 4.1.1. *Let $n = m + 1$. Let $a \in [\mathrm{On}_M]^{<\omega}$. Let $N = M^{n,a}$. Let $\overline{\pi} : \overline{N} \longrightarrow_{\Sigma_j} N$ where $\overline{N}$ is a J-model and $j < \omega$. Then:*

(a) *There are unique $\overline{M}, \overline{a}$ such that $\overline{a} \in R_{\overline{M}}^n$ and $\overline{M}^{n,\overline{a}} = \overline{N}$.*

(b) *There is a unique* $\pi \supset \overline{\pi}$ *such that:*

$$\pi : \overline{M} \longrightarrow_{\Sigma_0^{(m)}} M \text{ strictly and } \pi(\bar{(a)}) = a.$$

(c) $\pi : \overline{M} \longrightarrow_{\Sigma_j^{(n)}} M$.

In §2.6. we also proved:

Lemma 4.1.2. *Let* $n = m + 1$*. Let* $a \in R_M^n$*. Then every element of* M *has the form* $F(\xi, a)$ *where* $\xi < \rho_M^n$ *and* F *is a good* $\Sigma_1^{(m)}$ *function.*

Corollary 4.1.3. *Let* $n, a, \overline{\pi}, \pi$ *be as in Lemma 4.1.1, wehere* $j > 0$*. Then*

$\mathrm{rng}(\pi) =$ *The set of* $F(\xi, a)$ *such that* F *is a good* $\Sigma_1^{(m)}$ *function and* $\xi \in \mathrm{rng}(\overline{\pi}) \cap \rho_M^n$

Proof.. Let Z be the set of such $F(\xi, a)$.

Claim 1. $\mathrm{rng}(\pi) \subset Z$.

Proof. Let $y = \pi(\overline{y})$. Then $\overline{y} = \overline{F}(\xi, \overline{a})$ where $\overline{F}$ is a good $\Sigma_1^{(n)}(\overline{M})$ function and $\xi < \rho_{\overline{M}}^n$ by Lemma 4.1.2. Hence $y = F(\pi(\xi), a)$, where F has the same good $\Sigma_1^{(n)}$ definition in M.

QED(Claim 1.)

Claim 2. $Z \subset \mathrm{rng}(\pi)$.

Proof. Let $y = F(\pi(\xi), a)$, where F is a good $\Sigma_1^{(m)}(M)$ function. Then the $\Sigma_1^{(n)}$ statement:

$$\bigvee y \, y = F(\pi(\xi), a)$$

holds in M. Hence, there is $\overline{y} \in \overline{M}$ such that $\overline{y} = \overline{F}(\xi, a)$ where $\overline{F}$ has the same good $\Sigma_1^{(m)}$ definition in $\overline{M}$. Hence

$$\pi(\overline{y}) = F(\pi(\xi), a) = y.$$

QED(Corollary 4.1.3)

Note. $\mathrm{rng}(\pi) \subset Z$ holds even if $j = 0$.

Lemma 4.1.1 shows that a great deal of the theory developed in §2.5.3 for sound structures actually generalizes to arbitrary acceptable structures. This is not true, however, for the concept of *standard parameter*.

In our earlier definition of standard parameter, we assumed the soundness of M (meaning that $P^n = R^n$ for $n < \omega$). We defined a well ordering $<_*$ of $[\mathrm{On}]^{<\omega}$ by:

$$a <_* b \longleftrightarrow \bigvee \xi(a \setminus \xi = b \setminus \xi \wedge \xi \in b \setminus a).$$

We then defined the n-th *standard parameter* p^n_M to be the $<_*$-least $a \in M$ with $a \in P^n$. This definition stil makes sense even in the absence of soundness. We know that $p^n \setminus \rho^i \in P^i$ for $i \leq n$. Hence by $<_*$-minimality we get: $p^n \setminus \rho^n = \varnothing$. For $i \leq n$ we clearly have $p^i \leq_* p^n \setminus \rho^i$ by $<_*$-minimality. However, it is hard to see how we could get more than this if our only assumption on M is acceptability.

Under the assumption of soundness we were able to prove:

$$p^n \setminus \rho^i = p^i \text{ for } i \leq n.$$

It turns out that this does still holds under the assumption that M is fully $\omega_1 + 1$ iterable. Moreover if $\pi : M \longrightarrow N$ is an iteration map, then $\pi(p^n_M) = P^n_N$. The property which makes the standard parameter so well behaved is called *solidity.* As a preliminary to defining this notion we first define:

Definition 4.1.1. Let $a \in M$ be a finite set of ordinals such that $\rho^\omega \cap a = \varnothing$ in M. Let $\nu \in a$. The ν-th *witness to a* in M (in symbols M^ν_a) is defined as follows:

Let $\rho^{i+1} \leq \nu < \rho^i$. Let $b = a \setminus (\nu + 1)$. Let $\overline{M} = M^{i,b}$ be the i-th reduct of M by b. Set: $X = h(\nu \cup (b \cap \overline{M}))$, i.e. $X =$ the closure of $\nu \cup (u \cap \overline{M})$ under $\Sigma_1(M)$ functions. Let:

$$\overline{\sigma} : \overline{W} \longleftrightarrow \overline{M}|X$$

be the transitivation of $\overline{M}|X$. By the extension of embedding lemma there are unique $W, n, \sigma \supset \overline{\sigma}$ such that:

$$\overline{W} = W^{i,\overline{b}}, \sigma : W \longrightarrow_{\Sigma^{(i)}_1} M, \sigma(\overline{b}) = b.$$

Set: $M^\nu_a = W$. σ is called the *canonical embedding* for a in M and is sometimes denoted by σ^ν_a.

Note. Using Lemma 4.1.3 it follows that $\mathrm{rng}(\pi)$ is the set of all $F(\vec{\xi}, b)$ such that $\xi_1, \ldots, \xi_n \subset \nu$, $b = a \setminus (\nu + 1)$ and F is good $\Sigma^{(i)}_1(M)$ function. This is a more conceptual definition of M^ν_a, σ.

Definition 4.1.2. M is *n-solid* iff $M^\nu_a \in M$ for $\nu \in a = p^n_M$ it is solid iff it is n-solid for all n.

p^n was defined as the $<_*$- least element of P^n. Offhand, this seems like a rather arbitrary way of choosing an element of P^n. Solidity, however, provides us with a structural reason for the choice. In order to make this clearer, let us define:

Definition 4.1.3. Let $a \in M$ be a finite set of ordinals. a is *solid for M* iff for all $\nu \in a$ we have

$$\rho_M^\omega \leq \nu \text{ and } M_a^\nu \in M$$

.

Lemma 4.1.4. *Let $a \in P^n$ such that $a \cap \rho^n = \varnothing$. If a is solid for M, then $a = p^n$.*

Proof. Suppose not. Then there is $q \in P^n$ such that $q <_* a$. Hence there is ν such that $q \setminus (\nu+1) = a \setminus (\nu+1)$ and $\nu \in a \setminus q$. But then $q \subset \nu \cup (a \setminus (\nu+1)) \subset \operatorname{rng}(\sigma)$ where $\sigma_a = \sigma_a^\nu$ is the canonical embedding. Let A be $\Sigma^{(n)}(M)$ in q such that $A \cap \rho^{n+1} \notin M$. Let $\overline{A}$ be $\Sigma_1^{(n)}(M_a^\nu)$ in $\overline{q} = \sigma^{-1}(q)$ by the same definition. Since $\sigma \upharpoonright \nu = \mathrm{id}$ and $\rho^n \leq \nu$, we have:

$$A \cap \rho^n = \overline{A} \cap \rho^n \in M,$$

since $A \in \underline{\Sigma}_1^n(M_a^\nu) \subset M$. Contradiction!

QED(Lemma 4.1.4)

The same proof also shows:

Lemma 4.1.5. *Let a be solid for M such that $a \cap \rho^n = \varnothing$ and $a \cup b \in P^n$ for some $b \subset \nu$ such that $ab \subset \nu$ for all $\nu \in a$. Then a is an upper segment of p^n (i.e. $a \setminus \nu = p^n \setminus \nu$ for all $\nu \in a$.)*

Hence:

Corollary 4.1.6. *If M is n-solid and $i < n$, then M is i-solid and $p^i = p^n \setminus \rho^i$.*

Proof. Set $a = p^n \setminus \rho^i$. Then $a \in P^i$ is M-solid. Hence $a = p^i$.

QED(Corollary 4.1.6)

We set $p_M^* =: \bigcup_{n<\omega} p_M^n$. Then $p^* = p^n$ where $\rho^n = \rho^\omega$.

p^* is called the *standard parameter* of M. It is clear that M is solid iff p^* is solid for M.

Definition 4.1.4. Let $a \in [\mathrm{On}_M]^{<\omega}, \nu \in a$ with $\rho^{i+1} \leq \nu < \rho^i$ in M. Let $b = a \setminus (\nu + 1)$. By a *generalized witness* to $\nu \in a$ we mean a pair $\langle N, c \rangle$ such that N is acceptable, $\nu \in N$ and for all $\xi_a, \ldots, \xi_r < \nu$ and all $\Sigma_1^{(i)}$ formulae φ we have:

$$M \models \varphi(\vec{\xi}, b) \longrightarrow N \models (\vec{\xi}, c).$$

Lemma 4.1.7. *Let $N \in M$ be a generalized witness to $\nu \in a$. Assume that $\nu \notin \mathrm{rng}(\sigma)$, where $\sigma = \sigma_a^\nu$ is the canonical embedding. Then $M_a^\nu \in M$.*

Proof. Let $W = M_a^\nu, \overline{W}, \overline{\sigma}$ be as in the definition of M_a^ν. Then $\overline{W} = W^{i,\overline{b}}$, where $\rho^{i+1} \leq \nu < \rho^i$ in M, $b = a \setminus (\nu + 1)$ and $\sigma(\overline{b}) = b$. Since $\sigma \upharpoonright \nu = \mathrm{id}$, we have:

$$\overline{W} \models \varphi(\vec{\xi}, \overline{b}) \longrightarrow N \models \varphi(\vec{\xi}, c),$$

for $\xi_1, \ldots, \xi_r < \nu$ and $\Sigma_1^{(i)}$ formulae φ. We can then define a map $\tilde{\sigma} : W \longrightarrow_{\Sigma_1^{(i)}} N$ by:

Let $x = F(\vec{\xi}, \overline{b})$ where $\xi_1, \ldots, \xi_r < \nu$ and F is a good $\Sigma_1^{(i)}(W)$ function. Then, letting $\dot{F}$ be a good definition of F we have:

$$W \models \bigvee x(x = \dot{F}(\vec{\xi}, \overline{b})); \text{ hence } N \models \bigvee x(x = \dot{F}(\vec{\xi}, c)).$$

We set $\tilde{\sigma}(x) = y$, where $N \models y = \dot{F}(\vec{\xi}, c)$.

If we set: $\overline{N} = N^{i,c}$, we have:

$$\tilde{\sigma} \upharpoonright \overline{W} : \overline{W} \longrightarrow_{\Sigma_0} \overline{N}.$$

Let $\gamma = \sup \tilde{\sigma}\text{"}\, \mathrm{On}_{\overline{N}}, \tilde{N} = \overline{N} | \gamma$. Then:

$$\tilde{\sigma} \upharpoonright \overline{W} : \overline{W} \longrightarrow_{\Sigma_1} \tilde{N} \text{ cofinally.}$$

Note that, since $\sigma(\nu) > \nu$ and $\sigma \upharpoonright \nu = \mathrm{id}$, we have: ν is regular in M_a^ν. Hence $\sigma(\nu)$ is regular in M and $H_{\sigma(\nu)}^M$ is a ZFC^- model. We now code $\overline{W}$ as follows. Each $x \in \overline{W}$ has the form: $h(j, \prec \xi, \overline{b} \succ)$ where $h = h_{\overline{W}}$ is the Skolem function of $\overline{W}$ and $\sigma < \nu$.

Set:

$$\dot{\in} = \{\prec\prec j, \xi \succ, \prec k, \zeta \succ\succ : h(j, \prec \xi, \overline{b} \succ) \in h(k, \langle \zeta, \overline{b} \rangle)\}$$
$$\dot{A} = \{\prec j, \xi \succ : h(j, \langle \xi, \overline{b} \rangle) \in A\}$$
$$\dot{B} = \{\prec j, \xi \succ : h(j, \langle \xi, \overline{b} \rangle) \in B\}$$

where $\overline{W} = \langle J_\gamma^A, B \rangle$. Let $D \subset \nu$ code $\langle \dot{\in}, \dot{A}, \dot{B} \rangle$. Then:

$$D \in \Sigma_\omega(\tilde{(}N)) \subset M,$$

since e.g.

$$\dot{\in} = \{\langle \prec j, \xi \succ, \prec k, \zeta \succ \rangle : h_{\tilde{N}}(j, \langle \xi, c \rangle) \in h_{\tilde{N}}(k, \langle \zeta, c \rangle)\}$$

But then $D \in H^M_{\sigma(\nu)}$ by acceptability. But $H^M_{\sigma(\nu)}$ is a ZFC^- model. Hence $\overline{W} \in H^M_{\sigma(\nu)}$ is recoverable from D in $H^M_{\sigma(\nu)}$. Hence $W \in H^M_{\sigma(\nu)} \subset N$ is recoverable from W in $H^M_{\sigma(\nu)}$.

QED(Lemma 4.1.7)

We note that:

Lemma 4.1.8. *Let $a \in P^n, \nu \in a, M^\nu_a \in M$. Then $\nu \notin \mathrm{rng}(\sigma^\nu_a)$.*

Proof. Suppose not. Then $a \in \mathrm{rng}(\sigma)$. Let A be $\Sigma_1(M)$ such that $A \cap \rho^n \notin M$. Let $\overline{A}$ be $\Sigma_1(M^\nu_a)$ in $\overline{a} = \sigma^{-1}(a)$ by the same definition. Then:

$$A \cap \rho^n = \overline{A} \cap \rho^n \in \underline{\Sigma}^*(M^\nu_a) \subset M.$$

Contradiction!

QED (Lemma 4.1.8)

But then:

Lemma 4.1.9. *Let $q \in P^n_M$. Let a be an upper segment of q which is solid for M. Let $\pi : M \longrightarrow_{\Sigma^*} N$ such that $\pi(q) \in P^n_N$. Then $\pi(a)$ is solid for N.*

Proof. Let $\nu \in a, W = M^\nu_a, \sigma = \sigma^\nu_a$. Set:

$$a' = \pi(a), \nu' = \pi(\nu), W' = N^{\nu'}_{a'}, \sigma' = \sigma^{\nu'}_{a'}.$$

We must show that $W' \in N$. We first show:

(1) $\nu' \notin \mathrm{rng}(\sigma')$.

Proof. Suppose not. Let $\rho^{i+1} \leq \nu < \rho^i$ in M. Then $\rho^{i+1} \leq \nu' < \rho^i$ in N. Then in N we have: $\nu' = F'(\xi, b')$ where $\xi < \nu', b' = a' \setminus (\nu' + 1)$, and F' is a good $\Sigma_1^{(i)}(N)$ function.

Let $\dot{F}$ be a good definition for F'. Then in N the $\Sigma_1^{(i)}$ statement holds:

$$\bigvee \xi' < \nu' (\nu' = \dot{F}(\xi', b')).$$

But then in M we have:

$$\bigvee \xi' < \nu (\nu = \dot{F}(\xi', b))$$

where $b = a \setminus (\nu + 1)$. Hence $\nu \in \mathrm{rng}(\sigma)$. Contradiction!

QED(1)

Now set: $W'' = \pi(W)$. In M we have:

$$\bigwedge \xi < \nu (M \models \varphi(\xi, b) \longrightarrow W \models \varphi(\xi, b))$$

for $\Sigma_1^{(i)}$ formulas φ. But this is a $\Pi_1^{(i)}$ statement in M about ν, b, W. Hence the corresponding statement holds in N:

$$\bigwedge \xi < \nu' (N \models \varphi(\xi, b') \longrightarrow W' \models \varphi(\xi, b'))$$

Hence W'' is a generalized witness for $\nu' \in a'$. Hence $W = N_a^{\nu'} \in N$.

QED(Lemma 4.1.9)

As a corollary we then have:

Lemma 4.1.10. *Let M be n-solid. Let $\pi : M \longrightarrow_{\Sigma^*} N$ such that $\pi(p_M^n) \in P_N^n$. Then N is n-solid and $\pi(P_M^n) = P_N^n$.*

Proof. Let $a = p_M^n$. Then $a' = \pi(a) \in P_N^n$ is solid for N by the previous lemma. Moreover, $a' \cap \rho_N^n = \varnothing$. Hence $a' = p_N^n$.

QED(Lemma 4.1.10)

This holds in particular if $\rho^n = \rho^\omega$ in M. But if $\pi : M \longrightarrow N$ is strongly Σ^*-preserving in the sense of §3.2.5, then $\rho^n = \rho^\omega$ in N and $\pi``(P_M^n) \subset P_M^n$. Hence:

Lemma 4.1.11. *Let M be solid. Let $\pi : M \longrightarrow N$ be strongly Σ^*-preserving. Then N is solid and $\pi(p^i_M) = p^i_N$ for $i < \omega$.*

QED(Lemma 4.1.11)

Corollary 4.1.12. *Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a normal iteration. Let $h = T(i+1)$ where $i+1 \leq_T j$. Assume that $(i+1, j]_T$ has no drop. If M^*_j is solid, then M_j is solid and $\pi_{h,j}(p^n_{M^*_i}) = p^n_{M_j}$ for $n < \omega_1$.*

Proof. $\pi_{h,j}$ is strongly Σ^*-preserving.

We now define:

Definition 4.1.5. Let M be acceptable. M is a *core* iff it is sound and solid. M is the *core of* N with *core map* iff M is a core and $\pi : M \longrightarrow_{\Sigma^*} N$ with $\pi(p^*_M) = p^*_N$ and $\pi \restriction \rho^\omega_M = \mathrm{id}$.

Clearly M can have at most one core and one core map.

Definition 4.1.6. Let $M = \langle J^E_\alpha, E_\alpha\rangle$ be a premouse. M is *presolid* iff $M||\xi$ is solid for all limit $\eta < \alpha$.

Lemma 4.1.13. *Let M be acceptable. The property "M is presolid" is uniformly $\Pi_1(M)$. Hence, if $\pi : M \longrightarrow_{\Sigma_1} N$, then N is presolid.*

Proof. The function:

$$\langle \Vdash_{M||\xi} : \xi \text{ is a limit ordinal}\rangle$$

is uniformly $\Sigma_1(M)$. But for each $i < \omega$ there is a first order statement φ_i which says that M is "solid above ρ^i", i.e.

$$M^\nu_{P^i_M} \in M \text{ for all } \nu \in p^i_M.$$

The map $i \mapsto \varphi_i$ is recursive. But M is presolid if and only if:

$$\bigwedge \xi \in M \bigwedge i (\xi \text{ is a limit } \longrightarrow \Vdash_{M||\xi} \varphi_i)$$

QED(Lemma 4.1.13)

We shall prove that every fully iterable premouse is solid. But if M is fully iterable, then so is every $M||\eta$. Hence M is presolid.

The *comparison Lemma* (Lemma 3.5.1) tells us that, if we coiterate two premice M^0, M^1 of cardinality less than a regular cardinal θ, then the coiteration will terminate below θ. If both mice are $\theta+1$-iterable, and we use successful strategies, then termination will not occur until we reach $i < \theta$ such that $M_i^0 \triangleleft M_i^1$ or $M_i^1 \triangleleft M_i^0$ ($M \triangleleft M'$ is defined as meaning $\bigvee \xi \leq \mathrm{On}_{M'}, M = M' || \xi$.) If $M_i^0 \triangleleft M_i^1$, we take this as making a statement about the original pair M^0, M^1 to the effect that M^1 contains at least as much information as M_0. However, we may have truncated on the man branch to M_i^1, in which case we have "thrown away" some of the information contained in M_1. If we also truncated on the main branch to M_0, it would be hard to see why the final result tell us anything about the original pair. We now show that, if M^0 and M^1 are both presolid, then this eventually cannot occur: If there is a truncation on the main branch of the M^1-side, there is no such truncation on the other side. (Hence no information was lost in passing from M^0 to M_i^0.) Moreover, we then have $M_i^0 \triangleleft M_1^1$.

Lemma 4.1.14. *Let $\theta > \omega$ be regular. Let $M^0, M^1 \in H_\theta$ be presolid premice which are normally $\theta+1$-iterable. Let:*

$$I^h = \langle \langle M_i^h \rangle, \langle \nu_i^h \rangle, \langle \pi_{ij}^h \rangle, T^h \rangle \ (h = 0, 1)$$

be the coiteration of length $i+1 < \theta$ by successful $\theta+1$ strategies S^0, S^1 (Hence $M_i^0 \triangleleft M_i^1$ or $M_i^1 \triangleleft M_i^0$.) Suppose that there is a truncation on the main branch of I^1. Then:

(a) $M_i^0 \triangleleft M_i^1$.

(b) *There is no truncation on the main branch of I^0.*

Proof. We first prove (a). Let $l_1 + 1 \leq i$ be the least point of truncation in T^1"$\{i\}$. Let $h_1 = T(l_1+1)$. Let $Q^1 = M_{l_1}^{1*}$. Then Q^1 is sound and solid. Let $\pi^1 = \pi_{h_1,i}^1$. By Lemma 4.1.12, M_i' is solid and $\pi^1(p_{Q^1}) = p_{M_i^1}$. Hence $Q^1 = \mathrm{core}(M_i^1)$ and π^1 is the core map. But $\pi^1 \neq \mathrm{id}$. Hence M_i^1 is not sound. If $M_i^0 \not\trianglelefteq M_i^1$, we would have: $M_i^1 = M_i^0 || \eta$ for an $\eta \in M_i^0$. But $M_i^0 || \eta$ is sound. Contradiction! This proves (a).

We now prove (b). Suppose not. Let $l_0 + 1$ be the last truncation point in $T^{00}\{i\}$. Let $h_0 = T^0(l_0+1)$. Let Q^0, π^0 be defined as before. Then $Q^0 = \mathrm{core}(M_i^0)$ and $\pi^0 \neq \mathrm{id}$ is the core map. Hence M_i^0 is not sound. Hence, as before, we have: $M_i^1 \triangleleft M_i^0$. Hence $M_i^0 = M_i^1$ and $Q = Q^0 = Q^1$ is the core of $M_i = M_i^0 = M_i^1$ with core map $\pi = \pi^0 = \pi^1$. Set:

$$F^h =: E_{\nu_{l_h}}^{M_{l_h}^h} \ (h = 0, 1).$$

It follows easily that there is κ defined by:

$$\kappa = \kappa_{l_h}^h = \mathrm{crit}(F^h) = \mathrm{crit}(\pi) \ (h = 0, 1)$$

Thus $\mathbb{P}(\kappa_\alpha) \cap M^h_{l_h} = \mathbb{P}(\kappa) \cap Q$. But:

$$\alpha \in F^h[X] \longleftrightarrow \alpha \in \pi(X)$$

for $X \in \mathbb{P}(\kappa) \cap Q, \alpha < \lambda_h = F^h(\kappa)$. Hence $l_0 \neq l_1$, since otherwise $\lambda_0 = \lambda_1$ and $F^0 = F^1$. Contradiction!, since ν_{l_h} is the first point fo difference. Now let e.g. $l_0 < l_1$. Then ν_{l_0} is regular in M^0_j for $l_0 < j \leq i$. But then it is regular in $M^1_{l_1}||\nu_{l_1}$, since $M^1_{l_1}||\nu_{l_1} = M^0_{l_1}||\nu_{l_1}$ and $\nu_{l_1} > \nu_{l_0}$.

But $F^0 = F^1|\lambda_{l_0}$ is a full extender. Hence $F^0 \in M_{l_1}||\lambda_{l_1}$ by the initial segment condition. But then $\tilde{\pi} \in M_{l_1}||\lambda_l$, where $\tilde{\pi}$ is the canonical extension of F^0. But $\tilde{\pi}$ maps $\overline{\sigma} = \kappa^{+Q}$ cofinally to ν_{l_0}. Hence ν_{l_0} is not regular in $M^1_{l_1}||\nu_{l_1}$. Contradiction!

Lemma 4.1.14

We remark in passing that:

Lemma 4.1.15. *Each J_α is solid.*

Proof. Suppose not. Let $M = J_\alpha, \nu \in a = p^i_M$, where $\rho^{i+1} \leq \nu < \rho^i$ in M. Let $M^\nu_a = J_{\overline{\alpha}}$ and let $\pi : J_{\overline{\alpha}} \longrightarrow J_\alpha$ be the canonical embedding. Then $\overline{\alpha} = \alpha$, since $J_{\overline{\alpha}} \notin J_\alpha$. Let $b = a \setminus (\nu + 1), \overline{b} = \overline{\pi}^{-1}(b)$. Set $\overline{a} = (a \cap \nu) \cup \overline{b}$. Then $\overline{a} \in P^i$ in M_i. But π"$(\overline{a}) = (a \cap \nu) \cup b <_* a$ where π is monotone. Hence $\overline{a} <_* a$. Hence $\overline{a} \notin P^i$ by the $<_*$-minimality of a. Contradiction!

QED(Lemma 4.1.15)

By virtually the same proof:

Lemma 4.1.16. *Let $M = J^A_\alpha$ be a constructible extension of J^A_β (i.e. $A \subset J^A_\beta$, where $\beta \leq \alpha$). Let $\rho^\omega_M \geq \beta$. Then M is solid.*

The solidity Theorem

We intend to prove:

Theorem 4.1.17. *Let M be a premouse which is fully $\omega_1 + 1$-iterable. Then M is solid.*

A consequence of this is:

Corollary 4.1.18. *Let M be a 1-small premouse which is normally $\omega_1 + 1$-iterable. Then M is solid.*

Proof. If M is restrained, then it has the minimal uniqueness property and is therefore fully ω_1+1-iterable by Theorem 3.6.1 amd Theorem 3.6.2. But if M is not restrained it is solid by Lemma 4.1.16.

QED(Corollary 4.1.18)

It will take a long time for us to prove Theorem 4.1.17. A first step is to notice that, if $M \in H_\kappa$, where $\kappa > \omega_1$ is regular and $\pi : H \prec H_\kappa$, with $\pi(\overline{M}) = M$, where H is transitive and countable, then M is solid iff $\overline{M}$ is solid, by absoluteness. Moreover, $\overline{M}$ is fully ω_1+1-iterable by Lemma 3.5.7. Hence it suffices to prove our Theorem under the assumption: M is countable. This assumption will turn out to be very useful, since we will employ the Neeman-Steel Lemma. It clearly suffices to prove:

(*) If M is presolid, then it is solid.

To see this, let M be unsolid and let η be least such that $M||\eta$ is not solid. Then $M||\eta$ is also fully ω_1+1-iterable and ν is also presolid. Hence $M||\eta$ is solid. Contradiction!

Now let N be presolid but not solid. Then there is a least $\lambda \in p^*_N$ such that $N^\lambda_a \notin N$, where $a = p^*_N$. Set: $M = N^\lambda_a$ and let $\sigma : M \longrightarrow_{\Sigma_1^{(n)}} N$, $\sigma \restriction \lambda = \mathrm{id}$ where $\rho^{n+1}_N \leq \lambda < \rho^n_N$ and $a \setminus (\lambda+1) \in \mathrm{rng}(\sigma)$. We would like to show: $M \in N$, thus getting a contradiction. How can we do this? A natural approach is to coiterate M with N. Let $\langle I^0, I^1\rangle$ be the coiteration, I^0 being the iteration of M. If we are lucky, it might turn out that $M_\mu \in N_\mu$, where μ is the terminal point of the coiteration. If we are ever luckier, it may turn out that no point below λ was moved in pairing from M to M_μ -i.e. $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. In this case it is easy to recover M from M_μ, so we have: $M \in N_\mu$, and there is some hope that $M \in N$. There are many "ifs" in this scenario, the most problematical being the assumption that $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. In an attempt to remedy this, we could instead do a "phalanx" iteration, iterating the pair $\langle N, M\rangle$ against M. If, at some $i < \mu$, we have $F = E^{M^0_i}_{\nu_i} \neq \varnothing$, we ask whether $\kappa^0_i < \lambda$. If so we apply F to N. Otherwise we apply it in the usual way to M_h, where h is least such that $\kappa^0_i < \lambda_h$. For the sake of simplicity we take: $N = M^0_0, M = M^0_1$. ν_i is only defined for $i \geq 1$. The tree of I^0 is then "double rooted", the two roots being 0 and 1. (In the normal iteration of a premouse, 0 is the single root, lying below every $i \geq 0$). Here, $i < \mu$ will be above 0 or 1, but not both.

If we are lucky it will turns out the final point μ lies above 1 in T^0. This will then ensure that $\mathrm{crit}(\pi^0_{0,\mu}) \geq \lambda$. It turns out that this -still improbable seeming- approach works. It is due to John Steel.

In the following section we develop the theory of Phalanxes.

4.2 Phalanx Iteration

In this section we develop the technical tools which we shall use in proving that fully iterable mice are solid. Our main concern in this book is with one small mice, which are known to be of type 1, if active. We shall therefore restrict ourselves here to structures which are of type 1 or 2. When we use the term "mouse" or "premouse", we mean a premouse M such that neither it nor any of its segments $M||\eta$ are of type 3.

We have hitherto used the word "iteration" to refer to the iteration of a single premouse M. Occasionally, however, we shall iterate not a single premouse, but rather an array of premice called a *phalanx*. We define:

By a *phalanx* of length $\eta + 1$ we mean:

$$\mathbb{M} = \langle \langle M_i : i \leq \eta \rangle, \langle \lambda_i : i < \eta \rangle \rangle$$

such that:

(a) M_i is a premouse $(i \leq \eta)$

(b) $\lambda_i \in M_i$ and $J_{\lambda_i}^{E^{M_i}} = J_{\lambda_i}^{E^{M_j}}$, $(i < j \leq \eta)$

(c) $\lambda_i < \lambda_j$ $(i < j < \eta)$

(d) $\lambda_i > \omega$ is a cardinal in M_j $(i < j \leq \eta)$.

A *normal iteration* of the phalanx $\mathbb{M}$ has the form

$$I = \langle \langle M_i : i < \mu \rangle, \langle \nu_i : i + 1 \in (\eta, \mu) \rangle, \langle \pi_{i,j} : i \leq_T j \rangle, T \rangle$$

where $\mu > \eta$ is the *length* of I. $\mathbb{M} = I|\eta + 1$ is the first segment of the iteration. Each $i \leq \eta$ is a minimal point in the tree T. As usual, η_i is chosen such that $\lambda_h < \lambda_i$ for $h < i$. If h is minimal such that $\kappa_i < \lambda_h$ then $h = T(i+1)$ and $E_{\nu_i}^{M_i}$ is applied to an apropiately defined $M_i^* = M_h||\gamma$. But here a problem arises. The natural definition of M_i^* is:

> $M_i^* = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$.

But is there such a γ? If λ_h is a limit cardinal in M_i, then $\tau_i < \lambda_h$ and hence λ_h is such a γ. For $i < \eta$ we have left the possibility open, however, that λ_h is a successor cardinal in M_i. We could then have: $\tau_i = \lambda_h$. In this case κ_i is the largest cardinal in $J_{\lambda_i}^{E^{M_h}}$. If $E_{\lambda_h} \neq \varnothing$ in M_h, it follows that $\rho^1_{M_h||\lambda_h} \leq \kappa_i < \tau_i$. Hence there is no γ with the desired property and M_i^* is undefined.

In practice, phalanxes are either defined with restrictions which prevent this eventuality, or -in the worst case- a more imaginative definition of M_i^* is applied. If $h = T(i+1)$ and M_i^* is given, then $M_{i+1}, T_{h,i+1}$ are, as usual, defined by:

$$\pi_{h,i+1} : M_i^* \longrightarrow^{(n)}_{E_{\nu_i}} M_{i+1},$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{M_i^*}$. In iterations of a single premouse, we were able to show that E_{ν_i} is always close to M_i^*, but there is no reason to expect this in arbitrary phalanx iterations.

We will not attempt to present a general theory of phalanxes, since in this section we use only phalanxes of length 2. We write $\langle N, M, \lambda\rangle$ as an abbreviation for the phalanx $\mathbb{M}$ of length 2 with $M_0 = N, M_1 = M$, and $\lambda_0 = \lambda$. We define:

Definition 4.2.1. The phalanx $\langle N, M, \lambda\rangle$ is *witnessed* (or verified) by σ iff the following hold:

(a) $\sigma : M \longrightarrow_{\Sigma_0^{(n)}} N$ for all $n < \omega$ such that $\lambda < \rho^n_M$

(b) $\lambda = \mathrm{crit}(\sigma)$

(c) σ is cardinal preserving and regularity preserving, i.e. if τ is a cardinal (regular) in M then $\sigma(\tau)$ is cardinal (regular) in N.

Note. (c) is superfluous if σ is Σ_1-preserving, since being a cardinal or regular is a Π_1 property.

Lemma 4.2.1. *Let $\langle N, M, \lambda\rangle$ be witnessed by σ. Then the following hold:*

(1) *Let $\alpha \in M$. Then α is a cardinal (regular) in M if and only if $\sigma(\alpha)$ is a cardinal (regular) in N.*

(2) *λ is regular in M.*

Proof. Suppose not. Then there is $f \in M$ such that $f\colon \gamma \longrightarrow \lambda$ and $\gamma < \lambda = \mathrm{lub}\, f''\gamma$. Hence $\sigma(\gamma) = \gamma$, $\sigma(f(\xi)) = f(\xi)$ for $\xi < \gamma$. Hence $\sigma(f) = f$ and $\sigma(\lambda) = \mathrm{lub}\, f''\gamma = \lambda$ in N. But $\sigma(\lambda) > \lambda$. Contradiction!

By acceptability it follows that:

(3) *If λ is a limit cardinal in M, then it is a limit cardinal in N. But if $\lambda = \gamma^+$ in M, then $\sigma(\lambda) = \gamma^+$ in N.*

Hence:

(4) $E^M_\lambda = \emptyset$.

Proof. This is trivial if λ is a limit cardinal in M. If $\lambda = \gamma^+$ in M, then $\rho^1_{M||\lambda} \leq \gamma$. Hence λ is not a cardinal in M. Contradiction! QED(4)

Hence:

(5) *Let $\kappa < \lambda$ be a cardinal in M. Set $\tau = \kappa^{+M}$. There is $\gamma \in N$ such that $\gamma > \tau$ and τ is a cardinal in $N||\gamma$.*

Proof. If $\tau < \lambda$, take $\lambda = \gamma$. Otherwise $\tau = \lambda$. But $E^N_\lambda = E^M_\lambda = \emptyset$ and λ is a cardinal in M. Hence $M||\lambda+\omega = N||\lambda+\omega = J^{E^M_\lambda}_{\lambda+\omega}$ and the assertion holds with $\gamma = \lambda + \omega$.

QED(Lemma 4.2.1)

Note. It will follow from (5) that if $h = T(i+1)$ is a normal iteration of $\langle N, M, \lambda\rangle$, then M^*_i is defined.

Following our earlier sketch, we define:

Definition 4.2.2. Let $\langle N, M, \lambda\rangle$ be a phalanx which is witnessed by σ. By a *normal iteration* of $\langle N, M, \lambda\rangle$ of length $\eta \geq 2$ we mean:

$$I = \langle\langle M_i : i < \mu\rangle, \langle\nu_i : i+1 \in (\eta, \mu)\rangle, \langle\pi_{i,j} : i \leq_T j\rangle, T\rangle$$

such that:

(a) T is a tree on η with $iTj \longrightarrow i < j$. Moreover $T"\{0\} = T"\{1\} = \varnothing$.

(b) M_i is a premouse for $i < \eta$. Moreover $M_0 = N, M_1 = N$.

(c) If $1 \leq i, i+1 < \eta$, then $M_i||\nu_i = \langle J^E_{\nu_i}, E_{\nu_i}\rangle$ with $E_{\nu_i} \neq \varnothing$. We define $\kappa_i, \tau_i, \lambda_i$ as usual. We also set: $\lambda_0 = \lambda$. We require: $\nu_i > \nu_h$ if $1 \leq h < i$ and $\lambda_h > \lambda$. (Hence $\lambda_i > \lambda_h$ for $h < i$).

(d) Let $i > 0$. Let h be least such that $h = i$ or $h < i$ and $\kappa_i < \lambda_h$. Then $h = T(i+1)$ and $J^{E^{M_h}}_{\tau_i} = J^{E^{M_i}}_{\tau_i}$.

(e) $\pi_{i,j}$ is a partial map of M_i to M_j for $i \leq_T j$. Moreover $\pi_{i,i} = \mathrm{id}$, $\pi_{i,j}\pi_{h,i} = \pi_{h,j}$.

(f) Let $h = T(i+1)$. Set: $M_i^* = M_h||\gamma$, where $\gamma \leq \mathrm{On}_{M_h}$ is maximal such that $\tau_i < \gamma$ is a cardinal in $M_h||\gamma$. (We call it a *drop point* in I if $M_i^* \neq M_k$). Then:

$$\pi_{h,i+1} : M_i^* \longrightarrow^{(n)}_{E_{\nu_i}} M_{i'+1},\ \text{where } n \leq \omega \text{ is maximal s.t.}$$
$$\lambda_h \leq \rho^n_{M_i^*} (\text{where } \lambda_0 = \lambda)$$

(g) If $i \leq_T j$ and $(i, j]_T$ has no drop point, then π_{ij} is a total function on M_i.

(h) Let $\mu < \eta$ be a limit ordinal. Then $T"\mu$ is a club in μ and contains at most finitely many drop points. Moreover, if $i < \mu$ and $(i, \mu)_T$ is drop free, then:

$$M_\mu,\ \langle \pi_{j,\mu} : i \leq_T j <_T \mu \rangle$$

is the transitivized direct limit of

$$\langle M_j : i \leq_T j \leq_T \mu \rangle, \langle \pi_{j,k} : i \leq_T j \leq_T k <_T \mu \rangle.$$

As usual we call M_μ, $\langle \pi_{j,\mu} : j <_T \mu \rangle$ the limit of $\langle M_i : i <_T \mu \rangle$, $\langle \pi_{j,k} : i \leq_T j \leq_T k <_T \mu \rangle$, since the missing points are given by:

$$\pi_{h,j} = \pi_{i,j}\pi_{h,i} \text{ for } h <_T i \leq_T j <_T \mu \rangle$$

This completes the definition. Note that the existence of M_i^* is guaranteed by Lemma 4.2.1(5). We define:

Definition 4.2.3. $i + 1$ is an anomaly in I if $i > 0$ and $\tau_i = \lambda$ (hence $0 = T(i+1)$).

Anomalies will cause us some problems. Just as in the case of ordinary normal iterations, we can extend an iteration of length $\eta + 1$ to a *potential iteration* of length $\eta + 2$ by appointing ν_η such that:

$$E^{M_\eta}_{\nu_\eta} \neq \varnothing, : \nu_\eta > \nu_i \text{ for } i \leq i < \eta, \lambda_\eta > \lambda.$$

This determines M_η^*. In ordinary iterations we know that E_{ν_η} is close to M_η^*. In the present situation this may fail, however, if $\eta + 1$ is an anomaly. We, nonetheless, get the following analogue of Theorem 3.4.4:

Theorem 4.2.2. *Let I be a potential normal iteration of $\langle N, M, \lambda \rangle$ of length $i + 1$. If $i + 1$ is not an anomaly, then $E^{M_i}_{\nu_i}$ is close to M_i^*. If $i + 1$ is an anomaly, then $E^{M_i}_{\nu_i,\alpha} \in N$ for $\alpha < \lambda_0$.*

We essentially repeat our earlier proof (but with one additional step). We show that if $A \subset \tau_i$ is $\underline{\Sigma}_1(M_i||\nu_i)$, then it is $\underline{\Sigma}_1(M_i^*)$ if $i+1$ is not an anomaly, and otherwise $A \in N$. Let I be a counterexample of length $i+1$ where i is chosen minimally. Let $h = T(i+1)$. Let $A \subset \tau_i$ be a counterexample. Then:

(1) $h < i$.

We then prove:

(2) $\nu_i = \mathrm{On}_{M_i}, \rho^1_{M_i} \leq \tau_i$.

The first equation is proven exactly as before. The second follows as before if $i+1$ is not an anomaly, since then $\tau_i < \lambda_h$. Now let $i+1$ be an anomaly. Assume $\rho^1_{M_i} > \tau_i$ and let $A \subset \tau_i$ be $\underline{\Sigma}(M_i)$. Then $A \in M_1$, since either $i = 1$ or $A \in J^{E^{M_i}}_{\lambda_1} = J^{E^{M_1}}_{\lambda_1}$ where λ_1 is a cardinal in M_i. Hence $A = \sigma(A) \cap \lambda \in N$. Contradiction!

QED(2)

In an extra step we then prove:

Claim. $i > 1$.

Proof. Suppose not. Then $i = 1$ and $h = 0$. Let:

$$\pi : J^E_{\tau_1} \longrightarrow J^E_{\nu_1},\ \pi' : J^{E'}_{\tau'_1} \longrightarrow J^{E'}_{\nu'_1}$$

be the extensions of M, N respectively. Then π, π' are cofinal and $\sigma\pi = \pi'\sigma$. If $\tau_1 < \lambda$ then $\sigma \upharpoonright \tau_1 + 1 = \mathrm{id}$ and σ takes M cofinally to N. Hence σ in Σ_1-preserving. If A is $\Sigma_1(M)$ in p, then A is also $\Sigma_1(N)$ in $\sigma(p)$, where $N = M_1^*$. Contradiction!

Now let $\tau_1 = \lambda$. Then $i+1$ is an anomaly. Then σ takes ν_1, non cofinally to ν'_1, since $\pi'(\lambda) > \pi(\xi) = \sigma\pi(\xi)$ for $\xi < \lambda$. Let $\tilde{\nu} =: \sup \sigma"\nu_1$. Then:

$$\sigma : M \longrightarrow_{\Sigma_1} \tilde{M} \text{ cofinally,}$$

where $\tilde{M} = \langle J^{E'}_{\tilde{\nu}}, E'_{\nu'_1} \cap J^{E'}_{\tilde{\nu}} \rangle$. Let A' be $\Sigma_1(\tilde{M})$ in $\sigma(p)$ by the same definition as A in p. Then $A' \in N$ and $A = A' \cap \lambda \in N$. Contradiction!

QED(Claim)

(3) i is not a limit ordinal.

Proof. Suppose not. Then as before, we can pick $l <_T i$ such that $\pi_{l,i}$ is a total function on M_l and $l > h$. Hence $\pi_{l,i}$ is Σ_1-preserving. Let $M_i = \langle J^E_{\nu_i}, F \rangle$. We can also pick l big enough that $p \in \mathrm{rng}(\pi_{l,i})$, where A is $\Sigma_1(M_i)$ in p. Hence $A \in \underline{\Sigma}_1(M_l)$, where $M_l = \langle J^{\tilde{E}}_{\tilde{\nu}}, \tilde{F} \rangle$, where $\tilde{\nu} = \mathrm{On}_{M_l} \geq \nu_l$. Extend $I|l+1$ to a potential iteration I' of length $l+2$ by setting: $\nu'_l = \tilde{\nu}$. Since $l > h$, it follows easily that:

$$\kappa'_l = \kappa_i, \tau'_l = \tau_i, h = T'(l+1), M_i^* = M_l'^*.$$

By the minimality of i it follows that $A \in \Sigma_1(M_l^*)$ if $i+1$ is not an anomaly and otherwise $A \in N$. Contradiction!

QED(3)

We then let: $i = j+1, \xi = \tau(i)$. By the claim we have: $j \leq 1$.

But:

$$\pi_{\xi,i} : M_j^* \longrightarrow^{(n)}_{E^{M_i}_{\nu_j}} M_i = \langle J^E_{\nu_i}, E_{\nu_i} \rangle.$$

If $n = 0$, this map is cofinal. Hence in any case $\pi_{\xi,i}$ is Σ_1-preserving. Hence:

(4) $M_j^* = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}} \rangle$ where $\overline{E}_{\overline{\nu}} \neq \varnothing$.

Hence:

(5) $\tau_i < \kappa_j$.

Proof. $\kappa_i < \lambda_h \leq \lambda_j$ where λ_j is inaccessible in M_i (since $j \geq 1$). Hence $\tau_i < \lambda_j$. Moreover, $\kappa_i, \tau_i \in \mathrm{rng}(\pi_{\xi,i})$ by (4). But:

$$\mathrm{rng}(\pi_{\xi,i}) \cap [\lambda_j, \lambda_j) = \varnothing.$$

QED(5)

Exactly as before we get:

(6) $\pi_{\xi,i} : M_j^* \longrightarrow_{E_{\nu_j}} M_i$ is a Σ_0 ultrapower. But then:

(7) i is not an anomaly.

Proof. Let $A \subset \tau_i$ be $\Sigma_1(M_i)$ in the parameter p. By (6) we have: $p = \pi_{\xi,i}(f)(\alpha)$, where $f \in M_j^*, \alpha < \lambda_j$.

Then:

$$A(\zeta) \longleftrightarrow \bigvee u \in M_j^* \bigvee y \in \pi_{\zeta,i}(u) A'(y, \zeta, p)$$

But then:

$$A(\zeta) \longleftrightarrow \bigvee u \in M_j^* \{\gamma < \kappa_j : \overline{A}'(y, \zeta, f(\gamma))\} \in (E_{\nu_j})_\alpha.$$

But since $j < i$ and $j+1$ is an anomaly, we have by the minimality of i that $(E_{\nu_j})_\alpha \in N$. Hence $A \in N$. Contradiction!

QED(7)

Since $j+1$ is not an anomaly, we have $(E_{\nu_j})_\alpha \in \underline{\Sigma}_1(M_j^*)$. Hence $A \in \underline{\Sigma}_1(M_j^*)$. Hence we have shown:

(8) $\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(M_j^*)$.

We know that $M_j^* = M_\xi || \overline{\nu} = \langle J^{\overline{E}}_{\overline{\nu}}, \overline{E}_{\overline{\nu}} \rangle$. Moreover, $\overline{\nu} > \nu_l$ for $l < \xi$, since $\lambda_l \leq \kappa_j < \lambda_{\overline{\xi}} < \overline{\nu}$; hence $\nu_l < \lambda_\xi < \overline{\nu}$. Thus we can extend $I|\xi+1$

to a potential iteration I' of length $\xi+2$ by setting: $\nu'_\xi = \overline{\nu}$. Since $\tau_i < \kappa_j$, we then have: $\kappa_i = \kappa'_\xi, \tau_i = \tau'_\xi$. Hence:

$$h = T(i+1) = T'(\xi+1) \text{ and } M_i^* = (M_\xi^*)'.$$

Suppose that $i+1$ is not an anomaly in I. Then neither is $\xi+1$ in I'. By the minimality of i we conclude:

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_\xi||\overline{\nu}) \subset \underline{\Sigma}_1(M_i^*)$$

where $M_\xi||\overline{\nu} = M_j^*$. Hence by (8):

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \underline{\Sigma}_1(M_i^*).$$

Contradiction!

Now let $i+1$ be an anomaly. Then so is $\xi+1$ in I'. But then just as before:

$$\mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_i) \subset \mathbb{P}(\tau_i) \cap \underline{\Sigma}_1(M_\xi||\overline{\nu}) \subset N.$$

Contradiction! QED(Theorem 4.2.2)

We now prove:

Lemma 4.2.3. *Let $h = T(i+1)$ in I, where I is a normal iteration of $\langle N, M, \lambda\rangle$. Then:*

$$\pi_{h,i+1} : M_i^* \longrightarrow_{\Sigma^*} M_{i+1} \textit{ strongly.}$$

Proof. If $i+1$ is not an anomaly, then $E_{\nu_i}^{M_i}$ is close to M_i^* and the result is immediate. Now let $i+1$ be an anomaly. Then $h = 0, M_i^* = N||\eta$ for an $\eta < \tau'_i = \sigma(\lambda)$, since $\tau_i = \lambda$. $\rho^\omega_{M_i^*} \leq \kappa_i$, since τ_i is not a cardinal in $N|\eta+\omega = J^{E^N}_{\eta+\omega}$. But then $\rho^\omega_{M_i^*} = \kappa_i$, since κ_i is a cardinal in N. Let $\rho^n_{M_i^*} > \kappa_i \geq \rho^{n+1}_{M_i^*}$, where $n < \omega$. Let $\pi = \pi_{h,i+1}$. Since M_{i+1} is the $\Sigma_0^{(n)}$ ultrapower of M_i^*, we know:

$$\pi\text{"}\rho^n_{M_i^*} \subset \rho^n_{M_{i+1}^*} \text{ and } \pi(\rho^j_{M_i^*}) = \rho^j_{M_{i+1}} \text{ for } j < n.$$

Since E_{ν_i} is weakly amenable, Lemma 3.2.16 gives us:

(1) $\sup \pi\text{"}\rho^n_{M_i^*} = \rho^n_{M_{i+1}}$ and π is $\Sigma_1^{(n)}$-preserving.

We now prove:

(2) Let $H =: |J_{\nu_i}^{E^{M_i}}| = |J_{\nu_i}^{E^{M_{i+1}}}|$. Then $\mathbb{P}(H) \cap \Sigma_1^{(n)}(M_{i+1}) \subset N$.

Proof. Let B be $\Sigma_1^{(n)}(M_{i+1})$ in q such that $B \subset H$. Let $q = \pi(f)(\alpha)$ where $f \in \Gamma^*(\kappa_i, M_i^*), \alpha < \lambda_i$. Let:

$$B(x) \longleftrightarrow \bigvee y \in H^n_{M_{i+1}} B'(y, x, q)$$

where B' in $\Sigma_0^{(n)}(M_{i+1})$. Let $\overline{B}'$ be $\Sigma_0^{(n)}(M_i^*)$ by the same definition. Then:

$$\begin{aligned} B(x) &\longleftrightarrow \bigvee u \in H^n_{M_i^*} \bigvee y \in \pi(u) B'(y, x, \pi(f)(\alpha)) \\ &\longleftrightarrow \bigvee u \in H^n_{M_i^*} \{\gamma < \kappa_i : \bigvee y \in u\, \overline{B}'(y, x, f(\gamma))\} \in (E_{\nu_i}^{M_i})_\alpha \end{aligned}$$

But $(E_{\nu_i}^{M_i})_\alpha \in N$. Hence $B \in N$.

QED(2)

Clearly, if $A \subset H$ is $\underline{\Sigma^*}(M_{i+1})$, then it is $\underline{\Sigma}_\omega(\langle H, B \rangle)$ where B is $\underline{\Sigma}_1^{(n)}(M_{i+1})$. Hence $A \in N$ and $\langle H, A \rangle$ is amenable, since $H = J_{\kappa_i}^{E^{M_i^*}} = J_{\kappa_i}^{E^N}$, and κ_i is regular in N. But then $\rho^\omega_{M_{i+1}} = \rho^\omega_{M_i^*} = \kappa_i$. It follows that:

(3) π is Σ^*-preserving.

Proof. By induction on j we show that if $R(\vec{x}^j, \vec{z})$ is $\Sigma_1^{(i)}(M_i^*)$ and $R'(\vec{x}^j, \vec{z})$ are $\Sigma_1^j(M_{i+1})$ by the same definition (where $\vec{z} = z_1^{h_1}, \ldots, z_m^{h_m}$ with $h_1, \ldots h_m < j$), then:

$$R(\vec{x}, \vec{z}) \longleftrightarrow R'(\pi(\vec{x}), \pi(\vec{z})).$$

For $j \leq n$ this holds by (1). Now let it hold for $j = m \geq n$. We show that it holds for $j = m + 1$. Then:

$$R(\vec{x}, \vec{z}) \longleftrightarrow H_{\vec{z}} \models \varphi[\vec{x}]$$

where φ is Σ_1 and:

$$H_{\vec{z}} = \langle H, \overline{Q}^1_{\vec{z}}, \ldots, \overline{Q}^P_{\vec{z}} \rangle$$

where $Q^l(\vec{w}, \vec{z})$ is $\Sigma_1^{(m)}(M_i^*)$ and:

$$\overline{Q}^l = \{\langle \vec{w} \rangle \in H : Q^l(\vec{w}, \vec{z})\} \text{ for } l = 1, \ldots, p.$$

Now let Q' be $\Sigma_1^{(m)}(M_{i+1})$ by the same definition and let $H'_{\vec{x}}$ be defined like $H_{\vec{x}}$ with $Q^{l'}$ in place of Q^l $(l = 1, \ldots, p)$. By the induction hypothesis we then have:

$$\begin{aligned} R(\vec{x}, \vec{z}) &\longleftrightarrow H_{\vec{z}} \models \varphi(\vec{x}) \\ &\longleftrightarrow H_{\pi(\vec{z})} \models \varphi(\vec{x}) \\ &\longleftrightarrow R'(\vec{x}, \pi(\vec{z})) \longleftrightarrow R'(\pi(\vec{x}), \pi(\vec{z})) \end{aligned}$$

since $\pi(\vec{x}) = \vec{x}$.

QED(3)

But this embedding π is also strong, since if $\rho^{m+1} = \kappa$ and A confirms $a \in P^m$ in M_i^*, then if A' is $\Sigma_{i+1}^{(m)}$ in $\pi(a)$ by the same definition, we have: $A \cap H = A' \cap H$, where $M_i^* \cap \mathbb{P}(H) = M_{i+1} \cap \mathbb{P}(H)$. Hence $A' \cap H \notin M_{i+1}$.

QED(Lemma 4.2.3)

But then:

Lemma 4.2.4. *Let $h = T(i+1)$, where $i+1 \leq_T j$ and $(i+1, j]$ has no drop point. Then:*

$$\pi_{h,j} : M_i^* \longrightarrow_{\Sigma^*} M_j \text{ strongly.}$$

Proof. By Lemma 3.2.27 and Lemma 3.2.28.

QED(Lemma 4.2.4)

Exactly as in Corollary 4.1.12, we conclude that if M_i^* is solid and $i = j+1$, then so is M_j and $\pi(p_i^m) = p_j^m$ for $m < \omega$.

We intend to do comparison iterations in which $\langle N, M, \lambda\rangle$ is coiterated with a premouse. For this we shall again need padded iteration. Our definition of a normal iteration of $\langle N, M, \lambda\rangle$ encompassed only strict iteration, but we can easily change that:

Definition 4.2.4. Let $\langle N, M, \lambda\rangle$ be a phalanx which is witnessed by σ. By a *padded normal iteration* of $\langle N, M, \lambda\rangle$ *of length* $\mu \geq 1$ we mean:

$$I = \langle\langle M_i : i < \mu\rangle, \langle \nu_i : i \in A\rangle, \langle \pi_{i,j} : i \leq_T j\rangle, T\rangle.$$

Where:

(1) $A = \{i :<\leq i+1 < \mu\}$ is the set of *active points*.

(2) (a)-(b) of the previous definition hold. However (f), (d) require that $i \in A$. Moreover:

  (i) Let $1 \leq h < j < \mu$ such that $[h, j) \cap A = \varnothing$. Then:
  - $h <_T j, M_h = M_j, \pi_{h,j} = \mathrm{id}$.
  - $i \leq h \longrightarrow (i \leq_T h \longleftrightarrow i <_T j)$ for $i < \mu$.
  - $j \leq i \longrightarrow (j \leq_T i \longleftrightarrow h <_T i)$ for $i < \mu$.

    (In particular, if $2 \leq i+1 < \mu, i \notin A$. Then $i = T(i+1), M_i = M_{i+1}, \pi_{i,i+1} = \mathrm{id}$).

Note. 0 plays a special role, behaving like an active point in that λ_0 exists, but ν_0 does not exist.

Our previous results go through *mutatis mutandis*. We shall say more about that later.

Definition 4.2.5. Let M^0 be a premouse and $M^1 = \langle M, N, \lambda\rangle$ a phalanx iteration witnessed by σ. By a *coiteration* of M^0, M^1 of length $\mu \geq 1$ with *coindices* $\langle \nu_i : 1 \leq i < \mu\rangle$ we mean a pair $\langle I^0, I^1\rangle$ such that:

(a) $I^h = \langle\langle M_i^h\rangle, \langle \nu_i^h : i \in A^h\rangle, \langle \pi_{i,j}^h\rangle, T^h\rangle$ is a padded normal iteration of M^h $(h = 0, 1)$.

(b) $M_0^0 = M_1^0$.

(c) $\nu_i =$ the least ν such that $E_\nu^{M_i^0} \neq E_\nu^{M_i^1}$.

(d) If $E_{\nu_i}^{M_i^n} \neq \varnothing$, then $i \in A^h$ and $\nu_i^h = \nu_j$. Otherwise $i \notin A_i^h$.

Note. We always have $M_0^0 = M_1^0$ whereas: $M_0^1 = N, M_1^1 = M$.

Definition 4.2.6. Let $M^0, M^1 \in H_\kappa$, where $\kappa > \omega$ is regular. Let S^h be a successful iteration strategy for M^h $(h = 0, 1)$. The *$\langle S^0, S^1\rangle$-coiteration of* length $\mu \leq \kappa + 1$ with coindices $\langle \nu_i : 1 \leq i < \mu\rangle$ is the coiteration $\langle I^0, I^1\rangle$ such that:

- I^h is S^h-conforming.
- Either $\mu = \kappa + 1$ or $\mu = i + 1 < \kappa$ and ν_i does not exist (i.e. $M_1^0 \lhd M_i^1$ or $M_0^1 \lhd M_i^0$).

  Note that $\lhd$ was defined by:

$$P \lhd Q \longleftrightarrow P = Q || \mathrm{On}_P$$

We leave it to the reader to show that the coiteration exists. This is spelled out in §3.5 for coiteration of premice. We obtain the following analogue of Lemma 3.5.1:

Lemma 4.2.5. *The coiteration of $M : M^1$ terminates below κ_1.*

The proof is virtually unchanged. We leave the details to the reader. Using Lemma 4.2.4, we get the following analogue of Lemma 4.1.14:

Lemma 4.2.6. *Let N, M^0 be presolid. (Hence M^1 is presolid). Let $\langle I^0, I^1\rangle$ be the coiteration of M^0, M^1 terminating at $j < \kappa$. Suppose there is a drop on the main branch of I^h. Then there is no drop on the main branch of I^{i-h}. Moreover, $M_i^{i-h} \lhd M_i^h$.*

The proof is virtually the same.

At the end of §4.1 we sketched an approach to proving that fully iterable mice are solid. The basic idea was to coiterate $\langle N, M, \lambda\rangle$ with N, where N is fully iterable and σ witnesses $\langle N, M, \lambda\rangle$. In order to do this, we must know that $\langle N, M, \lambda\rangle$ is normally iterable. (The notions "iteration strategy", "successful iteration strategy" and "iterability" are defined in the obvious way for phalanxes $\langle N, M, \lambda\rangle$. We leave this to the reader.) We prove:

Lemma 4.2.7. *If $\langle N, M, \lambda\rangle$ is witnessed by σ and N is normally iterable, then $\langle N, M, \lambda\rangle$ is normally iterable.*

For the sake of simplicity we shall first prove this under a *special assumption*, which eliminates the possibility of anomalies:

(SA) λ is a limit cardinal in M.

Later we shall prove it without SA.

In §3.4.5 we showed that if $\sigma : M \longrightarrow_{\Sigma^*} N$ and N is normally iterable, then M is normally iterable. Given a successful iteration strategy for N, we defined a successful strategy for M, based on the principle of *copying* the iteration of M onto N. In this case, we "copy" an iteration of $\langle N, M, \lambda\rangle$ onto an iteration of N. It suffices to prove it for strict iterations. Let

$$I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$$

be a strict normal iteration of $\langle N, M, \sigma\rangle$. Its copy will be an iteration of N:

$$I' = \langle\langle N_i\rangle, \langle\nu_i'\rangle, \langle\pi_{ij}'\rangle, T'\rangle$$

of the same length. We will have $N_0 = N_1 = N$. (Thus I' is a padded iteration, even if I is not). There will be *copying maps* $\sigma_i (i < \mathrm{lh}(I))$ with:

$$\sigma_i : M_i \longrightarrow N_i, \sigma_0 = \mathrm{id} \upharpoonright N, \sigma_1 = \sigma.$$

We shall have $\nu_i' \cong \sigma_i(\nu_i)$ for $1 \le i$. The tree T was "double rooted" with 0, 1 as its two initial points, T', on the other hand, has the sole initial point 0. We can define T' from T by:

$$iT'j \longleftrightarrow (iTj \ \vee \ i < 2 \le j)$$

In I each point $i < \mu$ has a unique origin $h \in \{0,1\}$ such that $h \leq_T i$. Denote this by: $\mathrm{or}(i)$. Using the function or we can define T from T' by:

$$iTj \longleftrightarrow (iTj \wedge \mathrm{or}(i) = \mathrm{or}(j))$$

Thus, each infinite branch b' in I' uniquely determines an infinite branch b in I defined by:

$$b = \bigcup_{i \in b' \setminus 2} \{\mathrm{or}(i), i\}$$

However, we cannot expect the copying map to always be Σ^*-preserving, since $\sigma_1 = \sigma$ is assumed to be $\Sigma_0^{(n)}$-preserving only for $\rho_M^n > \lambda$. In this connection it is useful to define:

$$\mathrm{depth}(M, \lambda) =: \text{ the maximal } n \leq \omega \text{ s.t. } \rho_M^n > \lambda.$$

Modifying our definition of "copy" in §3.4.5 appropiately we now define:

Definition 4.2.7. Let $\langle N, M, \lambda \rangle$ be witnessed by σ. Let

$$I = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi_{ij} \rangle, T \rangle$$

be a normal iteration of $\langle N, M, \lambda \rangle$ of length μ. Let:

$$I' = \langle \langle N_i \rangle, \langle \nu_i' \rangle, \langle \pi_{ij}' \rangle, T' \rangle$$

be a normal iteration of N of the same length. I' is a *copy of I onto N* with *copying maps* $\sigma_i (i < \mu)$ iff the following hold:

(a) $\sigma_i : M_i \longrightarrow_{\Sigma_*} N_i, \sigma_0 = \mathrm{id} \upharpoonright N, \sigma_1 = \sigma, N_0 = N_1 = N$.

(b) $iT'j \longleftrightarrow (iTj \vee i < 2 \leq j)$

(c) $\sigma_i \upharpoonright \lambda_h = \sigma_h \upharpoonright \lambda_h$ for $h \leq i < \mu$

(d) $\sigma_i \pi_{hi} = \pi_{hi}' \sigma_h$ for $i \leq_T h$.

(e) $\nu_i' \cong \sigma_i(\nu_i)$

(f) Let $1 \leq_T i$. If $(1, i]_T$ has no drop point in I, then σ_i is $\Sigma_0^{(n)}$-preserving for all n such that $\lambda \leq \rho_M^n$. If $(1, i]_T$ has a drop point in I. Then σ_i is Σ^*-preserving.

(g) If $0 \leq_T i$ then σ_i is Σ^*-preserving.

Note: $N_0 = N_1$, since $0 \notin A$.

Our notion of copy is very close to that defined in §3.4.5. The main difference is that σ_i need not always be Σ^*-preserving. Nonetheless we can imitate the theory developed in §3.4.5. Lemma 3.4.14 holds literally as before. In interpreting the statement, however, we must keep in mind that if $i \in A$ and $T(i+1) = 0$, then $T'(i+1) = 1$. In this case $\tau_i < \lambda$ is a cardinal in N. Hence $M_i^* = N$. Moreover $\tau_i' = \sigma(\tau_i) = \tau_i$. Hence τ_i' is a cardinal in $N^* = N$ and $N_i^* = N$. In all other cases $T'(i+1) = T(i+1)$. Clearly $\pi_{0j}' = \pi_{ij}'$ for all $j \geq 1$. Lemma 3.4.14 then becomes:

Lemma 4.2.8. *Let $I, I', \langle \sigma_i : i < \mu \rangle$ be as in the above definition. Let $h = T(i+1)$. Then:*

(i) *If $i+1$ is a drop point in I, then it is a drop point in I' and $N_i^* = \sigma_h(M_i^*)$.*

(ii) *If $i+1$ is not a drop point in I, then it is not a drop point in I' and $N_i^* = N_h$.*

(iii) *If $F = E_{\nu_i}^{M_i}, F' = E_{\nu_i'}^{N_i}$. Then:*

$$\langle \sigma_h \restriction M_i^*, \sigma_i \restriction \lambda_i \rangle : \langle M_i^*, F \rangle \longrightarrow \langle N_i^*, F' \rangle$$

(iv) $\sigma_{i+1}(\pi_{h,i+1}(f)(\alpha)) = \pi_{h,i+1}' \sigma_h(f)(\sigma_i(\alpha))$ *for $f \in \Gamma^*(\kappa_i, M_i^*), \alpha < \lambda_i$.*

(v) $\sigma_j(\nu_i) \cong \nu_i'$ *for $j > i$.*

(vi) σ_i *is cardinal preserving.*

Note. In the general case, where anomalies can occur, Lemma 3.4.14 will not translate as easily.

Proof. In §3.4.5 we proved this under the assumption that each σ_i is Σ^*-preserving. We must now show that the weaker degree of preservation which we have posited suffices. The proof of (i)-(ii) are virtually unchanged. We now show that Σ_0-preservation is sufficient to prove (iii). Set: $\overline{M} = M_i || \nu_i, \overline{N} = N_i || \nu_i'$. Then $\sigma_i \restriction \overline{M}$ is a Σ_0 preserving map to $\overline{N}$. Let $\alpha < \lambda, X \in \mathbb{P}(\kappa_i) \cap \overline{M}$. The statement $\alpha \in F(X)$ is uniformly $\Sigma_1(\overline{M})$ in α, X. But it is also $\Pi_1(\overline{M})$ since:

$$\alpha \in F(X) \longleftrightarrow \alpha \notin F(\kappa_i \setminus X)$$

Hence:

$$\alpha \in F(X) \longleftrightarrow \sigma(\alpha) \in F'(\sigma(X))$$

by Σ_0-preservation. Finally we note that $\sigma_i \restriction (M_i \restriction \lambda_i)$ embeds $M_i || \lambda_i$ elementarily into $\sigma_i(M_i || \lambda_i) = N_i || \lambda_i'$. Hence:

$$\sigma_i(\prec \vec{\alpha} \succ) = \prec \sigma_i(\vec{\alpha}) \succ \text{ for } \alpha_1, \ldots, \alpha_n < \lambda_i.$$

Thus all goes through as before, which proves (iii).

In our previous proof of (iv) we need that $\sigma_h \upharpoonright M_i^*$ is Σ^*-preserving. This can fail if $1 \leq_T h$ and $[1, h]_T$ has no drop point. But then σ_h is $\Sigma_0^{(n)}$-preserving for $\lambda < \rho^M$ in M, where $\lambda \leq \kappa_i$. Hence the preservation is sufficient. Finally, (v) is proven exactly as before.

(vi) is clear if σ_i is Σ_1-preserving. If not, then $1 \leq i$ and $(1, i]$ has no drop. Hence $\pi_{1,i}$ is cofinal, since only Σ_0-ultraproducts were involved. If α is a cardinal in M_i, then $\alpha \leq \beta$ for a β which is a cardinal in M. By acceptability it suffices to note that $\sigma_i \pi_{1i}(\beta) = \pi'_{1i}\sigma(\beta)$ is a cardinal in N_i.

QED(Lemma 4.2.8)

Exactly as before we get the analogue of Lemma 3.4.15:

Lemma 4.2.9. *There is at most one copy I' of I induced by σ. Moreover, the copy maps are unique.*

As before we define:

Definition 4.2.8. Let $\langle N, M, \lambda\rangle$ be a phalanx witnessed by σ. $\langle I, I', \langle\sigma\rangle\rangle$ is a *duplication induced by* σ iff I is a normal iteration of $\langle N, M, \lambda\rangle$ and I' is the copy of I induced by σ with copy maps $\langle \sigma_i : i < \mu\rangle$.

We also define:

Definition 4.2.9. $\langle I, I', \langle \sigma_i : i \leq \mu\rangle\rangle$ is a *potential duplication of length* $\mu + 2$ *induced by* σ iff:

- $\langle I|\mu + 1, I'|\mu + 1, \langle \sigma_i : i \leq \mu\rangle\rangle$ is a duplication of length $\mu + 1$ induced by σ.
- I is a potential iteration of length $\mu + 2$.
- I' is a potential iteration of length $\mu + 2$.
- $\sigma_\mu(\nu_\mu) = \nu'_\mu$.

To say that an actual duplication of length $\mu + 2$ is the *realization* of a potential duplication means the obvious thing. If it exists, we call the potential duplication *realizable*.

Our analogue of Theorem 3.4.16 is somewhat more complex. We define:

Definition 4.2.10. i is an exceptional point ($i \in \mathrm{EX}$) iff:

$$1 \leq_T i, (1, i]_T \text{ has no drop point, and } \rho^1 \leq \lambda \text{ in } M.$$

Note. Suppose $\rho^1 \leq \lambda$ in M. For $j \in \mathrm{EX}$ we have: $\rho^1_{M_j} \leq \lambda$, as can be seen by induction on j.

Our analogue of Theorem 3.4.16 reads:

Lemma 4.2.10. *Let $\langle I, I', \langle \sigma_i \rangle \rangle$ be a potential duplication of length $i+2$, where $h = T(i+1)$. Suppose that $i+1 \notin EX$. Then:*

$$\langle \sigma_h \restriction M_i^*, \sigma_i \restriction \lambda_i \rangle : \langle M_i^*, F \rangle \longrightarrow^* \langle N_i^*, F' \rangle$$

where $F = E^{M_i}_{\nu_i}, F' = E^{N_i}_{\nu'_i}$.

Before proving this we note some of its consequences. Just as in §3.4.5 it provides exact criteria for determining whether the copying process can be carried one step further. We have the following analogue of Lemma 3.4.17:

Lemma 4.2.11. *Let $\langle I, I', \langle \sigma_i : i \leq \mu \rangle \rangle$ be a potential duplication of length $\mu + 2$ (where $\mu \geq 1$). It is realizable iff N^*_μ is $*$-extendible by $E^{N_\mu}_{\nu'_\mu}$.*

Proof. If N^ν_μ is not $*$-extendable, then no realization can exist, so suppose that it is. Form the realization $\hat{I}'$ of I' by setting:

$$\pi'_{h,i+1} : N^*_\mu \longrightarrow^*_{F'} N_{\mu+1},$$

where $h = T(\mu+1), F' = E^{N_\mu}_{\nu'_\mu}$. We consider three cases:

Case 1. $\sigma_h \restriction M^*_\mu$ is Σ^*-preserving.

Bu Lemma 4.3.2 we have:

$$\langle \sigma_h \restriction M^*_\mu, \sigma_\mu \restriction \lambda_\mu \rangle \ \langle M^*_\mu, F \rangle \longrightarrow^* \langle N^*_\mu, F' \rangle,$$

where $\sigma_h \restriction M^*_h$ is Σ^*-preserving. By Lemma 3.2.23 this gives us:

$$\pi_{h,\mu+1} : M^*_\mu \longrightarrow^*_F M_{\mu+1},$$

and a unique:

$$\sigma_{\mu+1} : M_{\mu+1} \longrightarrow_{\Sigma^*} N_{\mu+1}$$

such that $\sigma_{mu+1}\pi_{h,\mu+1} = \pi'_{h,\mu+1}\sigma_h, \sigma_{\mu+1} \restriction \lambda_\mu = \sigma_\mu \restriction \lambda_\mu$.

The remaining verification are straightforward.

Case 2. Case 1 fails and $\eta+1 \notin$ EX.

By Lemma 4.3.2 we again have:

$$\langle \sigma_h, \sigma_\mu \upharpoonright \lambda_\mu \rangle : \langle M_h, F \rangle \longrightarrow^* \langle N_h, F' \rangle.$$

Moreover σ_h is $\Sigma_0^{(m)}$-preserving, where $m \leq \omega$ is maximal such that $\lambda < \rho^m$ in M. Now let $n \leq \omega$ be maximal such that $\kappa_i < \rho^n$ in M_h. Then $n \leq m$, since $\lambda \leq \kappa_i$. By Lemma 3.2.19 M_h is n-extendible by F. But then it is $*$-extendible, since F is close to M_h. Set:

$$\pi_{h,\mu+1} : M_h \longrightarrow_F^* M_{\mu+1}.$$

Since σ is $\Sigma_0^{(m)}$-preserving, it follows by Lemma 3.2.19 that there is a unique:

$$\sigma_{\mu+1} : M_{\mu+1} \longrightarrow_{\Sigma_0^{(n)}} N_{mu+1},$$

such that $\sigma'_{\mu+1}\pi_{h,\mu+1} = \pi'_{h,\mu+1}\sigma_h$ and $\sigma'\lambda_\mu = \sigma_n \upharpoonright \lambda_\kappa$. But σ' is, in fact, $\Sigma_0^{(m)}$-preserving. If $n = m$, this is trivial. If $n < m$, it follows by Lemma 3.2.24. We let $\sigma_{\mu+1} = \sigma'$. The remaining verification are straightforward.

QED(Case 2)

Case 3. The above cases fail.

Then $\mu+1 \in$ EX and $\rho^1 \leq \lambda$ in M. Thus $\rho^1 \leq \lambda \leq \kappa_i$ in M_h. By Lemma 4.2.8 we have:

$$\langle \sigma_h, \sigma_\mu \upharpoonright \lambda_\mu \rangle : \langle M_h, F \rangle \longrightarrow \langle N_h, F' \rangle.$$

Hence by Lemma 3.2.19, there are π, σ' with:

$$\pi : M_h \longrightarrow_F M_{\mu+1}, \sigma' : M_{\mu+1} \longrightarrow_{\Sigma_0} N_{\mu+1}$$

such that $\sigma'\pi = \pi'_{h,\mu+1}\sigma_h$ and $\sigma' \upharpoonright \lambda_\mu = \sigma_\mu \upharpoonright \lambda_\mu$. But $M_{\mu+1}$ is the $*$-ultrapower of M_h, since $\rho^1_{M_h} \leq \kappa_i$ and F is close to M_h. We set: $\pi_{h,\mu+1} = \pi, \sigma_{\mu+1} = \sigma'$. The remaining verifications are straightforward.

QED(Lemma 4.3.3)

Our analogue of Lemma 3.4.18 reads:

Lemma 4.2.12. *Let $\langle I, I', \langle\sigma_i : i < \mu\rangle\rangle$ be a duplication of limit length μ. Let b' be a well founded cofinal branch in I'. Let $b = \bigcup_{i \in b' \setminus 2}\{\text{or}(i), i\}$ be the induced cofinal branch in I. Our duplication extends to one of length $\mu + 1$ with:*

$$T\text{"}\{\mu\} = b, T\text{"}\{\mu\} = b'$$

and $\sigma_\mu \pi_{i,\mu} = \pi'_{i\mu}\sigma_i$ for $i \in b$.

The proof is left to the reader.

With these two lemmas we can prove Lemma 4.2.7:

Fix a successful normal iteration strategy for N. We construct a strategy S^* for $\langle N, M, \lambda\rangle$ as follows: Let I be a normal iteration of $\langle N, M, \lambda\rangle$ of limit length μ. If I has no S-conforming copy, then $S^*(I)$ is undefined. Otherwise, let I' be an S-conforming copy. Let $S(I') = b'$ be the cofinal well founded branch given by S. Set $S^*(I) = b$, where b is the induced branch in I. Clearly if I is S^*-conforming, then the S-conforming copy I' exists. If I is of length $\mu + 1 (\mu \geq 1)$, then by Lemma 4.3.3, if $\nu \in M_\mu, \nu > \nu_i$ for $i < \mu$, then I extends to an S^*-conforming iteration of length $\mu + 2$ with $\nu_\mu = \nu$. By Lemma 4.3.4, if I is of limit length μ, then $S^*(I)$ exists. Hence S^* is successful.

QED(Lemma 4.2.7)

We still must prove Lemma 4.3.2. This, in fact turns out to be a repetition of Lemma 3.4.16 in §3.4. As before we derive it from:

Lemma 4.2.13. *Let $\langle I, I', \langle\sigma_j\rangle\rangle$ be a potential duplication of length $i + 1$ where $h = T(i + 1)$. Suppose that $i + 1 \notin EX$. Let $A \subset \tau_i$ be $\Sigma_1(M_i||\nu_i)$ in a parameter p. Let $A' \subset \tau'_i$ be $\Sigma_1(N_i||\nu'_i)$ in $\sigma_i(p)$ by the same definition. Then A is $\Sigma_1(M_i^*)$ in a parameter q and A' is $\Sigma_1(N_i^*)$ in $\sigma_h(q)$ by the same definition.*

Proof. The proof is a virtual repetition of the proof of Lemma 3.4.20 in §3.4. As before we take $\langle I, I', \langle\sigma_j\rangle\rangle$ as being a counterexample of length $i + 1$, where i is chosen minimally for such counterexamples. The proof is exactly the same as before. The only difference is that σ_j may not be Σ^*-preserving if $j \in$ EX. But in the case where we need it, we will have that σ_j is $\Sigma_0^{(1)}$-preserving, which suffices.

QED(Lemma 4.3.5).

Hence Lemma 4.2.7 is proven.

However, we have only proven this on the special assumption that λ is a limit cardinal in M. We now consider the case: $\lambda = \kappa^+$ in M. This will require a radical change in the proof. Set:

$N^* =: N||\gamma$ where γ is maximal such that λ is a cardinal in $N||\gamma$.

Then $\lambda = \kappa^{+N^*} < \sigma(\lambda) = \kappa^{+N}$. An anomaly occurs at $i+1$ whenever $\tau_i = \lambda$. Then $0 = T(i+1)$ and $\kappa = \kappa_i$. Clearly $N^* = M_j^*$. Thus M_{i+1} is the ultraproduct of N^* by $F = E_{\nu_i}^{M_i}$ and N_{i+1} is the ultraproduct of N_i^* by $F' = E_{\nu_i}^{N_i}$. In order to define σ_{i+1}, we require:

$$\sigma(M_i^*) = N_i^*.$$

This is false however, since $\sigma_i \restriction \lambda_0 = \sigma \restriction \lambda_i$ where $\tau_i < \lambda_i$. Hence:

$$\tau_i' = \sigma_i(\tau_i) = \sigma(\tau_i) = \tau^{+N}.$$

Hence $N_i^* = N \ni \sigma(N^*)$.

The answer to this conundrum is to construct two sequences I' and $\hat{I}$. The sequence:

$$\hat{I} = \langle\langle \hat{N}_i\rangle, \langle \hat{\nu}_i : i \in A\rangle, \langle \pi_{ij} : \hat{i \leq_T j}\rangle, \hat{T}\rangle$$

will be a padded iteration of N of length μ in which many points may be inactive. The second sequence:

$$I' = \langle\langle N_i\rangle, \langle {\nu'}_i : i \in A\rangle, \langle \pi'_{ij} : i \leq_T j\rangle, T'\rangle$$

will have most of the properties it had before, but, in the presence of anomalies, it will not be an iteration . If no anomalies occurs, we will have: $I' = \hat{I}$. If $i+1$ is an anomaly, then $\pi_{0,i+1}$ will not be an ultrapower and N_i will be a proper segment of $\hat{N}_i = \hat{N}_{i+1}$. (Hence i is passive in $\hat{I}$). To see how this works, let $i+1$ be the first anomaly to occur in I, then $I'|_{i+1} = \hat{I}|_{i+1}$, but at $i+1$ we shall diverge. Under our old definition we would have taken $N_i^* = N$ and $\pi'_{i,i+1} = \pi''$, where:

$$\pi'' : N \longrightarrow_F^* N'',\ F = E_{\nu_i'}^{N_i}.$$

We instead take:

$$N_i^* = N^*,\ N_{i+1} = \pi''(N^*),\ \pi_{i,i+1} = \pi'' \restriction N^*.$$

Note that $\pi''(N^*) = \pi'(N^*)$, where π' is the extension of $\langle J_{\nu_i}^{E^{M_i}}, F\rangle$. But then N_{i+1} is a proper segment of $J_{\nu_i}^{E^{N_i}}$ hence of $N_i = \hat{N}_i$.

We can then define:

$$\sigma_{i+1} : M_{i+1} \longrightarrow N_{i+1}$$

by:

$$\sigma_{i+1}(\pi_{0,i+1}(f)(\alpha)) =: \pi'(f)(\sigma_i(\alpha))$$

for $f \in \Gamma^*(\kappa, N^*), \alpha < \lambda_i$. σ_{i+1} will then be $\Sigma_0^{(n)}$–preserving, where $n \leq \omega$ s maximal such that $\kappa < \rho^n$ in N^*. To see that this is so, let φ be a $\Sigma_0^{(n)}$ formula. Let $f_1, \ldots, f_n \in \Gamma^*(\kappa, N^*)$ and let $\alpha_1, \cdots, \alpha_n < \lambda_i$. Let:

$$x_j = \pi_{0,i+1}(f_j)(\alpha_j), y_j = \pi'(f_j)(\sigma_i(\alpha_j)) \ \ (j = 1, \ldots, n).$$

Let $X =: \{\prec \xi_1, \ldots, \xi_m \succ : N^* \models \varphi[f_1(\xi_1), \ldots, f_n(\xi_n)]\}$. Then $\sigma_i F(X) = F'(X)$, since $\sigma_i \restriction H_\lambda^M = \sigma_0 \restriction H_\lambda^M = \mathrm{id}$. Hence:

$$\begin{aligned} M_{i+1} \models \varphi[\vec{X}] &\longleftrightarrow \prec \vec{\alpha} \succ \in F(X) \\ &\longleftrightarrow \prec \sigma_i(\vec{\alpha}) \succ \in F'(X) = \pi'(X) \\ &\longleftrightarrow \sigma(N^*) \models \varphi[\vec{y}]. \end{aligned}$$

Since we had no need to form an ultraproduct at $i+1$, we set: $\hat{N}_{i+1} = \hat{N}_i$. i is then an inactive point in $\hat{I}$ and N_{i+1} is a proper segment of $\hat{N}_{i+1}$.

We continue in this fashion: The active points in $\hat{I}$ are just the points $i > 0$ such that $i + 1 < \mu$ is not an anomaly. If i is active, we set $\hat{\nu}_i = \nu'_i$. (This does not, however, mean that $\hat{N}_i = N'_i$.) If i is any non anomalous point, we will have: $N_i = \hat{N}_i$. If $h < i$ is also non anomalous, thus $\pi'_{hi} = \hat{\pi}_{hi}$. If i is an anomaly, we will have: N_i is a proper segment of $\hat{N}_i$. If μ is a limit ordinal it then turns out that any cofinal well founded branch b' in I', which, in turn, gives us such a branch b in I. This enables us to prove iterability.

We now redo our definition of "copy" as follows:

Definition 4.2.11. Let $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$ be a strict normal iteration of $\langle N, M, \lambda\rangle$, where $\langle N, M, \lambda\rangle$ is a phalanx witnessed by σ.

$$I' = \langle\langle M_i\rangle, \langle\nu'_i\rangle, \langle\pi'_{ij}\rangle, T'\rangle$$

is a copy of I with *copy maps* $\langle\sigma_i : i < \mu\rangle$ induced by σ if and only if the following hold:

(I) (a) T' is a tree such that $iT'j \longrightarrow i < j$.

(b) Let μ be the length of I. Then N_i is a premouse and

$$\sigma_i : M_i \longrightarrow_{\Sigma_0} N_i \text{ for } i < \mu$$

(c) $\pi'_{ij}(i \leq_T j)$ is a commutative system of partial maps from N_i to N_j.

(II) (a)-(f) of our previous definition hold. Moreover:

(g) Let $0 \leq_T j$. If $(0, i]_T$ have no anomaly, then σ_i is Σ^*-preserving.

(h) Let $h = T(i+1)$. Set:

$$N_i^* = \begin{cases} \sigma_h(M_i^*) & \text{if } M_i^* \in M_h \\ N_h & \text{if not} \end{cases}$$

Then $\pi'_{h,i+1} : N_i^* \longrightarrow_{\Sigma^*} N_{i+1}$.

(i) Let h, i be as above. If $i+1$ is not an anomaly, then:

$$\pi'_{h,i+1} : N_i^* \longrightarrow^*_{F'} N_{i+1}$$

where $F' = E^{N_i}_{\nu'_i}$.

(j) Let $i+1$ be an anomaly. (Hence $\tau_i = \lambda = \kappa^{+M}$, where $\kappa = \kappa_i$ is a cardinal in M, hence in N.)

We then have:

$$M_i^* = N^* =: N||\gamma,$$

where γ is maximal such that λ is a cardinal in $N||\gamma$. Let π be the extension of $N_i||\nu_i = \langle J^E_{\nu'}, F' \rangle$. Then:

$$N_{i+1} = \pi(N^*) \text{ and } \pi'_{0,i+1} = \pi \upharpoonright N^*.$$

Moreover, $\sigma_{i+1} : M_{i+1} \longrightarrow N_{i+1}$ is defined by:

$$\sigma_{i+1}(\pi_{0,i+1}(f)(\alpha)) = \pi'(f)(\sigma_i(\alpha))$$

where $f \in \Gamma^*(\kappa, N^*), \alpha < \lambda_i$. (Hence σ_{i+1} is $\Sigma_0^{(n)}$-preserving for $\kappa < \rho^n_{N^*}$.)

(k) Let $h \leq_T i$, where h is an anomaly. If $(h, i]_T$ has no drop point, then σ_i is $\Sigma_0^{(n)}$-preserving for $\kappa < \rho^n$ in N^*. If $(h, i]_T$ has a drop point, then σ_i is Σ^*-preserving.

(III) There is a *background iteration*:

$$\hat{I} = \langle \langle \hat{N}_i \rangle, \langle \hat{\nu}_i \rangle, \langle \hat{\pi}_{ij} \rangle, \hat{T} \rangle$$

with the properties.

(a) $\hat{I}$ is a padded normal iteration of length μ.

(b) $i < \mu$ is active in $\hat{I}$ iff $0 < i+1 < \mu$ and $i + \mu$ is not an anomaly in I. In this case: $\hat{\nu}_i = \nu'_i$.

(c) If i is not an anomaly in I, then $\hat{N}_i = N'_i$. Moreover, if $h < i$ is also not an anomaly, then:

$$h <_{\hat{T}} i \longleftrightarrow h <_{T'} i,\ \hat{\pi}_{h,i} = \pi'_{h,i} \text{ if } h <_{T'} i.$$

This completes the definition. In the special case that λ is a limit cardinal in M, we of course have: $I' = \hat{I}$ and the new definition coincides with the old one. We note some simple consequence of our definition:

Lemma 4.2.14. *The following hold:*

(1) *If $i < j < \mu$, then $\sigma_j(\lambda_i) = \lambda_i$. (Hence $\lambda'_i < \lambda'_j$ for $j + 1 < \mu$.)*

Proof. By induction on j. For $j = 0$ it is vacuously true. Now let it hold for j.

$$\sigma_{j+1}(\lambda_j) = \sigma_{j+1}\sigma_{h,i+1}(\kappa_j) = \pi'_{h,j+1}\sigma_h(\kappa_j) = \pi'_{h,j+1}(\kappa'_j) = \lambda_j.$$

(Here $\sigma_h(\kappa_j) = \sigma_j(\kappa_j) = \lambda'_j$, since $\kappa_j < \lambda_h$ and $\sigma_j || \lambda_h = \sigma_h \upharpoonright \lambda_h$.)

For $i < j$ we then have:

$$\sigma_{j+1}(\lambda_i) = \sigma_j(\lambda'_i)(\text{since } \lambda_i < \lambda_j).$$

QED(1)

(2) *σ_i is a cardinal preserving for $i < \mu$.*

Proof. If σ_i is Σ_1-preserving, this is trivial, so suppose not. Then one of two cases hold:

Case 1. $1 \leq_T i, (1, i]_T$ has no drop, and $\rho^1 \leq \lambda$ in M.

Then $\pi_{hj} : M_h \longrightarrow_{\Sigma^*} M_j$ is cofinal for all $h \leq_T j \leq_T i_\eta$ since each of the ultrapower involved is a Σ_0-ultrapower. Hence, if α is a cardinal in M_i, then $\alpha \leq \pi_{1,i}(\beta)$ where β is a cardinal in M_1. By acceptability it suffices to show that $\sigma_i\pi_{1,i}(\beta)$ is a cardinal in N_i. But $\sigma_i\pi_{1,i}(\beta) = \pi'_{1t}\sigma(\beta)$, where σ and π'_{1i} are cardinal preserving.

Case 2. $h \leq_T i$ where h is an anomaly, $(h, i]_T$ has no drop and $\rho^1 \leq k = k_i$ in N^*.

The proof is a virtual repeat of the proof in Case 1, with $(0, i]_T$ in place of $(1, i]_T$.

QED(2)

(3) I' behaves like an iteration at limits. More precisely:

Let $\eta < \kappa$ be a limit ordinal. Let $i_0 <_T \eta$ such that $b = (i_0, \eta)_T$ is free of drops. Then

$$N_\eta, \langle \pi_{i\eta} : i \in b \rangle$$

is the direct limit of:

$$\langle N_i : i \in b \rangle, \langle \pi_{ij} : i \leq j \text{ in } b \rangle.$$

Proof. No $i \in b \cup \{\eta\}$ is an anomaly since every anomaly is a drop point. Hence:

$$N'_i = \hat{N}_i, \pi'_{i,j} = \hat{\pi}_{i,j} \text{ for } i \leq j \text{ in } b \cup \{\eta\}.$$

Since I is an iteration, the conclusion is immediate.

QED(3)

(4) Let $i < \mu$. If $i + 1$ is an anomaly, then:

(a) N_{i+1} is a proper segment of $N_i || \nu'_i$. (Hence $\nu'_{i+1} < \nu'_i$).

(b) $\rho^\omega = \lambda'_i$ in N_{i+1}.

Proof. (a) is immediate by II (i) in the definition of "copy". But $N_{i+1} = \pi(N^*)$ where π is the extension of $N_i || \nu'_i$. By definition, $N^* = N || \gamma$, where $\gamma < \sigma(\lambda) = \kappa^{+N}$ is the maximal γ such that $\tau_i = \lambda$ is a cardinal in $N || \gamma$. Hence $\rho^\omega = \kappa$ in N^*. But then $\rho^\omega = \lambda'_i$ in N_{i+1}.

QED(4)

(5) Let $i < \mu$. There is a finite n such that $i + n + 1$ is not an anomaly. (This includes the case: $i + n + 1 = \mu$.)

Proof. If not then $\nu_{i+n+1} < \nu_{i+n}$ for $n < \mu$ by(4). Contradiction!

(6) Let $i < \mu$. There is a maximal $j \leq i$ such that j is not an anomaly.

Proof. Suppose not. Then $i \neq 0$ is an anomaly and for each $j < i$ there is $j' \in (j, i)$ which is an anomaly. But then i is a limit ordinal, hence not an anomaly.

By(5) and (6) we can define:

Definition 4.2.12. Let $i < \mu$. We define:

- $l(i) =$ the maximal $j \leq i$ such that j is not an anomaly.
- $r(i)$ the least $j \geq i$ such that $j + 1$ is not an anomaly.

Definition 4.2.13. An interval $[l, r]$ in μ is called *passive* iff i is an anomaly for $l < i \leq r$. A passive interval is called *full* if it is not properly contained in another passive interval.

It is then trivial that:

(7) $[l(i), r(i)] =$ the unique full I such that $i \in I$.

(8) Let $[l, r]$ be a full passive interval. Then, for all $i \in [l, r]$:

(a) $N_l = N_i$.

(b) If $j \leq l$ and $j \leq_{\hat{T}} i$, then $j \leq_{\hat{T}} l$.

(c) If $j \geq r$ and $i \leq_{\hat{T}} j$, then $r \leq_{\hat{T}} j$.

Proof. This follows by induction on j, using the general fact about padded iterations that if j is not active, then:

- $\hat{N}_j = \hat{N}_{j+1}$
- $h \leq_{\hat{T}} j \longleftrightarrow h <_{\hat{T}} j + 1$
- $j <_{\hat{T}} h \longleftrightarrow j + 1 \leq_{\hat{T}} h$. QED(8)

(9) Let b be a branch of limit length in $\hat{I}$. There are cofinally many $i \in b$ such that i is not an anomaly.

Proof. Let $j \in b$. Pick $i \in b$ such that $i > r(j)$. Then $l(i) > r(j)$, since $r(j)+1 \leq i$ is not an anomaly. Hence $l(i) \in b$ and $l(i) > j$ is not an anomaly.

QED(9)

We define N_i^* for $i < \mu$ exactly as if I' were an iteration: Let $h = T'(i + 1)$. Then:

$$N_i^* =: N_i||\gamma \text{ where } \gamma \text{ is maximal such that } \tau_i' \text{ is a cardinal in } N_i||\gamma.$$

We then get the following version of Lemma 4.2.8.

Lemma 4.2.15. *Let I' be a copy of I induced by σ. Let $h = T(i + 1)$. If $i + 1$ is not an anomaly. Then the conclusion (i)-(vi) of Lemma 4.2.8 hold. If $i + 1$ is an anomaly, then (v), (vi) continue to hold.*

Proof. If $i + 1$ is not an anomaly, the proof are exactly as before. Now let $i + 1$ be an anomaly. (iv) is immediate by II (j) in the definition of "copy". But then (vi) follows as before.

QED(Lemma 4.2.15)

Lemma 3.3.20 is strengthened to:

Lemma 4.2.16. *I has at most one copy I'. Moreover the background iteration $\hat{I}$ is unique.*

Proof. The first part is proven exactly as before (we imagine I'' to be a second copy and show by induction on i that $I'|i = I''|i$). The second part is proven similarly, assuming $\hat{I}'$ to be a second background iteration.

QED(Lemma 4.2.16)

The concept *duplication induced by* σ is defined exactly as before. Now let:

$$D = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$$

be a duplication of length $\eta + 1$. We turn this into a *potential duplication* D of length $\eta + 2$ by appointing a ν_ξ such that $\nu_\xi > \nu_i$ for $0 < i < \eta$.

By a *realization* of $\tilde{D}$ of length $\eta + 2$ by appointing a ν_η such that $\nu_\eta < \nu_i$ for $0 < i < \eta$. By a *realization* of $\tilde{D}$, we mean a duplication $\mathring{D} = \langle \mathring{I}, \mathring{J}, \langle \dot{\sigma}_i : i \leq \eta + 1 \rangle \rangle$ of length $\eta + 2$ such that $\mathring{D}|\eta + 1 = D$ and $\dot{\nu}_\eta = \nu_\eta$. It follows easily that $\tilde{D}$ has at most one realization.

Our analogue, Lemma 4.3.2, of Lemma 3.4.16 will continue to hold as stated if we enhance the definition of *exceptional point* as follows:

Definition 4.2.14. i is an *exceptional point* ($i \in EX$) iff either:

$$1 \leq_T i, (1, i]_T \text{ has no drop, and } \rho^1 \leq \lambda \text{ in } M$$

or there is an anomaly $h \leq_T i$ such that:

$$(0, i]_T \text{ has no drop, and } \rho^1 \leq \kappa \text{ in } N^*.$$

With this change Lemma 4.3.2 goes through exactly as before. As before, we derive this form Lemma 4.3.5. The proof is as before. As before the condition $i + 1 \notin EX$ guarantees that the map σ_i will always have sufficient preservation when we need it.

When we worked under the special assumption Lemma 4.3.3 was our analogue of Lemma 3.4.17. In the presence of anomalies the situation is somewhat more complex. We first note:

Lemma 4.2.17. *Let $\tilde{D} = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$ be a potential duplication of length $\eta + 2$. If $\eta + 1$ is an anomaly, then $\tilde{D}$ is realizable.*

Proof. Form $N_{\eta+1}, \pi_{0,\eta+1} : N^* \longrightarrow N_{\eta+1}$ and $\sigma_{\eta+1}$ as in II(j). Set: $\tilde{N}_{\eta+1} = N_\eta$. The verification of I, II, III is straightforward.

QED(Lemma 4.2.17)

Now suppose that $\eta + 1$ is not an anomaly. Let $h = T(\eta + 1)$. Then η is an active point is any realization of $\hat{I}$, so we set: $\hat{\nu}_\eta = \nu'_\eta$. In order to realize $\tilde{D}$, we must apply $F = E^{M_\eta}_{\nu_\eta}$ to M^*_η, getting:

$$\pi_{h,\eta} : M^*_\eta \longrightarrow^*_F M_{\eta+1}.$$

Similarly we apply $F' = E^{N_\eta}_{\nu'_\eta}$ to N^*_η getting:

$$\pi'_{h,\eta} : N^*_\eta \longrightarrow^*_{F'} N_{\eta+1}.$$

We then set:

$$\sigma_{\eta+1}(\pi_{h\eta}(f)(\alpha)) = \pi'_{h\eta}\sigma_h(f)(\sigma_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_{\dot{\eta}}, M^*_{\dot{\eta}}), \alpha < \lambda_\eta$.

We must also extend $\hat{I}$. Since $\hat{\nu}_\eta = \nu_\eta$ and N_η is an initial segment of $\hat{N}_\eta$, we have:

$$F' = E^{\hat{N}_\eta}_{\hat{\nu}_\eta}.$$

Now let: $k = \hat{T}(\eta + 1)$. (k can be different from h!) III constrains us to set:

$$\hat{\pi}_{k,\eta+1} : \hat{N}^*_\eta \longrightarrow^*_F \hat{N}_{\eta+1}.$$

However, III also mandates that $\hat{N}_{\eta+1} = N_{\eta+1}$. Happily, we can prove:

Lemma 4.2.18. *Let $\tilde{D} = \langle I, I', \langle \sigma_i : i \leq \eta \rangle \rangle$ be as above, where $\eta + 1$ is not an anomaly. Then:*

(a) $N^*_\eta = \hat{N}^*_\eta$.

(b) *$\tilde{D}$ is realizable iff N^*_η is $*$-extendible by F'.*

Proof. We first prove (a). Let $h = T'(\eta + 1)$. Set:

$$l = l(h), r = r(h).$$

Then $h \in [l, r]$ where l is not an anomaly, $j + 1$ is an anomaly for $l \leq j < r$, and $r + 1$ is not an anomaly. h is least such that $\kappa'_\eta < \lambda'$ or $h = \eta$. $k = T'(\eta + 1)$ is least such that $k + 1$ is not an anomaly and $\kappa'_\eta < \lambda'_k$. Since j is not an anomaly for $l < j \leq r$, we conclude that $k = r$. Then $N_l = \hat{N}_j$ for $l \leq j \leq r$.

Case 1. $h = l$.

Then $\hat{N}_h = N_h$ and:

$$N^*_\eta = \hat{N}_\eta = N_h||\gamma$$

where γ is maximal such that τ'_η is a cardinal in $N_h||\gamma$. QED(Case 1)

Case 2. $l < h$.

Then $h = j + 1$ where $l \leq j$. N_h is a proper segment of $\hat{N}_h$. We again have: $N^*_\eta = N_h||\gamma$ where $\gamma \leq \mathrm{On}_{N_h}$ is maximal such that τ'_η is a cardinal in $N_h||\gamma$. We have $r = \hat{T}(\eta + 1)$ and $\hat{N}^*_\eta = \hat{N}_r||\hat{\gamma}$, where $\hat{\gamma} \leq \mathrm{On}_{\hat{N}_r}$ is maximal such that τ'_η is a cardinal in $\hat{N}_r||\hat{\gamma}$. But $\rho^\omega_{N_h} = \kappa_j$, where $h = j + 1$ by Lemma 4.2.14 (4). Since $\lambda'_j \leq \kappa'_\eta < \tau'_\eta < \lambda'_h$ and N_h is a proper segment of $\hat{N}_h = \hat{N}_r$, we conclude that $\hat{\gamma} \leq \mathrm{On}_{N_h}$. Hence $\gamma = \hat{\gamma}$ and $N^*_\eta = \hat{N}^*_\eta$. QED(a)

We now prove (b). If $\hat{N}^*_\eta$ is not extendable by F', then no realization can exists, so assume otherwise. This gives us $N_{\eta+1}$ and $\pi'_{h,\eta+1}$, where $\hat{N}_{\eta+1} = N_{\eta+1}$ and $\hat{\pi}_{k,\eta+1} = \pi'_{h,\eta+1}$, where $k = T'(\eta + 1)$. $\sigma_{\eta+1}$ is again defined by:

$$\sigma_{\eta+1}(\pi_{h,\eta+1}(f)(\alpha)) = \pi'_{h,\eta+1}\sigma_h(f)(\sigma_\eta(\alpha))$$

for $f \in \Gamma^*(\kappa_\eta, M^*_\eta), \alpha < \lambda_\eta$. The verification of I, II, III is much as before. However Case 2 splits into two subcases:

Case 2.1. $1 \leq_T \eta + 1$.

This is exactly as before.

Case 2.2. $0 \leq_T \eta + 1$.

Then there is $j \leq_T h$ such that j is an anomaly and $(0, \eta + 1]_T$ has no drop. Moreover, $\rho^1 > \kappa$ in N^*. Then σ_h is a $\Sigma^{(m)}_0$-preserving where $m \leq \omega$ is maximal such that $\kappa < \rho^m$ in N^*. The rest of the proof is as before.

Case 3 also splits into two subcases:

Case 3.1. $1 \leq_T \eta + 1$.

We argue as before.

Case 3.2. $0 \leq_T \eta + 1$.

Then $j \leq_t h$, where j is an anomaly and $\rho^1 \leq \kappa$ in N^*. Hence $\rho^1 \leq \kappa_h$ in M_h and we argue as before. QED(Lemma 4.2.18)

Using Lemma 4.2.14 (9) we get:

Lemma 4.2.19. *Let $D = \langle I, I', \langle\sigma_i\rangle\rangle$ be a duplication of limit length μ. Let $\hat{b}$ be a cofinal well founded branch in $\hat{I}$. Let X be the set of $i \in \hat{b}$ which are not an anomaly. Let:*

$$b' = \{j : \bigvee i \in X j <_T i\}, b = \{j : \bigvee i \in X j <_T i\}.$$

Then D has a unique extension to a $\tilde{D}$ of length $\mu + 1$ such that:

$$\hat{T}\text{"}\{\mu\} = \hat{b}, T'\text{"}\{\mu\} = b', T\text{"}\{\mu\} = b.$$

The proof is left to the reader.

Now let S be a successful normal iteration strategy for N. We define an iteration strategy S^* for $\langle N, M, \lambda\rangle$ as follows:

Let I be an iteration of $\langle N, M, \lambda\rangle$ of limit length μ. We ask whether there is a duplication $\langle I, I', \langle\sigma_0\rangle\rangle$ induced by σ^*. If not, then $S^*(I)$ is undefined. Otherwise, we ask whether $S(\hat{I})$ is defined. If not, then $S^*(I)$ is undefined. If not, then $S^*(I)$ is undefined. If $\hat{b} = S(\hat{I})$, define b', b as above and set: $S^*(I) = b$. It is easily seen that if I is any S^*-conforming normal iteration of $\langle N, M, \lambda\rangle$, then the duplication $\langle I, I', \langle\sigma_i\rangle\rangle$ exists. Moreover $\hat{I}$ is S-conforming. In particular, if I is of limit length, then $S(I)$ is defined. Moreover, if I is of length $\eta+1$, and $\nu > \nu_i$ for $i < \eta$, then by Lemma 4.2.18, we can extend I to an $\tilde{I}$ of length $\eta + 2$ by setting: $\nu_\eta = \nu$. Hence S is a successful iteration strategy.

This proves Lemma 4.2.7 at last!

We note however, that our strategy S^* is defined only for strict iteration of $\langle N, M, \lambda\rangle$. We can remedy this in the usual way. Let:

$$I = \langle\langle M_i\rangle, \langle\nu_i : i \in A\rangle, \langle\pi_{ij}\rangle, T\rangle$$

be a padded iteration of $\langle N, M, \lambda\rangle$, of length μ. Let h be the monotone enumeration of:

$$\{i : i = 0 \vee i \in A \vee i + 1 = \mu\}.$$

The *strict pullback* of I is then:

$$\dot{I} = \langle\langle\dot{M}_i\rangle, \langle\dot{\nu}_i\rangle, \langle\dot{\pi}_{ij}\rangle, \hat{T}\rangle$$

where:

$$\dot{M}_i = M_{h(i)}, \dot{\nu}_i = \nu_{h(i)}, \dot{\pi}_{ij} = \pi_{h(i),h(i)}$$

and:

$$i\hat{T}j \longleftrightarrow h(i)Th(j).$$

$\dot{I}$ is a strict iteration and contains all essential information about I. We extend S^* to a strategy on padded iteration as follows: Let I be a padded iteration of limit length μ. If A is cofinal in μ, we form $\dot{I}$, which is then also of limit length. We set:

$$S^*(I) = b, \text{ where } S^*(\dot{I}) = \dot{b},$$

and $b = \{i : \bigvee j(i \leq_T h(j))\}$. If A is not cofinal in μ, there is $j < \mu$ such that $A \cap [j, \mu) = \varnothing$. We set:

$$S^*(I) = \{i < \mu : iTj \vee j \leq i\}.$$

It follows that I is S^*-conforming iff $\dot{I}$ is S^*-conforming.

Since $\dot{I}$ is strict, we have $I', \hat{I}, \langle \sigma_i : i < \dot{\mu} \rangle$, (where $\dot{\mu}$ is the length of $\dot{I}$). We shall make use of this machinery in analyzing what happens when we coiterate N against $\langle N, M, \sigma \rangle$. This will yield the "simplicity lemma" stated below.

Note. We could, of course, have defined $I', \hat{I}$ and $\langle \sigma_i : i < \mu \rangle$ for arbitrary padded I, but this will not be necessary.

Building upon what we have done thus far, we prove the following "simplicity lemma", which will play a central role in our further deliberations:

Lemma 4.2.20. *Let N be a countable premouse which is presolid and fully $\omega_1{+}1$ iterable. Let $\langle N, M, \sigma \rangle$ be witnessed by σ. Set $Q^0 = N, Q^1 = \langle N, M, \sigma \rangle$. There exist successful $\omega_1 + 1$ normal iteration strategies S^0, S^1 for Q^0, Q^1 respectively such that $\langle I^0, I^1 \rangle$ is the coiteration of Q^0, Q^1 by S^0, S^1 respectively with coiteration indices ν_i, then the coiteration terminates at $\mu < \omega_1$ with:*

$$I_0 = \langle \langle Q_i \rangle, \langle \nu_i \rangle, \langle \pi^0_{ij} \rangle, T^0 \rangle$$

$$I_1 = \langle \langle M_i \rangle, \langle \nu_i \rangle, \langle \pi^1_{ij} \rangle, T^1 \rangle$$

such that:

(a) $M_\mu \triangleleft Q_\mu$.

(b) $1 \leq_{T^1} \mu$ *in* I^1.

(c) *There is no drop point* $i + 1 \leq_{T^1} \mu$ *in* I^1.

In the next section we shall use this to derive the solidity lemma, which says that all mice are solid. We shall also us eit to derive a number of other structural facts about mice.

We now prove the simplicity lemma.

Let N be countable, presolid and fully $\omega_1 + 1$.iterable. Let $\langle N, M, \lambda\rangle$ be a phalanx witnessed by σ. (Recall that this entails $\lambda \in M$ and $\lambda = \mathrm{crit}(\sigma)$. Moreover, σ is $\Sigma_0^{(n)}$-preserving whenever $\lambda < \rho_M^n$). Fix an enumeration $e = \langle e(n) : n < \omega\rangle$ of $\mathrm{On} \cap N$. Suppose that $\sigma : N \longrightarrow_{\Sigma^*} N'$. We can define a sequence $e'_i \in N' (i < \omega)$ as follows. By induction on $i < \omega$ we define:

$$\begin{aligned} e'_i = {} & \text{the least } \eta \in N' \text{ s.t. there is some } \sigma' : N \longrightarrow_{\Sigma^*} N' \\ & \text{with } \sigma'(e_h) = e'_h \text{ for } h < i \text{ and } \eta = \sigma'(e_i). \end{aligned}$$

It is not hard to see that there is exactly one $\sigma' : N \longrightarrow_{\Sigma^*} N$ such that $\sigma'(e_i) = e'_i$ for $i < \omega$. We then call σ' the *e-minimal* embedding of N into N'. The *Neeman-Steel Lemma* (Theorem 3.5.8) says that N has an *e-minimal normal iteration strategy* S with the following properties:

- S is a successul $\omega_1 + 1$ normal iteration strategy for N.
- Let N' be an iterate of N by an S-conforming iteration I. Let $\sigma : N \longrightarrow_{\Sigma^*} M \lhd N'$. Then I has no drop on its main branch $M = N'$ and the iteration map $\pi : N \longrightarrow N'$ is the e-minimal embedding.

Hence, in particular, if M is a proper segment of N' or the main branch of I has a drop, then there is no Σ^*-preserving embedding from N to M.

From now on let e be a fixed enumeration of On_N and let S be an e-minimal strategy for N. Let S^* be the induced strategy for $\langle N, M, \lambda\rangle$. Coiterate $Q_0 = N$ against $M_0 = \langle N, M, \lambda\rangle$ using the strategies S, S^* respectively. Let $\langle I^0, I^1\rangle$ be the coiteration with:

$$I^1 = \langle\langle M_i\rangle, \langle\nu_i^0\rangle, \langle\pi_{ij}^0\rangle, T^0\rangle$$

$$I^0 = \langle\langle Q_i\rangle, \langle\nu_i^1\rangle, \langle\pi_{ij}^1\rangle, T^1\rangle$$

and coiteration indices $\langle\nu_i : 1 \le i \le \mu\rangle$ where $\mu + 1 < \omega_1$ is the length of the coiteration.

We note some facts:

(A) If N' is any S-iterate of N (i.e. the result of an S-conforming iteration), then there is no Σ^*-preserving map of N into a proper segment of N'.

(B) Call N' a *truncating* S-iterate of N iff it results from an S-conforming iteration with a truncation on its main branch. If N' is a truncating S-iterate, then there is no Σ^*-preserving embedding of N into N'.

(C) If N' is a *non truncating* S-iterate of N, then the iteration map $\pi : N \longrightarrow N'$ is the unique e-minimal map.

Now form the *strict pullback* $\dot{I}$ of I^1 as before. Let I be of length $\mu + 1$. $\dot{I}$ will then be of length $\dot{\mu} + 1$. Let $I', \hat{I}, \langle \sigma_i : i \leq \dot{\mu} \rangle$ be defined as before. Set: $N' =: N'_{\dot{\mu}}$, $\hat{N} =: \hat{N}_{\dot{\mu}}$, $\sigma' = \sigma'_{\dot{\mu}}$. The following facts are easily established:

(D) $\hat{N}$ is an S-iterate of N. Moreover: $\sigma' : M_\mu \longrightarrow_{\Sigma_0} N'$ where $N' \triangleleft \hat{N}$.

(E) If there is a drop point $i + 1 \leq_{T^1} \mu$ which is not an anomaly in I^1, then there is $i + 1 \leq_{T^0} \dot{\mu}$ which is not an anomaly in $\dot{I}$. Hence $\hat{N}$ is a truncating iterate of N and $\sigma' : M_\mu \longrightarrow_{\Sigma^*} \hat{N}$.

(F) If there is no anomaly $i + 1 \leq_{T^1} \mu$ in I, then there is no anomaly $i + 1 \leq_{\dot{T}} \dot{\mu}$ in $\dot{I}$.

(G) Suppose $0 \leq_{T^1} \mu$ and no $i + 1 \leq \mu$ is an anomaly. Hence the same situation holds in $\dot{I}$. Then $\hat{N}$ is an S-iterate of N by the iteration map $\sigma' \pi'_{0,\mu}$ (since $\dot{\sigma}_{\dot{\mu}} \dot{\pi}_{0,\dot{\mu}} = \hat{\pi}_{0,\dot{\mu}}$).

We now prove the simplicity lemma. We do this by eliminating all other possibilities.

Claim 1. Q_μ is not a proper segment of M_μ.

Proof. Suppose not. Then Q_μ is a non-truncating iterate of N with iteration map $\pi^0_{0,\mu}$. Hence $\sigma' \pi^0_{0,\mu} : N \longrightarrow_{\Sigma^*} \sigma_\mu(Q_\mu)$, where $\sigma_\mu(Q_\mu)$ is a proper segment of $\hat{N}$ and $\hat{N}$ is an S-iterate of N. Contradiction!

QED(Claim 1)

Claim 2. There is no truncation point $i + 1 \leq_{T^1} \mu$ such that $i + 1$ is not an anomaly in I^1.

Proof. Suppose not. Then $\sigma' : M_\mu \longrightarrow_{\Sigma^*} \hat{N}$, where $\hat{N}$ is a truncating S-iterate of N. I^0 is truncation free on its main branch, since I^1 is not. Hence $Q^0_\mu \triangleleft M_\mu$. Hence, $Q^0_\mu \triangleleft M'_\mu$ by Claim 1. Hence:

$$\sigma' \pi^0_{0,1} : N \longrightarrow_{\Sigma^*} \hat{N},$$

where $\hat{N}$ is a truncating iterate of N. Contradiction!

QED(Claim 2)

Claim 3. No $i + 1 \leq_{T^1} \mu$ is an anomaly in I^1.

Proof. Suppose not. Then $\kappa_i = \kappa$ and $\tau_i = \lambda$. Hence $\tau_i < \sigma(\lambda) = \kappa^{+N}$. Thus $M_i^* = N^*$, where $N^* = N||\eta$, η being maximal such that λ is a cardinal in $N||\eta$. By Claim 2, there is no drop point $j + 1 \leq_{T^1} \mu$ such that $i < j$. Hence:

$$\pi'_{0,\mu} : N^* \longrightarrow_{\Sigma^*} M_\mu.$$

$\kappa = \rho^\omega$ in N^*, since $\rho^\omega \leq \kappa$ by the definition of N^*, but $\rho^\omega \geq \kappa$ since $N^* \in N$ and κ is a cardinal in N. But $\kappa_i = \mathrm{crit}(\pi^1_{0,\mu})$. Hence $\kappa = \rho^\omega$ in M_μ.

$Q_\mu = M_\mu$ as above. Moreover the iteration I^0 is truncation free on its main branch, since I^1 is not. Thus:

$$\pi^0_{0,\mu} : N \longrightarrow_{\Sigma^*} M_\mu$$

Hence $\kappa^0_i \geq \rho^\omega_N$ for $i + 1 \leq_{T^0} \mu$, since otherwise $\rho^\omega_{M_\mu} \geq \lambda_i > \kappa$. Hence:

$$\rho^\omega_N = \rho^\omega_{Q_\mu} = \kappa$$

and:

$$\mathbb{P}(\kappa) \cap N = \mathbb{P}(\kappa) \cap Q_\mu = \mathbb{P}(\kappa) \cap M_\mu = \mathbb{P}(\kappa) \cap N^*.$$

This is clearly a contradiction, since $N^* \in N$ and $\mathrm{card}(N^*) = \kappa$ in N. Hence by a diagonal argument there is $A \in \mathbb{P}(\kappa) \cap N$ such that $A \notin N^*$.

QED(Claim 3)

It remain only to show:

Claim 4. $1 \leq_{T^1} \mu$.

Proof. Suppose not. Then $o <_{T^1} \mu$. By Claim 3 there is no anomaly on the main branch of I^1. Hence, if $\kappa_i < \lambda$ and $i + 1 \leq_{T^1} \mu$, we have $\tau_i < \lambda$. But then $M^*_{\nu^1_i} = N$. By claim 2 there is no drop on the main branch of I^1. Hence:

$$\pi^1_{0,\mu} : N \longrightarrow_{\Sigma^*} M_\mu.$$

$M_\mu \lhd Q_\mu$ by Claim 1. Hence $M_\mu = Q_\mu$, since otherwise $\pi^1_{0,\mu}$ would map N into a proper segment of an S-iterator of N. Thus we have:

$$\pi^0_{0,\mu}; N \longrightarrow_{\Sigma^*} M_\mu.$$

Set: $\pi^0 = \pi^0_{0,\mu}, \pi^1 = \pi^1_{0,\mu}$. We claim:

Claim. $\pi^0 = \pi^1$.

Proof. Suppose not. Let i be least such that $\pi^0(e_i) \neq \pi^1(e_i)$. Then $\pi^1(e_i) > \pi^0(e_i)$ since the map π^0, being an S-iteration map, is e-minimal. But $\sigma'\pi^1$

is the S-iteration map from N to $\hat{N}$. Hence $\sigma'\pi^1(e_i) < \sigma'\pi^0(e_i)$, since $\sigma'\pi^0 : N \longrightarrow_{\Sigma^*} \hat{N}$. Hence $\pi^1(e_i) < \pi^0(e_i)$. Contradiction!

QED(Claim)

Let $i_h + 1 \leq_{T^h} \mu$ with $o = T^h(i_h + 1)$ for $h = 0, 1$. Then $\kappa_{i_0} = \kappa_{i_1} = \text{crit}(\pi)$, where $\pi = \pi^0_{0,\mu} = \pi^1_{0,\mu}$. Set:

$$F^0 = E^{Q_0}_{\nu_{i_0}}, F^1 = E^{M_0}_{\nu_{i_1}}.$$

Then:

$$F^h(X) = \pi^h_{0,i_h+1}(X) \text{ for } X \in \mathbb{P}(\kappa_{i_h}) \cap N.$$

Thus:

$$\alpha \in F^h(X) \longleftrightarrow \alpha \in \pi(X) \text{ for } \alpha < \lambda_{i_h},$$

since $\pi = \pi^h_{i_h+1,\mu} \circ \pi^h_{0,i_h+1}$. But then $\nu_{i_0} \not< \nu_{i1}$, since otherwise $F^0 \in J^{E^{M_{i_1}}}_{\nu_{i_1}}$ by the initial segment condition, whereas ν_{i_0} is a cardinal in $J^{E^{M_{i_1}}}_{\nu_{i_1}}$. Contradiction! Similarly $\nu_{i_1} \not< \nu_{i_0}$. Thus $i_0 = i_1 = i$ and $F^0 = F^1$. But then ν_i is not a coiteration index! Contradiction.

QED(Claim 4)

This proves the simplicity lemma.

4.3 Solidity and Condensation

In this section we employ the simplicity lemma to establish some deep structural properties of mice. In §4.3.1 we prove the **Solidity Lemma** which says that every mouse is solid. In §4.3.2 we expand upon this showing that any mouse N has a unique core $\overline{N}$ and core map σ defined by the properties:

- $\overline{N}$ is sound.
- $\sigma :\longrightarrow_{\Sigma^*} N$.
- $\rho^\omega_{\overline{N}} = \rho^\omega_N$ and $\sigma \upharpoonright \rho^\omega_N := \text{id}$.
- $\sigma(p^i_{\overline{N}}) = p^i_N$ for all i.

In §4.3.3 we consider the condensation properties of mice. The condensation lemma for L says that if $\pi : M \longrightarrow_{\Sigma_1} J_\alpha$ and M is transitive, then $M \triangleleft J_\alpha$. Could the same hold for an arbitrary sound mouse in place of J_α? In that generality it certainly does not hold, but we discover some interesting instances of condensation which do hold.

We continue to restrict ourselves to premice M such that $M||\alpha$ is not of type 3 for any α. By a mouse we mean such a premouse which is fully iterable. (Though we can take this as being relativized to a regular cardinal $\kappa > \omega$, i.e. $\mathrm{card}(M) < \kappa$ and M is fully $\kappa + 1$-iterable.)

4.3.1 Solidity

The *Solidity lemma* says that every mouse is solid. We prove it in the slightly stronger form:

Theorem 4.3.1. *Let N be a fully ω_1+1-iterable premouse. Then N is solid.*

We first note that we may w.l.o.g. assume N to be countable. Suppose not. Then there is a fully $\omega_1 + 1$ iterable N which is unsolid, even though all countable premice with this property are solid. Let $N \in H_\theta$, where θ is a regular cardinal. Let $\sigma : \overline{H} \prec H_\theta$, $\sigma(\overline{N}) = N$, where $\overline{H}$ is transitive and countable. Then $\overline{H}$ is a ZFC^- model. Since $\sigma \restriction \overline{N} : \overline{N} \prec N$, it follows by a copying argument that $\overline{N}$ is a ω_1+1 fully iterable (cf. Lemma 3.5.6.). Hence $\overline{N}$ is solid. By absoluteness, $\overline{N}$ is solid in the sense of $\overline{H}$. Hence N is solid in the sense of H_θ. Hence N is solid. Contradiction!

Now let $a = p_N^n$ for some $n < \omega$. Let $\lambda \in a$. Let $M = N_a^\lambda$ be the λ–th witness to a as defined in §4.1. For the reader's convenience we repeat that definition here. Let:

$$\rho^{l+1} \leq \lambda < \rho^l \text{ in } N;\ b =: a \setminus (\lambda + 1)$$

Let $\overline{N} = N^{l,b}$ be the l-th reduct of N by b. Set:

$$X = h(\lambda \cup b) \text{ where } h = h_{\overline{N}} \text{ is the } \Sigma_1\text{-Skolem function of } \overline{N}.$$

Then $X = h``(\omega \times (\lambda \times \{b\}))$ is the smallest Σ_1-closed submodel of $\overline{N}$ containing $\lambda \cup b$. Let:

$$\overline{\sigma} : \overline{M} \longleftrightarrow \overline{N}|X \text{ where } \overline{M} \text{ is transitive.}$$

By the extension of embedding lemma, there are unique $M, \sigma, \overline{b}$ such that $\sigma \supset \overline{\sigma}$ and:

$$\overline{M} = M^{l,b},\ \sigma : M \longrightarrow_{\Sigma_1'} N \text{ and } \sigma(\overline{b}) = b.$$

Then $N_a^\lambda =: M$ and $\sigma_a^\lambda =: \sigma$.

It is easily seen that σ witnesses the phalanx $\langle N, M, \lambda\rangle$. Employing the simplicity lemma, we coiterate $\langle N, M, \lambda\rangle$ against N, getting $\langle I^N, I^M\rangle$, terminating at η, where:

- $I^N = \langle\langle N_i\rangle, \langle\nu_i^N\rangle, \langle\pi_{ij}^N\rangle, T^N\rangle$ is the iteration of N.
- $I^M = \langle\langle M_i\rangle, \langle\nu_i^M\rangle, \langle\pi_{ij}^M\rangle, T^M\rangle$ is the iteration of $\langle N, M\rangle$.
- $\langle\nu_i : i < \eta\rangle$ is the sequence of coiteration indices. We know that:
- $M\eta \triangleleft N_\eta$.
- I^M has no truncation on its main branch.
- $1 \leq_{T^M} \eta$.

It follows that $\kappa_i \geq \lambda$ for $i <_{T^M} \eta$. Moreover $\nu_i > \lambda$ for $i < \eta$, since $M|\lambda = N|\lambda$.

We consider three cases:

Case 1. $M_\eta = N_\eta$ and I^N has no truncation on its main branch.

We know that $\rho_M^{l+1} \leq \lambda$, since every $x \in M$ is $\Sigma_1^{(l)}(M)$ in $\lambda \cup \bar{b}$. But $\kappa_i \geq \lambda$ for $i <_{T^M} \eta$.

Hence:

(1) $\mathbb{P}(\lambda) \cap M = \mathbb{P}(\lambda) \cap M_\eta$ and $\rho_M^h = \rho_{M_\eta}^h$ for $h > i$. But then $\kappa_j \geq \rho_N^{l+1}$ for $j <_{T^N} \eta$, since otherwise:

$$\kappa_i < \sup \pi_{h,j+1}^N{}''\rho_N^{l+1} \leq \rho_{N_\eta}^{l+1} = \rho_{M_\eta}^{l+1} \leq \lambda < \kappa_j$$

where $h = T^N(j+1)$. Hence for $h > l$ we have:

(2) $\rho_M^h = \rho_N^h$ and $\mathbb{P}(\rho^h) \cap M = \mathbb{P}(\rho^h) \cap N$.

Recall, however, that $a = p_N^n$, where $m > l$. Since every $x \in M$ is $\Sigma_1^{(i)}(M)$ in $\lambda \cup \bar{b}$, there is a finite $c \subset \lambda$ such that $c \cup \bar{b} \in P_M^n$. Let $\overline{A}$ be $\Sigma_1^{(n)}(M)$ in $c \cup \bar{b}$ such that $\overline{A} \cap \rho^n \notin M$. Let A be $\Sigma_1^{(n)}(N)$ in $c \cup b$ by the same definition. Then:

$$\overline{A} \cap \rho^n = A \cap \rho^n \in N,$$

since $c \cup b <_* a = p_N^n$. Thus,

$$\mathbb{P}(\rho^n) \cap M \neq \mathbb{P}(\rho^n) \cap N,$$

contradiction! QED(Case 1)

Case 2. M_η is a proper segment of N_η.

Then M_η is sound. Hence M did not get moved in the iteration and $M = M_\eta$. But then N is not moved and $N = N_\eta, \eta = 0$, since otherwise ν_1 is a cardinal in N_η. But then $\lambda < \nu_1 \leq \mathrm{On}_M$ and $\rho_M^\omega \leq \lambda < \nu_1$, where M is a proper segment of N_η. Hence ν_1 is not a cardinal in N_η. Contradiction!

QED(Case 2)

Case 3. The above cases fail.

Then $M_\eta = N_\eta$ and I^N has a truncation on its main branch. We shall again prove: $M \in N$.

We first note the following:

Fact. Let Q. be acceptable. Let $\pi : Q \longrightarrow_F^* Q'$, where $\rho^{i+1} \leq \kappa < \rho^i$ in $Q, \kappa = \mathrm{crit}(F)$. Then:

$$\underline{\Sigma}_1^{(n)}(Q') \cap \mathbb{P}(\kappa) = \underline{\Sigma}_1^{(n)}(Q) \cap \mathbb{P}(\kappa) \text{ for } n \geq i.$$

Note. It follows easily that:

$$\underline{\Sigma}_1^{(n)}(Q') \cap \mathbb{P}(H) = \underline{\Sigma}_1^{(n)}(Q) \cap \mathbb{P}(H)$$

where $H = H_\kappa^Q = H_\kappa^{Q'}$.

We prove the fact. The direction $\supset$ is straightforward, so we prove $\subset$ by induction on $n \geq i$. The first case is $n = i$. Let $A \subset \kappa$ be $\Sigma_1^{(i)}(Q')$ in the parameter a. Then:

$$A_\xi \longleftrightarrow \bigvee z \in H_{Q'}^i\, B'(z, \xi, a)$$

where B' is $\Sigma_1^{(1)}(Q')$. But then π takes H'_Q cofinally to $H_{Q'}^i$. Hence:

$$A_\xi \longleftrightarrow \bigvee u \in H_Q^{i'} \bigvee z \in \pi(u) B'(\tau, \xi, a).$$

Let $a = \pi(f)\alpha$ where $f \in \Gamma^*(\kappa, Q)$ and $\alpha < \lambda(F) = F(\kappa)$. Let B be $\Sigma_0^{(i)}(Q)$ by the same definition as B'. Then:

$$A_\xi \longleftrightarrow \bigvee u \in H_Q^i \{\zeta < \kappa : \bigvee z \in u B(z, \xi, f(\alpha))\} \in F_\alpha,$$

where $F_\alpha \in \underline{\Sigma}_1(Q)$ by closeness.

This proves the case $n = i$. The induction step uses the fact that $\rho^n_Q = \rho^n_{Q'}$, for $n > i$. (Hence $H^n_Q = H^n_{Q'}$.)

Let $n = m + 1 > i$ and let it hold at m. Let $A \subset \kappa$ be $\underline{\Sigma}^{(m)}_1(Q')$. Then:

$$A_\xi \longleftrightarrow \langle H^n_{Q'}, B^1_\xi, \ldots, B^r_\xi \rangle \vdash \varphi$$

where φ is a Σ_1 sentence and:

$$B^h_\zeta = \{z \in H^n_Q : \langle \xi, z \rangle \in B^h\} \; (h = 1, \ldots, r)$$

and B^h is $\underline{\Sigma}^{(m)}_1(Q')$. We may assume w.l.o.g. that $B^h \subset H$. But then B^h is $\Sigma^{(m)}_1(Q)$. Hence A is $\underline{\Sigma}^{(n)}_1(Q)$.

QED(Fact)

Recall that $\rho^{l+1} \leq \lambda < \rho^l$ in M. Using this we get:

(1) There is a $\underline{\Sigma}^{(l)}_1(M)$ set $B \subset \lambda$ which codes M (in particular, if Q is a transitive ZFC^- model and $B \in Q$, then $M \in Q$.)

Proof. Recall from the definition of M that:

$$\overline{M} = M^{l,b} = h_{\overline{M}}(\omega \times (\lambda \times \{\overline{c}\})), \text{ where } \overline{c} = \overline{b} \cap \rho^l_M.$$

Thus we can set:

$$\dot{M} = \{\prec i, \xi \succ \in M : i < \omega, \xi < \lambda, \text{ and } h_{\overline{M}}(i, \langle \xi, \overline{c} \rangle) \text{ is defined}\}.$$

For $\prec i, \xi \succ \in \dot{M}$ set: $h(\prec i, \xi \succ) = h_{\overline{M}}(i, \prec \xi, \overline{c} \succ)$. Let $M = \langle J^E_\alpha, F \rangle$. We set:

- $\dot{\in} =: \{\langle x, y \rangle \in \dot{M}^2 : h(x) \in h(y)\}$
- $\dot{I} =: \{\langle x, y \rangle \in \dot{M}^2 : h(x) = h(y)\}$
- $\dot{E} =: \{x \in \dot{M} : h(x) \in E\}$
- $\dot{F} =: \{ x \in \dot{M} : h(x) \in F\}$

Then:

$$\langle \dot{M}, \dot{\in}, \dot{E}, \dot{F} \rangle / I \cong \langle J^E_\alpha, F \rangle = M.$$

Let B be a simple coding of $\langle \dot{M}, \dot{\in}, \dot{E}, \dot{F} \rangle$, e.g. we could take it as the set of $\prec \xi, j \succ$ such that one of the following holds:

- $j = 0 \wedge \xi \dot{\in} \dot{M}$
- $j = 1 \wedge \xi = \prec \xi_u, \xi_1 \succ$ with $\xi_0 \dot{\in} \xi_1$
- $j = 2 \wedge \xi = \prec \xi_0, \xi_1 \succ$ with $\xi_0 I \xi_1$

- $j = 3 \wedge \xi \in \dot{E}$
- $j = 4 \wedge \xi \in \dot{F}$.

It is clear that if $B \in Q$ and Q is a transitive ZFC^- model, then $\overline{M}$ is recoverable from B in Q by absoluteness. Hence $\overline{M} \in Q$. But $\overline{M} = M^{l,\bar{b}}$ and M is recoverable from $\overline{M}$ in Q by absoluteness. Hence $M \in Q$.

QED(1)

Let $j+1$ be the final truncation point on the main branch of I^N. Then:

(2) B is $\underline{\Sigma}_1^{(l)}(N_{j+1})$.

Proof. Let B be $\Sigma_1^{(l)}(M)$ in the parameter p. Let B' be $\Sigma_1^{(\theta)}(M_\eta)$ in $\pi(p)$ by the same definition, where $\pi = \pi_{1,\eta}^M$. Then $B = \lambda \cap B'$ is $\underline{\Sigma}_1^{(l)}(N_\eta)$. Let i be the least $i \geq_T j+1$ in I^N set. B is $\Sigma_1^{(l)}(N_i)$. i is not a limit ordinal, since otherwise $\mathrm{lub}\{\kappa_h : h \leq_{T^N} i\} = \mathrm{lub}\{k_h : h < i\} > \lambda$ and there is $h \leq_{T^N} i$ such that $\kappa_h > \lambda$ and $a \in \mathrm{rng}(\pi_{hi}^N)$, where B is $\Sigma_1^{(l)}(N_i)$ in the parameter a. Hence B is $\underline{\Sigma}_1^{(l)}(N_h)$. Contradiction! But then $i = k+1$. Let $t = T^N(k+1)$. If $k > j$, then $t \geq j+1$ and $\kappa_k \geq \lambda_j \geq \lambda > \rho_M^{l+1} = \rho_{N_\xi}^{l+1} = \rho_{N_t}^{l+1}$. By the above Fact we conclude that $B \in \underline{\Sigma}_1^{(l)}(N_t)$ where $t < i$. Contradiction! Hence $i = j+1$. QED(2)

We consider two cases:

Case 3.1. $\kappa_j \geq \lambda$.

By the Fact, we conclude that B is $\underline{\Sigma}_1^{(i)}(N_j^*)$ is a proper segment of N_t, where $t = T^N(j+1)$. Hence $B \in \underline{\Sigma}_1^{(i)}(N_j^*) \subset N$. But then $B \cap \mathbb{P}(\lambda) \cap N \subset J_{\sigma(\lambda)}^{E^N}$, since $\sigma(\lambda) > \lambda$ is regular in N. Hence $J_{\sigma(\lambda)}^{E^N}$ is a ZFC^- model and $M \in J_{\sigma(N)}^{E^N} \subset N$.

QED(Case 3.1)

Case 3.2. Case 3.1 fails.

Then $\kappa_j < \lambda$. But $\tau_j \geq \lambda$, since otherwise $\tau_j < \lambda$ is a cardinal in M, hence in N. Hence $N_j^* = N$ and no truncation would take place at $j+1$. Contradiction! Thus:

$$\lambda = \tau =: \tau_j,\ N_j^* = N^* = N||\gamma,\ \kappa_j = \kappa,$$

where κ is the cardinal predecessor of λ in M and $\gamma > \lambda$ is maximal such that τ is a cardinal in $N||\gamma$. Then:

(1) $\pi : N^* \longrightarrow_F^* N_{j+1}$ where $\pi = \pi_{0,j+1}^N$, $F = E_{\nu_j}^{N_j}$

Since:

$$\pi_{j+1,\eta} : N_{j+1} \longrightarrow_{\Sigma^*} M_\eta \text{ and } \operatorname{crit}(\pi_{j+1,\eta}) > \lambda,$$

we know that:

(2) $\rho^{l+1} < \lambda < \rho^l$ in N_{j+1}

By the definition of N^* we have: $\rho^\omega_{N^*} < \lambda$. But $\rho^\omega_{N^*} \geq \kappa$, since κ is a cardinal in N and $N^* \in N$. Hence:

(3) $\rho^\omega_{N^*} = \kappa$.

Now let: $\rho^{i+1} \leq \kappa < \rho^i$ in N^*. Then:

$$\rho^{i+1} \leq \kappa < \lambda \leq \rho^i \text{ in } N_{j+1},$$

since:

$$\lambda < \sup \pi\text{"}\lambda = \lambda(F) \leq \sup \pi\text{"}\rho^i_{N^*} = \rho^i_{N_{j+1}}.$$

Hence $i = l$ and:

(4) $\rho^{l+1} = \kappa < \rho^l$ in N_{j+1}.

We now claim:

(5) $B \in \operatorname{Def}(N^*)$, i.e. B is definable in parameters from N^*. Hence $B \in N$.

Proof. For $\xi < \lambda$ define a map $g_\xi : \kappa \longrightarrow \kappa$ as follows:

For $\alpha < \kappa$ set:

- $X_\alpha =$ the smallest $X \prec J^{E^{N^*}}_\lambda$ such that $\alpha \cup \{\xi\} \in X$.
- $C_\xi = \{\alpha < \kappa : X_\xi \circ k \subset \alpha\}$.

For $\alpha \in C_\xi$, let $\sigma_\xi : Q_\xi \overset{\sim}{\longleftrightarrow} X_\xi$ be the transitivator of X_ξ. Set:

$$g_\xi(\alpha) =: \begin{cases} \sigma_\xi^{-1}(\xi) & \text{if } \alpha \in C_\xi \\ \varnothing & \text{if not} \end{cases}$$

It is easily seen that:

$$\pi(g_\xi)(\kappa) = \xi \text{ where } \pi = \pi^N_{0,j+1}.$$

Since B is $\underline{\Sigma}^{(l)}_1(N_{j+1})$ we have:

$$B_\zeta \longleftrightarrow \bigvee z \in J^{E^{N_{j+1}}}_{\rho^l_{N_{j+1}}} B'(z, \zeta, a)$$

for some $a \in N_{j+1}$. But π takes cofinally to $\rho^l_{N_{j+1}}$. Hence:

$$B_\zeta \longleftrightarrow \bigvee u \in J^{E^{N^v}}_{\rho_{N^*}} \bigvee z \in \pi(u) B'(z, \zeta, u).$$

Let $f \in \Gamma^*(\kappa, N^*)$ such that $a = \pi(f)(\alpha), \alpha < \lambda$. We know that $\xi = \pi(g_\xi)(\kappa)$ for $\xi < \lambda$. But then the statement B_ζ is equivalent to

$$\bigvee u \in J^{E^{N^v}}_{\rho^f_{N^*}} \{\langle \mu, \delta \rangle : \bigvee x \in uB''(x, g_\zeta(\mu), f(\delta))\} \in F_{\langle K, \alpha \rangle}$$

where $F = E^{N_j}_{\nu_j}$ and B'' is $\Sigma^{(l)}_0(N^*)$ by the same definition. But $F_{\langle \kappa, \alpha \rangle}$ is $\underline{\Sigma}_1(N^*)$ by closeness. QED(5)

But then $B \in \text{Def}(N^*) \subset J^{E^N}_{\sigma(\lambda)} \subset N$. Hence $M \in N$.

QED(Lemma 4.3.1)

4.3.2 Soundness and Cores

Let N be any acceptable structure. Let $m < \omega$. In §2.5 we defined the set R^n_N of *v*ery good n-parameters. The definition is equivalent to:

> $a \in R^n$ iff a is a finite set of ordinals and for $i < n$, each $x \in N||\rho^i$ has the form $F(\xi, a)$ where F is a $\Sigma^{(i)}_1(N)$ map and $\xi < \rho^{i+1}$.

We said that N is n-sound iff $R^n_N = P^n_N$. It follows easily that N is n-sound iff $p^n \in R^n$, where $p^n = p^n_N$ is the $<_*$-least $p \in P^n$. We called N **sound** iff it is n-sound for all n. It followed that, if N is sound, then $\rho^n \setminus \rho^i = p^i$ for $i \leq n < \omega$.

We have now shown that, if N is a mouse then $p^n \setminus \rho^i = p^i$ for $i \leq n < \omega$, regardless of soundness. We set: $p^* = \bigcup_{n<\omega} p^n$. Then $p^* = p^n$ whenever $\rho^n = \rho^\omega$ in N. We know:

Lemma 4.3.2. *If N is a mouse and $\pi : \overline{N} \longrightarrow_{\Sigma^*} N$ strongly, then $\overline{N}$ is a mouse and $\pi(p^*_{\overline{N}}) = p^*_{N^*}$.*

Proof. $\overline{N}$ is a mouse by a copying argument. Hence $\overline{N}$ is solid. But then $\pi(p^i_{\overline{N}}) = P^i_N$ for all $i < \omega$, by Lemma 4.1.11.

QED(Lemma 4.3.2)

We know generalize the notion R^n_N as follows:

Definition 4.3.1. Let $\rho^\omega_N \leq \mu \in N, a \in R^{(\mu)}_N$ iff a is aa finite set of ordinals and for some n,

- $\rho^n \leq \mu < \rho^{n-1}$ in N.

- Every $x \in N||\rho^{n-1}$ has the form $F(\vec{\xi}, a)$, where $\xi_1, \ldots, \xi_r < \mu$ and F is $\Sigma_1^{(n-1)}(N)$.
- If $j < n-1$, then $a \in R_N^j$.

We also set:

Definition 4.3.2. N is sound above μ iff for some $n, \rho^n \leq \mu < \rho^{n-1}$ in N and whenever $p \in P_N^n$ then $p \setminus \mu \in R_N^{(\mu)}$.

(It again follows that N is sound above μ iff $p_N^n \setminus \mu \in R_N^{(\mu)}$.) We prove:

Lemma 4.3.3. *Let N be a mouse. Let $\rho_N^\omega \leq \mu \in N$. There is a unique pair σ, M such that:*

- $\sigma : M \longrightarrow_{\Sigma^*} N$
- *M is a mouse which is sound above μ*
- $\sigma \restriction \mu = \mathrm{id}$ *and* $\sigma(p_M^*) = p_N^*$.

Before proving this, we develop some of its consequences.

Definition 4.3.3. Let N be a mouse. If M, σ are as above, we call M the **μ-th core of** N, denoted by: $\mathrm{core}_\mu(N)$, and σ the **μ-th core map**, denoted by σ_μ^N.

We also set: $\mathrm{core}(N) = \mathrm{core}_{\rho_N^\omega}(N)$ and $\sigma^N = \sigma_{\rho_N^\omega}^N$, $M = \mathrm{core}(N)$ is the **core** of N, and σ^N is the **core map**.

We leave it to the reader to prove:

Corollary 4.3.4. *Let N be a mouse. Then:*

- $\mathrm{core}_\mu(\mathrm{core}_\mu(N)) = \mathrm{core}_\mu(N)$.
- *N is sound above μ iff* $N = \mathrm{core}_\mu(N)$.
- *Let* $M = \mathrm{core}_\mu(N), \overline{\mu} \leq \mu, \overline{M} = \mathrm{core}_{\overline{\mu}}(M)$. *Then* $\overline{M} = \mathrm{core}_{\overline{\mu}}(M)$ *and* $\sigma_\mu^N \sigma_{\overline{\mu}}^M = \sigma_{\overline{\mu}}^N$.

We now turn to the proof of Lemma 4.3.3. By Löwenheim-Skolem argument it suffices to prove it for countable N. We first prove uniqueness. Suppose not. Let M, π and M', π' both have the property. If $x \in M$, then $x =$

$F(\vec{\xi}, P_N^*)$ where F is good and $\xi_1, \ldots, \xi_r < \mu$, since M is sound above μ. Hence:

$$\pi(x) = \tilde{F}(\vec{\xi}, P_N^*)$$

where $\tilde{F}$ has the same good definition over N. But then in N the Σ^* statement holds:

$$\bigvee y\, y = \tilde{F}(\vec{\xi}, P_N^*).$$

(This is Σ^* since it results from the substitution of $\tilde{F}(\vec{\xi}, P_N^*)$ in the formula $\nu = \nu$.) Hence in M' we have:

$$\bigvee y\, y = F'(\vec{\xi}, P_N^*),$$

where F' has the same good definition over M'. Thus $\mathrm{rng}(\pi) \subset \mathrm{rng}\, \pi'^{-1}$ and $\pi'^{-1}\pi$ is a Σ^*-preserving map of M to M'. A repeat of this argument then shows that $\mathrm{rng}(\pi') \subset \mathrm{rng}(\pi^{-1})$ and $\pi'^{-1}\pi$ is an isomorphism of M onto M'. But M, M' are transitive. Hence $M = M'$ and $\pi = \pi'$.

QED

This prove uniqueness. We now prove existence. Let $a = p_N^*$. Let $\rho^{n+1} \leq \mu < \rho^n$. Set $\overline{N} = N^{n,a}$. Let $b = a \cap \rho_N^n$ and set:

$$X = h_{\overline{N}}(\mu \cup b) = \text{ the closure of } \mu \cup b \text{ under } \Sigma_1(\overline{N}) \text{ functions.}$$

Let $\overline{\sigma} : \overline{M} \overset{\sim}{\longleftrightarrow} \overline{N}|X$ be the transitivazation of $\overline{N}|X$. By the downward extension lemma, there are unique $M, \sigma \supset \overline{\sigma}, \overline{a}$ such that:

$$\overline{M} = M^{n,\overline{a}},\ \sigma : M \longrightarrow_{\Sigma_1^{(n)}} N,\ \sigma(\overline{a}) = a.$$

Clearly, $\sigma \restriction \mu = \mathrm{id}$. Moreover, $\overline{a} \in R_M^{(\mu)}$. It suffices to prove:

Claim. σ is Σ^*-preserving and $\overline{a} = p_M^*$.

If $\sigma = \mathrm{id}$ and $M = N$, there is nothing to prove, so suppose not. Let $\lambda = \mathrm{crit}(\sigma)$. (Hence $\mu \leq \lambda$.) There is then a $h \leq n$ such that $\rho^{h+1} \leq \lambda < \rho^h$ in N. λ is a regular cardinal in M, since $\sigma(\lambda) > \lambda$. It follows easily that σ witnesses the phalanx $\langle N, M, \lambda\rangle$. Note that $\rho_M^\omega \leq \mu \leq \lambda$, since $\overline{a} \in R_M^{(\mu)}$. We now apply the simplicity lemma, coiterating $N, \langle N, M\lambda\rangle$ with:

$$I^N = \langle\langle N_i\rangle, \langle\nu_i^N\rangle, \langle\pi_{i,j}^N\rangle, T^N\rangle$$

$$I^M = \langle\langle M_i\rangle, \langle\nu_i^M\rangle, \langle\pi_{i,j}^M\rangle, T^M\rangle$$

being the iteration of $N, \langle N, M, \lambda\rangle$ respectively. We assume that the iteration terminates at an $\eta < \omega_1$ and that $\langle\nu_i : 1 \leq i < \eta\rangle$ is the sequence of coindices.

It is now time to mention that some of the steps in the proof of solidity go through with a much weaker assumption on the phalanx $\langle N, M, \lambda\rangle$ and its witness σ. In particular:

Lemma 4.3.5. *Let σ witness $\langle N, M, \lambda\rangle$, where $R_M^{(\lambda)} \neq \varnothing$. If cases 2 or 3 hold, then $M \in N$.*

The reader can convince himself of this by an examination of the solidity proof. But the premiss of Lemma 4.3.5 is given. Hence:

(1) Case 1 applies.

Proof. Suppose not. Let A be $\Sigma_1^{(h)}(N)$ in a such that $A \cap \rho_N^{h+1} \notin N$. Let $\overline{A}$ be $\Sigma_1^{(h)}(M)$ in $\overline{a}$ by the same definition. Then $A \cap \rho_N^{h+1} = \overline{A} \cap \rho_N^{h+1} \in N$, since $\overline{A} \in \underline{\Sigma}_\omega(M) \subset N$. Contradiction!

QED(1)

Then $M_\eta = N_\eta$ and there is no truncation on the main branch of I^N. Then $\pi_{1,\eta}^M : M \longrightarrow_{\Sigma^*} M_\eta$. Hence, by a copying argument, M is a mouse, hence is solid. Since $\mathrm{crit}(\pi_{1,\eta}^M) \geq \lambda$, we have:

(2) $\mathbb{P}(\lambda) \cap M = \mathbb{P}(\lambda) \cap M_\eta$ and $\rho_M^i = \rho_{M_\eta}^i$ for $i > h$.

But:

(3) $\mathrm{crit}(\pi_{1,\eta}^N) \geq \rho^{h+1}$.

Proof. Suppose not. then there is $j + 1 \leq_{T^N} \eta$ such that $\kappa_j < \rho^{h+1}$. Let j be the least such. Let $t = T^N(j + 1)$. Then:

$$\kappa_j < \sup\ \pi_{t,j+1}{}^{"}\rho_N^{h+1} \leq \rho_{N_{j+1}}^{h+1} \leq \rho_{N_\eta}^{h+1} = \rho_M^{h+1} > \kappa_j.$$

Contradiction!

QED(3)

Hence:

(4) $\rho_N^i = \rho_M^i$ for $i > h$. Moreover if $\rho^i = \rho_N^i$, then $\mathbb{P}(\rho^i) \cap N = \mathbb{P}(\rho^i) \cap M$ for $i > h$.

Using this we get:

(5) $\sigma : M \longrightarrow_{\Sigma^*} N$.

We first show that σ is Σ^*-preserving. By induction on $i \geq h$ we show:

Claim. σ is $\Sigma_1^{(i)}$-preserving.

For $i = h$, this is given. Now let $i = k + 1 \geq h$ and let it hold for k. Let A be $\Sigma_1^{(i)}(M)$. then:

$$Ax \longleftrightarrow \langle H^i, B_x^1, \ldots, B_x^r \rangle \models \varphi$$

where φ is a Σ_1-sentence and:

$$B_x^i \{z \in H^i : \langle z, x \rangle \in B^l\},$$

where B^l is $\Sigma_1^{(k)}(M)$ for $l = 1, \ldots, r$. Let A' be $\Sigma_1^{(k)}(M)$ by the same definition. Then:

$$B_{zx}^l \longleftrightarrow B_{z\sigma(x)}^{l'} \text{ for } z \in H_M^i = H_N^i.$$

Hence $Ax \longleftrightarrow A'\sigma(x)$.

QED(5)

But

(6) σ is strongly Σ^*-preserving.

Proof. Let $\rho^m = \rho^\omega$ in M and N. Let A be $\Sigma_1^{(m)}(M)$ in x such that $A \cap \rho^m \notin M$. Let A' be $\Sigma_1^{(m)}(M)$ in $\sigma(x)$ by the same definition. Then $A \cap \rho^n = A' \cap \rho^m \notin N$, since $\mathbb{P}(\rho^m) \cap M = \mathbb{P}(\rho^m) \cap N$.

QED(6)

But then $\sigma(P_M^*) = P_N^*$. Hence $P_M^* = \overline{a} = \overline{\sigma}'(P_N^*)$. We know that $\overline{a} \in R_M^{(\mu)}$. Hence M is solid above μ.

QED(Lemma 4.3.5)

4.3.3 Condensation

The condensation lemma for L says that if M is transitive and $\pi : M \longrightarrow J_\alpha$ is a reasonable embedding, then $M \triangleleft J_\alpha$. It is natural to ask whether the dame holds when we replace J_α by an arbitrary sound mouse. In order to have any hope of doing this, we must employ a more restrictive notion of reasonable. Let us call $\sigma : M \longrightarrow N$ reasonable iff either $\sigma = \mathrm{id}$ or σ witnesses the phalanx $\langle N, M, \lambda \rangle$ and $\rho_M^\omega \leq \lambda$. We then get:

Lemma 4.3.6. *If N, M are sound mice and $\sigma : M \longrightarrow N$ is reasonable in the above sense, then $M \triangleleft N$.*

It ifs not too hard to prove this directly from the solidity lemma and the simplicity lemma. We shall, however, derive it from a deeper structural lemma:

Lemma 4.3.7. *Let N be a mouse. Let σ witness the phalanx $\langle N, M, \lambda\rangle$. Then M is a mouse. Moreover, if M is sound above λ, then one of the following hold:*

(a) $M = \mathrm{core}_\lambda(N)$ *and* $\sigma = \sigma_\lambda^N$.

(b) M *is a proper segment of* N.

(c) $\pi : N||\gamma \longrightarrow_F^* M$, *where* $F = F_\mu^N$ *such that:*

  (i) $\lambda < \gamma \in N$ *such that* $\rho_{N||\gamma}^\omega < \lambda$.

  (ii) $\lambda = \kappa^{+N||\gamma}$ *where* $\kappa = \mathrm{crit}(F)$.

  (iii) F *is generated by* $\{\kappa\}$.

Remark. In case (c) we say that M is one measure away from N. Then γ is maximal such that λ is a cardinal in $N||\gamma$. Hence $\rho_{N||\gamma} \leq \kappa$. But κ is a cardinal in N and $N||\gamma \in N$. Hence $\rho_{N||\gamma} = \kappa$. But $\pi \upharpoonright \kappa = \mathrm{id}$ and $\pi(p_{N|\gamma}^*) = p_M^*$. Hence $N||\gamma = \mathrm{core}(M)$ and π is the core map. Clearly, μ is least such that $E_\mu^M \neq E_\mu^N$.

Remark. Lemma 4.3.6 follows easily, since the possibilities (a) and (c) can be excluded. (a) cannot hold, since otherwise $M = \mathrm{core}_\lambda(N) = N$ by the soundness of N. Hence $\sigma_N^\lambda = \mathrm{id}$. Contradiction, since $\mathrm{crit}(\sigma_N^\lambda) = \lambda$. If (c) held, then $N^* = \mathrm{core}(M)$ where $N^* = N||\gamma$, and π is the core map. But M is sound. Hence $M = N^* = \mathrm{core}(M)$ and $\pi = \mathrm{id}$. Contradiction!

Remark. Lemma 4.3.7 has many applications, through mainly in setting where the awkward possibility (c) can be excluded (e.g. when λ is a limit cardinal in M). We have given a detailed description of (c) in order to facilitate such exclusions.

We now prove Lemma 4.3.7. We can again assume N to be countable by Löwenheim-Skolem argument. We again coiterate against $\langle N, M, \lambda\rangle$ getting the iterations:

$$I^N = \langle\langle N_i\rangle, \ldots, T^N\rangle,\ I^M = \langle\langle M_i\rangle, \ldots, T^M\rangle$$

with coiteration indices $\langle \nu_i : i < \eta\rangle$, where the coiteration terminates at $\eta < \omega_1$. Then $\pi_{1,\eta} : M \longrightarrow_{\Sigma^*} M_\eta$ and M is a mouse by a copying argument. Now let M be sound above λ. We again consider three cases:

Case 1. $M_\eta = N_\eta$ and I^N has no truncation on the main branch.

We can literally repeat the proof in cases of Lemma 4.3.5, getting:

$$\sigma \text{ is strongly } \Sigma^*\text{-preserving.}$$

Hence $\sigma(p^*_M) = p^*_N$ where M is sound above λ and $\sigma = \sigma^N_\lambda$.

QED(Case 1)

Case 2. M_η is a proper segment of N_η.

We can literally repeat the proof in Case 2 of the solidity Lemma, getting: M is a proper segment of N.

Case 3. The above cases fail.

Then $M_\eta = N_\eta$ and I^N has a truncation on the main branch. Let $j+1$ be the last truncation point on the main branch. Then M is a mouse and $\pi^M_{1,\eta}$ is strongly Σ^*-preserving. Hence $\pi^M_{1,\eta}(p^*_M) = p^*_{M_\xi}$. But $\kappa_i \geq \lambda$ for all $i \leq_{T^M} \eta$. Hence $\mathrm{crit}(\pi_{1,\eta}) \geq \lambda$. Hence:

$$M = \mathrm{core}_\lambda(M_\eta) \text{ and } \pi_{1,\eta} = \sigma^{M_\xi}_\lambda,$$

since M is sound above λ. We also know:

$$\kappa_i \geq \lambda_j \geq \lambda \text{ for } j+1 <_{T^N} i+1 <_{T^N} \eta.$$

Hence $\mathrm{crit}(\pi^N_{j+1,\eta}) \geq \lambda$ and $\pi^N_{j+1,\eta}(p^*_{N_{j+1}}) = p^*_{N_\eta} = p^*_{M_\eta}$. Hence:

$$M = \mathrm{core}_\lambda(N_{j+1}) \text{ and } \sigma^{N_{j+1}}_\lambda = (\pi^N_{j+1,\eta})^{-1} \circ \pi^M_{1,\eta}.$$

We consider two cases:

Case 3.1. $\kappa_j \geq \lambda$.

Then N^*_j is a proper initial segment of N_j, hence is sound. Since $\kappa_j \geq \lambda$, it follows as before that $M = \mathrm{core}_\lambda(N^*)$. Hence $M = N^*_j$ by the soundness of N^*_j. But this means that M was not moved in the iteration I^M up to $t = T^N(j+1)$, since if $h < t$ in the least point active in I^*, then $E^M_{\nu_h} \neq \varnothing$ and hence $E^{N_t}_{\nu_h} = E^{N^*_j}_{\nu_h} = \varnothing$. Hence $N^*_j \neq M$. Contradiction!

Thus $M_t = M = N^*_j$ is a proper segment of N_t. Hence the coiteration terminates at $t < \eta$. Contradiction!

QED(Case 3.1)

Case 3.2. Case 3.1 fails.

Then $\kappa_j < \lambda$. But $\tau_j \geq \lambda$, since otherwise τ_j is a cardinal in N and $N^*_j = N$. Hence $j+1$ is not a truncation point in I^N. Contradiction!

Thus $\tau_j = \lambda$. λ is regular in M, since $\sigma(\lambda) > \lambda$. But then $\lambda = \kappa_j^+$ in M and $\sigma(\lambda) = \kappa_j^+$ in N. Hence λ is not a cardinal in N. $E_\lambda^M = \emptyset$, since λ is a cardinal in M. But $E_\lambda^N = \emptyset$, since otherwise κ_j, being regular in N, would be regular in $N||\lambda$. Hence $N||\lambda$ would be an active premouse of type 3. By lemma 3.3.26 in §3.3, contradiction!

But 0 is inactive in I^N and $\nu_1 =$ the least ν such that $E_\nu^M \neq E_\nu^N$. Hence $\nu_i \geq \nu_1 > \lambda$ for all i which are active in I^N. Hence no $i < t$ is active in I^N, since otherwise $\kappa_j < \lambda_i$. But $t = T(j+1)$ is the least t such that t is active in I^N and $\kappa_j < \lambda_t$. Contradiction!

But then $N = N_t$ and $N_j^* = N^* = N||\gamma$, where γ is maximal such that $\tau = \lambda$ is a cardinal in $N||\gamma$. Hence $\kappa_j = \kappa =$ the cardinal predecesor of τ in N^*. $\kappa = \rho_{N^*}^\omega$, since κ is a cardinal in N and $N^* \in N$. We have:

$$\kappa_i \geq \lambda \text{ for } 1 \leq_{T^M} i+1 \leq_{T^M} \eta$$

Hence $\mathrm{crit}(\pi_{1,\eta}^M) \geq \lambda$. But:

$$\kappa_i \geq \lambda_t \geq \lambda \text{ for } j+1 <_{T^N} i+1 <_{T^N} \eta$$

Hence $\mathrm{crit}(\pi_{j+1,\eta}^N) \geq \lambda$. Hence:

$$M = \mathrm{core}_\lambda(N_{j+1}),\ (\pi_{j+1,\eta}^N)^{-1} \circ \pi_{1,\eta}^M = \sigma_\lambda^{N_{j+1}},$$

$\rho_{N^*}^\omega \leq \kappa$. But then $\rho_{N^*}^\omega = \kappa$ since κ is a cardinal in N and $N^* \in N$. Set $\langle \tilde{N}, \tilde{F} \rangle = N_j||\nu_j$. Then:

$$\pi_{t,j+1} : N_j^* \longrightarrow_{\tilde{F}}^* N_{j+1}$$

By closeness we have: $\tilde{F}_\kappa \in \underline{\Sigma}_1(N^*)$. Hence $\tilde{F}_\kappa \in \underline{\Sigma}_1(N^*) \subset N||\sigma(\tau)$, where $\sigma(\tau)$ is regular in N and $\gamma < \sigma(\tau)$. Set: $\bar{Q} = N||\tau$. By a standard construction there is a unique triple $\langle Q, F, \bar{\pi} \rangle$ such that F is a full extender at κ with base $\bar{Q}$, $\bar{\pi} \colon \bar{Q} \longrightarrow_F Q$ is the extension of $\langle \bar{Q}, F \rangle$, F is generated by $\{\kappa\}$ and $F_\kappa = \tilde{F}_\kappa$. (To see this we note that $\tilde{F}_\kappa$ is a normal ultrafilter on $\bar{Q}$ at κ. Hence we can form the ultraproduct $\bar{\pi} \colon \bar{Q} \longrightarrow_{\tilde{F}_\kappa} Q$. Q is well-founded , since each element of Q has the form $\bar{\pi}(f)(\kappa)$ where $f \in \bar{Q}$, $f \colon \kappa \longrightarrow \bar{Q}$ and:

$$\begin{aligned} \bar{\pi}(f)(\kappa) \in \tilde{\pi}(g)(\kappa) &\iff \{\xi \colon f(\xi) \in g(\xi)\} \in \tilde{F}_\kappa \\ &\iff \pi_{t,i+1}^N(f)(\kappa) \in \pi_{t,i+1}^N(g)(\kappa). \end{aligned}$$

Set: $F = \bar{\pi} \upharpoonright \mathbb{P}(\kappa)$. Then Q, F, π have the above properties.) The construction of Q, F, $\bar{\pi}$ can be carried out in the ZFC^- model $N||\sigma(\tau)$, since

$\bar{Q}, \tilde{F}_\kappa \in N||\sigma(\tau)$. Then $Q, F, \tilde{\pi} \in N$. It is easily seen that F is close to N^*. Hence we can form the Σ^* ultrapower:

$$\pi\colon N^* \longrightarrow^*_F M'.$$

M' is transitive, since each of its element has the form $\pi(f)(\kappa)$, where $f \in \Gamma^*(\kappa, N^*)$ and as before:

$$\pi(f)(\kappa) \in \pi(g)(\kappa) \iff \pi^N_{t,i+1}(f)(\kappa) \in \pi^N_{t,i+1}(g)(\kappa).$$

There is a $\Sigma_0^{(n-1)}$ preserving map $\sigma\colon M' \longrightarrow N_{i+1}$ defined by:

$$\sigma(\pi(f)(\kappa)) = \pi_{t,i+1}(f)(\kappa)$$

for $f \in \Gamma^*(\kappa, N^*)$. Since π takes $\rho^{n-1}_{N^*}$ cofinally to $\rho^{n-1}_{M'}$ and $\pi t, i+1$ takes $\rho^{n-1}_{N^*}$ cofinally to $\rho^{n-1}_{N_{j+1}}$, we know that σ' takes $\rho^{n-1}_{N^*}$ cofinally to $\rho^{n-1}_{N'}$. Hence σ is $\Sigma_1^{(n-1)}$-preserving. Since $\sigma \restriction \kappa = \mathrm{id}$ and $\kappa \geq \rho^n_{N^*}$, it follows easily that σ' is Σ^* preserving.

Claim 1. M' is sound above τ. Hence $M = M' = \mathrm{core}_\tau(N_{j+1})$.

Proof. Let $\rho^n \leq \kappa < p^{n-1}$ in N^*. Hence $\kappa = \rho^n = \rho^\omega$ in N^*. Let $x \in M'$. Then $x = \pi(f)(\kappa)$, where $f \in \Gamma^*(\kappa, N^*)$.

By the soundness of N^* we may assume:

$$f(\xi) = F(\xi, a, \vec{\zeta})$$

where F is a good $\Sigma_1^{(n-1)}(N^*)$ function, $a = p^n_{N^*}$ and $\zeta_1, \ldots, \zeta_r < \kappa$. Hence:

$$\pi(f)(\kappa) = F'(\kappa, \pi(a), \vec{\zeta})$$

where F' is $\Sigma_1^{(n-1)}(M')$ by the same good definition, $\pi(a) = p^n_{M'}$, and $\vec{\zeta} < \tau$. But $\kappa < \tau$, where $\rho^n < \tau < \rho^{n-1}$ in M'.

QED(Claim 1)

All that remains is to show:

Claim 2. $\langle Q, F\rangle = N||\mu$ for a $\mu \leq \gamma$.

Proof. We note that if $\langle Q, F\rangle = N||\mu$, then we automatically have $\mu \leq \gamma$, since τ is then a cardinal in $N||\mu$ and γ is maximal s.t. τ is a cardinal in $N||\gamma$.

(1) $\langle Q, F\rangle \in N$.

Proof.$(E^{N_\gamma}_{\nu_j})_\kappa = F_\kappa \in N||\sigma(\tau)$, where $N||\sigma(\tau)$ is a ZFC^- model. Hence $\langle Q, F\rangle \in N||\sigma(\tau)$ since the construction of $\langle Q, F\rangle$ can be carried out in $N||\sigma(\tau)$ by absoluteness.

(2) $\rho^1_{\langle Q,F\rangle} \leq \tau$.

Proof. As above, let $\overline{\pi} : N||\sigma(\tau) \longrightarrow Q$ be the extension map given by F. By §3.2 we know that $\overline{\pi}$ is $\underline{\Sigma}_1(\langle Q, F\rangle)$ and that $\langle Q, F\rangle$ is amenable. But then there is a $\underline{\Sigma}_1(\langle a, \pi\rangle)$ partial map G of $N||\tau$ onto Q defined by: $G(f) = \overline{\pi}(f)(\kappa)$ for $f \in N||\tau, : f : \kappa \longrightarrow N||\tau$.

QED(2)

Define a map $\tilde{\sigma} : \langle Q, F\rangle \longrightarrow N_j||\nu_j$ by:

$$\tilde{\sigma}(\overline{\pi}(f)(\kappa)) := \tilde{\pi}(f)(\kappa) \text{ for } f \in N|\tau, f : \kappa \longrightarrow N||\tau,$$

where $\tilde{\pi} = \pi^N_{t,i}\restriction(N||\tau)$ is the extension of $\langle N_j||\tau, F\rangle$.

Then:

(3) $\tilde{\sigma} : \langle Q, F\rangle \longrightarrow_{\Sigma_0} N_j||\nu_j$. In fact, it is also cofinal.

(4) $\tilde{\sigma}\restriction\tau + 1 = \mathrm{id}$.

Proof. Set:

$$\begin{aligned} &i^+ =: \text{ the least } \eta > i \text{ such that } \eta = \overline{\overline{\eta}} \geq \omega \text{ in } Q \\ &pl := \langle i^+ : i < \kappa\rangle. \end{aligned}$$

Then $\overline{\pi}(pl)(\kappa) = \kappa^{+Q} = \kappa^{+N_j||\nu_j} = \tilde{\pi}(pl)(\kappa)$.

Set:

$$\begin{aligned} &\Gamma =: \{f \in N||\tau : f : \kappa \longrightarrow \kappa \wedge f(i) < i^+ \text{ for } i < \kappa\} \\ &\dot{<} = \{\langle f, g\rangle \in \Gamma : \{i : f(i) \in g(i)\} \in F_\kappa\} \end{aligned}$$

Then every $\xi < \tau$ has the form $\overline{\pi}(f)(\kappa)$ fo an $f \in \Gamma$. Clearly, $f \dot{<} g \longleftrightarrow \overline{\pi}(f)(a) < \pi(g)(a)$ for $f, g \in \Gamma$. Hence by $\dot{<}$-induction on $g \in \Gamma$:

$$\pi(g)(\kappa) = \{\overline{\pi}(\kappa) : f \dot{<} g\}.$$

But $F_\kappa = (E^{N_j}_{\nu_j})_\kappa$. Hence the same holds for $\tilde{\pi}$ in place of $\overline{\pi}$. Thus, by $\dot{<}$-induction on $g \in \Gamma$:

$$\tilde{\pi}(g)(\kappa) = \{\tilde{\pi}(\kappa) : f \dot{<} g\} = \{\pi(\kappa) : f \dot{<} g\} = \overline{\pi}(f)(\kappa).$$

Hence $\tilde{\sigma}\restriction\tau = \mathrm{id}$. But:

$$\tilde{\sigma}(\tau) = \tilde{\sigma}(\overline{\pi}(pl)(\kappa)) = \overline{\pi}(pl)(\kappa) = \tau$$

QED(4)

Redoing the proof of (2) with more care, we get:

(5) $\varnothing \in R^{(\tau)}_{\langle Q,F\rangle}$.

Proof. $X \subset \kappa$ and $X = \kappa$ are both $\Sigma_1(\langle Q,F\rangle)$, since:

$$X \subset \kappa \longleftrightarrow X \in \mathrm{dom}(F),\ X = \kappa \longleftrightarrow X \in \mathrm{On} \cap \mathrm{dom}(F).$$

Thus this suffices to show that $\overline{\pi}$ is $\Sigma_1(\langle Q,F\rangle)$. We note that if $f : X \overset{\text{onto}}{\longrightarrow} u$ and u is transitive, then $\overline{\pi}(f) : \overline{\pi}(X) \overset{\text{onto}}{\longrightarrow} \overline{\pi}(u)$ and $\overline{\pi}(u)$ is transitive. But $\overline{\pi}(X) = F(X)$ for $X \subset \kappa$. Hence $y = \overline{\pi}(x)$ can be expressed by saying that there are:

$$X, Y, f, u, X', Y', f', u'$$

such that:

$$\begin{aligned}
&\bigvee u \bigwedge X, Y \in \mathrm{dom}(F) \wedge f : X \overset{\text{onto}}{\longrightarrow} u \wedge x = f(0)\\
&\wedge \bigwedge \xi, \zeta \in X (f(\xi) \in f(\zeta) \longleftrightarrow \prec \xi, \zeta \succ \in Y)\\
&\wedge X' = F(X) \wedge Y' = F(Y) \wedge f' : X' \overset{\text{onto}}{\longrightarrow} u' \wedge y = f'(0)\\
&\wedge \bigwedge \xi, \zeta \in X'(f'(\xi) \in f'(\zeta) \longleftrightarrow \prec \xi, \zeta \succ \in Y')
\end{aligned}$$

QED(5)

We then prove:

(6) One of the following holds:

(a) $\langle Q,F\rangle = \mathrm{core}_\tau(N_j||\nu_j)$ and $\tilde{\sigma}$ is the core map.

(b) $\langle Q,F\rangle$ is a proper segment of $N_j||\nu_j$

(c) $\rho^\omega > \tau$ in $\langle Q,F\rangle$.

Proof. If $\tilde{\sigma} = \mathrm{id}, \langle Q,F\rangle = N_j||\nu_j$, then (a) holds. Now let $\tilde{\sigma} \neq \mathrm{id}$. Let $\tilde{\lambda} = \mathrm{crit}(\tilde{\sigma})$. Then $\tilde{\lambda} > \tau$ by (4). We know $\rho^1 \leq \tau < \tilde{\lambda}$ in $\langle Q,F\rangle$. Moreover $\tilde{\sigma}$ is Σ_0-preserving. It follows easily that $\tilde{\sigma}$ verifies the phalanx $\langle N_j||\nu_j, \langle Q,F\rangle, \tilde{\lambda}\rangle$. $\langle Q,F\rangle$ is then a mouse. Moreover, it is sound above τ since $\varnothing \notin R^{(\sigma)}_{\langle Q,F\rangle}$. Hence it is sound above $\tilde{\lambda}$ since $\tau < \tilde{\lambda}$. We then coiterate $N_j||\nu_j$ against $\langle N_j||\nu_j, \langle Q,F\rangle, \tilde{\lambda}\rangle$, using all that we have learned up until now. We consider the same three cases. In case 1, (a) holds. In case 2, (b) holds. We now consider case 3, using what we have learned up to now. We know that $\tilde{\lambda}$ is a successor cardinal in $\langle Q,F\rangle$ and that its predecessor $\tilde{\kappa}$ is a limit cardinal in $\langle Q,F\rangle$. Since $\tau < \tilde{\lambda}$ is a successor cardinal in $\langle Q,F\rangle$, we conclude: $\tau < \tilde{\kappa} = \rho^\omega$.

(7) $\langle Q,F\rangle$ is a proper segment of N.

Proof. Suppose not. We derive a contradiction. (c) cannot hold, since $\rho^\omega \leq \tau$ in $\langle Q,F\rangle$. We now show that (b) cannot occur. In fact, we show:

Claim. There is no $i \leq \eta$ such that $\langle Q, F\rangle$ is a proper segment of N_i.

Proof. Suppose not, Then $N_i \neq N$. Hence there is a least $h < i$ which is active in I^N. Then $J^{E^{N_i}}_{\nu_h} = J^{E^N}_{\nu_h}$, where $\nu_h > \tau$ is regular in N_i. But $\rho^\omega_{\langle Q,F\rangle} \leq \tau$. Hence $\langle Q, F\rangle$ is a proper segment of $J^{E^N}_{\nu_h}$, hence of N. Contradiction! QED(Claim)

We now show that (a) cannot occur. If $\nu_j \in N_j$ then $N||\nu_j$ is sound, hence sound above τ. Hence:

$$\langle Q, F\rangle = \mathrm{core}_\tau(N_j||\nu_j) = N_j||\nu_j$$

is a proper segment of N_j. Contradiction! Thus $N_j = N_j||\nu_j$. If there is no truncation on the main branch on $I^M|j+1$, then $N = N_j$. But τ then a cardinal in N_j and not in N. Contradiction! Hence there is a least truncation point $(i+1) \leq_T j$. Let $h = T(i+1)$ and $\pi = \pi_{h,j}$. Then:

$$\pi\colon N^*_i \longrightarrow_{\Sigma^*} N_j, \kappa_i = \mathrm{crit}(\pi),$$

N_j has the form $\langle J^E_\gamma, F'\rangle$. Hence N_i has the form $\langle J^{E^*}_{\nu^*}, F^*\rangle$ where $\kappa_i = \mathrm{crit}(F^*)$, $\tau_i = \tau(F^*)$. But then $\pi(\tau_i) = \tau = \tau(F')$. Hence $\tau \in \mathrm{rng}(\pi)$. Hence $\kappa_i > \tau$, since $(\kappa_i, \lambda_i) \cap \mathrm{rng}(\pi) = \emptyset$. Since N^*_i is sound, being a proper segment of N_h. Hence it is sound above τ. Since $\pi(p^*_{N_i}) = p^*_{N_j}$ and $\pi \restriction \tau = \mathrm{id}$, we conclude:

$$N^*_i = \mathrm{core}(N_j) = \langle Q, F\rangle.$$

But then $\langle Q, F\rangle$ is a proper segment of N_h. Contradiction!

QED(7)

QED(Lemma 4.3.7)

Using the condensation lemma, we prove a sharper version of the initial segment condition for mice:

Lemma 4.3.8. *Let $N = \langle J^E_\nu, F\rangle$ be an active mouse. Let $\overline{\lambda} \in N$. Let $\overline{F} = F|\lambda$ be a full extender. Set:*

$$M = \langle J^E_{\overline{\nu}}, \overline{F}\rangle \text{ where } \overline{\pi} : J^E_\tau \longrightarrow J^E \text{ is the extension of } \vec{F}$$

. Then M is a a proper segment of N.

Proof. Let $\kappa = \mathrm{crit}(F)$. Define $\tau = \tau_F, \lambda = \lambda_F, \nu = \nu_F$ as usual. Hence: $\tau = \kappa^{+N}, \lambda = F(\lambda)$. Then $\overline{\tau} = \tau_{\overline{F}}, \overline{\lambda} = \lambda_{\overline{F}}, \overline{\nu} = \nu_{\overline{F}}$. Let $\pi : J^E_\tau : J^E_\nu$ be the extension of F. Define: $\sigma : J^E_{\overline{\tau}} \longrightarrow J^E_\tau$ by:

$$\sigma(\overline{\pi}(f)(\alpha)) = \pi(f)(\alpha) \text{ for } \alpha < \overline{\lambda}, f \in J^E_\tau, \mathrm{dom}(f) = u.$$

Then $\overline{\lambda} = \mathrm{crit}(\lambda), \sigma(\overline{\lambda})$ and σ is Σ_0-preserving, where:

$$\rho^\omega_M \leq \overline{\lambda} \text{ and } \varnothing \notin R^{(\overline{\lambda})}_M.$$

This is because $\overline{\pi}$ is $\Sigma_1(M)$ and each element of M has the form $\overline{\pi}(f)(\alpha)$ where $f \in J^E_\tau$ and $\alpha < \overline{\lambda}$. It follows easily that σ witnesses the phalanx $\langle N, M, \overline{\lambda}\rangle$. Applying the condensation lemma, we see that one of the possibilities (a), (b), (c) holds. (c) cannot hold since $\overline{\lambda}$ is a limit cardinal in M. (a) cannot hold, since $M \in N$ by the initial segment condition. If (a) holds, we would have: $\sigma(p^*_M) = p^*_N, \sigma \upharpoonright \overline{\lambda} = \mathrm{id}$, where σ is Σ^*-preserving. But then $\rho^\omega_M = \rho^\omega_N$. Let $\rho = \rho^\omega_N$. Let A be $\Sigma^*(N)$ in p^*_N such that $A \cap \rho \notin N$. Let $\overline{A}$ be $\Sigma^*(M)$ in p^*_M by the same defition. Then:

$$A \cap \rho = \overline{A} \cap \rho \in \underline{\Sigma}^*(M) \subset N.$$

Contradiction! Thus, only the possibility (b) remains.

QED(Lemma 4.3.8)

As a corollary of the proof of Lemma 4.3.7, we obtain a lemma which will be very useful in the next chapter. We first define:

Definition 4.3.4. Let M be a premouse. Set:

$$\rho = \rho_M =: \rho^\omega_M, \mu = \mu_M = \{\xi \in M \mid \mathrm{card}(\xi) \leq \rho \text{ in } M\}.$$

Lemma 4.3.9. *Let N be a fully iterable premouse. Let $M = \mathrm{core}(N)$. Let $\mu = \mu_M$. Then $\mu = \mu_N$ and $M||\mu = N||\mu$.*

Proof. If $N = M$ there is nothing to prove, so assume $N \neq M$. Let $\sigma : M \longrightarrow N$ be the core map. Since $\sigma \neq \mathrm{id}$, it has a critical point λ. Clearly $\lambda \geq \rho = \rho_M = \rho_N$, since $\sigma|\rho = \mathrm{id}$. It is easily seen that σ verifies the phalanx $\langle N, M, \lambda\rangle$. Note that the two possibilities (b), (c) in the condensation lemma(4.3.7) cannot hold, since (b) would require: $M \in N$ and (c) would imply that M is unsound. Coiterate $\langle N, M, \lambda\rangle$, N to get I^M, I^N as in the proof of lemma 4.3.7. Then the cases 2 and 3 cannot hold, since then either (b) or (c) would follow. Hence case 1 holds-i.e. $M_\zeta = N_\zeta$ and I^N has no truncation on its main branch. We know that I^M has no truncation on its main branch, where $\kappa_i \geq \lambda \geq \rho$ for i on the main branch. Thus $\rho = \rho_{N_\zeta}$ and $\kappa_i > \rho$ for all i.

Then $\mu = \mu_M = \rho^{+M} = \rho^{+N_\zeta}$ and $M||\mu = N_\zeta||\mu$. Now suppose $\kappa_i = \rho$, where $i + 1$ is the first point above 1 on the main branch. Then $\pi_{1,i+1} : M \longrightarrow_{E^{M_i}_{\nu_i}} M_{i+1}$ where $\rho = \rho_{M_{i+1}}$ and $\mu = \tau_i = \rho^{+M}$. But then $\tau_i = \rho^{+M_{i+1}}$ and $M||\tau_i = M_{i+1}||\tau_i$. Since $\kappa_j \geq \lambda_i$ for $j + 1$ on the main branch with

$j+1 >_T i+1$, we conclude: $\tau_i = \rho^{+M_\zeta} = \mu_{N_\zeta}$ and $M||\tau_i = N_\zeta||\tau_i$, since $\pi^M_{i+1,\zeta}|\lambda_i = \mathrm{id}$. We have shown:

Claim 1. $\mu = \mu_{N_\zeta}$ and $M||\mu = N_\zeta||\mu$.

But since $\rho = \rho^\omega_{N_\zeta}$ we must have $\kappa_i \geq \rho$ for all $i+1$ on the main branch of I^N, since otherwise $\pi^N_{0\zeta}(\rho) = \rho^\omega_{N_\zeta} > \rho$. Hence we can respect the above proof on the N-side to get:

Claim 2. Let $\mu = \mu_N$. Then $\mu = \mu_{N_\zeta}$ and $N||\mu = N_\zeta||\mu$.

QED(Lemma 4.3.9)

We have defined $\mu = \mu_M$ in such a way that $\mu \notin M$ is possible.In fact we could have: $\rho = \mu = \mathrm{ht}(M)$. However, by the above proof:

Lemma 4.3.10. *Let N be fully iterable and $N \neq M = \mathrm{core}(N)$. Then for all fully iterable N' with $M = \mathrm{core}(N')$ we have:*

Let $\mu' = \mu_{N'}$. Then $\mu' \in N'$ and $\mu = \rho^{+N'}$.

We also note:

Lemma 4.3.11. *Let J^A_α be a constructible extension of J^A_β(i.e. $\beta \leq \alpha$ and $A \subset J^A_\beta$). Assume: $\rho = \rho^{J^A_\alpha} \geq \beta$. Then $J^A_\alpha = \mathrm{core}(J^A_\alpha)$ and $\sigma = \mathrm{id}$ is the core map.*

4.4 Mouselikeness

In §3 we showed that every normally iterable premouse which has the unique branch property is fully iterable. In the present chapter we have derived several deep structural properties of fully iterable premice. We shall call a premouse which has these properites *mouselike*, be it iterable or not. We define:

Definition 4.4.1. Let N be a premouse. N is *condensable* if and only if

(i) N is solid

(ii) Let $M = \mathrm{core}(N)$, $\rho = \rho^\omega_M = \rho^\omega_N$ and $\mu = \rho^{+N}$. Then $\mu = \rho^{+M}$ and $M||\mu = N||\mu$.

(iii) Let σ witness the phalanx $\langle N, M, \lambda\rangle$, where M is sound above λ. Then one of the alternatives (a), (b), (c) in lemma 4.3.7 hold.

Definition 4.4.2. N is *mouselike* if and only if every initial segment $N' \triangleleft N$ is condensible.

Definition 4.4.3. N is *precondensible*(or *pre-mouselike*) if and only if every *proper* initial segment $N' \triangleleft N$ is condensible.

We have seen that every fully iterable premouse W is condensible. Since every $N' \triangleleft N$ is then also fully iterable, we conclude that N is mouselike.

The definition of "condensible" becomes simpler if we assume N to be sound and solid. The conditions (i), (ii) are then vacuously true. (iii) then says that, if σ witnesses $\langle N, M, \lambda\rangle$ and M is sound above λ, then either (b) or (c) hold. (If (a) holds, then $M = \text{core}_\lambda(M)$ and $\sigma = \sigma_M^\lambda$. But by soundness, $M = \text{core}(M)$ and $\sigma_M^\lambda = \sigma_M = \text{id}$, contradicting the fact that $\lambda = \text{crit}(\sigma)$.)

In §4.1 we defined a premouse to be *presolid* if and only if all of it's proper initial segments are solid. Lemma 4.1.13 said that the property of being presolid is uniformly $\Pi_1(M)$ for premice M. Hence:

Lemma 4.4.1. *Let M, N be premice. Then*

- *If M is presolid and $\pi : M \longrightarrow_{\Sigma_1} N$, then N is presolid.*
- *If N is presolid and $\pi : M \longrightarrow_{\Sigma_0} N$, then M is presolid.*

We shall prove:

Lemma 4.4.2. *The property of being pre-mouselike is uniformly $\Pi_1(M)$ for premice M.*

Hence:

Lemma 4.4.3. *Let M, N be premice. Then:*

- *If M is pre-mouselike and $\pi : M \longrightarrow_{\Sigma_1} N$, then N is pre-mouselike.*
- *If N is pre-mouselike and $\pi : M \longrightarrow_{\Sigma_0} N$, then M is pre-mouselike.*

As preparation for the proof of lemma 4.4.2, we list a series of facts which are implicit in what we have done this far, but may not always have been made explicit.

Definition 4.4.4. $M = \langle |M|, E, F\rangle$ is a set *model* if and only if $|M|$ is transitive and $E, F \subset |M|$.

(*Note* we can, of course, generalize this to models with more than two predicates.)

In the following let U be any set which is transitive and closed under rudimentary functions.

Fact 1. The set $\{M \in U : M \text{ is a model}\}$ is uniformly $\Delta_1(U)$.

Models have a first order language $\mathbb{L}$ with predicate symbols $\dot{\in}, \dot{=}, \dot{E}, \dot{F}$. $\dot{\in}, \dot{=}$ are interpreted by $\in, =$ respectively and $\dot{E}, \dot{F}$ by E, F. We assume an "arithmetization" of $\mathbb{L}$, whereby the formulae of $\mathbb{L}$ are identified with objects in ω or V_ω in such a way that the normal syntactic relation and operation become recursive. (In §1.4.1 we proposed an arithmetization of languages over an admissible set. If we take the admissible set as V_ω, we get a suitable arithmetization of $\mathbb{L}$.)

Definition 4.4.5. The *satisfaction relation* is defined as follows: $M \models \varphi[f]$ means:

- M is a model
- φ is a formula of $\mathbb{L}$.
- f is a *variable interpretation* -i.e. f is function such that $\mathrm{dom}(f)$ is a finite set of variables and $\mathrm{ran}(f) \subset M$
- All variables occurring free in φ lie in $\mathrm{dom}(f)$
- φ becomes a true statement in M if each $v \in \mathrm{dom}(f)$ is interpreted by $f(v)$.

(*Note* informally we write: $M \models \varphi[a_1, \ldots, a_m/v_1, \ldots, v_m]$ for $M \models \varphi[f]$ where $\mathrm{dom}(f) = \{v_1, \ldots, v_m\}$ and $a_i = f(v_i)$ for $i = 1, \ldots, n$. When the context permits, it is customary to omit the list of variables and write: $M \models \varphi[a_1, \ldots, a_m]$.)

Fact 2. $\{\langle M, \varphi, f\rangle \mid M \in U \wedge M \models \varphi[f]\}$ is uniformly $\Delta_1(U)$.

Definition 4.4.6. A model M is *amenable* if and only if $\bigwedge x \in M(E \cap x, F \cap x \in M)$.

Definition 4.4.7. M is a J-model if and only if M is amenable and $|M| = J_\alpha[E]$ where $\alpha = \mathrm{On} \cap |M|$.

(Note: we write $\mathrm{ht}(M)$ for $\mathrm{On} \cap |M|$.)

Fact 3. There is a Π_2 sentence φ such that

$$M \text{ is a } J\text{-model} \longleftrightarrow M \models \varphi.$$

(Hence $\{M \in U \mid M \text{ is a } J\text{-model}\}$ is uniformly $\Delta_1(U)$.)

Definition 4.4.8. M is acceptable if and only if it is a J-model and, whenever $\eta \geq \omega$ is a cardinal in M(i.e. $\eta < \mathrm{ht}(M)$ and for all $\xi < \eta$ there is no $f \in M$ mapping ξ onto η.), then:

$$\bigwedge \xi < \eta \ \mathbb{P}(\xi) \cap M \subset J^E_\eta.$$

Fact 4. There is a Π_2 sentence φ such that for J-model M:

$$M \text{ is acceptable} \longleftrightarrow M \models \varphi$$

(Hence $\{M \in U \mid M \text{ is acceptable}\}$ is uniformly $\Delta_1(U)$.)

In §1.6 we expanded the language $\mathbb{L}$ to a *many sorted* language $\mathbb{L}^*$ which is more suitable for analyzing acceptable structures N. $\mathbb{L}^*$ contains variables of type n for $n < \omega$, two original variables of $\mathbb{L}$ being of type 0. Variables of type i range over $N^i = J^E_{\rho^i_N}$, where $\rho^i \leq \mathrm{ht}(N)$ and $\rho^0 = \mathrm{ht}(N)$. We then defined an appropriate satisfaction relation for $\mathbb{L}^*$ formulae. $R(x_1^{i_1}, \ldots, x_n^{i_n})$ is an $\mathbb{L}^*$-*definable relation* on N(with arity $\langle i_1, \ldots, i_n\rangle$) if and only if there is an $\mathbb{L}^*$-formula $\varphi(v_1^{i_1}, \ldots, v_n^{i_n})$ with:

$$R(\vec{x})) \longleftrightarrow N \models \varphi[\vec{x}].$$

We defined a hierarchy $\Sigma_n^{(m)}(n = 0, 1)$ of $\mathbb{L}^*$-formulas and defined a $\Sigma_n^{(m)}(N)$ relation to be a relation which is N-definable by a $\Sigma_n^{(m)}$-formula. This hierarchy is better suited to acceptable structures than the Levy hierarchy.

The following fact is implicit in §2.6. As far as we can tell, however, we have hitherto not stated it explicitly, although we have made tacit use of it(for instance in the proof of Lemma 4.1.13).

Fact 5. Let N be acceptable. Let $\varphi(v_1^{i_1}, \ldots, v_m^{i_m})$ be any formula in the many sorted language $\mathbb{L}^*$ developed in §2.6. There is a formula $\tilde{\varphi}$ in the first order language $\mathbb{L}$ of N such that

$$N \models \varphi[x_1, \ldots, x_m] \longleftrightarrow N \models \tilde{\varphi}[x_1, \ldots, x_m]$$

for $x_j \in H_N^{i_j}(j = 1, \ldots, m)$. Moreover the function $\varphi \mapsto \tilde{\varphi}$ is recursive.

Proof(sketch). Let $\mathbb{L}^m$ consist of formulas with variables of type $i \leq m$. By induction on m, we construct the function $\varphi \mapsto \tilde{\varphi}$ for $\varphi \in \mathbb{L}^m$. It clearly suffices to have $\tilde{\rho}^i, \tilde{H}^i$ $(i \leq m)$, since we can then form $\tilde{\varphi}$ by replacing $\bigwedge v^i \ldots$ by $\bigwedge v(\tilde{H}^i v \rightarrow \ldots)$ everywhere. We proceed by induction on m. The case $m = 0$ is trivial, since $\mathbb{L}^0$ is the set of non sorted formulas in the language of N. Moreover we have: $\rho^0 = \mathrm{ht}(N)$, $H^0 = |N|$. Now let it hold at m.

Let $T^m(x_m, \ldots, x_0)$ be the predicate defined in §2.6 preceding the proof of lemma 2.6.17. Set:

$$T'(i, z, \vec{x}) \longleftrightarrow \langle J^E_{\rho^m}, T^{m,x_{m-1},\ldots,x_0}\rangle \models \varphi_i[z, x_m]$$

where $T^{m,x_{m-1},\ldots,x_0} = \{y \mid T^m(y, x_{m-1}, \ldots, x_0)\}$ and $\langle \varphi_i \mid i < \omega\rangle$ is a fixed enumeration of Σ_1 formulae with two free variables. Thus T' is $\Sigma_1^{(m)}(N)$. Moreover, it is *universal* in the sense that, if D is any $\underline{\Sigma}_1^{(m)}(N)$ subset of H^m, then there are $i < \omega$, $\vec{x}$ such that

$$D(z) \longleftrightarrow T'(i, z, \vec{x}).$$

But then:

$$\xi < \rho^{m+1} \longleftrightarrow \bigwedge i < \omega \bigwedge \vec{x}(T_i^{\vec{x}} \cap \xi) \cap \xi \in N$$

and:

$$x \in H^{m+1} \longleftrightarrow \bigvee \xi < \rho^m x \in J^E_\xi.$$

These definitions of ρ^n, H^n are by formulae lying in $\mathbb{L}^m$. That gives us $\tilde{\rho}^{m+1}, \tilde{H}^{m+1}$.

QED(Fact 5)

In §2.6.3 we introduced the class of *m-sound* acceptable models. N is *sound* if and only if it is m-sound for every $m < \omega$.

Fact 6. For $m < \omega$ there is an $\mathbb{L}$-sentence φ_m such that,

$$N \text{ is } n\text{-sound} \longleftrightarrow N \models \varphi_m.$$

Moreover $m \mapsto \varphi_m$ is a recursive function. Hence $\{N \in U \mid N \text{ is sound}\}$ is uniformly $\Pi_1(U)$.

In §3.3 we introduced the class of *premice* and proved:

Fact 7. There is an $\mathbb{L}$-sentence φ such that

$$N \text{ is a premouse } \longleftrightarrow N \models \varphi.$$

(Hence $\{N \in U \mid N \text{ is a premouse}\}$ is uniformly $\Delta_1(U)$.)

In §4.1 we defined the class of *m-solid* premice. We call N *solid* if and only if it is m-solid for all $m < \omega$. Using Fact 5:

Fact 8. For $m < \omega$ there is an $\mathbb{L}$-sentence φ_m such that

$$N \text{ is } m\text{-solid } \longleftrightarrow N \models \varphi_m.$$

Moreover $m \mapsto \varphi_m$ recursive. (Thus $\{N \in U \mid N \text{ is solid}\}$ is uniformly $\Pi_1(U)$.)

In §4.3.2 we defined what it means for a premouse N to be *sound above* λ, where $\lambda \in N$. The definition was equivalent to:

Definition 4.4.9. Let $\lambda \in N$. N is *m-sound above* λ if and only if

- $\rho^m \leq \lambda < \rho^{m-1}$ and N is i-sound for $i < m$.
- Let $a \in P_N^m$. Set $b = a \cap \rho_N^{m-1}$, $\bar{N} = N^{n,a \setminus \rho^{m-1}}$. Then every $x \in \bar{N}$ has the form $h(i, \langle \xi, b \rangle)$ where $i < \omega$, $\xi < \lambda$ and h is the canonical Σ_1-Skolem function for $\bar{N}$.

Definition 4.4.10. N is *sound above* λ if and only if it is m-sound above λ for some m.

By Fact 5 it follows that:

Fact 9. Let $\lambda \in N$. For each $m < \omega$ there is a formula $\varphi_m \in \mathbb{L}$ such that

$$N \text{ is } m\text{-sound above } \lambda \text{ if and only if } N \models \varphi_m[\lambda].$$

Moreover, the function $m \mapsto \varphi_m$ is recursive. Hence:

Fact 10.

- $\{\langle N, \lambda \rangle \in U \mid N \text{ is } m\text{-sound above } \lambda\}$ is $\Delta_1(U)$
- $\{\langle N, \lambda \rangle \in U \mid N \text{ is sound above } \lambda\}$ is $\Sigma_1(U)$

In §4.2 we defined what it means to say that σ witnesses the phalanx $\langle N, M, \lambda \rangle$. We aim to prove the following lemma, which in turn, implies lemma 4.4.2:

Lemma 4.4.4. *Let N be sound and solid. Let $N \in U$, where U is transitive and rudimentarily closed. 'N is condensible' is uniformly $\Pi_1(U)$ in the parameter N.*

The proof will stretch over several sublemmas. U could be quite small- e.g. it could be the closure of $|N| \cup \{N\}$ under rudimentary functions. We call $\langle \sigma, M, \lambda \rangle$ a *counterexample* to the condensibility of N if σ witnesses $\langle N, M, \lambda \rangle$, M is sound above λ, and (b), (c) both fail. At first glance it might seem that there could be a counterexample in V which is not in U. But this is not so:

Lemma 4.4.5. *Let σ witness $\langle N, M, \lambda\rangle$, where M is sound above λ. Then $M \in N$ and $\sigma \in U$.*

Proof. Let $\rho^n \leq \lambda < \rho^m$ in M, where $n = m+1$. Let $\bar{a} \in [\mathrm{ht}(M)]^{<\omega}$ such that, letting $\bar{a}^{(i)} = \bar{a} \cap \rho^i$ for $i = 0, \ldots, m$, we have:

- Every $x \in M^{m,\bar{a}}$ is $\Sigma_1(M^{m,a})$ in parameters $\bar{a}^{(m)}$, ξ such that $\xi < \lambda$
- $\bar{a}^{(i)} \in R^i_M$ for $i < m$.

Set: $a = \sigma(\bar{a})$, $a^{(i)} = \sigma(\bar{a}^{(i)}) = a \cap \rho^i_N$. Then $\sigma|M^{m,\bar{a}} : M^{n,\bar{a}} \longrightarrow_{\Sigma_0} N^{n,a}$ and $\bar{a}$, M is the unique pair b, Q such that $b \in R^m_Q$ and $Q^{m,b} = M^{m,\bar{a}}$. Moreover σ is the unique $\sigma \supset \sigma|M^{m,\bar{a}}$ such that $\sigma(\bar{a}) = a$ and $\sigma : M \longrightarrow_{\Sigma^{(n)}_0} N$ strictly. We consider two cases:

Case 1. $m = 0$(Hence $N = N^{m,a}$, $M = M^{m,\bar{a}}$)

We consider two subcases:

case 1.1. $\sup \sigma"\rho^0_M < \rho^0_N$. Set:

$$\tilde{\rho} = \sup \sigma"\rho^0_M; \tilde{N} = N|\tilde{\rho} = \langle J^{E^N}_{\tilde{\rho}}, E^N_\nu \cap J^{E^N}_{\tilde{\rho}}\rangle$$

where $\nu = \rho^0_N = \mathrm{ht}(N)$. Then $\tilde{N}$ is amenable and $\tilde{N} \in N$, since N is amenable. We have: $\sigma : M \longrightarrow_{\Sigma_1} \tilde{N}$ cofinally. Let $\tilde{h} = h_{\tilde{N}}$, $h = h_M$. Clearly $a = \sigma(\bar{a}) \in \tilde{N}$. Set:

$$\tilde{h}^a(\xi) \simeq \tilde{h}((\xi)_0, \langle(\xi)_1, a\rangle) \text{ for } \xi < \lambda,$$

where $\xi =: \prec(\xi)_0, (\xi)_1\succ$. Set:

$$h^{\bar{a}}(\xi) \simeq h_M((\xi)_0, \langle(\xi)_1, \bar{a}\rangle) \text{ for } \xi < \lambda.$$

Then $\sigma(h^{\bar{a}}(\xi)) \simeq \tilde{h}^a(\xi)$. Set: $\tilde{M} = \langle|\tilde{M}|, \tilde{\in}, \tilde{=}, \tilde{E}, \tilde{F}\rangle$, where:

- $|\tilde{M}| =: \mathrm{dom}(\tilde{h}^a)$
- $\xi \tilde{\in} \zeta \longleftrightarrow: \tilde{h}^a(\xi) \in \tilde{h}^a(\zeta)$
- $\xi \tilde{=} \zeta \longleftrightarrow: \tilde{h}^a(\xi) = \tilde{h}^a(\zeta)$
- $\tilde{E}\xi \longleftrightarrow: \tilde{h}^a(\xi) \in E^N$
- $\tilde{F}\xi \longleftrightarrow: \tilde{h}^a(\xi) \in E^N_\nu$

Then:

(1) $\tilde{M} \in N$, since $\tilde{N} \in N$. $h^{\bar{a}}$(hence M) is recoverable from $\tilde{M}$ by the recursion:

$$h^{\bar{a}}(\xi) = \{h^{bara}(\zeta) \mid \zeta \tilde{\in} \xi\} \text{ for } \xi \in \tilde{M}.$$

λ is easily seen to be a regular cardinal in M, since $\sigma(\lambda) > \lambda$. Hence $\sigma(\lambda)$ is a regular cardinal in N. Hence:

$$|\tilde{M}| \in \mathbb{P}(\lambda)_N \subset J^{E^N}_{\sigma(\lambda)}$$

by acceptability. Hence M can be recovered from $\tilde{M}$ in the ZFC^- model $J^{E^N}_{\sigma(\lambda)}$. Hence:

(2) $M \in N$

But then:

$$\sigma = \{\langle \tilde{h}^a(\xi), h^{\bar{a}}(\xi)\rangle \mid \xi \in |M|\}$$

where $\tilde{h}^a, h^{\bar{a}} \in N$. Thus:

(3) $\sigma \in \underline{\Sigma}_\omega(N) \subset U$.

QED(Case 1.1)

Case 1.2. Case 1.1 fails.

Then $\tilde{N} = N, \tilde{h}^a = h^a$, where $h^a(\xi) \simeq h_N((\xi)_0, \langle (\xi)_1, a\rangle)$ for $\xi < \lambda$. We have $\sigma : M \longrightarrow_{\Sigma_1} N$ cofinally.

Case 1.2.1. $\lambda < \rho^\omega_N$.

Then $\tilde{M} \in N$, since $\langle J^{E^N}_{\rho^\omega_N}, B\rangle$ is amenable whenever $B \subset J^{E^N}_{\rho^\omega_N}$ is $\underline{\sigma}^*(N)$. The rest of the proof is exactly like Case 1.1.

QED(Case 1.2.1)

Case 1.2.2. The above cases fail.

Then $\rho^\omega \leq \lambda$ in N. We conclude that:

(4) $p^* \setminus \lambda \not\subset a$, where $p^* = p^*_N$.

Proof. If not, $\rho^\omega \cup p^* \subset \operatorname{ran}(\sigma) \prec_{\Sigma_1} N$. But then $M = N$, $\sigma = \operatorname{id}$ by the soundness of N. Contradiction! Since $\lambda = \operatorname{crit}(\sigma)$.

QED(4)

Let $\eta \in (p^* \setminus \lambda) \setminus a$ be maximal.(Hence $\eta \geq \lambda$) Then $a \setminus \eta = p^* \setminus (\eta + 1)$. Let $\rho^{i+1} \leq \eta < \rho^i$ in N.(Since we core in Case 1, we know that $i = 0$, but we preserve the more general formulation for later use.) Let $X = h_{N^{i,a\setminus\eta}}(\eta \cup (a \setminus \eta))$. Let $\bar{\pi} : Q' \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Then $\bar{\pi} : Q' \prec_{\Sigma_1} N^{i,a\setminus\eta}$ and by solidity we have: $\bar{N} \in N$, where $\bar{N}, b$ are the unique objects such that $\bar{N}^{i,b} = Q'$. Moreover; there is unique $\pi \supset \bar{\pi}$ such that

$$\pi : \bar{N} \longrightarrow_{\Sigma_1^{(i)}} N \text{ and } \pi(b) = a \setminus \eta.$$

In the present case we know that $i = 0$ and $\eta \geq \lambda$. Let $\pi^{-1}(a) = b' = b \cup (a \cap (\lambda, \eta))$.

$$\pi^{-1}(a) = b' = b \cup (a \cap (\lambda, \eta)).$$

Set: $h^{b'}(\xi) \simeq h_{\bar{N}}((\xi)_0, \langle(\xi)_1, b'\rangle)$ for $\xi < \lambda$. Then $|\tilde{M}| = \mathrm{dom}(h^{b'})$ and:

$$\xi \tilde{\in} \zeta \longleftrightarrow h^{b'}(\xi) \in h^{b'}(\zeta) \text{ for } \xi, \zeta \in \lambda,$$

etc. Thus $\tilde{M} \in N$, since $\bar{N} \in N$. The rest of the proof is exactly as in Case 1.1.

QED(Case 1)

Case 2. $m > 0$. Let $m = r + 1$.

There is a good $\Sigma_1^{(m)}(M)$ function $\bar{G}$ such that each $x \in M$ has the form $\bar{G}(\zeta, \bar{a})$ for an $\zeta < \rho_M^m$. Let G be a good $\Sigma_1^{(m)}(N)$ function by the same good definition. Then:

$$\sigma(\bar{G}(\zeta, \bar{a})) \simeq G(\sigma(\zeta), a) \text{ for } \zeta < \rho_M^m.$$

Set: $\bar{Q} = M^{m,\bar{a}}$, $Q = N^{m,a}$. Then $\sigma|\bar{Q} : \bar{Q} \longrightarrow_{\Sigma_0^{(m)}} Q$. Let $\tilde{\rho} = \sup \sigma" \rho_M^m$. Set:

$$\tilde{Q} = Q|\tilde{\rho} =: \langle J_{\tilde{\rho}}^{E^N}, T_N^{m,a} \cap J_{\tilde{\rho}}^{E^N}\rangle.$$

Then $\sigma : \bar{Q} \longrightarrow_{\Sigma_1^{(m)}} \tilde{Q}$ cofinally. We now set:

- $h^{\bar{a}}(\xi) \simeq h_{\bar{Q}}((\xi)_0, \langle(\xi)_1, \bar{a}\rangle)$.
- $\tilde{h}^a(\xi) \simeq h_{\tilde{Q}}((\xi)_0, \langle(\xi)_1, a\rangle)$.
- $\bar{G}^{\bar{a}}(\xi) \simeq \bar{G}(\bar{h}^{\bar{a}}(\xi), \bar{a})$.
- $\tilde{G}^a(\xi) \simeq G(\tilde{h}^a(\xi), a)$.

Then $\sigma(\bar{G}^{\bar{a}}(\xi)) = \tilde{G}^a(\xi)$ for $\xi < \lambda$. Moreover, each $x \in M$ has the form $\bar{G}^{\bar{a}}(\xi)$ for an $\xi < \lambda$. Set:

- $\tilde{M} = \mathrm{dom}(\tilde{G}^a)$
- $\xi \tilde{\in} \zeta \longleftrightarrow \tilde{G}^a(\xi) \in \tilde{G}^a(\zeta)$ for $\xi, \zeta < \lambda$
- $\xi \tilde{=} \zeta \longleftrightarrow \tilde{G}^a(\xi) = \tilde{G}^a(\zeta)$ for $\xi, \zeta < \lambda$
- $\tilde{E}\xi \longleftrightarrow \tilde{G}^a(\xi) \in E^N$ for $\xi < \lambda$
- $\tilde{F}\xi \longleftrightarrow \tilde{G}^a(\xi) \in E^N_\nu$ for $\xi < \lambda$, where $\nu = \mathrm{ht}(N)$.

Then $\tilde{M}/\tilde{=}$ is isomorphic to M and the function $\tilde{G}^a$ is obtainable from $\tilde{M}$ by the recursion:

$$\bar{G}^a(\xi) = \{\tilde{G}^a(\zeta) \mid \zeta \tilde{\in} \xi\}.$$

Hence it suffices to prove:

Claim. $\tilde{M} \in N$.

Since just as before we will then have:

$$|\tilde{M}| \in \mathbb{P}(\lambda) \cap N \subset J^{E^N}_{\sigma(\lambda)}$$

and we can recover M from $\tilde{M}$ in the ZFC^- model $J^{E^N}_{\sigma(\lambda)}$ by the above recursion. But then: $\sigma = \{\langle \tilde{G}^a(\xi), \bar{G}^{\bar{a}}(\xi)\rangle \mid \xi \in |\tilde{M}|\}$. Hence $\sigma \in \underline{\Sigma}_\omega(N) \subset U$ by the above Fact. We prove the Claim by cases as before:

Case 2.1. $\tilde{\rho} < \rho^m_N$.

Then $\tilde{M} \in N$, since $N^{m,a}$ is amenable.

Case 2.2. Case 2.1 fails.

Case 2.2.1. $\lambda < \rho^\omega_N$.

Then $\tilde{M} \in N$ for the same reason as before.

Case 2.2.2. The above cases fail.

Just as before we conclude:

(5) $p^* \setminus \lambda \not\subset a$.

We again let η be maximal. Let $\rho^{i+1} \leq \eta < \eta^i$ in N. Hence $i \leq m$. As before let:

$$X = h_{N^{i,a\setminus\eta}}(\eta \cup (a \setminus \eta)).$$

Let $\bar{\pi}' : Q' \stackrel{\sim}{\longleftrightarrow} X$ be the transitivation of X. Then $\bar{\pi}' : Q' \prec_{\Sigma_1} N^{i,a\setminus\eta}$. But then as before, $N' \in N$, where N', b are the unique objects such that $N'^{i,b} = Q'$. Moreover, there is a unique $\pi' \supset \bar{\pi}'$ such that

$$\pi' : N' \longrightarrow_{\Sigma_1^{(i)}} N \text{ and } \pi'(b) = a \setminus \eta.$$

Let $\pi'^{-1}(a) = a' = b \cup (a \cap (\lambda, \eta))$. Now let $Q = N'^{m,a'}$. Let $G'(\zeta, a')$ be $\Sigma_1^{(m)}(N')$ by the same good definition as $G(\zeta, a)$. Then:

$$\pi'(G'(\zeta, a')) = G(\pi'(\zeta), a)$$

for $\zeta < \rho^m_{N'}$. Let $\rho' = \sup \pi'^{-1} \circ \sigma\text{''}\rho^m_M$. Set:

$$Q' = Q|\rho' =: \langle J^{E^{N'}}_{\rho'}, T^{m,a'}_{N'} \cap J^{E^{N'}}_{\rho'} \rangle.$$

$$h'^{a'}(\xi) \simeq h_{Q'}((\xi)_0, \langle(\xi)_1, a'\rangle)$$

for $\xi < \lambda$. Set:

$$G'^{a'}(\xi) \simeq G'(h'^{a'}(\xi), a') \text{ for } \xi < \lambda.$$

Then: $|\tilde{M}| = \mathrm{dom}(G'^{a'})$, $\xi \tilde{\in} \zeta \longleftrightarrow G'^{a'}(\xi) \in G'^{a'}(\zeta)$ for $\xi, \zeta < \lambda$, etc. But since $N' \in N$, we conclude $\tilde{M} \in N$.

QED(Lemma 4.4.5)

Tweaking this proof a bit, we get:

Lemma 4.4.6. *For each $n < \omega$ there is a formula $\varphi_n \in \mathbb{L}$ such that for all sound and solid N, $N \models \varphi_n[M, \lambda, \tilde{\lambda}]$ if and only if there is σ witnessing $\langle N, M, \lambda\rangle$ such that the following hold:*

- $\rho^{n+1} \leq \lambda < \rho^n$ *in* M
- M *is sound above* λ
- $\tilde{\lambda} = \sigma(\lambda)$

Proof. $N \models \varphi_n[M, \lambda, \tilde{\lambda}]$ says that there are $a, \bar{a}, b, \bar{b}$ such that

- $a \in [\rho^0_N]^{<\omega}, \bar{a} \in [\rho^0_M]^{<\omega}$
- $b = a \cap \rho^n_N, \bar{b} = \bar{a} \cap \rho^n_M$
- $\bar{a} \in P^{n+1}_{\bar{M}}$ and $\rho^{n+1} \leq \lambda < \rho^n$ in M

- M is sound above λ
- $M^{n,\bar{a}} \models \varphi[\vec{\xi}, \bar{b}] \rightarrow N^{n,a} \models \varphi[\vec{\xi}, \tilde{\lambda}, b]$ for all Σ_0 formulas φ and all $\xi_0, \ldots, \xi_{n-1} < \lambda$.
- $\tilde{\lambda} > \lambda$
- For $m = 0$: Let $h, \bar{h}$ be the Skolem function for N, M respectively. If $\bar{h}(i, \langle \xi, \bar{a} \rangle)$ is a cardinal in M, then $h(i, \langle \xi, a \rangle)$ is a cardinal in N(where $\xi < \lambda$).

We see that this can be expressed by an $\mathbb{L}$-formula φ_n using Fact 5 and the facts:

- M-satisfaction relation is uniformly $\Delta_1(N)$ in M
- $N^{n,a}$ satisfaction relation for Σ_0-formulae is uniformly $\Sigma_1(N^{n,a})$.

The direction $(\leftarrow)$ of an equivalence then follows easily by lemma 4.4.5. To prove the other direction we note that if h is the canonical Skolem function for $N^{n,a}$ and $\bar{h}$ is the Skolem function for $M^{n,\bar{a}}$, then for all $\xi < \lambda$:

$$\langle i, \langle \xi, \bar{b}, \lambda \rangle \rangle \in \mathrm{dom}(\bar{h}) \longrightarrow \langle i, \langle \xi, b, \tilde{\lambda} \rangle \rangle \in \mathrm{dom}(h).$$

Hence we can define $\bar{\sigma} : M^{n,\bar{a}} \longrightarrow_{\Sigma_0} N^{n,a}$ by:

$$\bar{\sigma}(\bar{h}(i, \langle \xi, \bar{b}, \lambda \rangle)) = \begin{cases} h(i, \langle \xi, b, \tilde{\lambda} \rangle), & \text{if } \bar{h}(i, \langle \xi, \bar{b}, \lambda \rangle) \text{ is defined;} \\ \text{otherwise undefined.} \end{cases}$$

Applying the downward extension lemma, we get:

There are unique M', a' with $M'^{n,a'} = M^{n,\bar{a}}$ and $a' \in R^n_{M'}$.

By uniqueness we conclude: $M' = M, a' = \bar{a}$. But then there is a unique $\sigma' \supset \bar{\sigma}$ such that $\sigma' : M \longrightarrow_{\Sigma_0^{(n)}} N$ and $\sigma'(\bar{a}) = a$. Thus, by uniqueness, $\sigma' = \sigma$.

QED(Lemma 4.4.6)

Condensability for N says that if σ,$\langle N, M, \lambda \rangle$ are as in lemma 4.4.3, then one of the conclusions (b), (c) hold.

Lemma 4.4.7. *Let $\sigma, \langle N, M, \lambda\rangle$ be as in lemma 4.4.6. Then there is a formula $\chi \in \mathbb{L}$ such that*

$$N \models \chi[M, \lambda, \sigma(\lambda)] \longleftrightarrow \text{ (b) or (c) hold.}$$

Proof. χ says that either $\bigvee \alpha \in N(M = N||\alpha)$, or that there are $\kappa, \gamma, \mu \in N$ such that

- λ is the cardinal successor of κ in M.
- $\rho^1_{N||\lambda} = \kappa$.
- $\mu \leq \gamma$, $E^N_\mu \neq \emptyset$ and $\mathrm{crit}(E^N_\mu) = \kappa$, E^N_μ is generated by $\{\kappa\}$.
- $(N||\tilde{\lambda}) \models$ There is π such that $\pi : N||\gamma \longrightarrow_{E^N_\mu} M$.

This can be written as an $\mathbb{L}$-formula by Fact 5 and the fact that for $Q \in N$, Q-satisfaction is uniformly $\Delta_1(N)$ in Q. The asserted equivalences then hold because statements of the form:

$$\bigvee \pi \quad \pi : Q \longrightarrow^*_F Q'$$

are absolute in transitive ZFC^- models.

QED(Lemma 4.4.7)

Set:

$$\psi_n =: \bigwedge u \bigwedge v \bigwedge w(\varphi_n(u, v, w) \longrightarrow \chi(u, v, w)).$$

Then obviously:

Lemma 4.4.8. *Let N be sound and solid. Then*

$$N \models \psi_n \longleftrightarrow N \text{ is condensable.}$$

It is apparent from the above proofs that the function $n \mapsto \psi_n$ is recursive. Hence, if N is sound and solid, then:

$$\bigwedge n \ N \models \psi_n \longleftrightarrow N \text{ is condensable.}$$

But $\bigwedge n \ N \models \psi_n$ is uniformly $\Pi_1(U)$ in N, since N-satisfaction is uniformly $\Delta_1(U)$ in N. This proves lemma 4.4.4.

Lemma 4.4.2 then follows, since it says:

$$\bigwedge \alpha \in M(\mathrm{Lim}(\alpha) \longrightarrow \bigwedge n \ (N||\alpha) \models \psi_n).$$

QED(Lemma 4.4.2)

4.4.1 Σ_1-acceptability

Definition 4.4.11. Let $N = \langle J^A_\alpha, B\rangle$ be a J-model. N is Σ_1-acceptable if and only if it is acceptable and whenever $\gamma > \omega$ is a limit cardinal in N, then $J^A_\gamma \prec_{\Sigma_1} J^A_\alpha$.

Lemma 4.4.9. *Every pre-mouselike premouse is Σ_1-acceptable.*

Proof. We proceed by induction on $\alpha = \mathrm{ht}(N)$. If $\alpha = \omega$, the assertion is vacuously true. If α is a limit of limit ordinals, then the assertion is trivial, since any cardinal γ in N is a cardinal in $N||\beta$ for $\beta > \gamma$. There remains the case: $\alpha = \beta + \omega$. Let $M = \langle J^E_\beta, F\rangle$, where $F = E_\beta$. Then $N = \langle J^{E'}_\alpha, \emptyset\rangle$, where

$$E' = E * F = E \cup (\{\beta\} \times F).$$

Let $\rho = \rho^\omega_M$. Then ρ is the largest cardinal in N. Let $\gamma > \omega$ be a limit cardinal in N. Then $\gamma \leq \rho$. If $\rho < \beta$, then γ, ρ are cardinals in M. Now let ψ be a Σ_1 formula such that

$$J^{E'}_\alpha \models \psi[x] \text{ where } x \in J^{E'}_\gamma.$$

We must prove:

Claim. $J^{E'}_\gamma \models \psi[x]$.

We first note that:

$$|N| = \mathrm{rud}(|M| \cup \{M\}) = \mathrm{rud}(|M| \cup \{E\} \cup \{F\}),$$

where $\mathrm{rud}(Y)$ is the closure of Y under rudimentary functions. Let $\psi = \bigvee v\psi'$, where ψ' is Σ_0 in the language of N. Then:

(1) $N \models \psi'[t, x]$ for a $t \in N$

Since $N = J^{E'}_\alpha$ and $E' = E * F$, (1) can be equivalently written as:

(2) $N \models \varphi[t, x, |M|, E, F]$, where φ is a Σ_0 formula containing only the predicate $\in$.

Let $t = f(x, z, |M|, E, F)$ where f is rudimentary and $z \in M$. Recall that rudimentary functions are *simple* in the sense of §2.2. This means that, given the function f: (2) reduces uniformly to:

(3) $N \models \varphi'[x, z, |M|, E, F]$, where φ' is a Σ_0 formula containing only the predicate $\in$.

But this can easily be converted into an equivalent statement of the form:

(4) $M \models \chi'[x,z]$, where χ' is a first order formula in the language of M. Set $\chi = \bigvee v\chi'$. Then:

(5) $M \models \chi[x]$.

In order to derive Claim 1, we show:

Claim 2. There is $\bar{\beta} < \gamma$ such that, letting $\bar{M} = M||\bar{\beta}$, $\bar{N} = M||\bar{\alpha}$, $\bar{\alpha} = \bar{\beta} + \omega$, we have: $\bar{M} \models \chi[x]$.

But then $\bar{M} \models \chi'[x,z]$ for a $z \in \bar{M}$. We then reverse the above chain of equivalent reductions to get: $\bar{N} \models \psi'[\bar{t},x]$, where $\bar{t} = f(x,z,|\bar{M}|,\bar{E},\bar{F})$ and f is the above mentioned rudimentary function. Thus: $\bar{N} \models \psi[x]$ and $J^E_\gamma \models \psi[x]$, since $\bar{N} \triangleleft J^E_\gamma$, proving Claim 1.

Our procedure will be to first define $\bar{M}$ and then, using the condensability of M, show that $\bar{M}$ is a proper segment of J^E_γ. We can assume that w.l.o.g. that the formula χ is a Σ_m-formula for some $m < \omega$. Choose $n < \omega$ such that $n \geq m$ and $\rho^\omega_M = \rho^n_M$. Since M is sound, it has a standard parameter a. Hence $a \in P^n_M$. Hence $a \in R^n_m$ by soundness. Now let δ' be the least cardinal in M such that $x \in J^E_{\delta'}$. Then δ' is a successor cardinal in M(hence in N). Let δ be the immediate successor cardinal of δ' in M (and N). Then $\delta < \gamma$. Let X be the smallest $X \prec_{\Sigma_1} M^{n,a}$ such that $(\delta'+1) \cup a \subset X$. Then $X = \tilde{h}``\delta'$, where

$$\tilde{h}(\prec i, \xi \succ) \simeq h(i, \langle \xi, \delta', a \rangle)$$

and h is the Skolem function for $M^{n,a}$. Let $\bar{\pi} : \bar{Q} \stackrel{\simeq}{\longleftrightarrow} X$ be the transitivation of X. Then $\bar{\pi} : \bar{Q} \longrightarrow_{\Sigma_1} M^{n,a}$. By the downward extension of embeddings lemma(Lemma 2.6.32) we conclude:

(a) There are unique $\bar{M}, \bar{a}$ such that $\bar{a} \in R^n_{\bar{M}}$ and $\bar{M}^{n,\bar{a}} = \bar{Q}$.

(b) There is a unique $\pi \supset \bar{\pi}$ such that $\bar{\pi} : \bar{M} \longrightarrow_{\Sigma^{(n)}_1} M$ and $\pi(\bar{a}) = a$.

But M is sound and a is its standard parameter. Hence $\bar{M}, \bar{a}, \pi$ are the unique objects given by our earlier downward extension lemma and we have:

(6) $\pi : \bar{M} \longrightarrow_{\Sigma_{n+1}} M$.

We now show:

(7) $\bar{M} \in J^E_\delta$.

Proof. $\tilde{h}$ is $\Sigma_1^{(n)}(M)$ in $a \cup \{\delta'\}$ and is a partial map of δ' unto X. Thus $\bar{h} = \bar{\pi}^{-1}\tilde{h}$ is $\Sigma_1^{(n)}(M)$ in $\bar{a} \cup \{\delta'\}$ and is a partial map of δ' onto $\bar{M}^{n,\bar{a}}$. Since $\bar{a} \in R^n_{\bar{M}}$, there is a partial map $\bar{g}$ of $\bar{M}^{n,\bar{a}}$ onto $\bar{M}$ which is $\Sigma_1^{(n)}(\bar{M})$ in $\bar{a}$. Let g be $\Sigma_1^{(n)}(M)$ in a by the same definition. Then $\bar{k} = \bar{g}\bar{h}$ is a $\underline{\Sigma}^*(\bar{M})$ map of δ' onto $\mathrm{ran}(\pi)$, since $g\bar{\pi} = \pi\bar{g}$. Set:

- $|\tilde{M}| =: \mathrm{dom}(k) \subset \delta'$.
- $x \tilde{\in} y \longleftrightarrow: k(x) \in k(y)$ for $x, y \in |\tilde{M}|$.
- $x \tilde{=} y \longleftrightarrow: k(x) = k(y)$ for $x, y \in |\tilde{M}|$.
- $\tilde{E}x \longleftrightarrow: k(x) \in E$, $\tilde{F}x \longleftrightarrow: k(x) \in F$ for $x \in |\tilde{M}|$.

Set: $\tilde{M} =: \langle |\tilde{M}|, \tilde{\in}, \tilde{=}, \tilde{E}, \tilde{F}\rangle$. Then $\tilde{M} \in J^E_\gamma$, since $\langle J^E_\rho, D\rangle$ is amenable for all $\underline{\Sigma}^*(M)$ sets D, and δ is a cardinal in J^E_ρ. But J^E_δ is a ZFC^- model, since δ is a successor cardinal in J^E_ρ. $\tilde{E}$ is well founded. Hence $j \in J^E_\delta$, where $j : \tilde{M} \longrightarrow \bar{M}$ is defined by the recursion: $j(x) = j\text{"}\tilde{\in}\text{"}\{x\}$ for $x \in |\tilde{M}|$. Hence $\bar{M} \in J^E_\delta$.

QED(7)

Set: $\bar{\delta} = \pi^{-1}(\delta)$. It follows easily that $\pi \restriction \delta = \mathrm{id}$. But $\pi(\bar{\delta}) = \delta > \bar{\delta}$, sicne $\bar{\delta} \in J^E_\delta$. Thus $\bar{\delta} = \mathrm{crit}(\pi)$. Using this, we show:

(8) π verifies the phalanx $\langle M, \bar{M}, \bar{\delta}\rangle$.

Proof.

- $\pi : \bar{M} \longrightarrow M$.
- π is $\Sigma_1^{(n)}$-preserving, where $\bar{\delta} < \rho^n_{\bar{M}}$.
- $\rho^{n+1}_{\bar{M}} < \bar{\delta}$, since $\bar{h}$ is a $\Sigma_1^{(n)}(\bar{M})$ partial map of $\delta' < \bar{\delta}$ onto $\bar{M}^{n,a}$.
- ξ is a cardinal in $\bar{M}$ if and only if $\pi(\xi)$ is a cardinal in M, by (6).

QED(8)

But M is condensable. Hence $\bar{M}$ satisfies one of the three conditions (a), (b), (c) in the condensation lemma. But:

(9) (a) does not hold, since otherwise:

$$\rho^n_{\bar{M}} = \mathrm{ht}(\bar{M}^{n,\bar{a}}) < \delta < \rho.$$

But we can also show:

(10) (c) does not hold.

Proof. Suppose not. Then there is $\eta \in M$ such that $\rho^{\omega}_{J^E_\eta} = \kappa < \delta$, where κ is the largest cardinal in J^E_δ. Moreover, there is $\mu \leq \eta$ such that $\sigma : J^E_\eta \longrightarrow_F \bar{M}$, where $F = E_\mu$ and $\kappa = \mathrm{crit}(F)$. But then $\kappa = \delta'$ would be a limit cardinal in $\bar{M}$. Contradiction!, since δ' is a successor cardinal.

QED(10)

Thus (b) holds, and $\bar{M} \triangleleft M$. Since $\bar{\beta} = \mathrm{ht}(\bar{M}) < \delta$, we have:

(11) $\bar{M} = M||\bar{\beta} = \langle J^{\bar{E}}_{\bar{\beta}}, \bar{F}\rangle$ where $\bar{\beta} < \delta$.

Moreover, if $\bar{\alpha} = \bar{\beta} + \omega$ and $\bar{N} = M||\bar{\alpha}$, we have:

(12) $\bar{N} = M||\bar{\alpha} = J^{\bar{E}*\bar{F}}_{\bar{\alpha}}$.

By (6) we know: $\bar{M} \models \chi[x]$, hence:

(13) $\bar{M} \models \chi'[x, z]$ for a $z \in \bar{M}$.

Reversing our earlier chain of equivalences, we see that (13) is equivalent to:

(14) $\bar{N} \models \varphi'[x, z, |\bar{M}|, \bar{E}, \bar{F}]$.

Set $\bar{t} = f(x, z, |\bar{M}|, \bar{E}, \bar{F})$ where f is the rudimentary function used above. Then (14) is equivalent to:

(15) $\bar{N} \models \varphi[\bar{t}, x, |\bar{M}|, \bar{E}, \bar{F}]$,

which is, in turn, equivalent to:

(16) $\bar{N} \models \psi'[\bar{t}, x]$.

Hence $\bar{N} \models \psi[x]$, where $\bar{N} \triangleleft J^E_\delta$.

QED(Lemma 4.4.9)

Call a premouse N *fully preiterable*. If every proper $M \triangleleft N$ is fully iterable. By lemma 4.4.9 we of course have:

Corollary 4.4.10. *Every fully preiterable premouse is Σ_1-acceptable.*

(Hence of course, every fully iterable premouse is Σ_1-acceptable.)

4.4.2 Mouselikeness in 1-small premice

The reader may wonder why we develop theory of mouselikeness and pre-mouselikeness in such detail, when we already know that these properties hold for all fully iterable mice. The reason is that we may encounter iterations where we can verify the mouselikeness of a structure without yet knowing it to be fully iterable. We give an example involving *1-small premice*, which were introduced in §3.8 and will be our main object of investigation in the ensuing chapters. We call a 1-small premouse N *unrestrained* if and only if

- $N = J^E_\alpha$ is a constructible extension of J^E_β, where $\beta \leq \rho^\omega_N$.
- β is Woodin in $J^{E^N}_{\alpha+\omega}$, where $\alpha = \mathrm{ht}(N)$.

Otherwise we call N *restrained.* Restrained premice have the unique branch property-i.e. any normal iteration of limit length has at most one cofinal well founded branch. Hence, by Theorem 3.6.1 and Theorem 3.6.2 we know that N is fully iterable if it is normally iterable. Happily, however, it turns out that if N is unrestrained and pre-mouselike, then it is mouselike. We, in fact, prove:

Lemma 4.4.11. *Let $N = J^E_\alpha$ be 1-small, where $\beta \leq \alpha$ is Woodin in $J^E_{\alpha+\omega}$. If J^E_β is pre-mouselike; then N is mouselike.*

Proof. Since β is Woodin in $J^E_{\alpha+\omega}$. We have $\beta \leq \rho^\omega_N$, N is then a constructible extension of J^E_β by 1-smallness,

(1) N is sound, by Lemma 2.5.22.

(2) N is solid, by Lemma 4.1.16.

Now let σ witness $\langle N, M, \lambda\rangle$ where M is sound above λ. By Lemma 4.4.5:

(3) $M \in N$, $\sigma \in \underline{\Sigma}_\omega(N)$.

Claim. One of the conditions (b), (c) holds.

(4) If $\lambda \geq \beta$, the (b) holds.

Proof. $\lambda \neq \beta$, since otherwise $\sigma(\lambda) > \beta$ is Woodin in N. Contradiction! But then $\sigma(\beta) = \beta$. Hence M is a constructible extension of J^E_β, since $\sigma : M \longrightarrow_{\Sigma_0} N$. But then $M \triangleleft N$ is a proper segment of N and (b) holds.

QED(4)

From now on assume: $\lambda < \beta$. Thus:

(5) $M \in J^E_\beta$.

Proof. Let $\gamma = \mathrm{ht}(M)$. There is $f \in N$ such that $f : \lambda \xrightarrow{\text{onto}} \gamma$, since M is sound above λ. Moreover M is coded by a $b \subset \lambda$. Hence $b \in J^E_\beta$, since β is a cardinal in N. But β is a regular limit cardinal in N. Hence J^E_β is a transitive model of ZFC. Hence b can be decoded in J^E_β. Hence $M \in J^E_\beta$.

QED(5)

(6) $\sigma(\lambda) \leq \beta$

Proof. Otherwise $\beta < \sigma(\lambda)$ is the unique Woodin cardinal in N. Hence some $\bar{\beta} < \lambda$ is the unique Woodin cardinal in M. Hence $\beta = \sigma(\bar{\beta}) = \bar{\beta} < \beta$, and $\bar{\beta} < \lambda$. Contradiction!

QED(6)

Let $\varphi_m \in \mathbb{L}$ be the formula in Lemma 4.4.6, where $\rho^{m+1} \leq \lambda < \rho^m$ in M. Without loss of generality, suppose φ_m to be Σ_r in the Levy hierarchy. Pick $n \geq r$ such that $\rho^n = \rho^\omega$ in N. Let $a \in P^n_N$. Let $Q = N^{n,a}$. Let h be the canonical Σ_1 Skolem function for Q. Working in $J^E_{\alpha+\omega}$, we define sequences $X_i \prec_{\Sigma_1} Q$, $\alpha_i < \alpha$ for $i < \omega$ as follows: let $\beta_0 < \beta$ such that $M \in J^E_{\beta_0}$ and $\sigma(\lambda) < \beta_0$ if $\sigma(\lambda) < \beta$. Set: $X_i = h(\beta_i) =: \{h(i, \xi) \mid \xi < \beta_i\}$, $\beta_{i+1} = \operatorname{lub} \beta \cap X_i$.

Since β is a regular limit cardinal in $J^E_{\alpha+\omega}$, it follows that $\beta_i < \beta$ for $i < \omega$, where the sequence $\langle \beta_i \mid i < \omega \rangle$ is defined from φ. Hence $\langle \beta_i \mid i < \omega \rangle$ is N-definable by Fact 5. Hence $\langle \beta_i \mid i < \omega \rangle \in J^E_{\alpha+\omega}$ and

$$\bar{\beta} =: \sup_{i<\omega} \beta_i < \beta.$$

Set $X = h(\bar{\beta}) = \bigcup_{i<\omega} X_i$. Then $X \in J^E_{\alpha+\omega}$. Let $\bar{\pi} : \bar{Q} \overset{\sim}{\longleftrightarrow} X$. Thus $\bar{\pi} : \bar{Q} \prec_{\Sigma_1} Q$ and by the downward extension Lemma there are unique $\bar{N}, \bar{a}$ such that $\bar{a} \in R^n_{\bar{N}}$ and $\bar{N}^{n,\bar{a}} = \bar{Q}$. Moreover there is a unique $\pi \supset \bar{\pi}$ such that $\pi(\bar{a}) = a$ and $\pi : \bar{N} \longrightarrow_{\Sigma_1} N$. Since $a \in R^n_N$, we then get: $\pi : \bar{N} \longrightarrow_{\Sigma_n} N$. But then $\bar{N} \models \varphi_m[M, \lambda, \tilde{\lambda}]$, where $\tilde{\lambda} = \sigma(\lambda)$ if $\sigma(\lambda) < \beta$ and $\tilde{\lambda} = \bar{\beta}$ is $\sigma(\lambda) = \bar{\beta}$. Hence:

(7) There is $\bar{\sigma}$ witnessing $\langle \bar{N}, M, \lambda \rangle$ where $\bar{\sigma}(\lambda) = \sigma(\lambda)$ if $\sigma(\lambda) < \beta$ and $\bar{\sigma}(\lambda) = \bar{\beta}$ if $\sigma(\lambda) = \beta$.

Clearly $\bar{N}$ is a constructible extension of $J^E_{\bar{\beta}}$ and $\bar{\beta}$ is Woodin in $\bar{N}$ if $\beta < \alpha$. Using this, we get:

(8) $\bar{N} \lhd J^E_{\beta}$, where $\mathrm{ht}(N) < \beta$.

Proof. Since $\bar{\beta} < \beta$, there is a least $\nu < \beta$ such that $E_\nu \neq \emptyset$. But then J^E_ν is a constructible extension of $J^E_{\bar{\beta}}$ and $\bar{\beta}$ is not Woodin in J^E_ν by 1-smallness. Hence $\bar{\alpha} < \nu$, where $\bar{\alpha} = \mathrm{ht}(\bar{N})$ and $\bar{N} = J^E_\beta || \bar{\alpha}$.

QED(8)

Since J^E_β is pre-mouselike, we conclude that $\bar{N} \models \chi[M, \lambda, \bar{\sigma}(\lambda)]$. We can w.l.o.g. assume n to be chosen so that χ is Σ_n in the Levy hierarchy. But then:

$$N \models \chi[M, \lambda, \sigma(\lambda)], \text{ since } \pi(\bar{\sigma}(\lambda)) = \sigma(\lambda).$$

Hence (b) or (c) hold.

QED(Lemma 4.4.11)

Chapter 5

The Model K^c

5.1 Introduction

From now on we make the assumption: There is no inner model with a Woodin cardinal.(However, we may from time to time, prove individual results under more general assumptions.) Under this assumption we define an inner model known as the *core model*, denoted by 'K', and examine its properties. K will be a Weasel -i.e. it will be a class $K = J^E_\infty = \langle L[E], \in, E\rangle$ such that $E \subset K$ and $K||\eta$ is a premouse for every limit ordinal η. Thus it remains quite "L-like" in its internal structure. It also satisfies a set of propositions which we collectively call the "covering lemma". They say that the global structure of cardinals and cofinalities in V is not very different from K, although huge local differences are possible. In addition, K has a definition which is absolute in all set generic extensions of V. Finally, K is normally α-iterable for all $\alpha < \infty$. If M is any (set) premouse which is ∞-iterable, then the coiteration of M and K will terminate below ∞ and there will be no truncation on the M-side(hence the K-side "absorbs" M), K is in this sense "universal".

Before attempting the construction of K, however, we shall construct an auxiliary model known as K^c. We shall "extract" K from K^c. $K^c$is universal in the same sense as K, but it lacks the covering properties and the absoluteness properties.

The investigation of K has a long history. The original construction by Jensen assumed that $0^\#$ does not exist, and K was L. The covering lemma for L had the simple form:

If X is a set of ordinals of cardinality $> \omega$, then it is covered by a set $Y \in L$ of the same cardinality.

This implies among other things that successors of singular cardinals are absolute in L -i.e. if β is a singular cardinal in V, then $\beta^+ = \beta^{+L}$.(This statement will continue to hold for the K constructed here.) Jensen then went a step further by constructing the core model under the weaker limiting assumption: There is no inner model with a measurable cardinal. In this version the covering lemma became somewhat weaker. In the sequel, Tony Dodd, Bill mitchell and Jensen did a variety of core model constructions, each with its own limiting assumption. Mitchell was the first to divide the construction into two parts: The construction of K^cfollowed by the "extraction" of K from K^c. Finally, after the discovery of Woodin cardinals, John Steel realized that an inner model with the properties listed above could not exist in the presence of an inner model with a Woodin cardinal. He then took the nonexistence of an inner model with a Woodin cardinal as his limiting assumption and proved the existence of the core model. However, he was still not able to do this within the theory ZFC. He needed a higher order set theory. Following this, Steel, Mitchell and Schindler, and Jensen independently proved the existence of K^cin ZFC, on the above limiting assumption. Steel and Jensen thereupon proved the full result, which is presented in this book.

We now develop some consequences of our assumption that there is no inner model with a Woodin cardinal. We define:

Definition 5.1.1. Let $M = \langle J_\nu^E, F\rangle$ be an active premouse. F is ω-complete in M if and only if the following hold:

Let $\mathcal{U} \subset \lambda = \lambda(F)$, $W \subset \mathbb{P}(\kappa) \cap M$ be countable sets(where $\kappa = \text{crit}(F)$, $\lambda = F(\kappa)$). Then there is a $g : \mathcal{U} \to \kappa$ such that whenever $(\alpha_1, \ldots, \alpha_n) \in \mathcal{U}$ and $X \in W$, then:

$$\prec g(\vec{\alpha}) \succ \in X \leftrightarrow \prec \vec{\alpha} \succ \in F(X).$$

We prove:

Lemma 5.1.1. *Let F be ω-complete in M. Then:*

$$M \models \textit{ there is no Woodin cardinal.}$$

(Hence M is 1-small and restrained *in the sense of §3.8.)*

Proof. We first define:

Definition 5.1.2. M is *top iterable* if and only if there is a sequence $\langle M_i : i < \infty\rangle$ and $\langle \pi_{ij} : i \leq j < \infty\rangle$ with:

- $M_i = \langle J^{E_i}_{\nu_i}, F_i\rangle$
- $M_0 = M$ and $\pi_{i,i+1} : M_i \longrightarrow_F M_{i+1}$
- $\pi_{ij} \circ \pi_{ki} = \pi_{kj}$ for $k \leq i \leq j$
- if η is a limit ordinal, then:

  $$M_\eta, \langle \pi_{i,\eta} : i < \eta\rangle$$

  is the transitivized direct limit of: $\langle M_i : i < \eta\rangle, \langle \pi_{ij} : i \leq j < \eta\rangle$.

(*Note* we have only Σ_0 ultrapowers in this definition.) We first prove:

Claim 1. If M is top iterable, then

$$M \models \text{ there is no Woodin cardinal.}$$

Proof. Suppose not. Let γ be Woodin in M. Then $\nu = \nu_0$ is a cardinal in M_i for $i > 0$. By acceptability it follows that γ is Woodin in M_i. Hence $W = \bigcup_{i<\infty} J^{E_i}_{\nu_i}$ is an inner model with a Woodin cardinal. Contradiction!

QED(Claim 1)

We then show:

Claim 2. If F is ω-complete in M, then M is top iterable.

Proof. Suppose not. Then M_α is not defined for some α. Let θ be regular such that $\alpha, M \in H_\theta$. Let $X \prec H_\theta$ be countable with $\alpha, M \in X$. Let $\sigma : \overline{H} \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Let $\sigma(\overline{\alpha}) = \alpha$, $\sigma(\overline{M}) = M$. Then $\overline{H} \models$ "$\overline{M}_{\overline{\alpha}}$ does not exist". By absoluteness $\overline{M}_{\overline{\alpha}}$ does not exist. But α is countable. We derive a contradiction by recursively constructing $\overline{M}_\xi, \sigma_\xi$ ($\xi \leq \alpha$) such that $\overline{M}_\xi$ exists and $\sigma_\xi : \overline{M}_\xi \longrightarrow_{\Sigma_0} M$. We proceed by cases as follows:

Case 1. $\overline{M}_0 = \overline{M}$, $\sigma_0 = \sigma \restriction \overline{M}$.

Case 2. $\overline{M}_i, \sigma_i$ are given. By ω-completeness there is $g : \lambda_i \to \kappa_i$ such that for all $\alpha_1, \ldots, \alpha_n < \lambda_i$ and all $X \in \mathbb{P}(\kappa_i) \cap \overline{M}_i$, we have :

$$\prec g(\vec{\alpha}) \succ \in X \longleftrightarrow \prec \vec{\alpha} \succ \in F_i(X).$$

We know by §3.2 that the transitivized ultrapowers:

$$\overline{\pi}_{i,i+1} : \overline{M}_i \longrightarrow_{F_i} \overline{M}_{i+1}$$

exists if and only if there are no sequences $\langle\alpha_n \mid n<\omega\rangle$, $\langle f_n \mid n<\omega\rangle$ such that $\alpha_n<\lambda_i$, $f_n \in \overline{M}_i$ maps κ_i into $\overline{M}_i$, and:

$$\langle\alpha_{i+1},\alpha_i\rangle \in F_i(\{\langle\xi,\zeta\rangle \mid f_{i+1}(\xi)\in f_i(\zeta)\}).$$

But there can be no such sequence, since otherwise:

$$f_{i+1}(g(\alpha_{i+1}))\in f_i(g(\alpha_i)),\ \text{for } i<\omega.$$

Contradiction! We then define σ_{i+1} by:

$$\sigma_{i+1}(\overline{\pi}_{i,i+1}(f)(\alpha))=\sigma_i(f)(g(\alpha))$$

for $\alpha<\lambda_i$, $f:\kappa_i\to\overline{M}_i$, $f\in\overline{M}_i$.

Case 3. η is a limit ordinal and $\overline{m}_i,\sigma_i$ are given for $i<\eta$. Let

$$\overline{M}_\eta,\langle\overline{\pi}_{i,\eta} \mid i<\eta\rangle$$

be a direct limit of:

$$\langle\overline{M}_i \mid i<\eta\rangle,\langle\overline{\pi}_{ij} \mid i\le j<\eta\rangle.$$

We can define $\sigma_\eta:\overline{M}_\eta \longrightarrow_{\Sigma_0} M$ by: $\sigma_\eta\overline{\pi}_{i\eta}=\sigma_i$ for $i<\eta$. Hence $\overline{M}_\eta$ is well founded and we can take it as being transitive.

QED(Lemma 5.1.1)

We recall that every 1-small mouse M either has a $\gamma\in M$ which is Woodin in M or is *restrained*. If M is restrained, it has the unique branches property. Moreover, if on the other hand, M is not restrained, then it is a constructible extension of $M||\rho^\omega_M$. We prove:

Lemma 5.1.2. *Suppose that M is restrained and countably normally iterable. Then M is normally ∞-iterable.(Hence M is fully α-iterable for all α)*

Note. Since M has the unique branches property, being normally ∞-iterable is the same as being normally α-iterable for all α.

Proof. Let $I=\langle\langle M_i\rangle,\langle\nu_i\rangle,\langle\pi_{ij}\rangle,T\rangle$ be a (potential) normal iteration of M. We must prove:

(A) If I is a potential iteration of length $i+1$, then it extends to an actual iteration of that length.

(B) If I is of limit length, then it has a cofinal well founded branch.

We first prove **(A)**. Let $I \in H$, where H is a transitive ZFC^- model. Let $X \prec H$ be countable with $I \in X$. Let $\sigma : \overline{H} \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Let $\sigma(\overline{I}) = I$. Then $\overline{I}$ being countable, does extend to an actual iteration. Letting:

$$\overline{I} = \langle \langle \overline{M}_i \rangle, \langle \overline{\nu}_i \rangle, \langle \overline{\pi}_{ij} \rangle, \overline{T} \rangle, \text{ be of length } i+1,$$

this means that the ultrapower

$$\pi : \overline{M}_i^* \longrightarrow_F^* \overline{M}_{i+1} \text{ exists, where } F = E_{\overline{\nu}_i}^{\overline{M}_i}.$$

That is equivalent to saying that there is no pair of sequences

$$\langle \alpha_n \mid n < \omega \rangle, \langle f_n \mid n < \omega \rangle$$

such that $f_n \in \Gamma^*(\overline{\kappa}_n, \overline{M}_n)$, $\alpha_n < \overline{\lambda}_n$ and

$$\prec \alpha_{n+1}, \alpha_n \succ \in F(\{ \prec \xi, \zeta \succ \mid f_{n+1}(\xi) \in f_n(\zeta)\}).$$

But the same holds of I.

QED(A)

(B) Let I be of limit length η. Let H be any transitive ZFC^- model containing I as an element. Let $\sigma : \overline{H} \prec H$, $\sigma(\overline{I}) = I$ be as above. Then $\overline{I}$ is a countable normal iteration of limit length $\overline{\eta}$, where $\sigma(\overline{\eta}) = \eta$. Hence it has a unique cofinal well founded branch $\overline{b}$. We consider two cases:

Case 1. $\overline{H}, H$ can be so chosen that $\mathrm{On}_{\overline{M}_{\overline{b}}} \in \overline{H}$. Let $M_{\overline{b}} \cap \mathrm{On} = \alpha$. We consider the following language $\mathbb{L}$ on the admissible set $\overline{H}$:

Predicate: $\dot{\in}$

Constants: $\underline{x}$ $(x \in \overline{H})$, $\dot{b}$

Axioms:

- ZFC^-
- $\wedge v(v \in x \longleftrightarrow \bigvee\limits_{z \in x} v = \underline{z})$ for $x \in \overline{H}$
- $\dot{b}$ is a cofinal branch in $\underline{I}$ yielding a limit model $\dot{M}_{\dot{b}}$ such that $\mathrm{On} \cap \dot{M}_{\dot{b}} = \underline{\alpha}$

$\mathbb{L}$ is obviously consistent, since $\langle H_{\omega_1}, \overline{b} \rangle$ is a model. But then the corresponding language $\mathbb{L}'$ on H is consistent(with $\underline{\sigma(\alpha)}$ playing the role of $\overline{\alpha}$). If we force to make H countable, then in the resulting generic extension $\mathbb{L}'$ has a model $\mathbb{A}$. Set $b = \dot{b}^{\mathbb{A}}$. Then b is a cofinal well founded branch in I. But M is still restrained. Hence b is the unique such branch. But then

$$b = \{i \mid \mathbb{L}' \vdash \underline{i} \in \dot{b}\} \in V,$$

since otherwise there would be a model of $\mathbb{L}'$ yielding a different cofinal well founded branch.

QED(Case 1)

Case 2. Case 1 fails. Let θ be a regular cardinal such that $\text{card}(I)^+ < \theta$. Let $\lambda = \underset{i<\eta}{\text{lub}}\lambda_i$ and $J^E_\lambda = \bigcup\limits_{i<\eta} J^{E^{M_i}}_{\lambda_i}$. Then $\lambda^{+L^E} < \theta$. Let $X \prec H_\theta$ be countable such that $I \in X$. Let $\sigma : \overline{H} \overset{\sim}{\longleftrightarrow} X$ be the transitivation of X. Let $\sigma(\overline{I}) = I$. Since $\overline{I}$ is countable, it has a unique cofinal well founded branch b. But $\text{On} \cap \overline{H} \leq \text{On}_{\overline{M}_b}$, where $\overline{M}_b$ is the limit model. Hence the following language $\mathbb{L}$ on $\overline{H}$ is consistent: The predicates and constants are as before. The axioms are:

- ZFC^-
- $\wedge v(v \in x \longleftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = \underline{z})$ for $x \in \overline{H}$
- $\dot{b}$ is a cofinal well founded branch in $\underline{\overline{I}}$
- Let $\dot{M}_{\dot{b}}$ be the limit model. Then $\underline{\xi} \in \dot{M}_{\dot{b}}$ for all $\xi \in \overline{H}$.

$\mathbb{L}$ is consistent, since if b is the unique cofinal branch, then $\langle H_{\omega_1}, b\rangle$ is a model. By §1.4 however, $\mathbb{L}$ then has an ill founded model $\mathbb{A}$ such that $\text{On} \cap \overline{H} = \text{wfcore}(\mathbb{A})$.(This is by lemma 1.4.11) Set $b' = \dot{b}^{\mathbb{A}}$. Then $b' \neq b$, since b' yields an ill founded limit model. Defining $\overline{\lambda}$, $J^{\overline{E}}_{\overline{\lambda}}$ from $\overline{I}$ and λ, J^E_λ from I, we have by theorem 3.8.12:

$$\overline{H} \models (\overline{\lambda} \text{ is Woodin in } L^{\overline{E}}).$$

Hence:

$$H_\theta \models (\lambda \text{ is Woodin in } L^E).$$

But $\lambda^{L^E} < \theta$. Hence λ is Woodin in $(L^E)^{H_\theta} = L^E_\theta$. But we can choose θ arbitrarily large. Hence λ is Woodin in the inner model L^E. Contradiction!

QED(5.1.2)

As a consequence:

Lemma 5.1.3. *Suppose that M is restrained and that whenever $\sigma : P \prec M$ and P is countable, then P is countably normally iterable . Then M is normally iterable.*

Proof. Suppose not. Let I be a normal iteration which cannot be continued. Let $I \in H = H_\theta$, where θ is regular. Let $X \prec H$ be countable such that $I \in X$. Transitivize X to get $\sigma : \overline{H} \overset{\sim}{\longleftrightarrow} X$. Let $\sigma(\overline{I}) = I$. Then $\overline{H}$ thinks that $\overline{I}$ is an iteration that cannot be continued. Hence, by absoluteness, it cannot be continued. Contradiction!, since $\overline{I}$ is a countable iteration of $P = \sigma^{-1}(M)$.

QED(5.1.3)

Note that every smooth iterate of a restrained premouse is restrained. Hence by lemma 3.6.2:

Corollary 5.1.4. *Let M be as above. Then M is smoothly iterable.*

Hence by Lemma 3.6.1:

Corollary 5.1.5. *Let M be as above. Then M is fully iterable.*

5.2 The Steel Array

In this chapter we employ our machinery to construct inner models of set theory. These models will present themselves as *weasels*. We define:

Definition 5.2.1. A *weasel* is a proper class $N = J^E_\infty = \langle |N|, E\rangle$ such that $N||\nu$ is a sound premouse for all limit $\nu \in \mathrm{On}$.

(In other words, a weasel is "a passive premouse of length ∞". The minimal inner model L is a weasel by lemma 2.5.21. A weasel can be defined inductively like the definition of L, except that we allow certain stages to be an active premouse. If $N_i = \langle J^{E^i}_{\nu_i}, E^i_{\nu_i}\rangle$ is the i-th stage, we have as before:

$$N_0 = \langle J_\omega, \emptyset\rangle.$$

At successor stages, however, we can have either:

$$N_{i+1} = \langle J^{E^{i+1}}_{\nu_{i+1}}, \emptyset\rangle = \langle \mathrm{Def}(N_i), E_i, \emptyset\rangle$$

or, if possible:

$$N_{i+1} = \langle J^{E^i}_{\nu_i}, F\rangle, \text{ where } \langle J^{E^i}_{\nu_i}, F\rangle \text{ is an active premouse.}$$

In the choice of F we are guided by a "background condition" which tells us whether F is viable. For smaller weasels, it suffices that F is *ω-complete*. For the "fully backgrounded" construction, the requirement is that $F = F^* \cap |N_i|$, where F^* is an extender on V at $\kappa = \text{crit}(F)$(hence κ is inaccessible in V). We shall require that $\langle J_{\nu_i}^{E^i}, F\rangle$ satisfy a condition called *robustness*, which is intermediate between these extremes. However, the use of these background conditions means that $\langle J_{\nu_i}^{E^{i+1}}, F\rangle = N_{i+1}||\nu_i$ is *not* necessarily sound. If for instance F is the first extender inserted in the sequence, then ω-completeness requires that N_{i+1} is a rather long iterate of $0^{\#}$, hence is unsound. In order to rectify this, we must, having searched a given N_i, ask whether N_i is *solid*. If so, replace N_i with the sound structure:

$$M_i = \text{core}(N_i).$$

If not, we must discontinue the construction.

But this is no longer a linear construction. We are now constructing a double sequence M_i, N_i. Given M_i, we construct N_{i+1} from M_i by one of the above two options and then "core down" N_{i+1} to M_{i+1} if necessary. At limit points λ we *cannot* take:

$$N_\lambda = \bigcup_{i<\lambda} M_i$$

since M_i is not necessarily a submodel of M_j for $i < j < \lambda$. Instead we take:

$$N_\lambda = \bigcup_{i<\lambda} M_i||\mu_i$$

where μ_i is a carefully chosen point such that

$$M_i||\mu_i = M_j||\mu_i \text{ for } i \leq j < \lambda.$$

However, we ensure:

$$\wedge i < \lambda \ \vee j < \lambda \ \mu_i < \mu_j.$$

Thus, if $\lambda = \kappa$ is regular, then N_κ will have length κ. Similarly, N_∞ has length ∞ and is, therefore a weasel. The succession of models M_i, N_i generated by this process is called a *Steel array*. We now turn to the formal definition.

We shall, in fact, require that each of the models M_i, N_i in the array be not only solid but *mouselike* in the sence of §4.4. Our construction will guarantee that N_i is pre-mouselike if all previous stages were mouselike. (Hence N_i will be Σ_1-*acceptable* by §4.4.) If we assume that there is no inner model with a Woodin cardinal, then all premice are 1-small. If N_i is 1-small and *unrestrained*, then by §4.4 it will be mouselike. If, on the other hand, N_i is restrained, then it suffices to show that whenever $\sigma\colon P \prec N_i$ and P is a countable premouse, then P is normally $\omega_1 + 1$ iterable. By 1-smallness P is then uniquely iterable and hence by §3.8 is fully $\omega_1 + 1$ iterable. If $N_i \in H_\theta$, where θ is a regular cardinal and $\sigma\colon H \prec H_\theta$, $\sigma(P) = N_i$, where N is countable and transitive, then we can conclude that N_i is mouselike, since P is.

We define:

Definition 5.2.2. Let N be a premouse, $\eta \leq \mathrm{On} \cap N$. We let:

$$\mu_N(\eta) = \{\alpha \in N \mid \overline{\overline{\alpha}} \leq \eta \text{ in } N\}$$

If N is mouselike, then it is sound. Moreover, if $\rho = \rho_N^\omega$, $\mu = \mu_N(\rho)$ and $M = \mathrm{core}(N)$, then $\mu = \mu_M(\rho)$ and $N||\mu = M||\mu$. We have shown in §4 that if N is of type 1 or 2 (which we shall always assume in this chapter) and is fully $\omega_1 + 1$ iterable, then it is mouselike.

We sometimes write SA for "Steel array".

Definition 5.2.3. By a *quasi* SA we mean a sequence $\langle M_i \mid i < \Omega \rangle$ $(\Omega \leq \infty)$ of premice $\langle J_{\nu_i}^{E^i}, F^i \rangle$ such that

(a) M_i is sound and mouselike

(b) $M_0 = \langle J_\omega^\emptyset, \emptyset \rangle$

(c) Let $i + 1 < \Omega$. Then $M_{i+1} = \mathrm{core}(N)$ where N is mouselike and satisfies one of the following options:
Option 1. $N = \langle J_{\nu_i+\omega}^E, \emptyset \rangle$ where:

$$E = E^i \cup \{\langle x, \nu_i \rangle \mid x \in F^i\}.$$

Option 2. $N = \langle J_{\nu_i}^{E^i}, F \rangle$ is an active premouse, where $F^i = \emptyset$.

(d) Let $i \leq j < \Omega$. Set:

$$\kappa_{ij} =: \min\{\rho_{M_n}^\omega \mid i \leq n \leq j\}, \mu_{ij} =: \mu_{M_i}(\kappa_{ij})$$

Let $i < n \leq j$. Then κ_{in} is a cardinal in M_n. Moreover:

$$M_i||\mu_{ij} = M_n||\mu_{ij}$$

Lemma 5.2.1. *Let $\langle M_i \mid i < \Omega \rangle$ be a quasi SA. Then:*

1. *$\kappa_{ij} \leq \kappa_{nj}$, $\kappa_{in} \geq \kappa_{ij}$ for $i \leq n \leq j$.*

2. $\mu_{ij} \leq \mu_{nj}$ *for* $i \leq n \leq j$.

Proof. (1) is immediate. We prove (2). If $\alpha < \mu_{ij}$ then $\overline{\overline{\alpha}} \leq \kappa_{ij}$ in M_i. By acceptability then: $\overline{\overline{\alpha}} \leq \kappa_{ij}$ in $M_i||\mu_{ij}$. If $i \leq n \leq j$, it follows that $\overline{\overline{\alpha}} \leq \kappa_{ij}$ in $M_n||\mu_{ij}$, hence $\overline{\overline{\alpha}} \leq \kappa_{ij}$ in M_n. If $\kappa_{ij} < \kappa_{nj}$, then $\alpha < \kappa_{ij}^{+M_n} \leq \kappa_{nj} \leq \mu_{nj}$. If $\kappa_{ij} = \kappa_{nj}$, then $\alpha < \mu_{M_n}(\kappa_{nj}) = \mu_{nj}$.

QED(Lemma 5.2.1)

Lemma 5.2.2. *Let* $\langle M_n \mid n \leq i\rangle$ *be a quasi SA. Let* N *be formed from* M_i *as in Option 1 or in Option 2. Suppose that* N *is mouselike. Set:* $M_{i+1} =:$ core(N). *Then* $\langle M_n \mid n \leq i+1\rangle$ *is a quasi SA.*

Proof. (a), (b), (c) in the definition of quasi SA hold trivially. We prove (d). We must show that if $l < n \leq i+1$, then $\kappa_l =: \kappa_{l,i+1}$ is a cardinal in M_n and $M_l||\mu_l = M_n||\mu_l$, where $\mu_l =: \mu_{l,i+1}$.

Case 1. $l = i$.

Set: $\rho = \rho^\omega_N = \rho^\omega_{M_{i+1}}$. Then $\rho \leq \rho^\omega_{M_i}$. If N is obtained by Option 1 in (c) of the definition of quasi SA, then this holds by: $M_i \in N$. If Option 2 was used, then $\rho^\omega_{M_i} = \nu_i$ and $\rho < \nu_i$ is a cardinal in N, hence in M_i. But then $\rho = \kappa_i =: \kappa_{i,i+1}$. Let $\mu_i =: \mu_{i,i+1} = \mu_{M_i}(\rho)$ and $\mu = \mu_N(\rho)$. Clearly $\mu_i \leq \mu$. By mouselikeness we have $N||\mu = M_{i+1}||\mu$. Hence $M_i||\mu_i = (N||\mu)||\mu_i = (M_{i+1}||\mu)||\mu_i = M_{i+1}||\mu_i$.

QED(Case 1)

Case 2. $l < i$

Set: $\kappa_l =: \kappa_{l,i+1}$. Then $\kappa_l = \min\{\kappa_{li}, \rho\}$, where ρ is defined as in Case 1.

Case 2.1. $\rho > \kappa_{l,i}$

Then $\kappa_l = \kappa_{l,i}$. It suffices to show that κ_l is a cardinal in M_{i+1} and $M_l||\mu_l = M_{i+1}||\mu_l$, where $\mu_l =: \mu_{l,i+1} = \mu_{li}$. κ_l is a cardinal in $M_i||\rho$ where ρ is a cardinal in M_{i+1} by acceptability. But then:

$$M_l||\mu_l = M_i||\mu_l = (M_i||\rho)||\mu_l = (M_{i+1}||\rho)||\mu_l = M_{i+1}||\mu_l$$

QED(Case 2.1)

Case 2.2. $\rho = \kappa_i$

Then $\kappa_l = \kappa_{li} = \rho$. κ_l is trivially a cardinal in M_{i+1}, since ρ is. Then

$\mu_l =: \mu_{l,i}$ as before. Then:

$$M_l||\mu_l = M_i||\mu_l = (M_i||\mu_i)||\mu_l = M_{i+1}||\mu_l,$$

since $\mu_i = \mu_{i,i+1} = \mu_{M_{i+1}}(\rho)$.

QED(Case 2.2)

Case 2.3. $\rho < \kappa_{li}$

Then $\kappa_l = \rho$. For $l < n \leq i$, we have: ρ is a cardinal in $M_n||\kappa_{li}$ where κ_{li} is a cardinal in M_n. Hence ρ is a cardinal in M_n. But:

$$M_l||\rho = (M_l||\kappa_l)||\rho = (M_n||\kappa_l)||\rho = M_n||\rho$$

Now let $n = i$. Then $\rho = \kappa_i =: \kappa_{i,i+1}$ and $\mu_{M_i}(\rho) = \mu_i = \mu_{i,i+1}$, as we have seen in Case 1. $\rho = \rho^\omega_{M_i} \in M_{i+1}$ is clearly a cardinal in M_{i+1}: moreover $M_i||\mu_i = M_{i+1}||\mu_i$. But $\mu_l = \mu_{M_l}(\rho) \leq \mu_{M_i}(\rho)$, since $\rho < \kappa_l$ and $M_l||\kappa_l = M_i||\kappa_l$. Hence:

$$M_l||\mu_l = M_i||\mu_l = (M_i||\mu_i)||\mu_l = (M_{i+1}||\mu_i)||\mu_l = M_{i+1}||\mu_l.$$

QED(Lemma 5.2.2)

We now consider quasi SA's of the form $\langle M_i \mid i < \eta\rangle$ where η is a limit ordinal.

Lemma 5.2.3. *Let $\langle M_i \mid i < \eta\rangle$ be a quasi SA where η is a limit ordinal. Set:*

$\tilde{\kappa}_i = \tilde{\kappa}_{i,\eta} =: \min\{\rho^\omega_{M_i} \mid i < \eta\}$

$\tilde{\mu}_i = \tilde{\mu}_{i,\eta} =: \mu_{M_i}(\tilde{\kappa}_i)$. *Then:*

(1) $\tilde{\kappa}_i = \min\{\kappa_{ij} \mid i \leq j < \eta\}$ *is a cardinal in M_j for $i \leq j < \eta$.*

(2) $\tilde{\mu}_i = \min\{\mu_{ij} \mid i \leq j < \eta\}$ *is a cardinal in M_j for $i \leq j < \eta$.*

(3) $\tilde{\mu}_i \leq \tilde{\mu}_j$ *for $i \leq j < \eta$.*

(4) $M_i||\tilde{\mu}_i = M_j||\tilde{\mu}_i$ *for $i \leq j < \eta$.*

(5) $\bigwedge i < \eta \bigvee j < \eta\ \tilde{\mu}_i < \tilde{\mu}_j$.

Proof. It is easily seen that:

$$\tilde{\kappa}_i = \kappa_{ij} \text{ for sufficiently large } j < \eta.$$

Hence

$$\tilde{\mu}_i = \mu_{ij} \text{ for sufficiently large } j < \eta.$$

(1)-(4) follow easily from this and 5.2.2. We prove (5).

Case 1. $\bigwedge i < \eta \bigvee j < \eta\ \tilde{\kappa}_i < \tilde{\kappa}_j$.

Given i, pick n, j such that $\tilde{\kappa}_i < \tilde{\kappa}_n < \tilde{\kappa}_j$. If $\alpha < \tilde{\kappa}_i$, then $\overline{\overline{\alpha}} \leq \tilde{\kappa}_i$ is M_i. Hence $\overline{\overline{\alpha}} \leq \tilde{\kappa}_i$ in $M_i||\tilde{\kappa}_i$ by acceptability. Hence $\overline{\overline{\alpha}} \leq \tilde{\kappa}_i$ in $M_j||\tilde{\kappa}_i$, hence M_j. But then $\overline{\overline{\alpha}} \leq \tilde{\kappa}_n < \tilde{\kappa}_j \leq \tilde{\mu}_j$, since $\tilde{\kappa}_n$ is a cardinal in M_j.

QED(Case 1)

Case 2. $\tilde{\kappa}_i = \tilde{\kappa}_j$ for $i \leq j < \eta$.

Given i pick $j > i$ such that $\tilde{\kappa}_j = \rho^{\omega}_{M_j}$. Consider M_{j+1}. If N is derived from M_i by Option 1 of (c) in the definition of quasi SA, then $\rho^{\omega}_{M_{j+1}} = \tilde{\kappa}_j$, since $\rho^{\omega}_{M_{j+1}} \leq \rho^{\omega}_{M_j}$. But then $\tilde{\mu}_{j+1} = \mu_N(\tilde{\kappa}_j) = \nu_j + \omega > \nu_j \geq \tilde{\mu}_j \geq \tilde{\mu}_i$. Now suppose that Option 2 of (c) was used. Then $N = \langle J^E_{\nu_j}, F\rangle$, where $F \neq \emptyset$ and $M_j||\nu_j = \emptyset$. Hence M_j is a ZFC model and $\rho^{\omega}_{M_j} = \tilde{\kappa}_i = \nu_j$. But then $\tilde{\kappa}_i \leq \rho^{\omega}_{M_{j+1}} = \rho^{\omega}_N < \nu_j$, contradiction!

QED(Lemma 5.2.3)

But then $N = \langle \bigcup\limits_{i<\eta} J^{E^i}_{\tilde{\mu}_i}, \emptyset\rangle$ is a premouse. If N is mouselike, we can extend the sequence $\langle M_i \mid i < \eta\rangle$ by setting: $M_\eta = \text{core}(N)$.

Lemma 5.2.4. *Let $\langle M_i \mid i < \eta\rangle$ be a quasi SA, where η is a limit ordinal. Let N be defined as above and let $M_\eta = \text{core}(N)$. Then $\langle M_i \mid i \leq \eta\rangle$ is a quasi SA.*

Proof. (a), (b), (c) in the definition of quasi SA hold trivially. We prove (d). Set:

$$\kappa_i = \kappa_{i,\eta}, \mu_i = \mu_{i,\eta} \text{ for } i \leq \eta$$

$$\tilde{\kappa}_i = \tilde{\kappa}_{i,\eta}, \tilde{\mu}_i = \tilde{\mu}_{i,\eta} \text{ for } i < \eta$$

$$\rho = \rho^{\omega}_{M_\eta} = \rho^{\omega}_N$$

Then $\kappa_\eta = \rho$, $\kappa_i = \min\{\tilde{\kappa}_i, \rho\}$ for $i < \eta$. Clearly:

$$\rho = \text{On} \cap N \text{ and } N = M_\eta \text{ or } \rho \text{ is a cardinal in } M_\eta.$$

We must show:

Claim. If $i < n \leq \eta$, then κ_i is a cardinal in M_n and $M_i||\mu_i = M_n||\mu_i$.

Proof.

Case 1. $\tilde{\kappa}_i < \rho$. Then $\kappa_i = \tilde{\kappa}_i$ and it suffices to prove the claim for $n = \eta$. $M_i||\tilde{\kappa}_i = N||\tilde{\kappa}_i$ where $\tilde{\kappa}_i < \rho$. Hence $\tilde{\kappa}_i$ is a cardinal in $N||\rho = M_\eta||\rho$, hence in M_η. But $\mu_i = \tilde{\mu}_i$ and: $M_i||\tilde{\mu}_i = N||\tilde{\mu}_i = (N||\rho)||\tilde{\mu}_i = (M_\eta||\rho)||\tilde{\mu}_i = M_\eta||\tilde{\mu}_i$.

QED(Case 1)

Case 2. $\tilde{\kappa}_i = \rho$. Hence $\tilde{\kappa}_i = \rho \in N$ is a cardinal in N, hence in M_η, But then $\mu_i = \mu_{M_i}(\rho) = \tilde{\mu}_i$. Set:

$$\mu = \mu_{\eta,\eta} = \mu_N(\rho) = \mu_{M_\eta}(\rho).$$

Then $M_i||\tilde{\mu}_i = N||\tilde{\mu}_i = (N||\mu)||\tilde{\mu}_i = (M_\eta||\mu)||\tilde{\mu}_i = M_\eta||\tilde{\mu}_i$.

QED(Case 2)

Case 3. $\rho < \tilde{\kappa}_i$. Then $\kappa_i = \rho < \tilde{\kappa}_i$. Let $i < n \leq \eta$. If $n < \eta$, then ρ is a cardinal in $M_i||\tilde{\kappa}_i = M_n||\tilde{\kappa}_i$, where $\tilde{\kappa}_i$ is a cardinal in M_n. Hence ρ is a cardinal in M_n. But $\mu_i = \mu_{M_i}(\rho) \leq \tilde{\kappa}_i$ and:

$$M_i||\mu_i = (M_i||\tilde{\kappa}_i)||\mu_i = (M_n||\tilde{\kappa}_i)||\mu_i = M_n||\mu_i.$$

Now let $n = \eta$. Then ρ is a cardinal in N, hence in M_η. Let $\mu = \mu_N(\rho) = \mu_{M_\eta}(\eta) = M_\eta$. Then:

$$M_i||\mu_i = (M_i||\tilde{\kappa}_i)||\mu_i = (N||\tilde{\kappa}_i)||\mu_i = N||\mu_i = (N||\mu)||\mu_i = (M_\eta||\mu)||\mu_i = M_\eta||\mu_i.$$

QED(Lemma 5.2.4)

We can now define:

Definition 5.2.4. A *Steel array* is a sequence $\langle M_i \mid i < \Omega\rangle$ such that $\Omega \leq \infty$ and:

(1) $\langle M_i \mid i < \Omega\rangle$ is a quasi SA.

(2) Let $\lambda < \Omega$ be a limit ordinal. Set:

$$N = \langle \bigcup_{i<\lambda} J^{E^i}_{\tilde{\mu}_{i,\lambda}}, \emptyset\rangle.$$

Then N is mouselike and $M_\lambda = \text{core}(N)$.

Now suppose that $\langle M_i \mid i < \Omega\rangle$ is a Steel array and $\Omega > \omega$ is a regular cardinal. By induction on i we have: $\overline{\overline{M}}_i < \Omega$ for $i < \Omega$. If we then set: $N = \bigcup_{i<\Omega} J^{E^i}_{\mu_{i\Omega}}$, then $N = J^E_\Omega$ is of height Ω. But then:

Lemma 5.2.5. *Let $\langle M_i \mid i < \Omega\rangle$, N be as above, where $\Omega > \omega$ is regular. Then N models* ZFC^-.

Proof. We first show that N satisfies the comprehension axiom: Let $u \in N$ and $a = \{z \in u \mid N \models \varphi(z)\}$.

Claim. $a \in N$

Proof. Let $u \in N_i =: J^{E^i}_{\mu_i,\Omega}$. Let X be the smallest elementary submodel of N with $N_i \subset X$. By regularity we have $X \subset N_j$ for a $j > i$. But by induction on the formula ψ we can prove:

$$X \models \psi[\vec{x}] \longrightarrow N_j \models \psi[\vec{x}] \text{ for } x_1, \ldots, x_m \in X.$$

Hence $X \prec N_j$ and a is N_j definable. Hence $a \in N$ since $N_j \in N$.

QED(Claim)

It follows easily by the regularity of Ω that the replacement axiom holds in the form: $\wedge x \in u \vee y\, \varphi \rightarrow \vee \sigma \wedge x \in u \vee y \in \sigma \varphi$. Hence N models ZFC^-.

QED(Lemma 5.2.5)

N is then sound with: $\rho^\omega_N = \Omega$. Hence N is mouselike and we can set: $M_\Omega = N$. If Ω is inaccessible-i.e. $2^\kappa < \Omega$ for $\kappa < \Omega$, then N models full ZFC. By virtually the same argument it follows that if $\Omega = \infty$, then N is and inner model of ZFC. We can then set: $M_\infty =: N$.

Thus the Steel array can be a tool for creating inner models. The simplest inner model is obtained by using only the first option in (c) of the definition of quasi SA. We then get $\langle M_i \mid i < \infty\rangle$ with:

$$M_i = \langle J_{\omega i}, \emptyset\rangle$$

Hence $N = M_\infty = L$.

Larger inner models can be obtained by making judicious use of the second option(in (c) of the definition of quasi SA). There are two ways of ensuring that the construction does not break down before ∞. The first is to ensure that an extender used in Option 2 satisfy a “background condition” which normally says that the extender is very large. The second is to restrict the complexity of the premice M_i, which makes it harder to apply Option 2. This chapter is devoted to the construction of a specific inner model called K^c. Our background condition is called *robustness*. We shall require that all of the premice M_i be 1-small.

clearly for every Steel array $\langle M_i \mid i < \Omega\rangle$, there is a unique associated sequence $\langle N_i \mid i < \Omega\rangle$ defined by:

Definition 5.2.5. Let $\langle M_i \mid i < \Omega\rangle$ be a Steel array. By recursion on $i < \Omega$ we define:

- $N_0 = M_0 = \langle J_\omega^\emptyset, \emptyset\rangle$
- N_{i+1} is defined from M_i by Option 1 or Option 2(in (c) of the definition of quasi SA) and $M_{i+1} = \text{core}(N_{i+1})$
- If $i = \eta$ is a limit ordinal, then:

$$N_\eta = \langle \bigcup_{n<\eta} J^{E^n}_{\tilde{\mu}_{n,\eta}}, \emptyset\rangle \text{ and } M_\eta = \text{core}(N_\eta)$$

Obviously $\langle M_i \mid i < \Omega\rangle$ is definable from the associated sequence $\langle N_i \mid i < \Omega\rangle$ and we shall often commit the sin of referring to $\langle N_i \mid i < \Omega\rangle$ as a Steel array. We also define:

Definition 5.2.6. $\langle N_i \mid i \leq \Omega\rangle$ is a *putative Steel array* if and only if $\langle M_i \mid i < \Omega\rangle$ is a Steel array, where $M_i = \text{core}(N_i)$, and either $\Omega = i+1$ and N_Ω is obtained from M_i by Option 1 or 2, or else Ω is a limit ordinal and N_Ω is the canonical completion: $N_\Omega = \bigcup_{i<\Omega} J^{E^i}_{\tilde{\mu}_{i,\Omega}}, \emptyset\rangle$.

Thus a putative Steel array $\langle N_i \mid i \leq \Omega\rangle$ is a Steel array of length $\Omega + 1$ if and only if N_Ω is mouselike. N_Ω is obviously pre-mouselike

Let M be a premouse with: $\nu \in M$, $E^M_\nu \neq \emptyset$. Set:

Definition 5.2.7. $B = B(M, \nu) =:$ the set of $\beta \in M$ such that:

$$\rho^\omega_{M||\beta} < \nu \leq \beta \text{ and } \rho^\omega_{M||\gamma} > \rho^\omega_{M||\beta} \text{ for all } \gamma \in [\nu, \beta).$$

Then $\nu \in B$ since $\rho^1_{M||\nu} < \nu$. Moreover, if $\gamma, \beta \in B$, then:

$$\gamma < \beta \longrightarrow \rho^\omega_{M||\beta} < \rho^\omega_{M||\gamma}.$$

Hence B is finite. Set:

Definition 5.2.8. $\beta = \beta(M, \nu) =: \max B(M, \nu)$.

Lemma 5.2.6. *Let $\beta = \beta(M, \nu)$. Then $\rho^\omega_{M||\beta}$ is a cardinal in M.*

Proof. Suppose not. Let M be a counterexample with $\text{ht}(M)$ chosen minimally. Then $\text{ht}(M) > \beta$ and $\text{ht}(M)$ is not a limit of limit ordinals, since otherwise $\rho^\omega_{M||\beta}$ would fail to be a cardinal in $M||\gamma$ for a $\gamma \in (\beta, \text{ht}(M))$. Hence $\text{ht}(M) = \gamma + \omega$, where $\gamma \geq \beta$. Since $M||\gamma$ is sound, we have:

$$\underline{\Sigma}_\omega(M||\gamma) = \underline{\Sigma}^*(M||\gamma).$$

But $|M|$ is the rudimentary closure of $|M||\gamma| \cup \{M||\gamma\}$. Hence:

$$\mathbb{P}(M||\gamma) \cap M = \underset{\sim}{\Sigma}_\omega(M||\gamma).$$

Since $\rho = \rho^\omega_{M||\gamma}$ is not a cardinal in M, there is an $f \in M$ mapping an $\alpha < \rho$ onto ρ. But then $f \in \Sigma^*(M||\gamma)$. Hence $\rho = \rho^\omega_{M||\gamma} < \alpha < \rho$. Contradiction!

QED(Lemma 5.2.6)

Using this we prove:

Lemma 5.2.7. *Let $\nu \in N_\xi$, $\beta = \beta(N_\xi, \nu)$. Then $N_\xi||\beta = M_\eta$ for an $\eta < \xi$.*

Proof. Suppose not. Let N_ξ be a counterexample with ξ chosen minimally. We derive a contradiction as follows:

Case 1. ξ is a limit ordinal. Then $N_\xi = \bigcup_{i<\xi} M_i||\tilde{\mu}_{i\xi}$ and $M_i||\tilde{\mu}_{i\xi} = M_j||\tilde{\mu}_{i\xi} = N_\xi||\tilde{\mu}_{i\xi}$ for $i \leq j < \xi$.

Case 1.1. There is $i < \xi$ such that $\beta < \tilde{\kappa}_i =: \tilde{\kappa}_{i,\xi}$. Then $\beta = \beta(M_i||\tilde{\kappa}_i, \nu)$ since $M_i||\tilde{\kappa}_i = N_\xi||\tilde{\kappa}_i$ and $\beta = \beta(N_\xi, \nu)$. Hence $\rho = \rho^\omega_{N_\xi||\beta}$ is a cardinal in $M_i||\tilde{\kappa}_i$. Let $\sigma : M_i \longrightarrow N_i$ be the core map. Since $\rho^\omega_{M_i} \geq \tilde{\kappa}_i$, we conclude that ρ is a cardinal in N_i. Hence $\beta = \beta(N_i, \nu)$, where $i < \xi$ and $N_i||\beta = N_\xi||\beta$. Hence ξ was not minimal.

Case 1.2. Case 1.1 fails. Pick i such that $\beta < \tilde{\mu}_i =: \tilde{\mu}_{i,\xi}$. Then $\beta = \beta(M_i||\tilde{\mu}_i, \nu)$, since $M_i||\tilde{\mu}_i = N_\xi||\tilde{\mu}_i$ and $\beta = \beta(N_\xi, \nu)$. Clearly $\tilde{\kappa}_i \leq \beta_i$. $\tilde{\kappa}_i$ is the largest cardinal in $M_i||\tilde{\mu}_i$ and $\rho = \rho^\omega_{M_i||\beta}$ is a cardinal in $M_i||\tilde{\mu}_i$. Hence $\rho \leq \tilde{\kappa}_i$. But ρ is a cardinal in M_i by acceptability since $\tilde{\kappa}_i$ is a cardinal in M_i. Let $\sigma : M_i \longrightarrow N_i$ be the core map. Then $\mathrm{crit}(\sigma) \geq \rho^\omega_{M_i} \geq \tilde{\kappa}_i$. Hence $\rho < \nu$ is a cardinal in N_i and $\beta = \beta(N_i||\tilde{\mu}_i, \nu)$. Hence $\beta = \beta(N_i, \nu)$, where $i < \xi$ and $N_i||\beta = N_\xi||\beta$. Thus, ξ was not chosen minimally. Contradiction!

QED(Case 1)

Case 2. $\xi = i + 1$.

Case 2.1. Option 1 was used at i. Then $N_\xi = \langle J^{E^i}_{\gamma+\omega}, \emptyset\rangle$ where $M_i = \langle J^{E^i}_\gamma, E^i_\gamma\rangle$. Then $\beta \leq \gamma$, since γ is the largest limit ordinal in N_ξ. If $\beta = \gamma$, then $N_\xi||\beta = M_i$ where $i < \xi$. Hence ξ is not a counterexample. Contradiction! Hence $\beta < \gamma$. But then $\beta < \mu =: \mu_{M_i}(\rho)$ where $\rho = \rho^\omega_{M_i||\beta}$ and let $\sigma : M_i \longrightarrow N_i$ be the core map. Then $\mathrm{crit}(\sigma) \geq \rho$. Hence ρ is a cardinal in N_i and $M_i||\mu = N_i||\mu$. Hence $N_i||\beta = N_\xi||\beta$ where $i < \xi$. Hence ξ was not minimal. Contradiction!

QED(Case 2.1)

Case 2.2. Option 2 was applied. Then $N_\xi = \langle J^{E^i}_\gamma, F\rangle$ where $M_i = \langle J^{E^i}_\gamma, \emptyset\rangle$ is a ZFC^- model. Hence $N_i = M_i$, since, letting $\rho =: \rho^\omega_{N_i} = \rho^\omega_{M_i}$, we have: $\mu_{N_i}(\rho) = \mu_{M_i}(\rho) = \mathrm{ht}(M_i)$. In particular, $N_i||\beta = M_i||\beta$ where $i < \gamma$. Hence γ is not minimal.

QED(Lemma 5.2.7)

Now let $\langle N_i \mid i < \Omega\rangle$ be a (putative) Steel array. Let $N = N_\Omega$ and let $E^N_\nu \neq \emptyset$. It seems clear that $N||\nu$ "originated" at a stage $i+1 \leq \Omega$ and $N_{i+1} = \langle J^E_{\nu_i}, F\rangle$ where $J^E_{\nu_i} = M_i$. Using 5.2.5 we can trace back to the origin in a finite sequence of steps. Following Steel, we call this the *resurrection sequence*, since it "resurrects" the original ancestor of $N||\nu$.

Definition 5.2.9. Let $N = N_\Omega$ and let $N||\nu$ be an active premouse. The *resurrection sequence* for $\langle N, \nu\rangle$ is a finite sequence $\langle \eta_i, \nu_i\rangle(i \leq p)$ such that $N_{\eta_i}||\nu_i$ is active and $\eta_{i+1} < \eta_i$ for $i < p$. We define:

- $\eta_0 = \Omega$, $\nu_0 = \nu$.
- If $\nu_i \notin N_{\eta_i}$, then $i = p$ and the sequence terminates.
- If $\nu_i \in N_{\eta_i}$, let $\beta = \beta(N_{\eta_i}, \nu_i)$. Then:

$$\eta_{i+1} =: \text{ that } \eta' \text{ such that } N_{\eta_i}||\beta = M_\eta.$$

- Let $k : M_{\eta_{i+1}} \longrightarrow N_{\eta_{i+1}}$ be the core map. Then

$$\nu_{i+1} = \begin{cases} k(\nu_i) & \text{if } \nu_i \in M_{\eta_{i+1}} \\ \mathrm{On} \cap N_{\eta_{i+1}} & \text{if not} \end{cases}$$

$N_p||\nu_p$ is then the origin which we sought. We define:

Definition 5.2.10. $\overline{\beta}_0 = \mathrm{On} \cap N$, $\overline{\beta}_{i+1} \simeq \beta(N||\beta_i)$.

It follows easily that $\overline{\beta}_i$ is defined for $i \leq p$ and that there are unique maps:

$$k_i : N||\overline{\beta}_i \longrightarrow_{\Sigma^*} N_{\eta_i}$$

defined by $k_0 = \mathrm{id}$; $k_{i+1} = k \cdot k_i$ where:

$$k : M_{\eta_{i+1}} \longrightarrow N_{\eta_{i+1}} \text{ is the core map.}$$

k_p is then called the *resurrection map* for $\langle N, \nu\rangle$. It is easily seen that if $i \leq p$, then $k_p = k \cdot k_i$, where k is the resurrection map for $N_{\eta_i}||\nu_i$. Moreover, $\langle \eta_{i+n}, \nu_{i+n}\rangle(n \leq p - i)$ is the resurrection sequence for $\langle N_{\eta_i}, \nu_i\rangle$.

A proof similar to that of lemma 5.2.7 shows:

Definition 5.2.11. Let N be a premouse, $\alpha \in N$ a limit ordinal. α is *cardinally absolute* if and only if for all $\beta < \alpha$:

$$N||\alpha \models \beta \text{ is cardinal} \longrightarrow N \models \beta \text{ is a cardinal}.$$

Lemma 5.2.8. *Let $\alpha \in N_\xi$ be cardinally absolute such that $E_\alpha^{N_\xi} = \emptyset$. Then there is $i < \xi$ such that $N_\xi||\alpha = M_i$ and N_{i+1} is formed by Option 1.*

Proof. Suppose not. Let N_ξ be a counterexample with ξ chosen minimally. We derive a contradiction.

Case 1. ξ is a limit ordinal.

Case 1.1. There is $i < \xi$ such that $\alpha < \tilde{\kappa}_i =: \tilde{\kappa}_{i,\xi}$. Then $M_i||\tilde{\kappa}_i = N_\xi||\tilde{\kappa}_i$. Thus $\alpha < \tilde{\kappa}_i$ is cardinally absolute in M_i and $E_\alpha^{M_i} = \emptyset$. Let $\sigma : M_i \to N_i$ be the core map. Then $\text{crit}(\sigma) \geq \rho_{M_i}^\omega \geq \tilde{\kappa}_i$. Hence α is cardinally absolute in N_i and $E_\alpha^{N_i} = \emptyset$. Moreover, $N_i||\alpha = M_i||\alpha$. Thus i is a counterexample and ξ was not chosen minimally. Contradiction!

Case 1.2. Case 1.1 fails. Pick $i < \xi$ such that $\alpha < \tilde{\mu}_i =: \tilde{\mu}_{i,\xi}$. Then $\tilde{\kappa}_i \leq \alpha < \tilde{\mu}_i$ and $M_i||\tilde{\mu}_i = N_\xi||\tilde{\mu}_i$. Let $\sigma : M_i \longrightarrow N_i$ be the core map. Then $\text{crit}(\sigma) \geq \rho_{M_i}^\omega \geq \tilde{\kappa}_i = \alpha$. Hence $\tilde{\mu}_i = \mu_{N_i}(\tilde{\kappa}_i)$ and $\alpha \in N_i||\tilde{\mu}_i = M_i||\tilde{\mu}_i$. Thus $N_i||\alpha = M_i||\alpha$. Hence $E_\alpha^{N_i} = \emptyset$. But then i is a counterexample, where $i < \xi$. Hence ξ was not minimal. Contradiction!

Since α is a limit ordinal, we know that $\alpha \notin N_0 = J_\omega^\emptyset$, so there remains only the case:

Case 2. $\xi = i + 1$

Case 2.1. N_ξ is formed by option 2. Then $N_\xi = \langle M_i, F\rangle$ and $\alpha \in M_i$. But $M_i = N_i$ is a ZFC^- model. Hence $N_i||\alpha = M_i||\alpha = N_\xi||\alpha$. Hence $E_\alpha^{N_i} = \emptyset$. Thus $i < \xi$ is a counterexample and ξ is not minimal. Contradiction!

Case 2.2. $\alpha \in M_i$ and N_ξ is formed by option 1. Let:

$$\tau = \sup\{\beta < \alpha \mid \beta \text{ is a cardinal in } N_\xi||\alpha\}.$$

Then τ is a cardinal in N_ξ. Hence $\tau < \rho$ where $\rho = \rho_{M_i}^\omega = \rho_{N_i}^\omega$. Let $\mu = \mu_{M_i}(\rho) = \mu_{N_i}(\rho)$. Then $N_i||\mu = M_i||\mu$ and $\alpha \leq \mu$, since $\tau \leq \rho$. Clearly α is then cardinally absolute in N_i, since $\tau \leq \rho$ is a cardinal in N_i. But $E_\alpha^{N_i} = E_\alpha^{M_i} = E_\alpha^{N_\xi} = \emptyset$. Hence $i < \xi$ is a counterexample.

Case 2.3. The above cases fail. Then $\alpha = \text{ht}(M_i)$ is the largest limit ordinal in N_ξ, where $E_\alpha^{N_\xi} = \emptyset$. Then $M_\xi||\alpha = M_i||\alpha$, where $i < \alpha$ and N_{i+1} is formed by option 1. Hence ξ is not a counterexample. Contradiction!

QED(Lemma 5.2.8)

5.3 Robust Premice

5.3.1 The Chang hierarchy

The logician C. C. Chang proposed a modification of the constructible hierarchy in which, when passing to the next level, we include not only the previous level as a set but also the set:

$$\alpha^\omega =: \{f \mid f : \omega \longrightarrow \alpha\}$$

where α is the previous level. There are various ways of organizing this hierarchy (although any of them ultimately reaches the same inner model). We shall construct the hierarchy, indexing the level by the limit ordinals. We define:

Definition 5.3.1. The *Chang* hierarchy

$$\langle \bar{C}_\alpha \mid \alpha \text{ is a limit ordinal}\rangle$$

is defined inductively by:

$$\begin{aligned} \bar{C}_\omega &= J_\omega = H_\omega \\ \bar{C}_{\alpha+\omega} &= \mathrm{rud}(\bar{C}_\alpha \cup \{\bar{C}_\alpha\} \cup \alpha^\omega) \\ \bar{C}_{\omega\lambda} &= \bigcup_{\xi<\lambda} \bar{C}_{\omega\xi} \text{ for limit } \lambda \end{aligned}$$

(Here: $\mathrm{rud}(X)$=the closure of X under rud functions).

Then each $\bar{C}_\alpha$ is transitive and rudimentarily closed. Moreover, $\alpha = \mathrm{On} \cap \bar{C}_\alpha = \mathrm{rank}(\bar{C}_\alpha)$. Using the methods developed in Chapter 2 we get:

- $\langle \bar{C}_\xi \mid \xi \in \mathrm{Lim} \cap \eta\rangle \in \bar{C}_\alpha$ for $\eta < \alpha$
- $\langle \bar{C}_\xi \mid \xi \in \mathrm{Lim} \cap \alpha\rangle$ is uniformly $\bar{C}_\alpha$-definable for

α a limit of limit ordinals. (Hence: Lim=:the class of limit ordinals.) However, the definition of $\langle \bar{C}_\xi \mid \xi \in \mathrm{Lim} \cap \alpha\rangle$ is *not* necessarily $\Sigma_1(\bar{C}_\alpha)$. In order to remedy this we set:

Definition 5.3.2. $C_\alpha = \langle \bar{C}_\alpha ; \langle \bar{C}_\xi \mid \xi \in \mathrm{Lim} \cap \alpha\rangle\rangle$.

Then C_α is amenable and we trivially have:

$$\langle C_\xi \mid \xi \in \mathrm{Lim} \cap \alpha\rangle \text{ is uniformly } \Sigma_1(C_\alpha).$$

We shall often write $\langle C_\xi \mid \xi < \alpha\rangle$ as an abbreviation for $\langle C_\xi \mid \xi \in \mathrm{Lim} \cap \alpha\rangle$. The *condensation lemma* for the C-hierarchy has a much stronger hypothesis than the condensation lemma for L, to wif:

Lemma 5.3.1. *Let α be a limit ordinal. Let $X \prec C_\alpha$ such that $(X \cap \alpha)^\omega \subset X$. Then $X \simeq C_{\bar{\alpha}}$ for an $\bar{\alpha} \leq \alpha$.*

Note If α is closed under Gödel pairing, we can replace $(X \cap \alpha)^\omega \subset X$ by: $[X \cap \alpha]^\omega \subset X$, where $[Y]^\omega =$:the set of countable subsets of Y. This simplification is possible since if $f : \omega \longrightarrow X \cap \alpha$, then f is recoverable from: $\{\prec \delta, \xi \succ \mid f(\delta) = \xi\}$, which is a countable subset of $X \cap \alpha$.

We leave the proof of Lemma 5.3.1 to the reader. If we wished, we could define the Chang hierarchy *relative to a class* E by:

Definition 5.3.3. For limit ordinals α such that:

$$\begin{aligned}
&\bar{C}_\omega[E] = J_\omega = H_\omega \\
&\bar{C}_{\alpha+\omega}[E] = \mathrm{rud}(\bar{C}_\alpha[E] \cup \{\bar{C}_\alpha[E]\} \cup \{E \cap \bar{C}_\alpha[E]\} \cup \alpha^\omega) \\
&\bar{C}_{\omega\lambda}[E] = \bigcup_{\xi<\lambda} \bar{C}_{\omega\xi}[E] \text{ for limit } \lambda
\end{aligned}$$

We can then define:

$$C^E_\alpha = \langle \bar{C}_\alpha[E], E \cap \bar{C}_\alpha[E], \langle \bar{C}_\xi[E] \mid \xi \in \mathrm{Lim} \cap \alpha\rangle\rangle.$$

We leave it to the reader to formulate the condensation for the C^E-hierarchy. We shall, however, be more interested in a different modification of the Chang hierarchy: Let e be a set or class. Let τ, η be limit ordinals with $\tau \leq \eta$. $C^e_{\tau,\eta}$ then denotes the result of first constructing from e up to τ, getting J^e_τ, and therefore applying the operations of the Chang hierarchy without reference to e. We define:

Definition 5.3.4. Let e be any class or set. Let τ be a limit ordinal. For limit $\alpha \geq \tau$ we define $C^e_{\tau,\alpha}$ by induction on α as follows:

$$\begin{aligned}
&\bar{C}^e_{\tau,\tau} = J^e_\tau \\
&\bar{C}^e_{\tau,\alpha+\omega} = \mathrm{rud}(\bar{C}^e_{\tau,\alpha} \cup \{\bar{C}^e_{\tau,\alpha}\} \cup \alpha^\omega) \\
&\bar{C}^e_{\tau,\tau+\omega\lambda} = \bigcup_{i<\lambda} \bar{C}^e_{\tau,\tau+\omega i}.
\end{aligned}$$

Clearly $\bar{C}^e_{\tau,\eta}$ is rudimentarily closed and transitive. Moreover:

$$\eta = \mathrm{On} \cap \bar{C}^e_{\tau,\eta} = \mathrm{rank}(\bar{C}^e_{\tau,\eta}.$$

We set:

Definition 5.3.5. $C^e_{\tau,\eta} = \langle \bar{C}^e_{\tau,\eta}, e \cap J^e_\tau, \langle \bar{C}^e_{\tau,\xi} \mid \tau \leq \xi < \eta \rangle \rangle$

Note When using this notation we will often tacitly assume that $e = e \cap J^e_\tau$. In most cases, we will also assume that η is much greater than τ.

The condensation lemma for $C^e_{\tau,\eta}$ reads:

Lemma 5.3.2. *Let $X \prec_{\Sigma_1} C^e_{\tau,\eta}$ such that $\tau \in X$ and $(X \cap \eta)^\omega \subset X$. Then $X \simeq C^{\bar{e}}_{\bar{\tau},\bar{\eta}}$ for a $\bar{\tau} \leq \tau$ and an $\bar{\eta} \leq \eta$. Moreover, if $\tau \subset X$, then $\bar{\tau} = \tau$ and $\bar{e} = e$. (if η is closed under Gödel pairing we can again replace $(X \cap \eta)^\omega \subset X$ by: $[X \cap \eta]^\omega \subset X$.)*

5.3.2 Robustness

Without further ado we can now define:

Definition 5.3.6. Let $N = \langle J^E_\nu, F \rangle$ be an active premouse, as usual set: $\kappa = \kappa_\nu =: \mathrm{crit}(F)$, $\tau = \tau_\nu =: \kappa^{+N}$, $\lambda = \lambda_\nu =: F(\kappa)$. F is *robust in N* if and only if whenever $\mathcal{U} \subset \lambda$, $W \subset \mathbb{P}(\kappa) \cap N$ are countable sets, then there is $g\colon \mathcal{U} \longrightarrow \kappa$ such that

(a) $\prec g(\vec{\alpha}) \succ \in X \longleftrightarrow \prec \vec{\alpha} \succ \in F(X)$ for $\alpha_1, \ldots, \alpha_n \in \mathcal{U}$, $X \in W$.

(b) Let $\tau = \mathrm{lub}(\mathcal{U})$, $\vec{\tau} = \mathrm{lub}(g\text{"}\mathcal{U})$. Let φ be a Σ_1 formula. Then for all $v_1, \ldots, v_m \subset \mathcal{U}$ we have:

$$C^E_{\bar{\tau},\kappa} \models \varphi(g\text{"}v_1, \ldots, g\text{"}v_m) \longleftrightarrow C^E_{\tau,\infty} \models \varphi(v_1, \ldots, v_m).$$

Remark. It follows easily that if $\alpha_1, \ldots, \alpha_n \in \mathcal{U}$, then:

$$C^E_{\bar{\tau},\kappa} \models \varphi(g\text{"}\vec{v}, g(\vec{\alpha})) \longleftrightarrow C^E_{\tau,\infty} \models \varphi(\vec{v}, \vec{\alpha}).$$

Note. In the following we shall use the notation, if N is a premouse, set:

$$E^N =: \text{that } E \text{ such that } N = \langle J^E_\alpha, F \rangle = \langle J_\alpha[E], E, F \rangle.$$

(Recall that J^E_α is defined to be $\langle J_\alpha[E], E \cap J_\alpha[E] \rangle$.)

If $\nu \leq \alpha$ is a limit ordinal, we write:

$$E_\nu^N = \text{that } F \text{ such that } N||\nu = \langle J_\nu^E, F\rangle.$$

Note. If we omitted (b) in the definition of robustness, we would have the familiar condition of *ω-completeness.*

We now refine our definition as follows:

Definition 5.3.7. Let $N = \langle J_\nu^E, F\rangle$ be an active premouse. Let $\kappa \leq \gamma \leq \lambda$, where κ, λ are as above. F is *robust up to γ in N* if and only if whenever $\mathcal{U} \subset \lambda$, $W \subset \mathbb{P}(\kappa) \cap N$ are countable, then there is $g\colon \mathcal{U} \longrightarrow \kappa$ such that

(a) $\prec g(\vec{\alpha}) \succ \in X \longleftrightarrow \prec \vec{\alpha} \succ \in F(X)$ for $\alpha_1, \ldots, \alpha_n \in \mathcal{U}$, $X \in W$.

(b) Let $\tau = \text{lub}(\mathcal{U} \cap \gamma)$, $\vec{\tau} = \text{lub}(g``(\mathcal{U} \cap \gamma)$. Let φ be a Σ_1 formula. Then $\bar{\tau} < \kappa$ for all $v_1, \ldots, v_m \subset \mathcal{U} \cap \gamma$ we have:

$$C^E_{\bar{\tau},\kappa} \models \varphi(g``v_1, \ldots, g``v_m) \longleftrightarrow C^E_{\tau,\infty} \models \varphi(v_1, \ldots, v_m).$$

We then define:

Definition 5.3.8. A premouse M is *robust* if and only if whenever $M||\nu = \langle J_\nu^E, F\rangle$ is active and $\gamma \in [\kappa_F, \lambda_F]$ is a cardinal in M, then F is robust up to γ in $M||\nu$.

As usual, let: $\kappa = \kappa_\nu$, $\tau = \tau_\nu$, $\lambda = \lambda_\nu$. Let $\gamma \in [\kappa, \lambda]$ be a cardinal in N. We note the following consequences:

(1) Let $\mathcal{U} \subset \lambda$, $W \subset \mathbb{P}(\kappa) \cap N$ be countable and let $g\colon \mathcal{U} \longrightarrow \kappa$ be as in the above definition. Let ψ be a Σ_1 formula. Let $\bar{\gamma} = \text{lub}(g``\gamma)$. Let $\alpha_1, \ldots, \alpha_m \in \mathcal{U} \cap \gamma$, $v_1, \ldots, v_n \subset \mathcal{U} \cap \gamma$. Then:

$$C^{E^N}_{\gamma,\infty} \models \psi[\vec{\alpha}, \vec{v}] \longleftrightarrow C^{E^N}_{\bar{\gamma},\kappa} \models \psi[g(\vec{\alpha}), g``\vec{v}].$$

(2) If, in addition, we assume:

$$\omega \subset \mathcal{U}, \{\xi\} \in W \text{ for } \xi \in \mathcal{U} \cap \kappa,$$

then

$$g(\xi) = \xi \text{ for } \xi \in \mathcal{U} \cap \kappa.$$

To see this note that:

$$g(\alpha) \in \{\xi\} \longleftrightarrow \alpha \in F(\{\xi\}) = \{\xi\} \text{ for } \alpha \in \mathcal{U}.$$

But then $g"b = b$ for $b \subset \omega$. Hence: if $b_1, \ldots, b_l \subset \omega$, then:

$$C^{E^N}_{\gamma,\infty} \models \psi[\vec{b}, \vec{\alpha}, \vec{v}] \longleftrightarrow C^{E^N}_{\bar{\gamma},\kappa} \models \psi[\vec{b}, g(\vec{\alpha}), g"\vec{v}].$$

Taking $\gamma = \kappa$ we have:

(3) If κ is a cardinal in N and $\mathcal{U} \subset \kappa$ is countable and $\bar{\gamma} = \mathrm{lub}(\mathcal{U})$, then $\bar{\gamma} < \kappa$ and

$$C^{E^N}_{\bar{\gamma},\infty} \models \psi[\vec{b}, \vec{\alpha}, \vec{v}] \longleftrightarrow C^{E^N}_{\bar{\gamma},\kappa} \models \psi[\vec{b}, \vec{\alpha}, \vec{v}]$$

for $b_1, \ldots, b_l \subset \omega$, $\alpha_1, \ldots, \alpha_m \in \mathcal{U} \cap \kappa$, $v_1, \ldots, v_n \subset \mathcal{U} \cap \kappa$. Thus $\mathrm{cf}(\kappa) > \omega$. Hence every hereditarily countable set x lies in C_κ and is coded by a $b \subset \omega$ such that the Σ_1 statement "b codes x" holds in C_κ. Hence by (2):

(4) Let $x_1, \ldots, x_r$ be hereditarily countable. Let the assumption of (2) be given. Let $\alpha_1, \ldots, \alpha_m \in \mathcal{U} \cap \gamma$, $v_1, \ldots, v_n \subset \mathcal{U} \cap \gamma$. Then:

$$C^{E^N}_{\gamma,\infty} \models \psi[\vec{x}, \vec{\alpha}, \vec{v}] \longleftrightarrow C^{E^N}_{\bar{\gamma},\kappa} \models \psi[\vec{x}, g(\vec{\alpha}), g"\vec{v}].$$

By (3) we have:

Lemma 5.3.3. *Let N be robust, $F = E^N_\nu \neq \emptyset$ and let $\kappa = \kappa_\nu$ be a cardinal in N. Let $x_1, \ldots, x_r$ be hereditarily countable. Let $\mathcal{U} \subset \kappa$ be countable. Set: $\bar{\gamma} = \mathrm{lub}(\mathcal{U})$. Then $\bar{\gamma} < \kappa$. Let $\alpha_1, \ldots, \alpha_r \in \mathcal{U}$, $v_1, \ldots, v_n \subset \mathcal{U}$. Let ψ be a Σ_1 formula. Then:*

$$C^{E^N}_{\bar{\gamma},\infty} \models \psi[\vec{x}, \vec{\alpha}, \vec{v}] \longleftrightarrow C^{E^N}_{\bar{\gamma},\kappa} \models \psi[\vec{x}, \vec{\alpha}, \vec{v}].$$

In the usual application of robustness, we assume that there is a countable premouse $\bar{N} = \langle J^{\bar{E}}_{\bar{N}}, \bar{F}\rangle$ and a map $\sigma\colon \bar{N} \longrightarrow_{\Sigma_0} N||\nu$ such that:

$$\mathcal{U} = \mathrm{rng}(\sigma) \cap \lambda, W = \mathrm{rng}(\sigma) \cap \mathbb{P}(\kappa) \cap N.$$

Note that the assumptions in (2) are then automatically satisfied. Then by (4) we have

Lemma 5.3.4. *Let N be robust, $F = E^N_\nu \neq \emptyset$ and $\kappa = \kappa_\nu$, $\tau = \tau_\nu$, $\lambda = \lambda_\nu$ in N. Let:*

$$\sigma\colon \bar{N} \longrightarrow_{\Sigma_\omega} N||\nu$$

where $\bar{N} = \langle J^{\bar{E}}_{\bar{N}}, \bar{F}\rangle$ is a countable premouse. Let $\bar{\kappa} = \kappa_{\bar{\nu}}$, $\bar{\lambda} = \lambda_{\bar{\nu}}$ in N. There is $g\colon \bar{\lambda} \longrightarrow \kappa$ such that

(a) Let $\alpha_1, \ldots, \alpha_m < \bar{\lambda}$.

$$\prec g(\vec{\alpha}) \succ \in \sigma(x) \longleftrightarrow \prec \vec{\alpha} \succ \in \bar{F}(x) \text{ for } x \in \mathbb{P}(\bar{\kappa}) \cap \bar{N}$$

(b) Let $\gamma \in [\kappa, \lambda]$ be a cardinal in N. Let $x_1, \ldots, x_r$ be hereditarily countable. Let $\alpha_1, \ldots, \alpha_m < \lambda$ such that $g(\alpha_i) < \gamma$ $(i = 1, \ldots, m)$. Let $v_1, \ldots, v_m \subset \lambda$ such that $g\text{"}v_i \subset \gamma$ $(i = 1, \ldots, n)$. Let ψ be a Σ_1 formula. Then:

$$C^{E^N}_{\gamma,\infty} \models \psi[\vec{x}, \sigma(\vec{\alpha}), \sigma\text{"}\vec{v}] \longleftrightarrow C^{E^N}_{\bar{\gamma},\kappa} \models \psi[\vec{x}, g(\vec{\alpha}), g\text{"}\vec{v}].$$

Lemma 5.3.3 and 5.3.4 are our main lemmas on robustness.

Definition 5.3.9. A (putative) Steel array is *robust* if and only if whenever $N_{i+1} = \langle J^{E^i}_{\nu_i}, F\rangle$ is obtained by Option 2, then F is robust in N_{i+1}.

Lemma 5.3.5. *Let $\langle N_i\rangle$ be a (putative) robust Steel array. Then each N_i is a robust premouse.*

Proof. Let i be the least counterexample. Then $i > 0$.

Case 1. $i = j + 1$ and N_i is formed according to Option 1. Let $N_i||\nu = \langle J^E_\nu, F\rangle$ be active. Let $\kappa \leq \gamma \in N_i||\nu$, where γ is a cardinal in N_i.

Claim. F is robust up ti γ in $N_i||\nu$.

We know that $\nu \leq \mathrm{On} \cap M_j$, since N_i is passive and $\mathrm{On} \cap N_i = (\mathrm{On} \cap M_j) + \omega$. Hence $M_j||\nu = N_i||\nu$ and γ is a cardinal in M_j. Hence $\gamma \leq \rho^\omega_{M_j}$, since otherwise it would not be a cardinal in N_i. Let $\sigma\colon M_j \longrightarrow N_j$ be the core map. Then $\sigma \restriction \gamma = \mathrm{id}$ and $\sigma(\gamma)$ is a cardinal in N_j, where

$$\sigma(\kappa) \leq \sigma(\gamma) \in N_j||\sigma(\nu) = \langle J^{E'}_{\sigma(\nu)}, F'\rangle.$$

Hence F' is robust up to $\sigma(\gamma)$ in $N_j||\sigma(\nu)$, since N_j is robust. It follows easily that F is robust up to γ in $M_j||\nu$. QED(Case 1)

Case 2. $i = j + 1$ and Option 2 applied.

Let $N_i||\nu = \langle J^E_\nu, F\rangle$ be active. Let $\kappa \leq \gamma \in N_i||\nu$ where γ is a cardinal in N_i.

Claim. F is robust up to γ in $N_i||\nu$.

If $\nu \in N_i$ this is trivial, since $N_j||\nu = N_i||\nu$ and γ is a cardinal in $N_j = M_j$, where N_j is robust. Now let $\nu = \mathrm{On} \cap N_i$. Then $N_i = \langle N_j, F\rangle$ where F is robust in $N_i||\nu$. QED(Case 2)

Case 3. $i = \eta$ is a limit ordinal.

Then N_η is passive. Let $N_\eta||\nu = \langle J^E_\nu, F\rangle$ be active where $\kappa \leq \gamma \in N_\eta||\nu$ and γ is a cardinal in N_η. The definition of N_η tells that $N_\eta||\nu = N_j||\nu$

and $\gamma \in N_j||\nu$ is a cardinal in N_j for sufficiently large $j < \eta$ (it suffices that $\nu < \tilde{\mu}_{j,\eta}$ and $\gamma < \tilde{\kappa}_{j,\eta}$). But then F is robust up to γ, since N_j is robust.

QED(Lemma 5.3.5)

We shall prove:

Lemma 5.3.6. *Assume there is no inner model with a Woodin cardinal. Let $\langle N_i \mid i \leq \mu\rangle$ be a putative robust Steel array. Then it is a Steel array (i.e. N_μ is mouselike).*

It will suffice to show:

Lemma 5.3.7. *Let N_μ be restrained. Let $\sigma\colon P \longrightarrow_{\Sigma^*} N_\mu$, where P is a countable premouse. Then P is countably normally iterable.*

We first show that Lemma 5.3.7 implies Lemma 5.3.6. Suppose not. Let Ω be least such that Lemma 5.3.6 fails. Then $\langle N_i \mid i < \Omega\rangle$ is a robust Steel array. Hence N_Ω is pre-mouselike.

Case 1. N_Ω is restrained. We first show that N_Ω is mouselike. Let $N_\Omega \in H_\theta$, where $\theta > \Omega$ is regular. Let $X \prec H_\theta$ be countable such that $N_\Omega \in X$. Let $\sigma\colon \bar{H} \stackrel{\sim}{\longleftrightarrow} X$ be the transitivation of X. Let $\sigma(P) = N_\Omega$. Then P is pre-mouselike and restrained. Moreover, $\sigma \restriction P\colon P \longrightarrow_{\Sigma^*} N_\Omega$. By Lemma 5.3.7, P is then uniquely normally iterable. Hence P is mouselike. Hence, by absoluteness, P is mouselike in $\bar{H}$. Hence N_Ω is mouselike in H_θ. Hence N_Ω is mouselike, by absoluteness. But then $M_\Omega = \mathrm{core}(N_\Omega) \in X$ and $P' = \mathrm{core}(P) \in \bar{H}$. Hence $\sigma'\sigma\colon P' \longrightarrow_{\Sigma^*} N_\Omega$, where $\sigma' = \sigma_{M_\Omega}$ is the core map. Hence P' is fully iterable by Lemma 5.3.6. Hence P' is mouselike. But then $M_\Omega = \sigma(P')$ is mouselike. QED(Case 1)

Case 2. $N = N_\Omega$ is unrestrained. Then N is a constructible extension of $N||\alpha$ for an $\alpha \leq \mathrm{ht}(N)$. Moreover, α is Woodin in $N' = J^E_{\beta+1}$, where $N = J^E_\beta$. (Hence $\rho^\omega_N \geq \alpha$ and $E \subset J^E_\alpha$.) By Lemma 4.4.11 it follows that N is mouselike. But since N is a constructible extension of J^E_α and $\rho^\omega_N \geq \alpha$, it follows that N is sound and $\mathrm{core}(N) = N = M_\Omega$. QED(Case 2)

(*Note*: we can actually prove stronger result. By Corollary 5.1.4 and 5.1.5 we have:

Lemma 5.3.8. *Let N_μ be restrained. Then N_μ itself is smoothly ∞-iterable and fully α-iterable for all $\alpha < \infty$.*)

Before tackling this, however, we shall prove a much weaker theorem which will enable us to display some of our methods:

Lemma 5.3.9. *Let N be a robust premouse which is pre-mouselike. Let $\sigma\colon P \longrightarrow_{\Sigma^*} N$, where P is a countable premouse. Let:*

$$I = \langle\langle P_i\rangle, \langle v_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$$

be a non truncating normal iteration of P of length ω. Then I has a cofinal well founded branch b. (In fact, there is a map $\sigma'\colon P_b \longrightarrow_{\Sigma_0} N$ such that $\sigma'\pi_{0,b} = \sigma$.)

Before beginning the proof of Lemma 5.3.9, we establish the following *iteration fact*, which we will employ frequently:

Lemma 5.3.10. *Let P be pre-mouselike. Let $I = \langle\langle P_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ be a potential iteration of P. Let $i < \mathrm{lh}(I)$. There is a ν such that $P_i^*||\nu = \langle J_\nu^E, F\rangle$ with $F \neq \emptyset$, $\kappa_i = \mathrm{crit}(F)$, $\tau_i = \tau(F)$.*

Proof. We first recall that the statement: P is pre-mouselike is uniformly $\Pi_1(P)$ by Lemma 4.4.2. Moreover, if P is pre-mouselike, then every $Q \triangleleft P$ is trivially pre-mouselike. It follows easily that every P_i is pre-mouselike. By Lemma 4.3.11 it then follows that every P_i is Σ_1-acceptable.

Assume the Lemma to be fails. Let I be a counterexample with i chosen minimally. We derive a contradiction. Let $h = T(i+1)$. Then:

(1) $h \neq i$.

Proof. If not, take $\nu = \nu_i$. Then i is not a counterexample.

(2) $\nu_i \notin P_i$.

Proof. λ_h is a cardinal in P_i by (1).

$$P_i||\lambda_h = P_h||\lambda_h = P_i^*||\lambda_h.$$

In P_i we have:

$$\bigvee \nu V \models (E_\nu = F \wedge \kappa_i = \mathrm{crit}(F) \wedge \tau_i = \tau(F) = \tau^{+J_\nu^E})$$

This is a Σ_1 statement about κ_i, π_i where $\kappa_i, \tau_i < \lambda_h$. Hence by a Σ_1-acceptability the statement holds in $P_i||\lambda_h = P_h^*||\lambda_h$. Hence i is not a counterexample. Contradiction!

(3) i is not a limit ordinal.

Proof. Suppose not. Since $\lambda_i = \mathrm{lub}\{\kappa_j \mid j+1 <_T i\}$, we can choose $j+1 <_T i$ such that $\kappa_j > \kappa_i$ and $(j,i]_T$ has no truncation. $F = E_{\nu_i}$ is then the top extender of P_i, by (2). Since $\pi_{j,i}\colon P_j \longrightarrow P_i$, $\kappa_j = \mathrm{crit}(\pi_{j,i})$, then P_j has a top extender F'. Then $\kappa_i = \mathrm{crit}(F')$ since $\kappa_i = \pi_{j,i}(\kappa_i) = \mathrm{crit}(F)$. Then i is not a minimal counterexample, since, letting I' be defined by $I'|j{+}1 = I|j{+}1$ and $\nu'_j = \mathrm{ht}(P_j)$, then I' is a counterexample of length $j+2$ where $j < i$. Contradiction. QED(3)

Now let $i = k+1$, $t = T(k+1)$. Then $\pi_{t,i}\colon P^*_k \longrightarrow P_i$. Hence P^*_k has a top extender F^*. Let $\kappa^* = \mathrm{crit}(F^*)$. Then $\pi_{t,i}(\kappa^*) = \kappa_i$. But then

(4) $\kappa^* < \kappa_i$

Proof. Suppose not. Let $F^* = E^{P_k}_\nu$. Then $\kappa_i = \mathrm{crit}(F^*)$, where $\nu > \nu_j$ for all $j < t$. Define a potential iteration I' pf length $t+2$ by:

$$I'|t+1 = I|t+1, \nu_t = \nu.$$

Then I' is a counterexample where $t < i$. Hence i was not minimal. Contradiction! QED(4)

But then $\kappa_k \leq \kappa^*$, since otherwise $\pi_{t,i}(\kappa^*) = \kappa^* < \kappa_i$. Hence $\kappa_i = \pi_{t,i}(\kappa^*) \leq \lambda_k$. But $\kappa_i < \lambda h$. Hence $h = i$. Contradiction! by (1)

QED(Lemma 5.3.10)

5.4 Worlds

Our main tool in the proof of lemma 5.3.6 is the concept of *world*. Prior to defining this we let:

Definition 5.4.1. ZFC^* is the theory ZFC^- together with the additional axiom: $\bigwedge x\,([x]^\omega$ is a set).

Recall that we defined:

$$L^{A_1,\ldots,A_n}_\alpha = J^{A_1,\ldots,A_n}_\alpha =: \langle J_\alpha[\vec{A}],\, \in,\, A_1 \cap J_\alpha[\vec{A}],\, \ldots,\, A_n \cap J_\alpha[\vec{A}]\rangle$$

where $\langle J_\alpha[\vec{A}] \mid \alpha < \infty\rangle$ is the constructible hierarchy relative to $A_1, \ldots, A_n$.

We now define:

Definition 5.4.2. A *world of height* α is a set $W = L^A_\alpha$ such that $A \subset \alpha$ and:

- $W \models \mathsf{ZFC}^*$

- W is *reflexive* in the sense that there are arbitrarily large $\beta < \alpha$ with: $L^A_\beta \prec L^A_\alpha$.
- $W \in V[G]$ for some G which is set generic over V.
- $[\alpha]^\omega \cap W = [\alpha]^\omega \cap V$

Remark. We think of a world as being an ideal object, whose properties we can discuss in V, although it might not actually be present in V. Note that neither direction of the above final equation is vacuous.

Lemma 5.4.1. *Let W be a world of height α. Then:*

(a) $\mathrm{cf}(\alpha) > \omega$ *in V. Moreover, if $\beta \in W$, then:*

$$\mathrm{cf}(\beta) = \omega \text{ in } V \longleftrightarrow \mathrm{cf}(\beta) = \omega \text{ in } W.$$

(b) *Let $e, \tau \in W$. Then $C^e_{\tau,\xi} = (C^e_{\tau,\xi})^W$ for $\xi \in W$. (Hence $C^e_{\tau,\alpha} = (C^e_{\tau,\infty})^W$.)*

(c) *Let $a_1, \ldots, a_m \in W$. Let $t \subset \omega$ code the complete theory of $\langle W, \in, a_1, \ldots, a_m\rangle$. Then $t \in W$ (hence $t \in V$).*

Proof.

(a) By $[\alpha]^\omega \cap W = [\alpha]^\omega \cap V$

(b) By induction on $\xi \in W$

(c) By reflectivity, t codes the complete theory of $\langle L^{\vec{A}}_\beta, \in, \vec{a}\rangle$ for a $\beta < \alpha$. Hence $t \in W$.

QED(Lemma 5.4.1)

Note. Taking $\tau = 0$ in (b) we have: $C_\xi = C^W_\xi$ for $\xi \in W$.

Note. Let $\mathrm{coll}(\omega, \gamma)$ be the canonical set of finite conditions for collapsing γ to ω. It is known that any complete Boolean algebra is a complete subalgebra of the algebra generated by the condition $\mathrm{coll}(\omega, \gamma)$ for a sufficiently large γ. Thus:

$$W \in V[G] \text{ for a set generic } G$$

means the same as

$$W \in V[G] \text{ where } G \text{ is } \mathrm{coll}(\omega, \gamma)\text{-generic}$$

and γ is sufficiently large.

We shall often make statements of the form:

There is a potential world W with property $\ldots$,

meaning that, for sufficiently large γ, the existence of such a world is forced by $\mathrm{coll}(\omega, \gamma)$. It is often convenient to reformulate such statements using Barwise theory. For instance:

Lemma 5.4.2. *Let $\alpha < \nu$, where C_ν is admissible. There is a language $\mathbb{L} = \mathbb{L}_\alpha$ on C_ν such that*

$$\mathbb{L} \textit{ is consistent} \longleftrightarrow \textit{ there is a potential world of height } \alpha.$$

(Note:"$\mathbb{L}_\alpha$ is consistent" will be uniformly $\Pi_1(C_\nu)$ in α.)

Proof. The language $\mathbb{L}$ has:

Predicate: $\in$

Constants: $\underline{x}\,(x \in C), \dot{A}, \dot{W}$

Axioms:

(a) ZFC^-

(b) $\bigwedge v(v \dot{\in} \underline{x} \longleftrightarrow \bigvee\!\!\!\bigvee_{z \in x} v = z)$ for $x \in C_\nu$

(c) $\dot{W} = J_{\underline{\alpha}}^{\dot{A}}$ where $\dot{A} \subset \alpha$

(d) $\dot{W} \models \mathsf{ZFC}^*$, $\dot{W}$ is reflexive.

(e) $\underline{[\alpha]^\omega} = ([\mathrm{On}]^\omega)^{\dot{W}}$(where $[\alpha]^\omega = \{u \subset \alpha \mid \overline{\overline{u}} = \omega\}$.)

Note. (a), (b) constitute the "standard axioms". They will be present in every language on an admissible structures which we consider. (c) says that $\dot{W}$ has height α. Given (c), (d) and (e) then say that $\dot{W}$ is a world. Note that (e) implies: $\mathrm{cf}(\alpha) > \omega$.

We now prove the lemma. We first prove $(\longrightarrow)$. Let $\mathbb{L}_\alpha$ be consistent. If G is $\mathrm{coll}(\omega, \gamma)$-generic for a sufficiently large γ, then C_ν is countable and $\mathbb{L}_\alpha$ has a model $\mathbb{M}$. Set: $W = \dot{W}^{\mathbb{M}}$, $A = \dot{A}^{\mathbb{M}}$. Then $W = J_\alpha^A \in V[G]$ is a world of height α. Conversely, suppose $W \in V[G]$ to be such a world. Let $\kappa > \nu$ be regular. Then $\mathbb{L}$ has a model $\mathbb{M} = \langle H_\kappa[G], W, \ldots \rangle$(with $\underline{x}^{\mathbb{M}} = x$ for $x \in C_\nu$). Hence $\mathbb{L}$ is consistent.

QED(Lemma 5.4.2)

The proof of lemma 5.4.2 is a template for many similar proofs. For instance:

Lemma 5.4.3. *Let $\alpha < \nu$ where C_ν is admissible. Let $\varphi(v_1, \ldots, v_m)$ be a first order formula. Let $x_1, \ldots, x_m \in C_\nu$. There is a language $\mathbb{L} = \mathbb{L}_{\alpha,\vec{x}}$ on C_ν such that $\mathbb{L}$ is consistent if and only if there is a potential world W of height α with: $W \models \varphi[\vec{x}]$.*

Proof(sketch). $\mathbb{L}_{\alpha,\vec{x}}$ is $\mathbb{L}_\alpha$ with the additional axiom: $\dot{W} \models \varphi[\vec{x}]$. We leave the details to the reader.

QED(Lemma 5.4.3)

Lemma 5.4.4. *Let $\alpha < \nu$, where C_ν is admissible. Let $t \in C_\omega$. There is a language $\mathbb{L} = \mathbb{L}_{\alpha,t}$ on C_ν such that $\mathbb{L}$ is consistent if and only if there is a potential world W of height α with $a_1, \ldots, a_m \in W$ and:*

$$t = \text{ the complete theory of } \langle W, a_1, \ldots, a_m\rangle.$$

Proof(sketch). Add to $\mathbb{L}_\alpha$ the constants $\dot{a}_1, \ldots, \dot{a_m}$ and the axioms:

- $\dot{a}_1, \ldots, \dot{a_m} \in \dot{W}$
- $\underline{t} =$ the complete theory of $\langle \dot{W}, \dot{a}_1, \ldots, \dot{a_m}\rangle$.

QED(Lemma 5.4.4)

Another variant is:

Lemma 5.4.5. *Let $\gamma < \alpha < \nu$. Let $C^e_{\gamma,\nu}$ be admissible. There is a language $\mathbb{L} = \mathbb{L}^e_{\gamma,\alpha}$ on $C^e_{\gamma,\nu}$ such that $\mathbb{L}$ is consistent if and only if there is potentially a world W of height α such that $L^e_\gamma \in W$.*

Proof(sketch). The standard axiom (b) is now formulated for $x \in C^e_{\gamma,\nu}$ instead of C_ν. We add the additional axiom: $\underline{L^e_\gamma} \in \dot{W}$. The rest is left to the reader.

QED(Lemma 5.4.5)

All the lemmas relativize to an arbitrary world W' in place of V. The relativization of lemma 5.4.2 for instance reads:

Lemma 5.4.6. *Let W' be a world. Let $\alpha < \nu \in W'$ such that C_ν is admissible. There is a language $\mathbb{L} = \mathbb{L}_\alpha$ on C_ν such that*

$\mathbb{L}$ is consistent if and only if $W' \models$ there is a potential world W of height α.

(Note that "$\mathbb{L}$ is consistent" is absolute to W'.)

Proof(sketch). $\mathbb{L} = \mathbb{L}_\alpha$ is defined exactly as before. The direction $(\longrightarrow)$ is exactly as before. We prove $(\longleftarrow)$. Let $W \in W'[G]$ be a world of height α. Then $\mathbb{L}$ has a model $\mathbb{M} = \langle W'[G], W, \ldots \rangle$(with $\underline{x}^{\mathbb{M}} = x$ for $x \in C_\nu$).

QED(Lemma 5.4.6)

Note. If W' has a largest cardinal it might not be possible to find a $\kappa > \gamma, \nu$ which is regular in W'.

The other lemmas stated above can be similarly relativized to a world W'. We leave this to the reader.

5.4.1 Good Worlds

Definition 5.4.3. A world $W = L_\alpha^A$ is *good* if and only if there is $\beta < \alpha$ such that in W the following hold:

- β is the largest cardinal
- $\beta = \mathrm{card}(V_\beta)$, $L_\beta^A = V_\beta$
- $\mathrm{cf}(\beta) > \omega_1$

$\beta = \beta^W$ is then uniquely determined.

Definition 5.4.4. Let W be good. Let $\beta_i = \beta_i^W$ be the monotone enumeration of the $\gamma \leq \alpha$ such that $\gamma > \beta^W$ and $W|\gamma =: L_\gamma^A$ is a world.(Note that $\mathrm{cf}(\gamma) > \omega$ if $W|\gamma$ is a world. Hence the sequence β_i can be discontinuous at places.) By the *rank* of W we mean that i such that $\beta_i = \alpha$.

Suppose now that $\beta = \mathrm{card}(V_\beta)$ and $\mathrm{cf}(\beta) > \omega_1$ in V. Choose $A \subset \beta^+$ such that $L_\beta[A] = V_\beta$ and β is the largest cardinal in $L_{\beta^+}[A]$. Then $W = L_{\beta^+}^A$ is a good world and β_i^W is defined for $i \leq \beta^+$. However, we shall often be interested in good worlds which are present in $V[G]$ for a set generic G, but not necessarily in V.

Lemma 5.4.7. *Let $\alpha < \nu$ where C_ν is admissible. Let $i \leq \alpha$. Then there is a language $\mathbb{L} = \mathbb{L}_\alpha$ such that $\mathbb{L}$ is consistent if and only if there is a potential good world W such that $i \leq \mathrm{rank}(W)$.*

Proof(sketch). Add to $\mathbb{L}_\alpha$ a constant $\dot\beta$ and the axioms:

- $\dot W \models \dot\beta$ is the largest cardinal

- $\dot{W} \models \dot{\beta} = \mathrm{card}(V_\beta) \wedge L_{\dot{\beta}}[\dot{A}] = V_{\dot{\beta}} \wedge \mathrm{cf}(\dot{\beta}) > \underline{\omega_1}$
- $\dot{\beta}_{\underline{i}}$ exists.

The rest is left to the reader.

QED(Lemma 5.4.7)

We now turn to the proof of lemma 5.3.9.

5.4.2 The Relation R

We are assuming that:

$$I = \langle\langle P_i\rangle, \langle\nu_i\rangle, \langle\pi_{ij}\rangle, T\rangle$$

is a nontruncating normal iteration of length ω. Moreover $P = P_0$ is countable and there are σ, N such that

$$N \text{ is robust and pre-mouselike and } \sigma : P \longrightarrow_{\Sigma^*} N.$$

From this we wish to derive that I has a wellfounded branch. We define:

Definition 5.4.5. $\sigma \in D_i$ if and only if $i < \omega$ and the following hold:

- $\sigma : P_i \longrightarrow N$
- Let $n \leq_T i$(hence $\sigma\pi_{ni} : P_n \longrightarrow N$). Let $m \leq \omega$ be maximal such that $\lambda_j < \rho^m_{P_n}$ for all $j < m$. Then $\sigma\pi_{ni}$ is $\Sigma_0^{(m)}$-preserving.

We set: $D = \bigcup_{i<\omega} D_i$.

Note that for each $\sigma \in D$ there is a unique $i = i(\sigma)$ such that $\sigma \in D_i$.(We are assuming that $D_0 \neq \emptyset$.) We then define a relation $R \subset D^2$ as follows:

Definition 5.4.6. $\langle\sigma', \sigma\rangle \in R$ if and only if for some i we have: $\sigma' \in D_i$ and $\sigma = \sigma'\pi_{ni}$ for an $n <_T i$.

It will suffice to prove:

Claim. R is illfounded.

To see that this suffices, let $\sigma^{n+1}R\sigma^n$ for $n < \omega$, where $\sigma^n \in D_{i_n}$. Set:

$$b = \{j \mid \vee n j \leq_T i_n\}.$$

Then b is a cofinal branch. For $j \in b$ such that:

$$\sigma_j = \sigma^n \pi_{j,i_n} \text{ for } j \leq i_n.$$

Then $\sigma_j R \sigma_i$ for $i < j$ in b. b is wellfounded, since there is $\tilde{\sigma} : P_b \longrightarrow_{\Sigma_0} N$ defined by:

$$\tilde{\sigma}\pi_{ib} = \sigma_i \text{ for } i \in b.$$

Thus we shall assume R to be well founded and derive a contradiction. This assumption implies that each $\sigma \in D$ has a *level* defined by:

$$\text{level}(\sigma) = \text{lub}\{\text{level}(\sigma') \mid \sigma' R \sigma\}.$$

Note. The relation R is easier to think of if we imagine that N, P_0 are ZFC^- models. Then each P_i is a ZFC^- model and $\pi_{ij} : P_i \prec P_j$ for $i \leq_T j$. D_i is simply the set of σ such that $\sigma : P_i \prec N$ for some $i < \omega$, and $\sigma' R \sigma$ says that $\sigma : P_i \prec N$ for some i and $\sigma' : P_n \prec N$ for some $n <_T i$. In the general case, the maps π_{ij} will still be Σ^*-preserving, but the degree of preservation of $\sigma : P_i \prec N$ such that $\sigma \in D_i$ may drop as i increases, and may eventually fall to Σ_0. However, this still will suffice to prove lemma 5.3.9.

Now choose (in V) a cardinal β such that

$$\beta = \text{card}(V_\beta), N \in V_\beta \text{ and } \text{cf}(\beta) > \omega_1.$$

Since $\text{card}(N) < \beta$ and β is a limit cardinal, it follows easily that $\text{card}(D) = \text{card}(R) < \beta$. Hence $\text{level}(\sigma) < \beta$ for $\sigma \in D$. Then choose $A' \subset \beta$ such that $L_\beta[A'] = V_\beta$. Pick $A'' \subset [\beta, \beta^+)$ such that β is the largest cardinal in $L_{\beta^+}[A]$, where $A = A' \cup A''$. (To do this, we could pick $f_\xi : \xi \xrightarrow{\text{onto}} \beta$ for $\xi \in [\beta, \beta^+)$ and set:

$$A'' = \{\langle \xi, i, j\rangle \mid f_\xi(i) < f_\xi(j) \wedge \xi \in [\beta, \beta^+)\}.)$$

We then set: $W_0 = L^A_{\beta^+}$, $N_0 = N$. It is easily seen that $W_0 \in V$ is a good world of rank β^+.

Starting with this, we construct a sequence

$$\langle\langle W_i, N_i\rangle \mid i < \omega\rangle$$

such that for all $i < \omega$ we have:

(A) $W_i = L^{A_i}_{\alpha_i}$ is a good world

(B) $\langle W_0, N_0\rangle \equiv \langle W_i, N_i\rangle$

where $\equiv$ means 'elementarily equivalent'. However, we will *not* necessarily have: $W_i, N_i \in V$. The construction will take place in $V[G]$, where G is (β^+, ω)-generic.

Now define $\langle \beta^i_j \mid j \leq \mathrm{rank}(W_i)\rangle$ from W_i as $\langle \beta_j \mid j \leq \mathrm{rank}(W_0)\rangle$ was defined from W_0. Set:

$$W_i|\beta^i_j =: L^{A_i}_{\beta^i_j}.$$

Then by reflexiveness:

$$W_i|\beta^i_n \prec W_i|\beta^i_j \ (n \leq j \leq \mathrm{rank}(W_i)).$$

It follows that $N_i \in W_i|\beta^i_0$ and:

$$\langle W_0, N_0\rangle \equiv \langle W_i|\beta^i_j, N_i\rangle \text{ for } j \leq \mathrm{rank}(W_i).$$

Let R_i be defined from W_i, N_i as R was defined from W_0, N_0. Let

$$D^i = \bigcup_{j<\omega} D^i_j$$

be defined in W_i from N_i as $D = \bigcup\limits_{j<\omega} D_j$ was defined in W_0 from N_0.

Note that if $\sigma : P_n \longrightarrow_{\Sigma_0} N_i$, then $\sigma \in W_i$. This is because, letting $\alpha = \mathrm{On} \cap P_n$ and $\tilde{\alpha} = \mathrm{On} \cap N_n$, $\sigma|\alpha \in C^{W_i}_\infty \subset W_i$, since $\sigma|\alpha$ is a countable subset of $\tilde{\alpha} \times \alpha \in C^{W_i}_\infty$. But σ is the unique $f : P_n \longrightarrow_{\Sigma_0} N_i$ such that $f|\alpha = \sigma|\alpha$.

The i-th *level function*, $\langle \mathrm{level}^i(\sigma) \mid \sigma \in D^i\rangle$ is defined in W_i from N_i as the original level function was defined in W_0 from N_0. We shall construct $\langle \sigma_i \mid i < \omega\rangle$ such that

(C) $\sigma_i \in D^i_i$ and $\mathrm{level}^i(\sigma_i) \leq \mathrm{rank}(W_i)$

(D) $\alpha_i < \alpha_n$ for $n < i$ (where $\alpha_n = \mathrm{On} \cap W_n$)

(D) gives the desired contradiction. We set:

$$\gamma_i =: \text{ the largest } \gamma \in (\kappa_i, \lambda_i] \text{ which is a cardinal in } P_i.$$

Since I is a nontruncating iteration, τ_i will always be a cardinal in P_n, where $n = T(i+1)$. But τ_i is then a cardinal in P_i, since either $n = i$ or $\tau_i < \lambda_n$, where λ_n is inaccessible in P_i. Hence $\tau_i \leq \gamma_i$. We ensure that for $i < \omega$:

(E) $\sigma_n|\gamma_n = \sigma_i|\gamma_n$ for $n \leq i$

(F) $J^{E^{N_n}}_{\tilde{\gamma}_n} = J^{E^{N_i}}_{\tilde{\gamma}_n}$ for $n \leq i$, where $\tilde{\gamma}_n =: \mathrm{lub}\, \sigma_n{}''\gamma_n$.

Note. (E) seems paradoxical at first glance. This is because, if we assume:

$$\sigma_0 : P_0 \longrightarrow N, \sigma_1 : P_1 \longrightarrow N, \sigma_1\pi_{01} = \sigma_0,$$

then $\sigma_1(\kappa_0) < \sigma_1\pi_{01}(\kappa_0) = \sigma_0(\kappa_0)$, where $\kappa_0 < \gamma_0$. In fact, (E), (F) are only possible because N_n, N_i are different premice in different worlds for $n \neq i$.

W_0, N_0, σ_0 are given. Moreover (A)-(F) are vacuously true for $i = 0$. Now let W_i, N_i, σ_i be given such that (A)-(F) hold. We construct $W_{i+1}, N_{i+1}, \sigma_{i+1}$ and verify (A)-(F) at $i+1$. Let:

$$t =: \text{ the complete theory of } \langle W_0, N_0\rangle.$$

Let $\kappa, \tau, \lambda = \sigma_i(\kappa_i, \tau_i, \lambda_i)$. Let $\nu = \sigma_i(\nu_i)$ if $\nu_i \in P_i$ and $\nu = \mathrm{On}_N$ if not. By lemma 5.3.4 there is $g : \lambda_i \longrightarrow \kappa$ such that

(a) Let $\alpha_1, \ldots, \alpha_n < \lambda_i$. Let $X \in \mathbb{P}(\kappa_i) \cap P_i$, then

$$\langle g(\vec{\alpha})\rangle \in \sigma_i(X) \longleftrightarrow \langle\vec{\alpha}\rangle \in E^{N_i}_{\nu_i}(X).$$

(b) Let $\gamma_i \in [\kappa_i, \lambda_i]$ be maximal such that γ_i is a cardinal in P_i. Let $\alpha_1, \ldots, \alpha_n < \gamma_i$. Let $v_1, \ldots, v_n \subset \gamma_i$. Let $x_1, \ldots, x_n$ be hereditarily countable. Let Ψ be Σ_1. Then in W_i:

$$C^{E^{N_i}}_{\tilde{\gamma}_i,\infty} \models \Psi[\vec{x}, \sigma_i(\vec{\alpha}), \sigma_i\text{''}\vec{v}] \longleftrightarrow C^{E^{N_i}}_{\bar{\gamma}_i,\kappa} \models \Psi[\vec{x}, g(\vec{\alpha}), g\text{''}\vec{v}],$$

where $\tilde{\gamma}_i = \operatorname{lub} \sigma_i\text{''}\gamma_i$, $\bar{\gamma}_i = \operatorname{lub} g\text{''}\gamma_i$.

Since $(C^{E^{N_i}}_{\tilde{\gamma}_i,\infty})^{W_i} = C^{E^{N_i}}_{\tilde{\gamma}_i,\alpha_i}$, we have:

$$C^{E^{N_i}}_{\tilde{\gamma}_i,\alpha_i} \models \Psi[\vec{x}, \sigma_i(\vec{\alpha}), \sigma_i\text{''}\vec{v}] \longleftrightarrow C^{E^{N_i}}_{\bar{\gamma}_i,\kappa} \models \Psi[\vec{x}, g(\vec{\alpha}), g\text{''}\vec{v}].$$

Now let $n = T(i+1)$. Let $\pi = \pi_{n,i+1}$. Then:

$$\pi : P_n \longrightarrow^*_{E^{p_i}_{\nu_i}} P_i + 1.$$

It follows easily that:

$$\kappa_i < \rho^m_{P_n} \longleftrightarrow \lambda_i < \rho^m_{P_{i+1}} \text{ for } m \leq \omega.$$

Every element of P_{i+1} has the form:

$$\pi(f)(\alpha) \text{ where } \alpha < \lambda_i, f \in \Gamma^*(\kappa_i, P_n).$$

Using this we prove:

(1) There is $\sigma : P_{i+1} \longrightarrow N_n$ such that

- σ is $\Sigma_0^{(m)}$-preserving for $\lambda_i < \rho_{P_{i+1}}^m$
- $\sigma|\lambda_i = g$
- $\sigma\pi = \sigma_n$

Proof. Let $m \leq \omega$ be maximal such that $\lambda_i < \rho_{P_{i+1}}^m$. Let A be $\Sigma_0^{(m)}(P_{i+1})$. Let $\bar{A}$ be $\Sigma_0^{(m)}(P_n)$ by the same definition. Let $\tilde{A}$ be $\Sigma_0^{(m)}(N_n)$ by the same definition. Write e.g. $\bar{A}(\vec{f}(\vec{\xi}))$ as an abbreviation for $\bar{A}(f_1(\xi_1), \ldots, f_m(\xi_m))$. We make use of lemma 2.7.13. Note that both of the embeddings:

$$\pi : P_n \longrightarrow P_{i+1}, \sigma_n : P_n \longrightarrow N_n$$

are $\Sigma_0^{(m)}$-preserving.

Set: $X = \{\langle\vec{\xi}\rangle < \kappa_i \mid \bar{A}(\vec{f}(\vec{\xi}))\}$. Then $X \in \mathbb{P}(\kappa_i) \cap P_n = \mathbb{P}(\kappa_i) \cap P_i$ and $\sigma_n(X) = \sigma_i(X)$. Then, if $\alpha_1, \ldots, \alpha_m < \lambda_i$, we have:

$$\begin{aligned} A(\pi(\vec{f})(\vec{\alpha})) &\longleftrightarrow \langle\vec{\alpha}\rangle \in E_{\nu_i}^{N_i}(X) \\ &\longleftrightarrow \langle g(\vec{\alpha})\rangle \in \sigma_n(X) \\ &\longleftrightarrow \tilde{A}(\sigma_n(\vec{f})(g(\vec{\alpha}))). \end{aligned}$$

Hence there is a unique $\sigma : P_i \longrightarrow_{\Sigma_0^{(m)}} N_n$ defined by:

$$\sigma(\pi(f)(\alpha)) = \sigma_n(f)(g(\alpha)) \text{ for } \alpha < \lambda_i, f \in \Gamma^*(\kappa_i, P_n).$$

The conclusion follows easily.

QED(1).

But then $\sigma \in W_n$. Since $\sigma\pi_{n,i+1} = \sigma_n \in D^n$ and $\sigma : P_{i+1} \longrightarrow_{\Sigma_0^{(m)}} N_n$, we have: $\sigma \in D_{i+1}^n$, $\sigma R^n \sigma_n$. Hence:

$$\text{level}^n(\sigma) < \text{level}^n(\sigma^n) \leq \text{rank}(W_n).$$

Pick $j < \text{rank}(W_n)$ such that $\text{level}^n(\sigma) \leq j$. Set:

$$\alpha' =: \beta_j^n, W' =: W_n|\beta_j =: J_{\alpha'}^{A_n}.$$

Pick $\nu > \alpha'$ such that $\nu \in W_n$ and $C_{\bar{\gamma}_i,\nu}^E$ is admissible, where $E = E^{N_n}$.(Note that: $C_{\bar{\gamma}_i,\nu}^E = C_{\bar{\gamma}_i,\nu}^{E^{N_i}}$, since $\bar{\gamma}_i < \kappa = \sigma_i(\kappa_i)$.) Let t be the complete theory of $\langle W_0, N_0\rangle$. Let $\mathbb{L}' = \mathbb{L}_{\alpha',I,t,g"\gamma_i}$ be the following language on $C_{\bar{\gamma}_i,\nu}^E$:

Predicate: $\dot{\in}$

Constants: $\underline{x}(x \in C_{\bar{\gamma}_i,\nu}^E)$, $\dot{W}$, $\dot{A}$, $\dot{N}$, $\dot{\sigma}$

Axioms:

Standard axioms:

- ZFC^-
- $\bigwedge v(v\dot{\in}\underline{x} \longleftrightarrow \bigvee\!\!\!\bigvee_{z\in x} v = z)$ for $x \in C^E_{\bar{\gamma}_i,\nu}$

$\dot{W}$ is a world of height α':

- $\dot{W} = J^{\dot{A}}_{\underline{\alpha'}}$
- $\dot{W} \models \mathsf{ZFC}^*$
- $\dot{W}$ is reflexive
- $\underline{[\alpha']^\omega} = ([\mathrm{On}]^\omega)^{\dot{W}}$

Axioms about $\dot{\sigma}$:

- $\dot{\sigma} : \underline{P_{i+1}} \longrightarrow_{\Sigma_0^{(m)}} \dot{N}$
- $\dot{\sigma}\underline{\pi_{n,i+1}} : \underline{P_n} \longrightarrow_{\Sigma_0^{(m)}} \dot{N}$
- $\dot{\sigma}"\underline{\gamma_i} = \underline{g"\gamma_i}$
- $\underline{J^E_{\bar{\gamma}_i}} = J^{E^{\dot{N}}}_{\underline{\bar{\gamma}_i}}$, where $\bar{\gamma}_i = \mathrm{lub}\, g"\gamma_i$.

The elementary equivalence axiom:

- $\underline{t}$ = the complete theory of $\langle \dot{W}, \dot{N}\rangle$

  (By this, it follows that $\dot{W}$ is a good world and $\dot{\beta} = \beta_{\dot{W}}$ is defined as the largest cardinal in $\dot{W}$. Hence $\mathrm{rank}(\dot{W})$ is defined. Define $\dot{D}, \dot{R}$ in $\langle \dot{W}, \dot{N}\rangle$ as D_0, R_0 were defined in $\langle W_0, N_0\rangle$. It follows that: "$\dot{R}$ is wellfounded" holds in $\dot{W}$. Hence the level function, level$'$ is definable in $\langle \dot{W}, \dot{N}\rangle$ as level^0 was definable in $\langle W_0, N_0\rangle$. Our final axioms read:

- $\dot{\sigma}\underline{\pi_{n,i+1}} \in \dot{D}_n$ (Hence $\dot{\sigma} \in \dot{D}_{i+1}$)
- $\mathrm{level}(\dot{\sigma}) \leq \mathrm{rank}(\dot{W})$.

It is obvious that $\langle W_n, W_n|\alpha', A\cap\alpha', \sigma, \ldots\rangle$ is a model of $\mathbb{L}$. Hence $\mathbb{L}$ is consistent. The statement that there are α, ν such that $\alpha < \nu$, $C^E_{\bar{\gamma}_i,\nu}$ is admissible and $\mathbb{L}_{\alpha',I,t,g"\gamma_i}$ is consistent, is in W_n a $\Sigma_1(C^E_{\bar{\gamma}_i,\infty})$ statement about $I, t, g"\gamma_i$. By the *iteration fact* (lemma 5.3.10) there is $\nu > \kappa_i$ such that $E^{P_n}_\nu \neq \emptyset$ and $\kappa_i = \mathrm{crit}(E^{P_n}_\nu)$, where κ_i is a cardinal in P_n. Since N_n is robust in W_n we have $E^{N_n}_{\sigma_n(\nu)} \neq \emptyset$ and $\kappa = \mathrm{crit}(E^{N_n}_{\sigma_n(\nu)})$, where $\kappa = \sigma_n(\kappa_i) = \sigma_i(\kappa_i)$ is a cardinal in N_n. By lemma 5.3.3 it then follows that the same Σ_1 statement about $I, t, g"\gamma_i$ is true in $C^E_{\bar{\gamma}_i,\kappa} = C^{E^{N_i}}_{\bar{\gamma}_i,\kappa}$. Since N_i is robust in W_i, it follows by our assumption on g that the same statement holds in $(C^{E^{N_i}}_{\bar{\gamma}_i,\infty})^{W_i}$ of $I, t, \sigma_i"\gamma_i$. Hence there are $\alpha, \nu \in W_i$ such that $\alpha < \nu$, $C^{E^{N_i}}_{\bar{\gamma}_i,\nu}$ is admissible and (2) $\mathbb{L}_{\alpha,I,t,\sigma_i"\gamma_i}$ is consistent.

Set: $\alpha_{i+1} = \alpha$. Let $\mathbb{M}$ be a model of $\mathbb{L}_{\alpha,I,t,\sigma_i"\gamma_i}$. Set $W_{i+1} = \dot{W}^{\mathbb{M}}$, $A_{i+1} = \dot{A}^{\mathbb{M}}$, $\sigma_{i+1} = \dot{\sigma}^{\mathbb{M}}$. It is straightforward to see that (A)-(D) hold at $i+1$.

But $\sigma_i|\gamma_l = \sigma_l|\gamma_l$ for $l \leq i$ and $\sigma_{i+1}|\gamma_i = \sigma_i|\gamma_i$ and $J^{E^{N_{i+1}}}_{\tilde{\gamma}_i} = J^{E^{N_i}}_{\tilde{\gamma}_i}$ since in $\mathbb{L}_{\alpha,I,t,\sigma_i``\gamma_i}$, the axioms:

$$\dot{\sigma}``\gamma_i = \underline{\sigma_i``\gamma_i}, J^{E^{N_i}}_{\underline{\tilde{\gamma}_i}} = J^{E^{\dot{N}}}_{\underline{\tilde{\gamma}_i}}$$

hold. Hence (E), (F) hold at $i+1$. This completes the contradiction.

QED(Lemma 5.3.9).

Lemma 5.3.9 proves a special case of Lemma 5.3.7, which says that if N_μ is restrained and $\sigma\colon P \longrightarrow_{\Sigma^*} N_\mu$, P being countable, then P is countably normally iterable i.e. each countable normal iteration I of P of limit length has a cofinal well foun ded branch. If I happens to have length ω and be truncation free, this follows by applying Lemma 5.3.9 to $N = N_\mu$. But what if I has length ω and is *not* truncation free? We can, in fact, still carry out a similar proof but we must utilise the entire Steel array $\mathbb{N} = \langle N_i \mid i \leq \mu\rangle$ rather than just the model N_μ. In the old proof, D_i was a set of maps $\sigma\colon P_i \longrightarrow N_\mu$. For each $h \leq_T i$ we could then define a unique $\sigma_h \in Dh$ by:

$$\sigma_h = \sigma_i \pi_{h,i}.$$

If, however, there is a truncation point in $(h,i]_T$, we cannot recover σ_h in this way, since $\pi_{h,i}$ is only a partial function on P_h. We shall instead define D_i to be a set of $\sigma = \langle \sigma_j \mid j \leq_T i\rangle$ such that $\sigma \upharpoonright (j+1) \in D_j$ for $j \leq_T i$. Such $\sigma \in D_i$ is called a *realisation* of P_i in $\mathbb{N}$. We shall have: $\sigma_j\colon P_j \longrightarrow N_{\mu_j}$ for $j \leq_T i$, where μ_j is uniquely determined by $\sigma \leq j$, given I. We inductively define D_i. For $i = 0$ we set: $\mu_0 = \mu$ and D_0 =the set of $\sigma = \{\langle \sigma_0, 0\rangle\}$ such that $\sigma_0\colon P_0 \longrightarrow_{\Sigma^*} N_\mu$. Now let D_j be given for $j \leq i$. Let $h = T(i+1)$. If $i+1$ is not a truncation point, we set: $\mu_{i+1} = \mu_h$ and D_{i+1} is the set of $\sigma = \langle \sigma_j \mid j \leq_T i\rangle$ such that the following hold:

- $\sigma_{i+1}\colon P_{i+1} \longrightarrow_{\Sigma_0^{(n)}} N_{\mu+1}$ where $n \leq \omega$ is maximal such that $\lambda_i < \rho^n_{P_{i+1}}$
- $\sigma \upharpoonright h+1 \in D_h$
- $\sigma_h = \sigma_{i+1}\pi_{h,i+1}$.

Now suppose that $i+1$ is a truncation point. Let $\sigma = \langle \sigma_j \mid j \leq_T i+1\rangle$ and suppose that $\sigma \upharpoonright (h+1) \in D_h$. Then $P^*_i = P_h||\beta$, where $\beta \in P_h$ is maximal such that τ_i is a cardinal in $P_h||\beta$. Let $\bar{\beta} = \sigma_h(\beta)$, $\bar{\tau}_i = \sigma_h(\tau_i)$. Clearly $\nu_h \in P_h$, since otherwise λ_h is a cardinal in P_h, hence so is τ_i, since $\tau_i < \lambda_h$ is a cardinal in $J^{E^{P_h}}_{\lambda_h} = J^{E^{P_i}}_{\lambda_h}$. Contradiction! Set $\bar{\nu}_h = \sigma_h(\nu_h)$, $\bar{\kappa}_h = \sigma_h(\kappa_h)$. We know that $\sigma_h\colon P_h \longrightarrow_{\Sigma_0} N_{\mu_h}$. $\bar{\tau}_i$ is a cardinal in $N_{\mu_h}||\bar{\beta}$, but not in $N_{\mu_h}||\bar{\beta}+\omega$. Hence $\rho^\omega_{N_{\mu_h}||\bar{\beta}} < \bar{\tau}_i < \rho^\omega_{N_{\mu_h}||\xi}$ for $\xi < \bar{\beta}$ such that $\bar{\kappa}_h \leq \xi$.

Hence, letting $\langle \eta_i, \nu_i \rangle$ be the resurrection sequence for $\langle N_{\mu_i}, \bar{\nu_i} \rangle$, we see that $\bar{\beta} = \beta_j$ for some $j \leq p$, where $\langle \bar{\beta}_j, k_j \rangle$ is the associated sequence defined in §5.2. Then $k_j : N_{\mu_h} || \bar{\beta} \longrightarrow_{\Sigma^*} N_{\eta_j}$. We set: $\mu_{i+1} = \eta_j$, $\sigma_i^* = k_j \cdot \sigma_h$. Then $\sigma_i^* : P_i^* \longrightarrow_{\Sigma^*} N_{\mu_{i+1}}$. Note that μ_{i+1} is defined only from σ_h and τ_i, where τ_i is given by I. We then define D_{i+1} as the set of $\sigma = \langle \sigma_j \mid j \leq_T i+1 \rangle$ such that the following hold:

- $\sigma_{i+1} : P_{i+1} \longrightarrow_{\Sigma^{(n)}0} N_{\mu_{i+1}}$ where $n \leq \omega$ is maximal such that $\lambda_i < \rho^n_{P_{i+1}}$
- $\sigma \upharpoonright (h+1) \in D_h$
- $\sigma^* i = \sigma_{i+1} \pi_{h,i+1}$.

This defines D_i and $D = \bigcup_{i<\omega} D_i$. We again have that $\sigma \in D$, there is exactly one i such that $\sigma \in D_i$ (since $P_i = \text{dom}(\sigma_i)$).

Definition 5.4.7. Let b be an infinite branch in T. We call $\sigma = \langle \sigma_i \mid i \in b \rangle$ a *realization of* b if and only if σ_{i+1} realizes P_i for $i \in b$.

It follows as before that every realizable branch is wellfounded.

Definition 5.4.8. $\sigma' R \sigma$ if and only if for some i, σ' realizes P_i and $\sigma = \sigma' | n+1$ for an $n \leq_T i$.

It follows as before that if R is illfounded, then I has a wellfounded cofinal branch.

Thus we again assume R to be wellfounded in order to derive a contradiction. To this end we again construct(in a suitable generic $V[G]$) a sequence:

$$\langle \langle W_i, \mathbb{N}_i, \sigma_i \rangle \mid i < \omega \rangle, \text{ where:}$$

- W_i is a world
- $\mathbb{N}_i = \langle N_l^i \mid l \leq \mu^i \rangle$ is a Steel array in W_i
- $\sigma_i = \langle \sigma_n^i \mid n \leq_T i \rangle$ is a realization of P_i in $\mathbb{N}_i$.

$\mathbb{N}_0 = \mathbb{N}$ is our original array and W_0 is defined as before, pick β such that

$$\beta = \text{card}(V_\beta), \mathbb{N} \in V_\beta, \text{cf}(\beta) > \omega_1.$$

The sequence $\langle W_i, \mathbb{N}_i \rangle$ will satisfy:

(A) W_i is a world and $\mathbb{N}_i \in W_i$

(B) $\langle W_0, \mathbb{N}_0 \rangle \equiv \langle W_i, \mathbb{N}_i \rangle$

Thus each W_i is a good world and $\mathbb{N}_i$ is a robust Steel array in W_i. Just as before we define the sequence

$$\beta^i_j \, (j \leq \mathrm{rank}(W_i))$$

such that each $W_i|\beta^i_j$ is a component world of W_i and $\beta^i_j = \alpha_i = \mathrm{On}_{W_i}$ for $j = \mathrm{rank}(W_i)$. D_i is the set of realizations of P_j for $j < \mathrm{lh}(T)$. R_i is defined from $\mathbb{N}_i$ in W_i as R was defined from $\mathbb{N}$ in W_0. We ensure that:

(C) σ_i is a realization of P_i in $\mathbb{N}_i$ and: $\mathrm{level}^i(\sigma_i) \leq \mathrm{rank}(W_i)$

(D) $\alpha_i < \alpha_n$ for $n < i$ (where $\alpha_i = \mathrm{On} \cap W_i$)

(D) gives the desired contradiction. Now let σ_i have the form:

$$\sigma_i = \langle \sigma^i_n \mid n \leq_T i \rangle \text{ where } \sigma^i_n : P_n \longrightarrow N^i_{\mu^i_n},$$

σ^i_n being Σ^m_0-preserving where $m \leq \omega$ is maximal such that $\rho^m_{P_n} > \lambda_l$ for $l < n$. Let:

$$\hat{N}_i = N^i_{\mu^i_i}, \hat{\nu}_i = \sigma^i_i(\nu_i) \text{ for } i < \omega.$$

Set $\tilde{\sigma}_i = k \cdot \sigma^i_i$, where k is the resurrection map for $\hat{N}_i||\hat{\nu}_i$. Then:

$$\tilde{\sigma}_i : P_i||\nu_i \longrightarrow \tilde{N}_i = \langle J^E_{\tilde{\nu}_i}, F \rangle$$

where $\langle J^E_{\tilde{\nu}_i}, F \rangle$ is the origin of $\hat{N}_i||\hat{\nu}_i$. Set $\tilde{\lambda}_i = \tilde{\sigma}_i(\lambda_i)$. In place of the previous conditions (E), (F), we have:

(E) $\tilde{\sigma}_n{}^{"}\lambda_n = \tilde{\sigma}_i{}^{"}\lambda_n$ for $n \leq i$

(F) $J^{\tilde{E}^n}_{\tilde{\lambda}_n} = J^{\tilde{E}^i}_{\tilde{\lambda}_n}$ for $n \leq i$, where $\tilde{N}_i = \langle J^{\tilde{E}^i}_{\tilde{\nu}_i}, F^i \rangle$ and $\tilde{\lambda}_i = \mathrm{lub}\, \tilde{\sigma}_i{}^{"}\lambda_i$.

Without going into further detail, we mention that $\mathbb{E} = \langle \langle W_i, \mathbb{N}_i, \sigma_i \rangle \mid i < \omega \rangle$ will be what we shall call an *enlargement* of I. It will enable to essentially carry out our previous proof in a new setting . In the next section we develop the theory of enlargement and use it to prove Lemma 5.3.7.

5.5 Enlargements

In this section, we prove Lemma 5.3.7. We are given a putative Steel array $\mathbb{N} = \langle N_i \mid i \leq \mu \rangle$, where N_μ is a restrained 1-small premouse. Since $\mathbb{N}$ is a putative Steel array, we know that N_μ is pre-mouselike. We are also given a countable premouse P and a map $\sigma \colon P \longrightarrow_{\Sigma^*} N_\mu$. Hence P is restrained and pre-mouselike. Because P is restrained, we know that P satisfies the unique branch condition – i.e. if I is any countable normal iteration of P of limit length, then I has at most one cofinal well founded branch. But P is also pre-mouselike. Hence I satisfies the "iteration fact" (Lemma 5.3.10). We must show that P is countably normally iterable – i.e. that any countable normal iteration I of P can be continued in the following sense:

(*) If I is of length $i+1$ and ν_i such that $E^{P_i}_{\nu_i} \neq \emptyset$ is so chosen that it extends I to a potential iteration of length $i+2$, then there is a map $\pi\colon P^*_i \longrightarrow^n_F P_{i+1}$ where $F = E^{P_i}_{\nu_i}$ and $n \leq \omega$ is maximal such that $\rho^n_{P^*_i} > \kappa_i$.

(**) If I is of limit length, then I has a cofinal well founded branch.

Lemma 5.3.9 gave a positive answer to that question in the special case that I has length ω and is truncation free. That case is very special. Nonetheless, the reader should keep that proof in mind, since it contained the seed of the proof of the full Lemma 5.3.7.

In the proof of Lemma 5.3.9, we defined for $i < \omega$ the set D_i of what we call *realization* of P_i in N_μ: D_0 was the set of all $\sigma\colon P_0 \longrightarrow_{\Sigma^*} N_\mu$. D_{i+1} was then the set of $\sigma\colon P_{i+1} \longrightarrow N_\mu$ such that σ is $\Sigma^{(n)}_0$-preserving for all n such that $\lambda_i < \rho^n_{P_{i+1}}$ and has the further property that $\sigma\pi_{h,i+1}$ lies in D_h, here $h = T(i+1)$.

If, however, we drop the requirement that I be drop free, then this definition will not work, since $\pi_{h,i+1}$ is only a *partial* function on P_h if $i+1$ is a drop point. Hence $\sigma\pi_{h,i+1}$ is a partial function on P_h and it will not be possible to recover an element of D_h from σ alone. In fact, in order to handle this case, we must give up the requirement that σ map P_{i+1} into N_μ. It will map P_{i+1} into some smaller $N_{\mu_{i+1}}$ where $\mu_{i+1} < \mu$ and we shall have:

$$\sigma\pi_{h,i+1}\colon P^*_i \longrightarrow_{\Sigma^*} N_{\mu_{i+1}}.$$

The right notion of *realisation of* P_i is then a sequence $\sigma = \langle\langle\sigma_j, \mu_j\rangle \mid j \leq_T i\rangle$ such that $\sigma_j\colon P_j \longrightarrow N_{\mu_j}$. This encompasses not only the map σ_i but also its "history", which cannot be recovered from σ_i alone. Without further ado we give the full definition of "realization"

Let $I = \langle\langle P_i\rangle, \langle\nu_i\rangle, \langle\pi_i\rangle, T\rangle$ be any countable normal iteration of P of length η. By induction on $i < \eta$ we define the set D_i of *realization* of P_i in $\mathbb{N}$. Each $\sigma \in D_i$ will be a sequence :

$$\sigma = \langle\langle\sigma_j, \mu_j\rangle \mid j \leq_T i\rangle$$
$$\text{such that } \sigma_j\colon P_j \longrightarrow N_{\mu_j} \text{ for } j \leq_T i.$$

We shall inductively verify:

- $\mu_i \leq \mu_j$ for $j \leq_T i$.
- $\sigma_i \colon P_i \longrightarrow_{\Sigma_0^{(n)}} N_{\mu_i}$ whenever $\lambda_j < \rho^n_{P_i}$ for all $j < i$.
- If $(j,i]_T$ is drop free, then $\mu_j = \mu_i$ and $\sigma_j = \sigma_i \pi_{j,i}$.

We define D_i by the cases as follows:

Case 1 $i = 0$. D_0 is the set of $\sigma = \{\langle \sigma_0, \mu_0 \rangle\}$ such that $\mu_0 = \mu$ and $\sigma_0 \colon P_0 \longrightarrow_{\Sigma^*} N_\mu$.

Case 2 $i = j+1$. Let $h = T(i+1)$. We split into two subcases:

Case 2.1 $j+1$ is not a drop point.

Then $\sigma = \langle \langle \sigma_l, \mu_l \rangle \mid l \leq_T i \rangle \in D_i$ if and only if the following hold:

- $\sigma \upharpoonright h+1 \in D_h$
- $\mu_i = \mu_h$ and $\sigma_h = \sigma_i \pi_{h,i}$
- $\sigma_i \colon P_i \longrightarrow_{\Sigma_0^{(n)}} N_{\mu_i}$ whenever $\lambda_j < \rho^n_{P_i}$.

Case 2.2 $j+1$ is a drop point.

Then $P^*_j = P_h || \beta$ where $\beta =$the maximal $\beta \in P_h$ such that τ_j is a cardinal in $P_h || \beta$. Set: $\bar{\beta} = \sigma_h(\beta)$. Then $\bar{\beta}$ is the maximal $\bar{\beta} \in N_{\mu_h}$ such that $\sigma_h(\tau_j) = \sigma_j(\tau_j)$ is a cardinal in $N_{\mu_h} || \bar{\beta}$. Note that $\nu_h \in P_h$, since τ_j is a cardinal in $P_h || \lambda_h$ but not in P_h. Hence $\beta \in B(P_h, \nu_h)$ as defined in §5.2. Hence $\bar{\beta} \in B(N_{\mu_h}, \sigma_h(\nu_h))$. Let $\langle \eta_l, \nu_l \rangle$ $(l \leq p)$ be the *ressurection sequence* for $\langle N_{\mu_h}, \nu_h \rangle$ as defined in §5.2. Let $\langle \bar{\beta}_l, k_l \rangle$ be the auxiliary sequence defined there. Then $\langle \bar{\beta}_i \mid i \geq 1 \rangle$ is the enumeration of $B(N_{\mu_h}, \sigma_h 8 \nu_h)$ in descending order. Let $\bar{\beta} = \bar{\beta}_l$, $l \geq 1$. Then $k_l \colon N_{\mu_h} || \bar{\beta} \longrightarrow_{\Sigma^*} N_{\eta_l}$. We set:

$$\sigma^*_j =: k_l \sigma_h.$$

Then $\sigma^*_j \colon P^*_j \longrightarrow^{\Sigma^*} N_{\eta_l}$. We define: $\sigma = \langle \langle \sigma_\xi, \mu_\xi \rangle \mid \xi \leq_T i \rangle \in D_i$ if and only if the following hold:

- $\sigma \upharpoonright h+1 \in D_h$
- $\mu_i = \eta_l$ and $\sigma^*_h = \sigma_i \pi_{h,i}$
- $\sigma_i \colon P_i \longrightarrow_{\Sigma_0^{(n)}} N_{\mu_i}$ whenever $\lambda_j < \rho^n_{P_i}$.

(**Note:** If the Case 2.1 holds we also set: $\sigma_j^* = \sigma_h$. Hence we will always have: $\sigma_j^* = \sigma\pi_{h,j+1}$ for $j + 1 < \eta$.)

Case 3 $i = \eta$ is a limit ordinal.

Then $\sigma = \langle \sigma_j \mid j \leq_T \eta \rangle \in D_\eta$ if and only if the following hold:

- $\sigma \restriction j + 1 \in D_j$ for $j \leq_T \eta$
- If $i <_T \eta$ such that $(i, \eta]_T$ is frop free, then $\mu_i = \mu_\eta$, $\sigma\eta \colon P_\eta \longrightarrow N_{\mu_\eta}$, and $\sigma_i = \sigma_\eta \pi_{i,\eta}$.

The verification is straightforward.

Definition 5.5.1. Let b be a branch in I. We call a sequence $\sigma = \langle\langle \sigma_i, \mu_i \rangle \mid i \in b\rangle$ a *realisation of* b in $\mathbb{N}$ (in symbols $\sigma \in D_b$), if and only if $\sigma \restriction (i+1) \in D_i$ for $i \in b$.

Note that the existence of a realization σ for b means that b has only finitely drop points, since, if i_n $(n \in \omega)$ were an ascending sequence of drop points, then $\mu_{i_{n+1}} < \mu_{i_n}$. Contradiction! Hence:

(1) Let b be a realizable branch in I of limit length. Then it is well founded.

Proof. Let $j \in b$ such that no $i \in b \setminus (j + 1)$ is a drop point. Define $\sigma_b \colon P_b \longrightarrow N_{\mu_j}$ by: $\sigma_b \pi_{i,b} = \sigma_i$ for $i \in b \setminus (j + 1)$. Then P_b is well founded, since N_{μ_j} is. QED(1)

In the proof of 5.3.9, we assumed that I was of length ω and used a natural relation R on the set $D = \bigcup_{i<\omega} D_i$ of all realization to prove that I has a cofinal well founded branch. We now require only that the length η of I be at most countable.

We now define a new relation R on $D = \bigcup_{i<\omega} D_i$ which will play a role similar to that of the old relation R.

Definition 5.5.2. Let n^* be an injection of $\mathrm{lh}(I)$ into ω. Set :

$$n(i) = \min\{n^*(j) \mid i \leq_T j\}$$

(Hence $n(i) = n(j) \longrightarrow i \leq_T j$ for $i \leq j$ in I.)

Definition 5.5.3. i survives at j if and only if

$$i \leq j \wedge n(i) = n(j) \wedge n(h) \leq n(j) \text{ whenever } h \in [i, j].$$

Definition 5.5.4. $\sigma' R \sigma$ if and only if:

$$i <_T j \wedge \sigma' \text{ realizes } P_j \wedge \sigma = \sigma' \upharpoonright i+1 \wedge i \text{ does not survive at } j.$$

Then:

(2) If R is ill founded, then I is of limit length and has a cofinal well founded branch.

Proof. Let $\sigma_{n+1} R \sigma_n$ for $n \in \omega$. Let σ_n realize P_{j_n}. Set: $b = \{h \mid \wedge n h \leq_T j_n\}$, $\sigma = \bigcup_n \sigma_n$. Then b is of limit length $\eta = \text{lub}\{j_n \mid n \in \omega\}$. Moreover, σ is realization of b. If $\eta = \text{ht}(I)$, we are done. If not, then $\eta < \text{ht}(I)$. Then b and $b' = \{j \mid j <_T \eta\}$ are both well founded branches of height η. Since P is restrained, I is an iteration by unique branches. Hence $b = b'$. From this, we derive a contradiction. Let $n = n(\eta)$. For sufficient $i < \eta$ we then have: $n(i) \geq n$ and $n(i) = n$ if $i \in b = b'$. Now let $i < j_m$. Then j_m does not survive at j_{n+1}. Hence either $n = n(j_m) < n(j_{m+1})$ or there is $h \in (j_m, j_{m+1})$ such that $n(h) < n$. Contradiction! QED(2)

From now on we assume:

(***) R is well founded.

If I is of successor length, then (***) is simply true by (2), and we shall use this in proving (*). If $\text{lh}(I)$ is a limit ordinal, we deliberately posit (***) in hope of deriving a contradiction. Thus proving (**).

The sequence $\langle\langle w_i, \mathbb{N}_i, \sigma_i\rangle \mid i < \omega\rangle$ which we constructed in §5.4 was the first example of a class of structures which we call *enlargemnts.* We define

Definition 5.5.5. Let $\mathbb{P}$ be a set of conditions and let G be a $\mathbb{P}$-generic over V. Let $0 < l \leq \text{lh}(E)$. By an enlargement of $I|l$ in $V[G]$ we mean any structure:

$$\mathbb{E} = \{\langle W_i, \mathbb{N}_i, \sigma_i\rangle \mid i < l\} \in V[G]$$

which satisfies the following conditions:

(A) W_i is a good world.

(B) $\mathbb{N}_i = \langle N^i_h \mid h \leq \mu^i\rangle$ is a putative robust Steel array in the sense of W_i for $i \leq l$.

(C) $\sigma_i \in W_i$ is a realisation of P_i in $\mathbb{N}_i$ for $i < l$.

Thus $\sigma_i = \langle\langle \sigma^i_h, \mu^i_h\rangle \mid h \leq_T i\rangle$ where $\sigma^i_h \colon P_h \longrightarrow N^i_{\mu^i_h}$ for $h \leq_T i$. Set:

$$\hat{N}_i =: N^i_{\mu^i_i}, \hat{\sigma}_i = \sigma^i_i.$$

(D) $\hat{\sigma}_i(\lambda_j)$ is a cardinal in $\hat{N}_i$ for $j < i < l$.

Now suppose that $i < l$ such that $i + 1 < \mathrm{lh}(I)$. Then I gives us the point ν_i such that $E^{P_i}_{\nu_i} \neq \emptyset$. Let k be the resurrection map for $\langle \hat{N}_i, \hat{\sigma}_i(\nu_i)\rangle$. Then:

$$k\colon \hat{N}_i||\hat{\sigma}_i(\nu_i) \longrightarrow_{\Sigma^*} N^i_\eta = \langle J^E_\nu, F\rangle$$

where $\langle J^E_\nu, F\rangle$ is the "origin" of $\hat{N}_i||\hat{\sigma}_i(\nu_i)$ in $\mathbb{N}_i$. Set:

$$\tilde{N}_i =: N^i_\eta, \quad \tilde{\sigma}_i =: k \cdot \hat{\sigma}_i, \quad \tilde{\lambda}_i =: \mathrm{lub}\, \tilde{\sigma}_i\text{"}\lambda_i.$$

(**Note** If $\nu_i = \mathrm{ht}(P_i)$, we let $\hat{\sigma}_i(\nu_i)$ denote $\mathrm{ht}(\hat{N}_i)$. In this case, we have: $k =: \mathrm{id}$, $\tilde{N}_i = \hat{N}_i$, $\tilde{\sigma}_i = \hat{\sigma}_i$, $\tilde{\lambda}_i = \mathrm{lub}\, \hat{\sigma}_0\text{"}\lambda_i$.)

The next axioms read:

(E) $\tilde{\sigma}_h \restriction \lambda_h = \hat{\sigma}_i \restriction \lambda_h$ for $h < i < l$.

(F) $J^{E^{\tilde{N}_h}}_{\tilde{\lambda}_h} = J^{E^{\hat{N}_i}}_{\tilde{\lambda}_h}$ for $h < i < l$.

Note If we define:

$$\langle J^E_\alpha, F\rangle|\beta =: J^E_\beta \text{ for limit } \beta \leq \alpha,$$

we can express (F) by:

$$\tilde{N}_h|\tilde{\lambda}_h = \hat{N}_i|\tilde{\lambda}_h \text{ for } h < i < l.$$

Note The iteration I assigns a ν_i with $E^{P_i}_{\nu_i} \neq \emptyset$ if and only if $i + 1 < \mathrm{lh}(I)$. Hence we shall sometimes write "ν_i exists" or "ν_i is defined" to mean: $i + 1 < \mathrm{lh}(I)$.

(3) Let $h \leq i < l$ such that ν_i exists. Then: $\tilde{\sigma}_h \restriction \lambda_h = \tilde{\sigma}_i \restriction \lambda_h$.

Proof. $h = i$ is trivial. Now let $h < i$. Then $\hat{\sigma}_i(\lambda_h)$ is a cardinal in $\hat{N}_i$. Thus if k is the resurrection map for $\langle \hat{N}_i, \hat{\sigma}_i, N_i\rangle$, then $k \restriction \hat{\sigma}_i(\lambda_h) = \mathrm{id}$. Hence $\hat{\sigma} \restriction \lambda_i = \tilde{\sigma}_i \restriction \lambda_i$ QED(3)

Let R_i be defined in W_i from $\mathbb{N}_i$, I, n^* as R was defined in V form $\mathbb{N}$, I, n^*.

(G) R_i is well founded in w_i.

But then we can define the *level* function in W_i:

$$\mathrm{level}^i(\sigma) =: \mathrm{lub}\{\mathrm{level}^i(\sigma') \mid \sigma' R_i \sigma\}.$$

The next axiom reads:

(H) $\text{level}(\sigma_i) \leq \text{rank}(W_i)$

We shall impose on $\mathbb{E}$ an additional requirement which we did not impose in the previous section. In order to formulate this requirement we define:

Definition 5.5.6. For $i < l$ set:

- $\delta_i = \delta_i(\mathbb{E}) = \begin{cases} \tilde{\sigma}_i \restriction \lambda_i & \text{if } \nu_i \text{ exists} \\ \tilde{\sigma}_i \restriction \text{ht}(P_i) & \text{if not} \end{cases}$
- $t_i = t_i(\mathbb{E}) =$ the complete theory of

$$\langle W_i, \mathbb{N}_i, \sigma_i, I, l, \delta \restriction i, t \restriction i \rangle$$

The *trace* of $\mathbb{E}$ is defined by:

$$\text{trace}(\mathbb{E}) =: \langle \delta, t \rangle,$$

where $\delta = \langle \delta_i \mid i < l \rangle$, $t = \langle t_i \mid i < l \rangle$.

Our final axiom reads:

(I) $\text{trace}(\mathbb{E}) \in V$.

This completes the definition of "enlargement".

Note $\mathbb{E}$ is an ideal object, which might not exist V. Its trace, however, does lie in V and encodes vital information about $\mathbb{E}$.

Note The axiom (I) is only needed in the case that $\mathbb{E}$ is of limit length. This follows by:

Lemma 5.5.1. *let $\mathbb{E}$ be of length $i+1$ satisfying (A)-(H) and let $\mathbb{E} \restriction i$ satisfy (A)-(I). Then $\mathbb{E}$ satisfies (I).*

Proof. $\text{rng}(\delta_i)$ is a countable set of ordinals in W_i. Hence $\text{rng}(\delta_i) \in V$. is countable in V, since W_i is a world. Hence $\delta_i \in V$, since δ_i is the monotone enumeration of $\text{rng}(\delta_i)$. But $t_i \in W_i$ by reflexivity, Moreover, t_i is hereditarily countable in W_i. Hence $t_i \in C_{\omega_1}^{W_i} = C_{\omega_1} \subset V$. QED(Lemma 5.5.1)

Definition 5.5.7. Let $\mathbb{P}$, G be as above. Let $e \in V[G]$. $\mathbb{E} \in V[G]$ is an e-enlargement of $I|l$ if and only if the following hold:

- ν_i exists for $i < l$

- $\mathbb{E}$ is an enlargement of $I|l$
- $J^{E^{\tilde{N}_i}}_{\tilde{\lambda}_i} = J^{e}_{\tilde{\lambda}_i}$ for $i < \lambda$.

We leave it to the reader to prove the following two lemmas:

Lemma 5.5.2. *Let $\mathbb{E} \in V[G]$ be an enlargement of $I \upharpoonright l$ with trace $\langle \delta, t \rangle$. Let $0 < i < l$. Then $E \upharpoonright i$ is an $E^{\hat{N}_i}$-enlargement of $I \upharpoonright i$. Moreover,* $\mathrm{trace}(\mathbb{E}\upharpoonright i) = \langle \delta \upharpoonright i, t \upharpoonright i \rangle$.

Lemma 5.5.3. *Let $\mathbb{E} \in V[G]$ be an enlargement of $I \upharpoonright l$. Let $e \in V[G]$. Let $i < l$ such that $\mathbb{E} \upharpoonright i$ is an e-enlargement of $I \upharpoonright i$. Let $\mathbb{F} \in V[G]$ such that $\mathbb{F}$ is an e-enlargement of $I \upharpoonright i$ and* $\mathrm{trace}(\mathbb{F}) = \mathrm{trace}(\mathbb{E}\upharpoonright r)$. *Set:*

$$\mathbb{E}' = \mathbb{F} \cup \mathbb{E} \upharpoonright [i, l).$$

Then $\mathbb{E}'$ is an enlargement of $I \upharpoonright l$ and $\mathrm{trace}(\mathbb{E}') = \mathrm{trace}(\mathbb{E})$.

Lemma 5.5.3 is called the *interpolation lemma*. Both lemmas will be used frequently (though sometimes tacitly).

Definition 5.5.8. $\langle \delta, t \rangle$ is a *trace* if and only if there is a set of conditions $\mathbb{P}$ which forces that , if G is $\mathbb{P}$-generic over V, then there is an enlargement $\mathbb{E} \in V[G]$ such that $\langle \delta, t \rangle = \mathrm{trace}(\mathbb{E})$.

In fact, we only need to consider the sets of conditions $\mathrm{Col}(\gamma, \omega)$ where $\mathrm{Col}(\gamma, \omega)$ is the set of finite conditions for collapsing γ to ω. If $\mathbb{P}$ is any set of conditions and γ is sufficiently large, $\mathrm{Col}(\gamma, \omega)$ will force the existence of a set G which is $\mathbb{P}$-generic over V. Hence we can always take $\mathbb{P}$ in the above definition as being of the for $\mathrm{Col}(\gamma, \omega)$.

The verification that something is a trace is greatly simplified by:

Lemma 5.5.4. *There is a Σ_1 formula φ such that*

$$\langle \delta, t \rangle \text{ is a trace } \longleftrightarrow C_\infty \models \psi[\delta, t, I, n^*].$$

In order to prove this, we first define:

Definition 5.5.9. An enlargement $\mathbb{E} = \langle \langle W_i, \mathbb{N}_i, \sigma_i \rangle \mid i < l \rangle$ is *α-bounded* if and only if $\mathrm{ht}(W_i) < \alpha$ for $i < l$.

Definition 5.5.10. $\langle \delta, t \rangle$ is an *α-bounded trace* if and only if there is a set of conditions $\mathbb{P}$ which forces the existence of an α-bounded enlargement $\mathbb{E}$ with trace $\langle \delta, t \rangle$.

Definition 5.5.11. $\langle\delta, t\rangle$ is a *potential trace* if and only if δ, t are functions and : $0 < \mathrm{dom}(\delta) = \mathrm{dom}(t) \leq \mathrm{lh}(I)$ and:

- $\mathrm{rng}(\delta_i)$ is a set of ordinals for $i < \mathrm{dom}(\delta)$
- t_i is hereditarily countable for $i < \mathrm{dom}(\delta)$.

Lemma 5.5.4 follows easily from:

Lemma 5.5.5. *Let $\langle\delta, t\rangle$ be a potential trace. Let $\omega_1 < \alpha < \nu$ such that C_ν is admissible and $\delta, t \in C_\alpha$. There is a language $\mathbb{L} = \mathbb{L}_{\alpha,I,\delta,t}$ on C_ν such that*

$$\mathbb{L} \textit{ is consistent if and only if } \langle\delta, t\rangle \textit{ is an } \alpha\textit{-bounded trace.}$$

To derive Lemma 5.5.4 from Lemma 5.5.5, we let φ be the Σ_1 formula such that $C_\infty \models \varphi[I, \delta, t]$ says that there are α, ν with C_ν is admissible, $\nu > \alpha$, $\langle\delta, t\rangle \in C_\alpha$ is a potential trace, and $\mathbb{L}_{\alpha,I,\delta,t}$ is consistent.

We prove Lemma 5.5.5. We first describe the language $\mathbb{L}$. $\mathbb{L}$ has:

Predicate: $\dot{\in}$

Constants: $\underline{x}$ $(x \in C_\nu)$, $\dot{\mathbb{E}}$, $\dot{W}$, $\dot{A}$, $\dot{\mathbb{N}}$, $\dot{\sigma}$, $\dot{\alpha}$

Axioms:

(1) The standard axioms:

- ZFC^-
- $\bigwedge v(v \in \underline{x} \longleftrightarrow \bigvee_{z \in x} v = \underline{z})$ for $x \in C_\nu$

(2) $\dot{\mathbb{E}} = \langle\langle\dot{W}, \dot{\mathbb{N}}, \dot{\sigma}_i\rangle \mid i < \underline{l}\rangle$ and $\mathrm{dom}(\dot{W}) = \mathrm{dom}(\dot{N}) = \mathrm{dom}(\dot{\sigma}) = \underline{l}$ where $l = \mathrm{dom}(\delta) = \mathrm{dom}(t)$.

(3) $\dot{W}_i$ is a world for $i < \underline{l}$ -i.e.

- $\dot{W}_i \models \mathsf{ZFC}^* \wedge \dot{W}_i = J^{\dot{A}_i}_{\dot{\alpha}_i}$, where $\dot{\alpha}_i < \underline{\alpha}$
- $\dot{W}_i$ is reflexive
- $\bigwedge x(\dot{W}_i \models x \in [\mathrm{On}]^\omega) \longleftrightarrow (x \in \underline{[\alpha]}^\omega \wedge x \in \dot{W}_i)$

(4) $\dot{W}_i$ is a good world $(i < \underline{l})$ -i.e. there is $\dot{\beta}_i$ such that

- $\dot{W}_i \models \dot{\beta}_i$ is the largest cardinal
- $\dot{W}_i \models (V_{\beta_i} = L_{\dot{\beta}_i}[\dot{A}_i] \wedge \mathrm{cf}(\beta_i) > \omega_1)$

(5) For $i < \underline{l}$ the following hold in $\dot{W}_i$:

- $\mathbb{N}_i = \langle N^i_h \mid h \leq \mu_i \rangle$ is a putative robust Steel array of length μ_i.
- Each N^i_h is 1-small
- $N^i_{\mu_i}$ is restrained

(6) For $i < \underline{l}$ the following hold in $\dot{W}_i$:

- $\dot{\sigma}_i$ is a realization of $\underline{P}_i$ in $\mathbb{N}_i$ (where $P = \langle P_i \mid i < l \rangle$.)
- $\dot{\sigma}_i = \langle \langle \sigma^i_n, \mu^i_n \rangle \mid n <_T i \text{ in } \underline{I} \rangle$

It follows that:

$$\mu^i_i = \mu_i, \quad \sigma^i_n : \underline{P}_n \longrightarrow N^i_{\mu^i_i} \text{ for } n \leq_T i \text{ in } I.$$

Set:

$$\hat{N}_i = N^i_{\mu^i_i}, \quad \hat{\sigma}_i = \sigma^i_i, \quad \hat{\lambda}_i = \operatorname{lub} \hat{\sigma}_i \text{“} \underline{\lambda}_i,$$

where $\lambda = \langle \lambda_i \mid i < l \rangle$.

(7) $\hat{\sigma}_i(\underline{\lambda}_h)$ is a cardinal in $\hat{N}_i$ $h < i < \underline{l}$

For $i < \underline{l}$ such that $i + 1 < \underline{\mathrm{lh}(I)}$ let $k_i \colon \hat{N}_i || \hat{\sigma}_i(\underline{\nu}_i) \longrightarrow N^i_{\eta_i}$ be the resurrection map for $\langle \hat{N}_i, \hat{\sigma}_i(\underline{\nu}_i) \rangle$ in the sense of $\dot{W}_i$, where $\nu = \langle \nu_j \mid j + 1 < \mathrm{lh}(I) \rangle$.

Set: $\tilde{N}_i = N^i_{\eta_i}$, $\tilde{\sigma}_i = k_i \tilde{\sigma}_l$, $\tilde{\lambda}_i = \operatorname{lub} \tilde{\sigma}_i \text{“} \underline{\lambda}_i$ where $\lambda = \langle \lambda_i \mid i+1 < \mathrm{lh}(I) \rangle$.

(8) $\tilde{\sigma}_h \text{“} \underline{\lambda}_h = \hat{\sigma}_i \text{“} \underline{\lambda}_h$ for $h < i < \underline{l}$

(9) $\tilde{N}_h | \tilde{\lambda}_h = \hat{N}_i | \tilde{\lambda}_h$ for $h < i < \underline{l}$

(10) $\dot{R}_i$ is well founded

(where $\dot{R}_i$ is defined in $\dot{W}_i$ from $\dot{\mathbb{N}}_i, \underline{I}, \underline{n}^*$ as R was defined in V from $\mathbb{N}$, I, n^*.)

(11) $\mathrm{level}^i(\dot{\sigma}_i) \leq \mathrm{rank}(\dot{W}_i)$ for $i < \underline{l}$

(12)
- $\underline{\delta}_i = \hat{\sigma}_i \text{“} \underline{\lambda}_i$ for $i < \underline{l}$
- $\underline{\delta}_i = \hat{\sigma}_i \restriction \mathrm{ht}(P_l)$ for $i = \underline{l}$
- $\underline{t}_i =$ the complete theory of: $\langle \dot{W}_i, \dot{\mathbb{N}}_i, \dot{\sigma}_i, \underline{I}, \underline{\delta} \restriction i, \underline{t} \restriction i \rangle$ for $i < \underline{l}$

This describes the language $\mathbb{L}$. If $\mathbb{L}$ is consistent and $\gamma \geq \mathrm{card}(C_\nu)$, then forcing with $\mathrm{Col}(\gamma, \omega)$ yields a model $\mathbb{A}$ of $\mathbb{L}$. $\mathbb{E} = \dot{\mathbb{E}}^{\mathbb{A}}$ is then an enlargement with $\langle \delta, t \rangle = \mathrm{trace}(\mathbb{E})$. Moreover, $\mathbb{E} \in V[G]$ where G is $\mathrm{Col}(\gamma, \omega)$-generic. Conversely, if there is such an $\mathbb{E} \in V[G]$, then G is set generic over H_θ for a regular θ. $\langle H_\theta[G], \mathbb{E}, \cdots \rangle$ is a model of $\mathbb{L}$. $\mathbb{L}$ is therefore consistent. This proves Lemma 5.5.5 and with it Lemma 5.5.4.

Many of the arguments we have been making can be carried out if we replace V with an arbitrary world W. We have seen that:

- If $\alpha = \mathrm{ht}(W)$, then $C^W_\xi = C_\xi$ for $\xi < \alpha$. (Hence $C^W_\infty = C_\alpha$.)
- $I \in W$, since $I \in C_{\omega_1} \subset W$.

  We leave it to the reader to show:
- Let $W' \subset W$. Then:

  If G is set generic over W, we can relativize the definition of "enlargement" to $W[G]$, letting W play the role of V. Axiom (I) in the definition of "enlargement" thus becomes:

  (I) $\mathrm{trace}(\mathbb{E}) \in W$.

  However, by the definition of trace, we know that $\mathrm{trace}(\mathbb{E}) \in C^W_\infty = C_\alpha \subset V$.

  Relativizing the definition of "trace" to a world W, we have:

  Let W be a world. Let $\langle \delta, t\rangle \in W$. Them $W \models (\langle \delta, t\rangle$ is a trace) if and only if there is a set of conditions $\mathbb{P} \in W$ which forces that if G is $\mathbb{P}$-generic over W, then there is an enlargement $\mathbb{E} \in W[G]$ with $\langle \delta, t\rangle = \mathrm{trace}(\mathbb{E})$.

But then Lemma 5.5.4 and 5.5.5 relativize to an arbitrary world, yielding:

Lemma 5.5.6. *Let W be a world. Let $\langle \delta, t\rangle \in W$. There is a Σ_1 formula φ such that in W we have:*

$$\langle \delta, t\rangle \textit{ is a trace } \longleftrightarrow C_\infty \models \varphi[\delta, t, I].$$

This follows from:

Lemma 5.5.7. *Let W be a world. Let $\langle \delta, t\rangle \in W$ be a potential trace. Let $\omega_1 < \alpha < \nu \in W$ such that C_ν is admissible and $\delta, t \in C_\nu$. There is a language $\mathbb{L} = \mathbb{L}_{\alpha, I, \delta, t}$ on C_ν such that*

$$\mathbb{L} \textit{ is consistent } \longleftrightarrow W \models \langle \delta, t\rangle \textit{ is an } \alpha - \textit{bounded trace.}$$

Lemma 5.5.6 follows from Lemma 5.5.7 exactly as before. We prove Lemma 5.5.7:

($\longrightarrow$) in exactly as before. If $\mathbb{L}$ is consistent, then in $\mathrm{Col}(\gamma, \omega)$-generic over W, then $\mathbb{L}$ has a model $\mathbb{A}$ in $W[G]$. $\dot{\mathbb{E}}^{\mathbb{A}}$ is then α-bounded enlargement with trace $\langle \delta, t \rangle$ in $W[G]$.

($\longleftarrow$) Let G be set generic over W such that there is an enlargement $\mathbb{E} \in W[G]$ which is α-bounded and $\langle \delta, t \rangle =^t race(\mathbb{E})$. Then $\langle W[G], \mathbb{E}, \dots \rangle$ models $\mathbb{L}$. Hence $\mathbb{L}$ is consistent. QED(Lemma 5.5.7)

The following definition seems natural:

Definition 5.5.12. Let $\langle \delta, t \rangle, e \in V$. $\langle \delta, t \rangle$ is an e-trace if and only if there is a set of conditions $\mathbb{P} \in V$ which forces that, if G is $\mathbb{P}$-generic over V, then there is $\mathbb{E} \in V[G]$ which is an e-enlargement with trace $\langle \delta, t \rangle$.

We can of course relativize this definition to an arbitrary world W with $\langle \delta, t \rangle, e \in W$. The relativization is then of course in interest, since e is not necessarily an element of V. We can also state and prove the version of Lemma 5.5.4 and Lemma 5.5.5 for e-trace. These also relativize to an arbitrary world. We now state and prove the relativized versions of these lemmas for e-traces, since the relativized version is the more useful one.

Lemma 5.5.8. *There is a Σ_1 formula φ such that whenever W is a world, $e, \delta, t \in W$, then in W we have:*

$$\langle \delta, t \rangle \textit{ is an e-trace for } I|l \textit{ if and only if } C^e_{\lambda,\infty} \models \varphi[\delta, t, I, n^*]$$

where $l = \mathrm{dom}(\delta)$ and $\lambda = \mathrm{lub}\{\lambda_i \mid i < l\}$.

Note We have seen that if W is a world and $\xi \in W$, then $C^W_\xi = C_\xi$. Similarly, if W, W' are worlds, $e \in W \cap W'$ and $\lambda < \xi \in W \cap W'$, then:

$$(C^e_{\lambda,\xi})^W = (C^e_{\lambda,\xi})^{W'}.$$

Lemma 5.5.8 follows in the usual way from:

Lemma 5.5.9. *Let $\langle \delta, t \rangle$ be a potential trace, where $l = \mathrm{dom}(\delta) = \mathrm{dom}(t)$ and $l < \mathrm{lh}(I)$. Let $\tilde{\lambda} = \mathrm{lub}_{j<l}\, \delta_j"\lambda_j$. Let $\tilde{\lambda} < \alpha < \nu \in W$ such that $C^e_{\tilde{\lambda},\nu}$ is admissible. There is a language $\mathbb{L} = \mathbb{L}_{\alpha,I,\delta,t}$ on $C^e_{\tilde{\lambda},\nu}$ such that $\mathbb{L}$ is consistent if and only if $\langle \delta, t \rangle$ is e-trace.*

To prove Lemma 5.5.9 we add to our previous language $\mathbb{L} = \mathbb{L}_{\alpha,\delta,t}$ from Lemma **??** the axiom:

$$\wedge i < \underline{l} i + 1 < \underline{\mathrm{lh}(I)}, J^{\underline{e}}_{\tilde{\lambda}_i} = J^{E^{\tilde{N}_i}}_{\tilde{\lambda}_i} \text{ for } i < \underline{l}.$$

The proof is just as before.

In passing, we mention the following lemma:

Lemma 5.5.10. *Let W, W' be worlds such that $e, \alpha \in W \cap W'$. Let G be set genetic over W'. Let $\mathbb{E} \in W'[G]$ such that*

- $\mathbb{E}$ *is an α-bounded e-enlargement*
- $\langle \delta, t \rangle = \text{trace}(\mathbb{E})$.

Then $W \models \langle \delta, t \rangle$ is an α-bounded e-trace.

Proof. Let $\lambda = \text{lub}\{\tilde{\lambda}_i \mid i < l\}$ in $\mathbb{E}$. Hence $\lambda < \alpha$. Let ν be limit such that $\alpha < \nu$, $\delta, t \in C^e_{\lambda,\nu}$ such that $C^e_{\lambda,\nu}$ is admissible in W. Then $\nu \in W'$ and

$$(C^e_{\lambda,\nu})^W = (C^e_{\lambda,\nu})^{W'}.$$

But then $\langle W'[G], \mathbb{E}, \dots \rangle$ models $\mathbb{L}_{\alpha,\delta,t}$ on $C^e_{\lambda,\nu}$ in W. hence $\mathbb{L}$ is consistent. QED(Lemma 5.5.10)

We have seen that if $\mathbb{E}$ is an enlargement of $I \restriction l$ with trace $\langle \delta, t \rangle$, then for $0 < i < l$ we have: $\mathbb{E} \restriction i$ is an $E^{\hat{N}_i}$-enlargement of $I \restriction i$ with trace $\langle \delta \restriction i, t \restriction i \rangle$. Since $E^{\hat{N}_i} \in W_i$, it is natural to ask whether W_i thinks $\langle \delta \restriction i, t \restriction i \rangle$ is an $E^{\hat{N}_i}$-trace. In general, we do not know the answer to this question, but the question suggests the following definition:

Definition 5.5.13. Let $\mathbb{E}$ be an enlargement of $I|l$ with trace $\langle \delta, t \rangle$. $\mathbb{E}$ is *neat* (or *self justifying*) if and only if for $0 < i < l$ we have

$$W_i \models \langle \delta \restriction i, t \restriction i \rangle \text{ is an } E^{\hat{N}_i}\text{-trace.}$$

Definition 5.5.14. $\langle \delta, t \rangle$ is a *neat trace* if and only if there is a set of conditions $\mathbb{P}$ which forces, if G is $\mathbb{P}$-generic over V, then there is a neat enlargement $\mathbb{E} \in V[G]$ with trace $\langle \delta, t \rangle$.

It is apparent that any neat trace must satisfy a syntactical condition of the form: $x_i \in t_i$ for $0 < i < l$.

But then any enlargement with trace $\langle \delta, t \rangle$ will be a neat enlargement. Thus, $\langle \delta, t \rangle$ is neat if and only if it is a trace and satisfies the syntactical condition: $x_i \in t_i$ for $0 < i < l$.

A similar question is the following: Let $\mathbb{E}$ an enlargement of $I \restriction i+1$, where $i+1 < \text{lh}(I)$. Then ν_i is given and $\mathbb{E}$ is an $E^{\tilde{N}_i}$-enlargement of $I \restriction i+1$, where $E^{\tilde{N}_i} \in W_i$. Set: $\langle \delta, t \rangle = \text{trace}(\mathbb{E})$. It follows easily that $\langle \delta, t \rangle \in W_i$. Does W_i think that $\langle \delta, t \rangle$ is an $E^{\tilde{N}_i}$-trace? The answer will be yes if W_i has a property which we call *pride*. We define:

Definition 5.5.15. Let W be a good world. Let $W = J_\alpha^A$ and let $\beta = \beta^W$ be the largest cardinal in W. W is *proud* if and only if for all $\gamma < \beta$ there is $\bar{W} \in J_\beta^A$ such that

(a) $\bar{W} = J_{\bar{\alpha}}^{\bar{A}}$

(b) $W \models \bar{W}$ is a good world

(c) $\mathrm{rank}(\bar{W}) \geq \min(\gamma, \mathrm{rank}(W))$

(d) If $\xi_1, \cdots, \xi_n < \gamma$ and φ is any first-order formula, then:

$$\bar{W} \models \varphi[\vec{\xi}] \longleftrightarrow W \models \varphi[\vec{\xi}].$$

(Note; (b) implies that $\bar{W}$ is a good world, (d) implies that $J_\gamma^A = J_\gamma^{\bar{A}}$.)

Lemma 5.5.11. *Let G be generic over V. Let $\mathbb{E} \in V[G]$ be a neat enlargement of $I|i+1$ such that W_i is proud and $i+1 < \mathrm{lh}(I)$. Let $\mathrm{ht}(W_i)$ be collapsed to ω in $V[G]$. Let $\langle \delta, t \rangle = \mathrm{trace}(\mathbb{E})$. Then:*

$$W_i \models \langle \delta, t \rangle \textit{ is an } E^{\tilde{N}_i}\textit{-trace.}$$

Proof. Let $e = E^{\tilde{N}_i}$. For $\beta = \beta^{W_i}$ we know that $V_\beta = L_\beta^{A_i}$ and $\mathrm{cf}(\beta) > \omega_1$ in W_i. Hence there is $\gamma < \beta$ such that $L_\gamma^{A_i} = V_\gamma$ and $\mathbb{N}_i \in V_\gamma$ in W_i. Let $\tilde{W} \in W_i$ be as above with respect to γ. It follows easily that:

(1) $\bar{W} \models \varphi[\vec{x}] \longleftrightarrow W \models \varphi[\vec{x}]$,

whenever φ is a first-order formula and $x_1, \ldots, x_n \in J_\gamma^{A_i}$. Note that among the elements of $L_\gamma^{A_i}$ are:

$$I, N_i, x_i, R_i, \mathrm{level}^i, \hat{\sigma}_i, \tilde{\sigma}_i, \hat{N}_i, \tilde{N}_i$$

where:

$$W_i \models (x_i \text{ is the set of all realizations of some } P_j \text{ in } \mathbb{N}_i).$$

Clearly level^i maps X_i into γ. Since $\mathrm{level}^i(\sigma_i) \leq \mathrm{rank}(W_i)$ and $\mathrm{level}^i(\sigma_i) < \gamma$, we have:

$$\mathrm{rank}(\bar{W}) \geq \min(\gamma, \mathrm{rank}(W_i) \geq \mathrm{level}^i(\sigma_i).$$

Using (1) it follows easily that $\mathbb{E}'$ is an e-enlargement of $I \restriction i+1$, where

$$\mathbb{E}' \restriction i = \mathbb{E} \restriction i, \mathbb{E}'_i = \langle \bar{W}, \mathbb{N}, \sigma_i \rangle$$

But $\mathbb{E}' \in V[G]$. Clearly $\langle\delta, t\rangle = \text{trace}(\mathbb{E}')$. Since $\langle\delta, t\rangle$ is neat we have:

$$\bar{W} \models \langle\delta \restriction i, t \restriction i\rangle \text{ is an } \hat{e} - \text{trace}, \text{ where } \hat{e} = E^{\hat{N}_i}.$$

Hence there is $\delta \in \bar{W}$ large enough that $\text{Col}(\delta, \omega)$ forces the existence of an $\hat{e}$-enlargement with trace $\langle\delta \restriction i, t \restriction i\rangle$. Since W_i is countable in $V[G]$ there is $\bar{G} \in V[G]$ which is $\text{Col}(\delta, \omega)$-generic over W_i (hence over $\bar{W}$). Let $\bar{\mathbb{E}}' \in \bar{W}[\bar{G}]$ be an $\hat{e}$-enlargement of $I \restriction i$ with trace $\langle\delta \restriction i, t \restriction i\rangle$. However, $\bar{\mathbb{E}}'$ is an e-enlargement, since we know: $\tilde{\lambda}_j = \text{lub}\, \delta_i\text{"}\lambda_j$ and $J^{\hat{e}}_{\tilde{\lambda}_j} = J^{e}_{\tilde{\lambda}_j}$ for $j < i$. (This is because $\mathbb{E}$ is an e-enlargement with trace $\langle\delta, t\rangle$.) Since $\bar{\mathbb{E}} \in V[G]$ we can apply the interpolation lemma to form: $\bar{\mathbb{E}} = \bar{\mathbb{E}}' \cup \mathbb{E}' \restriction [0, i+1)$. Then $\bar{\mathbb{E}} \in V[G]$ is an e-enlargement of $I \restriction i+1$ with trace $\langle\delta, t\rangle$. But $\bar{\mathbb{E}} \in W_i[\bar{G}]$ is α-bounded, where $\alpha = \text{ht}(\bar{W}) + 1$. Hence by Lemma 5.5.10 we have: $W_i \models \langle\delta, t\rangle$ is an α-bounded e-trace.

QED(Lemma 5.5.11)

Note Lemma 5.5.11 relativizes to any world W' in place of V.

Lemma 5.5.12. *Let $W = J^A_\alpha$ be a good world. Let W' be a proper segment of W (i.e. $W' = W|\alpha_j$ for a $j < \text{rank}(W)$). Then W' is proud.*

Proof. By reflectivity there is an $\alpha^* < \alpha$ such that

$$W^* = J^A_{\alpha^*} \prec N \text{ and } W' \in W^*.$$

Working in W, we define a sequence $\langle X_i \mid i \leq \omega_1\rangle$ as follows:

- $X_0 = \gamma \cup \{W'\}$
- X_{2i+1} =the smallest $X \prec W^*$ such that $X_{2i} \subset X$.
- $X_{2i+2} = X_{2i+1} \cup [\text{On} \cap W' \cap X_{2i+1}]^\omega$
- $X_\lambda = \bigcup_{i<\lambda} X_i$ for limit λ.

Then $X_{\omega_1} \prec W^*$. Let $\gamma < \delta < \beta$ such that δ is a cardinal in W. By induction on $i \leq \omega_1$ we get:

$$\text{card}(X_i) \leq 2^\delta < \beta \text{ for } i \leq \omega_1.$$

Let $\sigma\colon \bar{W}^* \overset{\sim}{\longleftrightarrow} X_{\omega_1}$, where $\bar{W}^* = J^{A^*}_{\bar{\alpha}^*}$. Then $\alpha^* < (2^\delta)^+ < \beta$ and $\bar{W}^* \in J^A_\beta$. Let $\sigma(\bar{W}) = W'$. Then $\bar{W} = J^{\bar{A}}_{\bar{\alpha}}$ where $\bar{A} = A^* \cap \bar{W}$. But then:

(1) $[\bar{\alpha}]^\omega = ([\text{On}]^\omega)^{\bar{W}}$, hence $\bar{W}$ is a world.

Proof. We know that $X_{\omega_1} \in W$, since it was defined in W as a subset of W'.

($\subset$) Let $a \in [\bar{\alpha}]^\omega$. Then $W \models a \in [\bar{\alpha}]^\omega$ since W is a world and $\bar{\alpha} \leq \mathrm{ht}(W)$. Hence $W \models \sigma(a) \in [\alpha']^\omega$, where $\alpha' = \sigma(\bar{\alpha}) = \mathrm{ht}(W')$, since $\sigma(a) = \sigma"a$ and $\sigma \in W$ is bijective. Hence $\sigma(a) \in [\alpha']^\omega$, since W is a world. Hence $W' \models \sigma(a) \in [\mathrm{On}]^\omega$, since W' is a world and $\alpha' = \mathrm{ht}(W')$. Hence $\bar{W} \models a \in [\mathrm{On}]^\omega$, since $\sigma(\bar{W}) = W'$.

($\supset$) Let $\bar{W} \models a \in [\mathrm{On}]^\omega$. Then $W' \models \sigma(a) \in [\mathrm{On}]^\omega$, since $\sigma(\bar{W}) = W'$. Hence $\sigma(a) \in [\alpha']^\omega$, since W' is a world. Hence $W \models \sigma(a) \in [\alpha']^\omega$, since W is a world and $\alpha' < \mathrm{ht}(W)$. Hence $W \models a \in [\bar{\alpha}]^\omega$, since $\sigma(\bar{\alpha}) = \alpha'$ and $\sigma(a) = \sigma"a$ and $\sigma \in W$ is bijective. Hence $a \in [\bar{\alpha}]^\omega$, since W is a world.

QED(1)

Hence $\bar{W}$ is a world. Since $\sigma(\bar{W}) = W'$, it follows easily that W' is a good world. Moreover, $\sigma(\mathrm{rank}(\bar{W})) = \mathrm{rank}(W')$ and $\sigma \restriction \gamma = \mathrm{id}$. Hence:

$$\mathrm{rank}(\bar{W}) \leq \min(\gamma, \mathrm{rank}(W')).$$

But $\sigma \restriction \bar{W} \colon \bar{W} \prec W'$. Hence:

$$\bar{W} \models \varphi[\vec{\xi}] \longleftrightarrow W' \models \varphi[\vec{\xi}]$$

for $\xi_1, \cdots, \xi_n < \gamma$ and φ any first-order formula. QED(5.5.12)

Definition 5.5.16. An enlargement $\mathbb{E}$ of $I|i+1n$ is *proud* if and only if W_i is proud.

Definition 5.5.17. $\langle \delta, t \rangle$ is a *pride inducing e-trace* if and only if there is a set of conditions $\mathbb{P}$ which forces the existence of an enlargement $\mathbb{E}$ of $I|l+1$, where:

- $\mathbb{E}$ is a neat proud enlargement (hence $l < \mathrm{lh}(I)$).
- $\langle \delta, t \rangle = \mathrm{trace}(\mathbb{E} \restriction l)$
- $J^{E^{\tilde{N}_j}}_{\tilde{\lambda}_j} = J^e_{\tilde{\lambda}_j}$ for $j < l$.

This definition can be relativized to any world. as can the following definition:

Definition 5.5.18. $\langle \delta, t \rangle$ is an α-*bounded* pride inducing e-trace if and only if there is a set of conditions $\mathbb{P}$ forcing the existence of an α-bounded enlargement $\mathbb{E}$ with the above properties.

Lemma 5.5.13. *There is an Σ_1 formula ψ such that, whenever W is a world with $\langle\delta, t\rangle, e \in W$., then in W we have:*

$$\langle\delta, t\rangle \text{ is a pride inducing e-trace } \longleftrightarrow C^e_{\tilde{\lambda},\infty} \models \psi[\delta, t, I, n^*],$$

where $\tilde{\lambda} = \text{lub} \bigcup_{i<l} \delta_i$" λ_i.

Lemma 5.5.14. *Let W be a world such that $e, \langle\delta, t\rangle \in W$ where $\langle\delta, t\rangle$ is a potential trace of length $l < \text{lh}(I)$. Let $\alpha < \nu$ in W such that $C^e_{\lambda,\nu}$ is admissible and $\lambda = \text{lub} \bigcup_{i<l} \delta_i \lambda_i$. There is a language $\mathbb{L}_{\alpha,\delta,t}$ on $C^e_{\lambda,\nu}$ such that*

$$\mathbb{L} \text{ is consistent } \iff W \models \langle\delta, t\rangle \text{ is an } \alpha-\text{bounded pride inducing } e-\text{trace}.$$

Proof. We change the language $\mathbb{L}$ of Lemma 5.5.5 as follows:

(a) We add the axiom $\underline{l} < \underline{\text{lh}(I)}$.

(b) In (2) we change $i < \underline{l}$ to $i \leq \underline{l}$.

(c) In (3)–(11) we change the quantifier domain from $i < \underline{l}$ to $i \leq \underline{l}$.

(d) We change (12) to

- $\underline{\delta}_i = \tilde{\sigma}_i \restriction \underline{\lambda}_i$ for $i < \underline{l}$
- $\underline{t}_i$ =the complete theory of: $\langle \dot{W}_i, \dot{\mathbb{N}}_i, \dot{\sigma}_i, \underline{I}, \underline{\delta} \restriction i, \underline{t} \restriction i\rangle$ for $i < \underline{l}$.

(e) We add:

(13) $W_i \models \langle\underline{\delta} \restriction i, \underline{t} \restriction i\rangle$ is an $E^{\hat{N}_i}$-trace for $i \leq \underline{l}$.

(14) $W_{\underline{l}}$ is proud.

(15) $J^{\underline{e}}_{\tilde{\lambda}_i} = J^{E^{\tilde{N}_I}}_{\tilde{\lambda}_i}$ for $i < \underline{l}$.

If $\mathbb{L}$ is consistent, then forcing over W with a sufficient $\text{Col}(\gamma, \omega)$ ($\gamma \in W$) gives us a model $\mathbb{A}$. Set $\mathbb{E} = \dot{\mathbb{E}}^{\mathbb{A}}$. Then $\mathbb{E} \restriction l$ is an enlargement with trace $\langle\delta, t\rangle$. Moreover, $\mathbb{E}$ satisfies (A)–(H) in the definition of enlargement. Hence $\mathbb{E}$ is an enlargement by Lemma 5.5.1. $\mathbb{E}$ is neat by (15) and proud by (14). $\mathbb{E} \restriction l$ is an e-enlargement by (15).

Conversely, if

$$W \models \langle\delta, t\rangle \text{ is a pride inducing trace},$$

then forcing over W with a sufficient $\text{Col}(\gamma, \omega)$ gives an enlargement $\mathbb{E}$ of length $l + 1$ which is neat, proud and such that $\mathbb{E} \restriction l$ has trace $\langle\delta, t\rangle$. Hence $\langle W[G], \mathbb{E}, \ldots\rangle$ models: $\mathbb{L}$ where G is $\text{Col}(\gamma, \omega)$-generic. Hence $\mathbb{L}$ is consistent. QED(5.5.14)

Definition 5.5.19. Let G be set generic over V. An enlargement $\mathbb{E}$ is *bounded* in $V[G]$ if and only if $\mathbb{E} \in V[G]$ and $\mathbb{E}$ is bounded by an α which is collapsed to ω in $V[G]$.

Lemma 5.5.15. *Let $\mathbb{E} = \langle\langle W_i, \mathbb{N}_i, \sigma_i\rangle \mid i \leq \eta\rangle$ be a neat, proud enlargement of $I|\eta + 1$ which is bounded in $V[G]$. Let $\eta + 1 < \mathrm{lh}(I)$ (hence ν_η exists in I). Let $\gamma = T(\eta + 1)$. Then there is a neat enlargement $\mathbb{E}'$ of $I|\eta + 1$ such that $\mathbb{E}' \in V[G]$ and*

$$\mathbb{E}' = \langle\langle W'_j, \mathbb{N}'_j, \sigma'_j\rangle \mid j \geq \eta + 1\rangle \text{ where:}$$

(a) $\mathbb{E}' \restriction \gamma = \mathbb{E} \restriction \gamma$

(b) $\mathrm{ht}(W'_j) < \mathrm{ht}(W_\gamma)$ *for* $\gamma \leq j \leq \eta$

(c) $W'_{\eta+1} = W_\gamma$, $\mathbb{N}'_{\eta+1} = \mathbb{N}_\gamma$, $\sigma'_{\eta+1} \restriction \gamma + 1 = \sigma_\gamma$.

Proof. In W_η we have: F is robust in $\tilde{N}_\eta = \langle J^E_\nu, F\rangle$. Hence there is a $g\colon \lambda_i \longrightarrow \tilde{\kappa} = \tilde{\sigma}_\eta(\kappa_\eta)$ such that:

(A) Let $\alpha_1, \cdots, \alpha_n < \lambda_\eta$, $X \in \mathbb{P}(\kappa_\eta) \cap P_\eta$. Then:

$$\prec g(\vec{\alpha}) \succ \in \tilde{\sigma}_\eta(X) \longleftrightarrow \prec \vec{\alpha} \succ \in E^{P_\eta}_{\nu_\eta}(X).$$

(B) Let $a_1, \cdots, a_n \subset \lambda_\eta$. Let ψ be Σ_1. Then in W_η we have:

$$C^{E^{\tilde{N}_\eta}}_{\bar{\lambda}, \tilde{\kappa}_\eta} \models \psi[g\text{"}\vec{a}, \vec{u}] \longleftrightarrow C^{E^{\tilde{N}_\eta}}_{\tilde{\lambda}, \infty} \models \psi[\tilde{\sigma}_\eta\text{"}\vec{a}, \vec{u}]$$

where $u_1, \ldots, u_m$ are hereditarily countable and $\bar{\lambda} = \mathrm{lub}\, g\text{"}\lambda_\eta$, $\tilde{\lambda} = \mathrm{lub}\, \tilde{\sigma}_\eta\text{"}\lambda_\eta$.

Let $W = W_\gamma$, $\mathbb{N} = \mathbb{N}_\gamma$. Then:

(1) $g \in W_\gamma$

Proof. $g\text{"}\lambda_\eta$ is a countable set of ordinals in W_η, hence in V, hence in W. But g is the monotone enumeration of $g\text{"}\lambda_\eta$. QED(1)

Define N^*_η, σ^*_η in W as follows: If $P^*_\eta = P_\gamma$, set: $N^*_\eta = \hat{N}_\gamma$, $\sigma^*_\eta = \hat{\sigma}_\gamma$. Otherwise $P^*_\eta = P_\gamma||\beta$ where $\beta \in P_\gamma$ is maximal such that τ_η is a cardinal in $P_\gamma||\beta$. Then $\nu_\gamma \leq \beta \in P_\gamma$, since τ_η is a cardinal in $P_\gamma||\lambda_\gamma$, hence in $P_\gamma||\nu_\gamma$. Let $\langle\langle \eta_i, \nu_i\rangle \mid i \leq p\rangle$ be the resurrection sequence for $\langle \hat{N}_\gamma, \hat{\sigma}_\gamma(\nu_\gamma)\rangle$ with the associated sequence $\langle\langle k_i, \bar{\beta}_i\rangle \mid i \leq p\rangle$. Let $\bar{\beta} = \hat{\sigma}_\gamma(\beta)$. As we have seen, it follows that $\bar{\beta} = \bar{\beta}_j$ for a $j > 0$. We set: $N^*_\eta = N^\gamma_{\eta_j}$, $\sigma^*_\eta = k_j \hat{\sigma}_\eta$. Then:

$$\sigma^*_\eta\colon P^*_\eta \longrightarrow_{\Sigma^{(n)}_0} N^*_\eta, \text{ where } n \leq \omega \text{ is maximal such that } \kappa_\eta < \rho^n_{P^*_\eta}.$$

(2) Let $\tilde{\kappa} = \tilde{\sigma}_\eta(\kappa_\eta)$, $\kappa^* = \sigma^*_\eta(\kappa_\eta)$. Then $\tilde{\kappa} = \kappa^*$.

Proof. Let $k \in W$ be the resurrection map for $\langle N^*_\eta, \sigma^*_\eta(\nu_\gamma)\rangle$. (If $\nu_\gamma = \mathrm{ht}(P_\gamma)$, we let $\sigma^*_\eta(\nu_\gamma)$ denote $\mathrm{ht}(N^*_\eta)$ and we have: $k = \mathrm{id}$.) Then $\tilde{\kappa} = \tilde{\sigma}_\eta(\kappa_\eta) = \tilde{\sigma}_\gamma(\kappa_\eta)$ since $\tilde{\sigma}_\eta \upharpoonright \lambda_\gamma = \tilde{\sigma}_\gamma \upharpoonright \lambda_\gamma$. Hence:

$$\tilde{\kappa} = \tilde{\sigma}_\gamma(\kappa_\eta) = k\sigma^*_\eta(\kappa_\eta) = \sigma^*_\eta(\kappa_\eta) = \kappa^*.$$

QED(2)

(3) Let $e = E^{\hat{N}_\eta}$ in W_η, $e^* = E^{N^*_\eta}$ in W. Then: $J^{\tilde{e}}_{\tilde{\kappa}_\eta} = J^{e^*}_{\kappa^*_\eta} \in W$.

Proof. Let $k\colon N^*_\eta || \sigma^*_\eta(\nu_\gamma) \longrightarrow \tilde{N}_\gamma$ be the resurrection map. Then $k \upharpoonright \kappa^*_\eta = \mathrm{id}$, since κ^*_η is a cardinal in N^*_η. But $k\sigma^*_\eta = \tilde{\sigma}_\gamma$. Hence:

$$J^{e^*}_{\kappa^*_\eta} = N^*_\eta || \kappa^*_\eta = \tilde{N}_\gamma || \kappa^*_\eta = \hat{N}_\eta || \tilde{\kappa}_\eta = J^e_{\tilde{\kappa}_\eta}.$$

QED(3)

(4) Let $n \leq \omega$ be maximal such that $\kappa_\eta < \rho^n_{P^*_\eta}$. There is $\sigma\colon P_{\eta+1} \longrightarrow_{\Sigma^{(n)}_0} N^*_\eta$ such that

$$\sigma\pi_{\gamma,\eta+1} = \sigma^*_\eta, \sigma \upharpoonright \lambda_\eta = g.$$

Let $\pi = \pi_{\gamma,\eta+1}$. σ is defined by

$$\sigma(\pi(f)(\alpha)) = \sigma^*_\eta(f)(g(\alpha))$$

for $f \in \Gamma^*(\kappa_\eta, P^*_\eta)$, $\alpha < \lambda_\eta$.

Proof. Let φ be a $\Sigma^{(n)}_0$ formula. Let $f_1, \ldots, f_m \in \Gamma^*(\kappa_\eta, P^*_\eta)$ and $\alpha_1, \ldots, \alpha_m < \lambda_\eta$. Set: $X = \{\prec \vec{\xi} \succ < \kappa_\eta \mid P^* \models \varphi[f_1(\xi_1), \ldots, f_m(\xi_m)]\}$. Let $\pi = \pi_{\gamma,\eta+1}>$ Then:

$$\begin{aligned} P_{\eta+1} \models \varphi[\pi(\vec{f})(\vec{\alpha})] &\iff \prec \vec{\alpha} \succ \in E^{P_\eta}_{\nu_\eta}(X) \\ &\iff \prec g(\vec{\alpha}) \succ \in X \\ &\iff N^*_\eta \models \varphi[\sigma^*_\eta(\vec{f})(g(\vec{\alpha}))]. \end{aligned}$$

QED(4)

But σ is definable in W, since $g, \sigma^*_\eta \in W$. Hence

(5) $\sigma \in W$.

Define, in W, a realization σ' of $P_{\eta+1}$ by:

$$\sigma' \upharpoonright \gamma + 1 = \sigma_\gamma, \sigma'_{\eta+1} = \langle \sigma, \mu \rangle \text{ where } N^\gamma_\mu = N^*_\eta.$$

Let $\langle\delta, t\rangle = \text{trace}(\mathbb{E})$. Since W_η is proud, we know by Lemma 5.5.11 that:

(6) $W_\eta \models \langle\delta, t\rangle$ is an e-trace.

This means that in W_η we have:

$$C^e_{\tilde{\lambda},\infty} \models \psi[\delta, t, I, n^*]$$

where $\tilde{\lambda} = \text{lub}\,\tilde{\delta}_\eta\text{"}\lambda_\eta$ and ψ is a certain Σ_1 formula. But this can be rewritten as:

$$C^e_{\tilde{\lambda},\infty} \models \psi'[\delta_\eta\text{"}\lambda_\eta, t, I, n^*]$$

where t, I, n^* are hereditarily countable. Hence:

$$C^e_{\tilde{\lambda},\tilde{\kappa}} \models \psi'[g\text{"}\lambda_\eta, t, I, n^*]$$

which transforms back into:

$$C^e_{\tilde{\lambda},\tilde{\kappa}} \models \psi[\delta, t, I, n^*].$$

But since $C^e_{\tilde{\lambda},\tilde{\kappa}} = C^{e^*}_{\tilde{\lambda},\kappa^*} \in W$ we have:

(7) $W \models \langle\delta', t\rangle$ is an e^*-trace.

This means that if $\bar{\mathbb{P}} = \text{Col}(\delta, \omega)$ for a sufficiently large $\delta \in W$, then $\bar{\mathbb{P}}$ forces that, if $\bar{G}$ is $\bar{\mathbb{P}}$-generic over W, there is an $\mathbb{E}''' \in W[\bar{G}]$ which is an e^*-enlargement of $I \restriction \eta + 1$ with trace $\langle\delta', t\rangle$. We now extend $\mathbb{E}'''$ to a structure of length $\eta + 2$ by setting:

$$\mathbb{E}'' = \mathbb{E}''' \cup \{\langle\langle W, \mathbb{N}, \sigma'\rangle, \eta\rangle\}.$$

We claim:

(8) $\mathbb{E}''$ is an enlargement of $I \restriction \eta + 1$.

Proof. We verify (A)–(I) is the definition of "enlargement". (A)–(C) hold trivially for $i \le \eta + 1$. (D) holds trivially for $i \le \eta$. We prove (D) for $i = \eta + 1$. It is enough to see that $\hat{\sigma}_{\eta+1}(\lambda_\eta)$ is a cardinal in $\hat{N}_{\eta+1}$, since the rest follows by acceptability. We have:

$$\hat{\sigma}_{\eta+1} = \sigma^{\eta+1}_{\eta+1} = \sigma'_{\eta+1} \text{ and } \hat{N}_{\eta+1} = N^*_\eta \text{ where } \sigma\pi_{\gamma,\eta+1} = \sigma^*_\eta.$$

Thus:

$$\sigma(\lambda) = \sigma\pi_{\gamma,\eta+1}(\kappa_\eta) = \sigma^*_\eta(\kappa_\eta) = \kappa^*$$

is a cardinal in $N^*_\eta = \hat{N}_{\eta+1}$. QED(D)

(E) is trivial for $i \leq \eta$. Now let $i = \eta + 1$, $h \leq \eta$. Then

$$\hat{\sigma}_{\eta+1} \upharpoonright \lambda_h = \sigma \upharpoonright \lambda_h = g \upharpoonright \lambda = \delta'_h = \tilde{\sigma}''_h \upharpoonright \lambda_h.$$

QED(E)

(F) is trivial for $i \leq \eta$. We prove it for $i = \eta + 1$, $h \leq \eta$. $\mathbb{E}'' \upharpoonright \eta + 1$ is an e^*-enlargement of $I \upharpoonright \eta + 1$, where $e^* = E^{N^*_\eta}$. But $N^*_\eta = \hat{N}_{\eta+1}$. Hence for $h \leq \eta$ we have:

$$J^{E^{\tilde{N}_h}}_{\tilde{\lambda}_h} = J^{e^*}_{\tilde{\lambda}_h} = J^{E^{N_{\eta+1}}}_{\tilde{\lambda}_h} \text{ in } \mathbb{E}''.$$

QED(F)

(G) is immediate since $R = R_{\eta+1} = R_\gamma$ is well founded in W. But this gives us the level function:

$$\text{level} = \mathit{level}^\gamma = \text{level}^{\eta+1}$$

defined by:

$$\text{level}(\sigma) = \text{lub}\{\text{level}(\sigma') \mid \sigma' R \sigma\}.$$

(H) is trivial for $i \leq \eta$. Now let $i = \eta + 1$. If γ does not survive at $\eta + 1$, then $\sigma' R \sigma$. Hence:

$$\text{level}(\sigma') = \text{level}(\sigma_\gamma \leq \text{rank}(W).$$

If however, γ does survive at $\eta + 1$, it follows easily that $j <_T \eta + 1$ does not survive at $\eta + 1$ if and only if $j < \gamma$ and j does not survive at γ. Hence:

$$\text{level}(\sigma') = \text{level}(\sigma_\gamma) \leq \text{rank}(W).$$

QED(H)

But it then follows by Lemma 5.5.1 that $\mathbb{E}''$ is an enlargement. QED(8)

Now let: $\langle \delta'', t' \rangle = \text{trace}(\mathbb{E}'')$.

(9) $\langle \delta'', t' \rangle$ is a neat trace in W.

Proof. $\langle \delta'' \upharpoonright \eta + 1, t' \upharpoonright \eta + 1 \rangle$ is neat, since $\chi_i \in t_i$ for $i \leq \eta$. But $\chi_{\eta+1} \in t'_{\eta+1}$ by (7), since:

$$W \models \langle \delta \upharpoonright \eta + 1, t \upharpoonright \eta + 1 \rangle \text{ is an } e^* - \text{trace},$$

where $e^* = E^{N^*_\eta} = E^{\hat{E}_{\eta+1}}$. QED(9)

We now note that:

(10) $g(\alpha) = \tilde{\sigma}_\eta(\alpha)$ for $\alpha < \kappa_\eta$.

Proof.

$$g((\alpha) \in \tilde{\sigma}_\eta(\alpha) \iff \alpha \in E_{\nu_\eta}^{P_\eta}(\{\alpha\}).$$

Hence $g(\alpha) \in \tilde{\sigma}_\eta(\{\alpha\}) = \{\tilde{\sigma}_\eta(\alpha)\}$. QED(10)

But then for $j < \gamma$, we have:

$$\delta'_j = g \restriction \lambda_j = \tilde{\sigma}_\eta \restriction \lambda_j = \delta_j,$$

since $\lambda_j \leq \kappa_j$. Hence $\langle \delta \restriction \gamma, t \restriction \gamma \rangle = \langle \delta' \restriction \gamma, t \restriction \gamma \rangle$. Hence $\mathbb{E} \restriction \gamma$ is an e^*-enlargement of $I \restriction \gamma$ with $\text{trace}(\mathbb{E} \restriction \gamma) = \text{trace}(\mathbb{E}'' \restriction \gamma)$. Hence we can form:

$$\mathbb{E}' = \mathbb{E} \restriction \gamma \cup \mathbb{E}'' \restriction [\gamma, \eta + 2),$$

which has the desired properties. QED(Lemma 5.5.15)

The proof of Lemma 5.5.15 is actually more revealing than the statement, and we shall return to it later. One apparent weakness of Lemma 5.5.15 is that we need that proudness of $\mathbb{E}$ to prove it, but it does not follow that $\mathbb{E}'$ is proud. In fact, $\mathbb{E}'$ will be proud if and only if $\mathbb{E} \restriction \gamma + 1$ was proud, since $W'_{\eta+1} = W_\gamma$. Later we shall apply Lemma 5.5.15 only if γ survives at $\eta + 1$. If not, we shall apply the following lemma:

Lemma 5.5.16. *Let $\mathbb{E}$ be as in Lemma 5.5.15. Assume that γ does not survive at $\eta + 1$. Then $\mathbb{E}$ extends to a neat, proud enlargement $\mathbb{E}'$ of $I|\eta + 2$ such that $\mathbb{E}'$ is bounded in $V[G]$ and:*

(a) $\mathbb{E}' \restriction \eta + 1 = \mathbb{E}$

(b) $\text{ht}(W'_{\eta+1}) < \text{ht}(W_\eta)$.

Proof. Let $\mathbb{E}'''$ be as in the proof of Lemma 5.5.15. $\mathbb{E}'''$ was obtained by forcing over $W = W_\gamma$ with a $\bar{\mathbb{P}} = \text{Col}(\delta, \omega)$ where $\delta \in W$ was sufficiently large. But since W is collapsed in $V[G]$, there is a $\bar{G} \in V[G]$ which is $\bar{\mathbb{P}}$-generic over W. Hence $\mathbb{E}'''$ is bounded in $V[G]$, since $\mathbb{E}''' \in W[\bar{G}]$. We can form:

$$\mathbb{E}'' = \mathbb{E}''' \cup \{\langle \langle W, \mathbb{N}, \sigma' \rangle \eta + 1 \rangle\}.$$

$\mathbb{E}''$ is then a neat enlargement of $I \restriction \eta + 2$ which is bounded in $V[G]$. But γ does not survive at $\eta + 1$, where:

$$\sigma' \restriction \gamma + 1 = \sigma_\gamma, \gamma = T(\gamma + 1).$$

Hence $\sigma' R \sigma_\gamma$ in W. Hence:

$$\text{level}(\sigma') < \text{level}(\sigma_\gamma) \leq \text{rank}(W) \text{ in } W.$$

Let $j < \text{rank}(W)$ with $\text{level}(\sigma) \le j$. Then $\bar{W} =: W|\alpha_j$ is a proper segment of W. Hence $\bar{W} \in W$. Set:

$$\bar{\mathbb{E}}'' = \mathbb{E}''' \cup \{\langle\langle \bar{W}, \mathbb{N}, \sigma'\rangle, \eta+1\rangle\}.$$

Since $\mathbb{E}''$ was a neat enlargement of $I \restriction \eta + 2$, it follows easily that $\bar{E}''$ is a neat, proud enlargement of $I \restriction \eta + 2$. Moreover,

$$\text{trace}(\bar{\mathbb{E}}'' \restriction \eta + 1) = \text{trace}(\mathbb{E}''') = \langle \delta', t\rangle.$$

Since $\bar{\mathbb{E}}'' \in W[\bar{G}]$, we have shown:

$$W \models (\langle \delta', t\rangle \text{ is a pride inducing trace }).$$

This is expressed in W by:

$$C^{e^*}_{\tilde{\lambda},\infty} \models \psi[\delta', t, I, n^*],$$

where ψ is a certain Σ_1 formula. Since $\delta'_j = g \restriction \lambda_j$ for $j \le \eta$, this can be rewritten as:

$$C^{e^*}_{\tilde{\lambda},\infty} \models \psi'[g\text{"}\lambda_\eta, t, I, n^*],$$

where t, I, n^* are hereditarily countable. We now recall the Iteration Fact, which says that there is ν such that $P^*_\eta || \nu \neq \emptyset$ and,

$$\text{crit}(E^{P^*}_\nu) = \text{crit}(E^{P_\eta}_{\nu_\eta} = \kappa_\eta.$$

Hence $\text{crit}(E^{N^*_\eta}_{\sigma^*_\eta(\nu)} = \kappa^*$. Since κ^* is a cardinal in $N^*_\eta = \hat{N}''_{\eta+1}$ and N^*_η is a robust premouse, we conclude:

$$C^{e^*}_{\tilde{\lambda},\kappa^*} \prec_{\Sigma_1} C^{e^*}_{\tilde{\lambda},\infty} \text{ in } W.$$

Hence $C^{e^*}_{\tilde{\lambda},\kappa^*} \models \psi'[g\text{"}\lambda_\eta, t, I, n^*]$. But we know that $C^{e^*}_{\tilde{\lambda},\kappa^*} = C^{e}_{\tilde{\lambda},\tilde{\kappa}}$, where $e = E - \tilde{N}_\eta$. Hence in W_η we have by (B):

$$C^{e}_{\tilde{\lambda},\infty} \models \psi'[\tilde{\sigma}_\eta\text{"}\lambda_\eta, t, I, n^*],$$

which transforms into:

$$C^{e}_{\tilde{\lambda},\infty} \models \psi[\delta, t, I, n^*],$$

since $\delta_j = \tilde{\sigma}_\eta \restriction \lambda_j$ for $j \le \eta$. But this means that:

$$W_\eta \models \langle \delta, t\rangle \text{ is a pride inducing trace.}$$

Hence, if we force over W_η with a sufficient $\mathbb{P}^* = \text{Col}(\beta, \omega)$, there is $\mathbb{E}^* \in W[G^*]$ such that $\mathbb{E}^*$ is a neat, proud enlargement of $I \restriction \eta + 2$ and $\mathbb{E}^* \restriction \eta + 1$ is an e-enlargement of $I \restriction \eta + 1$. Since W_η is collapsed to ω in $V[G]$, there is a $G^* \in V[G]$ which is $\mathbb{P}^*$-generic over W_η. Hence $\mathbb{E}^* \in V[G]$. But $\mathbb{E} \in V[G]$, trace($\mathbb{E}^* \upharpoonright \eta + 1$) = trace($\mathbb{E}$) and $\mathbb{E}$ is an e-enlargement of $I \upharpoonright \eta + 1$. Hence we can set: $\mathbb{E}' = \mathbb{E} \cup \mathbb{E}^* \upharpoonright [\eta + 1, \eta + 2)$, which has the desired properties. QED(Lemma 5.5.16)

We now return to the proof of Lemma 5.5.15 in order to glean more information from it. In the following, let $V[G]$ be a set generic extension of V. We say that an enlargement $\mathbb{E}$ is *bounded in* $V[G]$ if and only if $\mathbb{E} \subset V[G]$ is α-bounded for an α which is collapsed to ω in $V[G]$. Similarly, we say that a world W is bounded in $V[G]$ if $W \in V[G]$ and ht(W) is collapsed to ω in $V[G]$.

Definition 5.5.20. $\langle W, \mathbb{N} \rangle$ is a *good pair* if and only if the following hold:

- W is a good world bounded in $V[G]$
- $\mathbb{N} = \langle N_i \mid i \leq h \rangle \in W$ such that

$$W \models \mathbb{N} \text{ is a putative Steel array.}$$

Define $R \in W$ from $\mathbb{N}, I, n^*$ in the usual way:

$$\begin{aligned} &\sigma' R \sigma \text{ if and only if for some } j, \sigma' \text{realizes } P_j \text{ in } \mathbb{N} \text{ and} \\ &\sigma \text{ realizes an } iTj \text{ in },\ \text{where } \sigma' \upharpoonright i + 1 = \sigma \\ &\ \text{and } i \text{ does not survive at } j. \end{aligned}$$

Then:

- R is well founded.

But then the level function level $\in W$ is defined in W by:

$$\text{level}(\sigma) = \text{lub}\{\text{level}(\sigma') \mid \sigma' R \sigma\}.$$

Definition 5.5.21. Let $\langle W, \mathbb{N} \rangle$ be a good pair. $\langle \sigma, \delta, t \rangle \in W$ is a *good triple* for $\langle W, \mathbb{N} \rangle$ at γ, lh(I) if and only if

(a) σ realizes P_γ in $\mathbb{N}$ and level(σ) $\leq$ rank(W)

(b) Let $\sigma_\gamma = \langle \hat{\sigma}, \mu \rangle$. Set $\hat{N} = N_\mu$. (Hence $\hat{\sigma} \colon P_\gamma \longrightarrow \hat{N}$.)

Let $e = E^{\hat{N}}$. Then:

$$W \models \langle \delta, t \rangle \text{ is a neat } e - \text{trace for } I \upharpoonright \gamma.$$

Lemma 5.5.17. *Let $\langle \sigma, \delta, t \rangle$ be a good triple for $\langle W, \mathbb{N} \rangle$ at γ. Let $\mathbb{E}$ be an e-enlargement of $I \upharpoonright \gamma$ which is bounded in $V[G]$ and $\langle \delta, t \rangle =$* trace($\mathbb{E}$). *Set:*

$$\mathbb{E}' = \mathbb{E} \cup \{\langle \langle W, \mathbb{N}, \sigma \rangle, \gamma \rangle\}.$$

Then $\mathbb{E}'$ is a neat enlargement of $I \upharpoonright \gamma + 1$ (and in obviously bounded in $V[G]$).

Proof. This is just like the proof of (8) in Lemma 5.5.15. The verification of (A)–(F) is straightforward. (G) is immediate, since R is well founded. (H) holds by (a). Since $\mathbb{E}$ is an enlargement with trace $\langle\delta, t\rangle$ and $\mathbb{E}'$ satisfies (A)–(H), it follows by Lemma 5.5.1 that $\mathbb{E}'$ is an enlargement of $I\restriction\gamma+1$. $\mathbb{E} = \mathbb{E}'\restriction\gamma$ is neat, since $\langle\delta, t\rangle$ is neat. But then $\mathbb{E}'$ is neat by (b). QED(Lemma 5.5.17)

Lemma 5.5.18. *There is such an* $\mathbb{E}$.

Proof. By (b), if $\bar{\mathbb{P}} = \mathrm{Col}(\beta, \omega)$ for a sufficiently large $\beta \in W$ and $\bar{G}$ is $\bar{\mathbb{P}}$-generic over W, then in $W[\bar{G}]$ there is an $\mathbb{E}$ with the above properties. But there is a $\bar{\mathbb{P}}$-generic $\bar{G}$ in $V[G]$, since $\mathrm{ht}(W)$ is collapsed to ω in $V[G]$. Hence $\mathbb{E} \in V[G]$ is bounded in $V[G]$. QED(Lemma 5.5.18)

Conversely, we have:

Lemma 5.5.19. *Let* $\mathbb{E}'$ *be a neat enlargement of* $I\restriction\gamma+1$ *which is bounded in* $V[G]$. *Let* $\mathbb{E}' = \langle W, \mathbb{N}, \sigma\rangle$. *Let* $\langle\delta, t\rangle = \mathrm{trace}(\mathbb{E}'\restriction\gamma)$. *Then* $\langle W, \mathbb{N}\rangle$ *is good and* $\langle\sigma, \delta, t\rangle$ *is a good triple at* γ.

Now let $\langle W, \mathbb{N}\rangle$ be good and let $\langle\sigma, \delta, t\rangle \in W$ be a good triple for $\langle W, \mathbb{N}\rangle$ at γ. Let $\gamma = T(\eta+1)$. To make things simple, we also suppose that $\eta+1$ is not a deop point of I (i.e. $P^*_\eta = P_\gamma$). $\langle, \sigma', \delta', t\rangle \in W$ is a *good continuation* of $\langle\sigma, \delta, t\rangle$ at $\eta+1$ if and only if the following hold:

(a) $\langle\sigma', \delta', t'\rangle \in W$ is a good triple for $\langle W, \mathbb{N}\rangle$ at $\eta+1$.

(b) $\delta'\restriction\gamma = \delta$, $t'\restriction\gamma = t$.

(c) $\sigma'\restriction\gamma+1 = \sigma$.

It follows that if $\sigma = \langle\hat{\sigma}, \mu\rangle$, $\hat{N} = N_\mu$, $\sigma' = \langle\hat{\sigma}', \mu'\rangle$, $\hat{N}' = N_{\mu'}$, then $\hat{\sigma}'\pi_{\gamma,\eta+1} = \hat{\sigma}$, $\mu = \mu'$, $\hat{N} = \hat{N}'$. Moreover:

$$\hat{\sigma}'(\lambda_\eta)0\hat{\sigma}\pi_{\gamma,\eta+1}(\kappa_\eta) = \hat{\sigma}(\kappa_\eta).$$

Hence $\delta'_\eta{}''\lambda_\eta = \hat{\sigma}'{}''\lambda_\eta \subset \hat{\sigma}(\kappa_\eta)$.

Lemma 5.5.20. *Let* $\langle W, \mathbb{N}\rangle$ *be good. Let* $\langle\sigma, \delta, t\rangle$ *be good at* γ. *Let* $\gamma = T(\eta+1)$ *where* $\eta+1$ *is not a drop point in* I. *Let* $\langle\sigma', \delta', t\rangle$ *be a continuation of* $\langle\sigma, \delta, t\rangle$ *at* $\eta+1$. *Let* $e = E^{\hat{N}} = E^{\hat{N}'}$. *There is* $\alpha < \beta = \beta^W$ *such that*

$$W \models \langle\delta', t'\rangle \text{ is a } \alpha\text{-bounded } e\text{-trace for } I\restriction\eta+1.$$

Proof. We know:

$$W \models \langle\delta', t'\rangle \text{ is a neat } e\text{-trace for } I\restriction\eta+1.$$

But this is expressed in W by:

$$C^e_{\bar{\lambda},\infty} \models \psi[\delta', t', I, n^*]$$

where ψ is a Σ_1 formula and:

$$\bar{\lambda} = \mathrm{lub}\, \delta'_\eta{}''\lambda_\eta < \hat{\sigma}(\kappa_\eta).$$

Since $\delta'_i = \delta'_\eta \restriction \lambda_i$ for $i < \eta$ we can rewrite this as:

$$C^e_{\bar{\lambda},\infty} \models \psi'[\delta'_\eta{}''\lambda_\eta, t', I, n^*],$$

where t', I, n^* are hereditarily countable. But $\hat{N}$ is a robust premouse in N and $\hat{\sigma}(\kappa_\eta) = \hat{\sigma}'(\kappa_\eta)$ is an inaccessible cardinal in $\hat{N}$. Hence $\mathrm{cf}(\hat{\sigma}(\kappa_\eta)) \geq \omega_1$ in W. Hence $\bar{\lambda} < \hat{\sigma}(\kappa_\eta)$. By the robustness of $\hat{N}$, we conclude that in W:

$$C^e_{\bar{\lambda},\hat{\sigma}(\kappa_\eta)} \prec_{\Sigma_1} C^e_{\bar{\lambda},\infty}.$$

Hence:

$$C^e_{\lambda,\tilde{\sigma}(\kappa_\eta)} \models \psi[\sigma'_\eta{}''\lambda_\eta, t', I, n^*]$$

which translates easily into:

$$C^e_{\bar{\lambda},\hat{\sigma}(\kappa_\eta)} \models \psi[\delta', t', I, n^*].$$

But this says there are α, ν such that $\bar{\lambda} < \alpha < \nu < \hat{\sigma}(\kappa_\eta)$ such that $C^e_{\bar{\lambda},\nu}$ is admissible and the language $\mathbb{L} = \mathbb{L}_{\alpha,\delta,t}$ on $C^e_{\bar{\lambda},\nu}$ is consistent. Hence:

$$W \models \langle \delta', t' \rangle \text{ is an } \alpha\text{-bounded } e\text{-trace},$$

where $\alpha < \beta = \beta^W$. QED(Lemma 5.5.20)

But the proof of Lemma 5.5.15 then gives us:

Lemma 5.5.21. *Let $\mathbb{E}$ be a neat, proud enlargement of $I \restriction \eta + 1$. Let $\gamma = T(\eta + 1)$, where $\eta + 1$ is not a drop point. Let:*

$$\langle W, \mathbb{N} \rangle = \langle W_\gamma, \mathbb{N}_\gamma \rangle, \sigma = \sigma_\gamma \text{ and } \langle \delta, t \rangle = \mathrm{trace}(\mathbb{E} \restriction \gamma).$$

Hence $\langle W, \mathbb{N} \rangle$ is good and $\langle \sigma, \delta, t \rangle \in W$ is a good triple for $\langle W, \mathbb{N} \rangle$ at γ. Then there exists a $\langle \sigma', \delta', t' \rangle \in W$ which is a good continuation of $\langle \sigma, \delta, t \rangle$.

In the proof of Lemma 5.5.21 we constructed a very specific good continuation which had strong properties (as witnessed by the proof of Lemma 5.5.16). However, there can be other continuations of $\langle \sigma, \delta, t \rangle$ in W, and we are free to choose which one we shall employ.

Without further ado, we turn to the proof of Lemma 5.3.7. In V we are given a putative Steel array $\mathbb{N} = \langle N_i \mid i \leq \mu\rangle$. We are also given a map $\sigma\colon P \longrightarrow_{\Sigma^*} N_\mu$, where P is a countable and restrained premouse. We want to show that P is countably iterable. To this end, we consider a countable normal iteration $I = \langle\langle M_i\rangle, \langle\nu_i\rangle, \langle\pi_{i,j}\rangle, T\rangle$ of I. We must prove (*), (**). We define the relation R and assume that R is well founded. We shall construct a sequence $\langle\mathbb{E}^{(i)} \mid i < \mathrm{lh}(I)\rangle$ such that

$$\mathbb{E}^{(i)} = \langle\langle W_j^{(i)}, \mathbb{N}_j^{(i)}, \sigma_j^{(i)}\rangle \mid j \leq i\rangle$$

is a neat proud enlargement of $I \restriction i+1$. Our first enlargement $\mathbb{E}^{(0)}$ is found in V: Let β be such that

$$\beta = \mathrm{card}(V_\beta), \mathbb{N} \in V_\beta, \mathrm{cf}(\beta) > \omega_1.$$

In V we can then find an $A \subset \beta^+$ such that

- $L_\beta[A] = V_\beta$
- $L_{\beta^+}[A] \models \beta$ is the largest cardinal.

Set $W = L^A_{\beta^+}$. Then W is a world of rank β^+, where $\beta = \beta^W$. Moreover, $\mathbb{N} \in W|\beta$.

We set:

$$\mathbb{E}^{(0)} = \{\langle\langle W, \mathbb{N}, \sigma'\rangle, 0\rangle\} \text{ where } \sigma' = \{\langle\sigma, \mu\rangle\}.$$

Now let G be set generic over V such that β^+ is collapsed to ω in $V[G]$. The rest of the construction takes place in $V[G]$ and each $\mathbb{E}^{(i)}$ will be bounded in $V[G]$.

We verify inductively that $\mathbb{E}^{(i)}$ is bounded in $V[G]$ and:

(a) $\mathbb{E}^{(i)}$ is neat and proud, where $\mathbb{E}_j^{(i)} = \langle W_j^{(i)}, \mathbb{N}_j^{(i)}, \sigma_j^{(i)}\rangle$ for $j \leq i$.

(b) If $h < i$ and $n_h < n_j$ for all $j \in (h, i]$, then:

$$\mathbb{E}^{(i)} \restriction h+1 = \mathbb{E}^{(h)} \restriction h+1 \text{ and } \mathrm{ht}(W_i^{(i)}) < \mathrm{ht}(W_h^{(h)}).$$

(c) If h survives at i, then $\mathbb{E}^{(h)} \restriction h = \mathbb{E}^{(i)} \restriction h$ and

$$W_h^{(h)} = W_h^{(i)}, \mathbb{N}_h^{(h)} = \mathbb{N}_h^{(i)}, \sigma^{(i)} \restriction h+1 = \sigma^{(h)}.$$

We define $\mathbb{E}^{(i)}$ by cases as follows:

Case 1 $i = 0$. $\mathbb{E}^{(0)} = \{\langle\langle W, \mathbb{N}, \sigma^{(0)}\rangle, 0\rangle\}$ as above. (a)–(c) then hold trivially.
Case 2 $i = \eta + 1$. Let $\gamma = T(\eta + 1)$. We split into two subcases:

Case 2.1 γ does not survive at $\eta + 1$.

By Lemma 5.5.16, there is a proud, neat enlargement $\mathbb{E}'$ of $I \restriction \eta + 2$ such that $\mathbb{E}' \restriction \eta + 1 = \mathbb{E}^{\eta}$, and:

$$\mathrm{ht}(W'_{\eta+1}) < \mathrm{ht}(W^{(\eta)}_{\eta}), \mathbb{E}^{\eta} \in V[G].$$

Hence $\mathbb{E}'$ is bounded in $V[G]$. We set: $\mathbb{E}^{\eta+1} = \mathbb{E}'$ and verify (a)–(c). (a) is immediate.

(b) If $h < i$ and $n_h < n_j$ for all $j \in (h, i)$, then $\mathbb{E}^{(i)} \restriction h + 1 = \mathbb{E}^{(h)}$ and $\mathrm{ht}(N(h)_h) > \mathrm{ht}(W^{(i)}_j)$ for $j \in (h, i)$. But $\mathbb{E}^{(i)} \restriction h + 1 = \mathbb{E}^{(i-1)} \restriction h + 1 = \mathbb{E}^{(h)}$ and $\mathrm{ht}(W^{(h)}_h) > \mathrm{ht}(W^{(i-1)}_{i-1}) > \mathrm{ht}(W^{(i)}_i)$.

(c) is vacuously true.

Case 2.2 γ survives at $\eta + 1$.

By Lemma 5.5.15, there is an $\mathbb{E}' \in V[G]$ such that $\mathbb{E}'$ is a neat enlargement of $I \restriction \eta + 2$ and:

$$\mathbb{E}' \restriction \gamma = \mathbb{E}^{\eta} \restriction \gamma, \mathrm{ht}(W'_{j+1}) < \mathrm{ht}(W^{(\gamma)}_{\gamma}) \text{ for } \gamma \leq j \leq \eta$$

and:

$$W'_{\eta+1} = W^{(\gamma)}_{\gamma}, \mathbb{N}'_{\eta+1} = \mathbb{N}^{(\gamma)}_{\gamma}, \sigma'_{\eta+1} \restriction \gamma + 1 = \sigma^{(\gamma)}_{\gamma}$$

We shall let $\mathbb{E}^{\eta+1}$ be such an $\mathbb{E}'$. Then it is clear that $\mathbb{E}^{\eta+1}$ is bounded in $V[G]$. We verify (a)–(c)

(a) $\mathbb{E}^{\eta+1}$ is neat. But $W^{(\eta+1)}_{\eta+1} = W^{(\gamma}_{\gamma}$. Hence $\mathbb{E}^{\eta+1}$ is proud.

(b) Let $h < \eta + 1$ such that $n_h < n_j$ for $j \in (h, \eta]$. Then $\mathbb{E}^{(j)} \restriction h + 1 = \mathbb{E}^{(h)} \restriction h + 1$ for $j \in (h, \eta]$. But then we have $h < \gamma$ and:

- $\mathbb{E}^{(\eta+1)} \restriction h + 1 = \mathbb{E}^{(h)} \restriction h + 1$
- $\mathrm{ht}(W^{(\eta+1)}_{\eta+1}) = \mathrm{ht}(W^{(\gamma)}_{\gamma}) < \mathrm{ht}(W^{(h)}_h)$.

(c) holds trivially at $\eta + 1$, since it holds at γ and no $h \in (\gamma, \eta)$ survives at $\eta + 1$.

However, we must specify $\mathbb{E}^{(\eta+1)}$ more carefully than we just did, if we are not to run into trouble at limit points of the induction. We therefore consider the subcase:

Case 2.1.1 $\gamma = T(\eta+1)$, γ survives at $\eta+1$, and $\eta+1$ is not a drop point (i.e. $P^*_\eta = P_\gamma$). We apply Lemma 5.5.20 and Lemma 5.5.21. $W = W^{(\gamma)}_\gamma$ has a definable well ordering. Let $\langle\sigma', \delta', t'\rangle$ be the W-least triple which is a good continuation of $\langle\sigma^{(\gamma)}, \delta, t\rangle$ at $\eta+1$, where $\langle\delta, t\rangle = \text{trace}(\mathbb{E}^{(\gamma)} \restriction \gamma)$. Such a $\langle\sigma', \delta', t'\rangle$ exists by Lemma 5.5.21. By Lemma 5.5.20, if we force over W with $\bar{\mathbb{P}} = \text{Col}(\beta, \omega)$ $(\beta = \beta^W)$, getting a $\bar{\mathbb{P}}$-generic $\bar{G}$, then in $W[\bar{G}]$ there is an $\mathbb{E}'$ which is an e-enlargement of $\langle\delta', t'\rangle$. But there is such a $\bar{G} \in V[G]$, since W is bounded in $V[G]$. Hence $\mathbb{E}' \in V[G]$. Set $\mathbb{E}^{(\eta+1)} = \mathbb{E}' \cup \{\langle\langle W, \mathbb{N}, \sigma'\rangle, \eta+1\rangle\}$. By Lemma 5.5.17, $\mathbb{E}^{(\eta+1)}$ is then a neat, proud enlargement of $I(\eta+2)$. QED(Case 2)

Case 3 $i = \eta$ is a limit ordinal.

Let $n = n(\eta)$. For each $m < n$ there is $i_m < \eta$ such that $n(j) \neq m$ for all $i_m \leq j < \eta$, since otherwise there would be unboundedly many $j < \eta$ such that $n(j) = m$. But $n(j) = m$ means that $j \leq_T h$, where $n^*(h) = m$. Hence, by closure, η lies on the branch $\{j \mid j \leq_T n\}$. Hence $n(\eta) \leq m < n$. Contradiction! Hence there is $\gamma < \eta$ such that $n(i) \geq n$ for all $i \in [\gamma, \eta]$. We can assume without loss of generality that $\gamma <_T \eta$ and that $[\gamma, \eta)_T$ does not contain a drop point. Set $b = [\gamma, \eta)_T$. Set $W = W^{(\gamma)}$, $\mathbb{N} = \mathbb{N}^{(\gamma)}$. If $j, k \in b$ and $j \leq k$, then:

$$\mathbb{E}^{(j)} \restriction j = \mathbb{E}^{(h)} \restriction j \text{ and } \mathbb{E}^{(j)}_j = \langle W, \mathbb{N}, \sigma^{(j)}\rangle$$

where: $\sigma^{(k)} \restriction j+1 = \sigma^{(j)}$. By definition, we have:

$$\hat{\sigma}^{(j)} = (\sigma^{(j)})^j_j \colon P_j \longrightarrow \hat{N}^{(j)}$$

where $\hat{N}^{(j)}$ is an element of $\mathbb{N}$. Since there are no drops in b, we have:

$$\hat{\sigma}^{(k)}\pi_{j,k} = \hat{\sigma}^{(j)} \text{ and } \hat{N}^{(k)} = \hat{N}^{(j)} = \hat{N}.$$

Set: $\tilde{\mathbb{E}}^j = \mathbb{E}^{(j)} \restriction j$ for $j \in b$. Set: $\tilde{\mathbb{E}} = \bigcup_{j\in b} \tilde{\mathbb{E}}^j$. It follows easily that $\tilde{\mathbb{E}}$ satisfies (A)–(H) in the definition of "enlargement". We must prove (I). Clearly, $\text{trace}(\tilde{\mathbb{E}}) = \langle\delta, t\rangle$ where:

$$\langle\delta \restriction i, t \restriction i\rangle = \text{trace}(\tilde{\mathbb{E}}^i) \text{ for } i \in b.$$

If we set: $s_i = \langle\sigma^{(i)}, \delta \restriction i, t \restriction i\rangle$ for $i \in b$, then $\langle s_i \mid i \in b\rangle$ can be recursively defined in W as follows:

- s_i is given

- If $i+1$ immediately succeeds h in b (hence $i+1 = T(h)$). Then s_{i+1} is the W-least good continuation of s_h at $i+1$.

- If $\mu \in b$ is a limit point, then:

$$\delta \upharpoonright \mu = \bigcup_{i \in b \cap \mu} \delta \upharpoonright i, t \upharpoonright \mu = \bigcup_{i \in b \cap \mu} t \upharpoonright i,$$

  and:

$$\sigma^{(\mu)} \upharpoonright \mu = \bigcup_{i \in h \cap \mu} \sigma^{(i)}.$$

  $\hat{\sigma}^{(\mu)} = (\sigma^{(\mu)})^{\mu}_{\mu}$ is then defined by:

$$\hat{\sigma}^{(\mu)} \pi_{i,\mu}(x) = \hat{\sigma}^{(i)}(x) \text{ for } i \in b \cap \mu.$$

Then $\langle s_i \mid i < \eta \rangle \in W$. Hence $\langle \delta, t \rangle \in W$. Hence $\tilde{E}$ satisfies (I) in W. Hence $\tilde{\mathbb{E}}$ is an enlargement of $I \upharpoonright \eta$. But

$$J^{E^{\tilde{N}_i}}_{\tilde{\lambda}_i} = J^{e}_{\tilde{\lambda}} \text{ for } i \in b,$$

where $e = E^{\hat{N}}$, since $\hat{N} = \hat{N}^i$ for all $i \in b$. Then $\tilde{E}$ is an e-enlargement of $I \upharpoonright \eta$. Clearly $\tilde{E}$ is neat, since every $\tilde{E}^i$ is neat. We now define in W a realization $\sigma^{(\eta)}$ of P_η by:

$$\sigma(\eta) \upharpoonright \eta = \bigcup_{i \in b} \sigma^{(i)} \text{ and } \hat{\sigma}^{(\eta)} \pi_{i,\eta}(x) = \sigma^{(i)}(x)$$

for $i \in b$. Set:

$$\mathbb{E}^{(\eta)} = \tilde{\mathbb{E}} \cup^{\{} \langle \langle W, \mathbb{N}, \sigma(\eta) \rangle, \eta \rangle \}.$$

We claim that $\mathbb{E}^{(\eta)}$ is an enlargement of $I \upharpoonright \eta + 1$. (A)–(F) is the definition of "enlargement" follows easily from the fact that each $\mathbb{E}^{(i)}$ is an enlargement of $I \upharpoonright i+1$ and $\mathbb{E}^{(i)}_i = \langle W_i, \mathbb{N}_i, \sigma^{(i)} \rangle$. (G) is clear, since we know that R is well founded. The level function for W is then defined by:

$$\text{level}(\sigma) = \text{lub}\{\text{level}(\sigma') \mid \sigma' R \sigma\}.$$

It is easily seen that if $h <_T \eta$ and it does not survive at η, then $h <_T \gamma$ and h does not survive at γ. Hence:

$$\text{level}(\sigma^{(\eta)}) = \text{level}(si^{(\gamma)} \leq \text{rank}(W),$$

and (H) holds. By Lemma 5.5.1 it follows that $\mathbb{E}^{(\eta)}$ is an enlargement of $I \upharpoonright \eta + 1$. $\mathbb{E}^{(\eta)}$ is proud, since $W = W^{(\eta)}_\eta$ is proud. However, we must still show that $\mathbb{E}^{(\eta)}$ is neat. The trace $\langle \delta, t \rangle$ of $\tilde{\mathbb{E}}$ is neat, since the syntactical condition $\chi_i \in t_i$ is satisfied. We must show that $\chi_\eta \in t_\eta$ or in other words:

$$W \models \langle \delta, t \rangle \text{ is an } e\text{-trace}.$$

This says that if we force over W with a sufficient $\bar{\mathbb{P}} = \mathrm{Col}(\delta, \omega)$ and $\bar{G}$ is $\bar{\mathbb{P}}$-generic over W, then there is an $\tilde{\mathbb{E}} \in W[\bar{G}]$ which is an e-enlargement if $I \upharpoonright \eta + 1$, where $e = E^{\hat{N}}$. Let $\bar{\mathbb{P}} = \mathrm{Col}(\beta, \omega)$ where $\beta = \beta^W$. Let $\bar{G}$ be $\bar{\mathbb{P}}$-generic over W. Then $W[\bar{G}]$ is a ZFC^- model, although all sets in $W[\bar{G}]$ are countable. Since $W = J^A_\alpha$ has a definable well ordering, $W[\bar{G}]$ has a well ordering definable in the parameter $\bar{G}$. For $i \in b$ set:

Definition 5.5.22. $\alpha(i)$ =:the least α such that

$$W \models \langle \delta \upharpoonright i, t \upharpoonright i \rangle \text{ is an } \alpha\text{-bounded } e\text{-trace.}$$

Then $\alpha(i) \le \alpha(j)$ for $i < j$ in b, since if $\tilde{\mathbb{E}}$ is an e-enlargement of $I \upharpoonright j$ with trace $\langle \delta \upharpoonright j, t \upharpoonright j \rangle$, then $\tilde{\mathbb{E}} \upharpoonright i$ is an e-enlargement of $I \upharpoonright i$ with trace $\langle \delta \upharpoonright i, t \upharpoonright i \rangle$. But $\tilde{E} \upharpoonright i$ is bounded by $\alpha(j)$. hence $\alpha(i) \le \alpha(j)$. But then $\alpha(i) < \beta$ for all $i \in b$, since $\alpha(i+1) < \beta$ for $i+1 \in \beta$ by Lemma 5.5.20. We now successively define $\tilde{\mathbb{E}}^i$ $(i \in b)$ such that

- $\tilde{\mathbb{E}}^i \in W[\bar{G}]$ is an $\alpha(i)$-bounded e-enlargement of $I \upharpoonright i$ with trace $\langle \delta \upharpoonright i, t \upharpoonright i \rangle$ for $i \in b$,
- $\tilde{\mathbb{E}}^j \upharpoonright i = \mathbb{E}^i$ for $i < j$ in b.

We let $\tilde{\mathbb{E}}_\gamma$ = the $W[\bar{G}]$-least e-enlargement of $I \upharpoonright \gamma$ which is $\alpha(\gamma)$-bounded and has trace $\langle \delta \upharpoonright \gamma, t \upharpoonright \gamma \rangle$. If $\tilde{\mathbb{E}}^h$ is given and $i+1$ is the immediate successor of h in b (hence $h = T(i+1)$), we first let $\mathbb{E}'$ be the $W[\bar{G}]$-least e-enlargement of $I \upharpoonright i+1$ which is $\alpha(i+1)$-bounded with trace $\langle \delta \upharpoonright i+1, t \upharpoonright i+1 \rangle$. We then set:

$$\tilde{\mathbb{E}}^{i+1} \ \tilde{E}^h \cup \mathbb{E}' \upharpoonright [h, i+1).$$

It follows as before that $\tilde{\mathbb{E}}^{i+1}$ is an e-trace of $I \upharpoonright i+1$-bounded. Now let μ be a limit point of b (hence $\mu \le \eta = \mathrm{lub}\, b$). Set: $\tilde{\mathbb{E}}^\mu = \bigcup_{i \in \mu \cap b} \tilde{\mathbb{E}}^i$. It follows as before that $\tilde{\mathbb{E}}^\mu$ satisfies (A)–(H) in the definition of enlargement. But $\langle \delta \upharpoonright \mu, t \upharpoonright \mu \rangle = \mathrm{trace}(\tilde{\mathbb{E}}^\mu)$ where $\langle \delta \upharpoonright \mu, t \upharpoonright \mu \rangle \in W$. Hence $\tilde{\mathbb{E}}^\mu$ is an enlargement of $I \upharpoonright \mu$. But since $\tilde{\mathbb{E}}^i$ is an e-enlargement for $i < \mu$, it follows that $\tilde{\mathbb{E}}^\mu$ ids an e-enlargement. Clearly:

$$\tilde{\mathbb{E}}^\mu \text{ is } \alpha = \sup\{\alpha(i) \mid i < \mu\}\text{-bounded.}$$

Hence $\alpha(\mu) = \sup\{\alpha(i) \mid i < \mu\}$. This gives m, in particular, $\tilde{\mathbb{E}} = \tilde{\mathbb{E}}^\eta \in W[\bar{G}]$. This proves that $\langle \delta, t \rangle$ is an e-trace in W. Hence $\mathbb{E}^{(\eta)}$ is neat and proud.

This completes the constrcution of $\mathbb{E}^{(i)}$ $(i \le \eta)$ in $V[G]$. We now turn to the proof of (*) and (**). We first prove (*). In this case, $\mathrm{lh}(I) = \eta + 1$. Hence it is certainly right that R is well founded. Let $\nu_\eta \in P_\eta$ be arbitrarily chosen such that $E_{\nu_\eta} \neq \emptyset$ in P_η and $\nu_\eta > \nu_i$ for $i < \eta$. This determines $\gamma = T(\eta+1)$ and with it P^*_η. We claim that the transitive ultrapower:

$$\pi \colon P^*_\eta \longrightarrow^n_F P'(F = E^{P_\eta}_{\nu_\eta})$$

exists where $n \leq \omega$ is maximal such that $\kappa_\eta < \rho^n_{P^*_\eta}$. But by §3.2 this is equivalent to saying that there is no sequence $\langle\langle \alpha_i, f_i\rangle \mid i < \omega\rangle$ such that

$$\prec \alpha_{i+1}, \alpha_i \succ \in E^{P_i}_{\nu_i}(X_i) \text{ where } X_i = \{\prec \xi, \zeta \succ \mid f_{i+1}(\xi) \in f_i(\zeta)\}\}$$

for $i < \omega$. Suppose not. Let k be the resurrection map for $\langle \hat{N}^{(\eta)}, \hat{\sigma}^{(\eta)}(\nu_\eta)\rangle$. Hence:

$$k \colon (\hat{N}^{(\eta)} || \hat{\sigma}^{(\eta)}(\nu_\eta) \longrightarrow \tilde{N}^{(\eta)}$$

where $\tilde{N}^{(\eta)} = \langle J^{\tilde{E}}_{\tilde{\nu}}, F\rangle$ and F is robust on $\tilde{N}^{(\eta)}$. But then there is $g \colon \lambda_\eta \longrightarrow \hat{\sigma}^{(\eta)}(\kappa_\eta)$ such that whenever $\alpha_1, \ldots, \alpha_m < \lambda_\eta$ and $X \in \mathbb{P}(\kappa_\eta) \cap P_\eta$, then:

$$\prec g(\vec{\alpha}) \succ \in \hat{\sigma}^{(\eta)}(X) \iff \prec \vec{\alpha} \succ \in E^{P_\eta}_{\nu_\eta}(X).$$

Hence:

$$\prec g(\alpha_{i+1}), g(\alpha_i) \succ \in \hat{\sigma}^{(\eta)}(X_i) \text{ for } i < \omega.$$

Hence:

$$f_{i+1}(g(\alpha_{i+1})) \in f_i(g(\alpha_i)) \text{ for } i < \omega.$$

Contradiction! QED(*)

We now prove (**). We have: $\mathrm{lh}(I)$ is a limit cardinal. We assume that R is well founded. Recall that n^* injects $\mathrm{lh}(I)$ into ω. Define a sequence j_n $(n \in \omega)$ by:

$$n^*(j_0) \text{ is minimal in } n^{*\text{''}}\, \mathrm{lh}(I)$$

$$n^*(j_{n+1}) \text{ is minimal in } \{n^*(h) \mid h > j_n\}.$$

Then $n(h) > n(j_n)$ for all $h > j_n$. Hence:

$$\mathrm{ht}(W^{(h)}_h) < \mathrm{ht}(W^{j_n}_{j_n}) \text{ for } h > j_n.$$

Hence:

$$\mathrm{ht}(W^{(j_{m+1})}_{j_{m+1}}) < \mathrm{ht}(W^{(j_m)}_{j_m} \text{ for } m \in \omega.$$

This is a contradiction. Hence we were mistaken in assuming that R is well founded. Hence R is ill founded and I has a cofinal well founded branch.

QED(Lemma 5.3.7)

Note: In this proof we have strongly used the assumption that there is no inner model with a Woodin cardinal. It may be of interest to see what is left of the proof if we relax this assumption. We still require of a putative Steel array $\mathbb{N} = \langle N_i \mid i \leq \mu \rangle$ that N_i be mouselike for $i < \mu$. Hence N_μ is pre-mouselike. Assume that $\sigma : P \longrightarrow_{\Sigma^*} \mathbb{N}_\mu$ where P is a countable premouse which has the unique branch property for countable normal iterations (I.e. a countable iteration of limit length has at most one cofinal well founded branch.) This is a much weaker assumption than our previous one. Since P is pre-mouselike, we still know that the Iteration Fact holds. Thus our proof still shows that P is countably normally iterable. However, we have *not* shown that P is $\omega_1 + 1$ normally iterable, which is what we would need to conclude that P is fully iterable and that N_μ is mouselike.

5.6 The Bicephalus

By lemma 5.3.6 the construction of a robust Steel array can be continued up to ∞, using:

$(**)$ If possible, we apply Option 2 at $i + 1$, if not we apply Option 1.

At limit points η we fomr N_η as usual. This includes the point ∞. It is easily seen that if $\kappa < \infty$ is regular in V, then N_κ is of height κ, is a ZFC^- model, and $\kappa = \kappa_{\kappa,\eta}$ for all $\eta \geq \kappa$. (cf. lemma 5.2.5.) Hence: $N_\infty = \langle \bigcup_{\kappa \text{ is regular}} N_\kappa, \emptyset \rangle$. Note that we had a choice for N_i only at successor i, and we restricted this choice by $(**)$. The structure N_∞ is then a weasel, having the form $\langle J^E_\infty, \emptyset \rangle$ and is an inner model of ZFC^-. It is denoted by $K^c$and is a preliminary to the construction of the core model K. However, we have not yet shown that K^c is uniquely defined. What if, in applying Option 2 at $i+1$, we have an embarrassment of riches and have two *different* robust mice $\langle J^E_\nu, F \rangle$, $\langle J^E_\nu, F' \rangle$ such that $J^E_\nu = M_i$, which could be applied. In this section, we show that that eventuality cannot occur.

$\langle J^E_\nu, F, F' \rangle$ is an example of what we call a *bicephalus*. This is defined by:

Definition 5.6.1. A bicephalus is a structure $\langle J^E_\nu, F^0, F^1 \rangle$ such that $\langle J^E_\nu, F^n \rangle$ is an active premouse for $n = 0, 1$.

Definition 5.6.2. A precephalus is a structure which is either a bicephalus or a premouse.

In §3.8.4 we noted that if $M = \langle J^E_\nu, F^0, F^1 \rangle$ is a bicephalus and $\pi : M \longrightarrow_G M'$, then $M' = \langle J^{E'}_\nu, F'^0, F'^1 \rangle$ is a bicephalus. (Note that here we are taking the Σ_0 ultrapower.) We also saw that, if M_0 is a bicephalus and $\pi_{i,j} : M_i \longrightarrow M_j$ $(i \leq j \leq \eta)$ such that

- $\pi_{i,i+1} : M_i \longrightarrow_{G_i} M_{i+1}$
- M_i is transitive and the $\pi_{i,j}$ commute
- If $\lambda \leq \eta$ is a limit ordinal then:

$$M_\lambda, \langle \pi_{i\lambda} \mid i < \lambda \rangle$$

is the transitivized direct limit of:

$$\langle M_i \mid i < \lambda \rangle, \langle \pi_{ij} \mid i \leq j < \lambda \rangle$$

then each M_i is a bicephalus.

We then defined the notion of a *normal iteration* of a bicephalus P. This has the form:

$$I = \langle \langle P_i \rangle, \langle \nu_i \rangle, \langle F_i \rangle, \langle \pi_{ij} \rangle, T \rangle.$$

Where $|\langle J_\nu^E, F \rangle| = |\langle J_\nu^E, F, F' \rangle| =: J_\nu^E$. I is like a normal iteration except that:

- If $P_i = \langle |P_i|, F_i^0, F_i^1 \rangle$ is a bicephalus and $\nu_i = \text{ht}(P_i)$, then $F_i \in \{F_i^0, F_i^1\}$.
- If P_i is a premouse or $\nu_i \in P_i$, then $F_i = E_{\nu_i}^{P_i}$.

The choice of F_i determines $\kappa_i = \text{crit}(F_i)$ and with it:

- $T(i+1) =$ the least n such that $\kappa_i < \lambda_n \vee n = i$.

Let τ_i, λ_i be defined as usual, P_i^* is defined by:

- If τ_i is a cardinal in P_n where $n = T(i+1)$, then $P_i^* = P_n$.
- If τ_i is not a cardinal in P_n, then $P_i^* = P_n||\beta$, where $\beta \in P_n$ is maximal such that τ_i is a cardinal in $P_n||\beta$.

F_i is then applied to P_i^*. However:

- If $P_i^* = P_n$ and P_n is a bicephalus, then

$$\pi_{n,i+1} : P_n \longrightarrow_{F_i} P_{i+1}.$$

(This is the Σ_0-ultrapower.)

- If P_i^* is a premouse, then:

$$\pi_{n,i+1} : P_i^* \longrightarrow_{F_i}^{n} P_{i+1}$$

where $n \leq \omega$ is maximal such that $\kappa_i < \rho_{P_i^*}^n$.

By a *precephalus* we mean a premouse or bicephalus. It follows by induction on i that:

Lemma 5.6.1. *If I is a normal iteration of a bicephalus, then:*

$$P_i \textit{ is a bicephalus } \longleftrightarrow [0, i)_T \textit{ has no drop point.}$$

If I is an iteration of length $i + 1$, we can extend it to a *potential iteration* of length $i + 2$ by appointing appropriate ν_i, F_i with $\nu_i > \nu_l$ for $l < i$. This determines $T(i+1), P_i^*$. (However, we do not know whether P_i^* is extendable by F_i.) Then:

Lemma 5.6.2. *Extend I of length $i + 1$ to a potential iteration of length $i + 2$ by appointing appropriate ν_i, F_i. Then F_i is close to P_i^*.*

The proof is virtually the same as that of Theorem 3.4.4, and we take it here as given. Applying this to $i + 1 < \mathrm{lh}(I)$, it follows that if P_i^* is a premouse, then $\pi_{n,i+1} : P_i^* \longrightarrow_{F_i}^{*} P_{i+1}$, where $n = T(i + 1)$. If, on the other hand, $P_i^* = P_n$ is a bicephalus, we ignore Lemma 5.6.2 and take the Σ_0-ultrapower.

Definition 5.6.3. Let P be a precephalus. I is a *padded iteration* of P of length μ if and only if

$$I = \langle \langle P_i \mid i < \mu \rangle, \langle \nu_i \mid i \in A \rangle, \langle F_i \mid i \in A \rangle, \langle \pi_{ij} \mid i \leq_T j \rangle, T \rangle$$

where the above holds with:

$$T(i + 1) = \text{ the least } n \in A \text{ such that } \kappa_i < \lambda_n \text{ or } i = n, \text{ for } i \in A$$

and:

$$\text{If } n < j \text{ and } [n, j) \cap A = \emptyset, \text{ then } nTj,\ P_n = P_j \wedge \pi_{n,j} = \mathrm{id}\,.$$

Lemma 5.6.2 continues to hold for padded iterations. Using padded iterations we can do a comparison iteration of a bicephalus with a premouse, another bicephalus, or even itself. We call the latter an *autoiteration*.

Definition 5.6.4. Let $P = \langle |P|, F^0, F^1 \rangle$ be a bicephalus. Let $F^0 \neq F^1$. Let $\mathrm{card}(P) < \theta$, where θ is regular. Suppose that P is $\theta + 1$-normally iterable. The *autoiteration* of P is a pair I^0, I^1 of padded iterations of P of length $\mu \leq \theta + 1$ and coiteration indices $\langle \nu_i \mid i < \mu \rangle$ such that

- $P_0^n = P$, $\nu_0 = \mathrm{ht}(P_0)$, $F_0^n = F^n$ for $n = 0, 1$.
- Let P_i^0, P_i^1 be given. For $\nu \leq \mathrm{ht}(P_i^0) \cap \mathrm{ht}(P_i^1)$ we define:

  $$\mathbb{F}_i^n(\nu) = \begin{cases} \{E_\nu^{P_i^n}\}, & \text{if } \nu \in P_i^n \text{ or } \nu = \mathrm{ht}(P_i^n) \text{ and } P_i^n \text{ is a premouse} \\ \{F^0, F^1\} & \text{if } \nu = \mathrm{ht}(P_i^n) \text{ and } P_i^n = \langle |P_i^n|, F^0, F^1 \rangle \text{ is a bicephalus.} \end{cases}$$

  Call ν *critical* at i if and only if $P_i^0|\nu = P_i^1|\nu$ and there exist $x^n \in \mathbb{F}_i^n(\nu)(n = 0, 1)$ such that $x^0 \neq x^1$. If so we set $\nu_i = \nu$. If $x^n = \emptyset$, then $x^{1-n} \neq \emptyset$ and we let $F_i^{1-n} = x^{1-n}$(hence $\nu_i \in A^{1-n}$), and $\nu_i \notin A^n$. If $x^0, x^1 \neq \emptyset$ we set, $F_i^n = x^n$ for $n = 0, 1$. This gives us P_{i+1}^0, P_{i+1}^1.
- If there is no critical ν, then $\mu = i+1$ and the autoiteration *terminates* at i.

Imitating the proof of Lemma 3.5.1 we get:

Lemma 5.6.3. *Let $P = \langle |P|, F^0, F^1 \rangle$ be a bicephalus. Let $\mathrm{card}(P) < \theta$ where θ is regular. If P is $\theta + 1$ normally iterable, then the autoiteration of P terminates below θ.*

However, if the autoiteration $\langle I^0, I^1 \rangle$ terminates at i, it could happen that both I^0 and I^1 have a truncation on the main branch. In this case, the result would tell us little about the original bicephalus P. If we assume, however, that P is *presolid* we get a better result. By the proof of lemma 4.1.14 we get:

Lemma 5.6.4. *Let P, θ be as above, where P is presolid. Let $\langle I^0, I^1 \rangle$ be the autoiteration of P, terminating at $i + 1 < \theta$. Then one of I^0, I^1 has no truncation on its main branch.*

But then:

Corollary 5.6.5. *If P, θ are as above and $P = \langle |P|, F^0, F^1 \rangle$, then $F^0 = F^1$.*

Proof. Suppose not. Then the autoiteration terminates at $i + 1 < \theta$, where $0 < i$. By lemma 5.6.4, we know that P_i^n is bicephalus for an $n = 0, 1$ and $\mathrm{ht}(P_i^n) \leq \mathrm{ht}(P_i^{1-n})$. Take e.g. $n = 0$. Then $P_i^0|\nu = P_i^1|\nu$, since otherwise we could continue the coiteration. Let $P_i^0 = \langle |P'|, F'^0, F'^1 \rangle$. Then $F'^0 = F'^1$, since otherwise there is $x \in \mathbb{F}_i^0 = \{F'^0, F'^1\}$ such that $x \neq y$ for a $y \in \mathbb{F}_i^1$. Hence:

$$F^0 = \pi_{0,i}^{-1}\text{"}F'^0 = \pi_{0,i}^{-1}\text{"}F'^1 = F^1.$$

Contradiction!

QED(Corollary 5.6.5)

An even stronger property than presolidity is pre-mouselikeness. As in the case of solidity, if $P = \langle J_\nu^E, F \rangle$ is a premouse or $P = \langle J_\nu^E, F^0, F^1 \rangle$ is a bicephalus, then pre-mouselikeness is a Π_1 property of J_ν^E. Hence, if I is a normal iteration of P, then every P_i in I will be pre-mouselike. By a virtual repetition of the proof of Lemma 5.3.10 we get:

Lemma 5.6.6 (Iteration Fact)**.** *Let I be a normal iteration of P, where P is pre-mouselike. Let $n = T(i+1)$. Let $\kappa_i = \text{crit}(F_i)$ and $\tau_i = \kappa_i^{+J_{\nu_i}^{E^{P_i}}}$. There is ν such that $P_i^* || \nu = \langle J_\nu^E, F \rangle$, $F \neq \emptyset$ and $\kappa_i = \text{crit}(F)$, $\tau_i = \kappa_i^{+J_\nu^{E^{P_i}}}$.*

We call a bicephalus $\langle |P|, F^0, F^1 \rangle$ one-small if and only if $\langle |P|, F^n \rangle$ is one-small for $n = 0, 1$. Note that in this case $\langle |P|, F^n \rangle$ is restrained for $n = 0, 1$. The proof of Lemma 5.1.2 can be adapted to show:

Lemma 5.6.7. *Let P be a one-small bicephalus. If P is countably normally iterable, then it is ∞-normally iterable.*

We now return to our original question. Let $\mathbb{N} = \langle N_i \mid i \leq \mu \rangle$ be a Steel array(hence every N_i is mouselike). Can there be two different extenders F^0, F^1 such that F^n is robust in $\langle M_\mu, F^n \rangle$ for $n = 0, 1$.(Hence $M_\mu = N_\mu$ is a ZFC^- model.) We want to show that this cannot occur, so we argue by contradiction. Set $N_{\mu+1} = \langle M_\mu, F^0, F^1 \rangle$. Then $N_{\mu+1}$ is a bicephalus. We then call $\mathbb{N}' = \langle N_i \mid i \leq \mu + 1 \rangle$ a *putative two headed Steel array*. Let us define:

Definition 5.6.5. Let $P = \langle |P|, F^0, F^1 \rangle$, $P' = \langle |P'|, F'^0, F'^1 \rangle$ be bicephali. We set:

$$\sigma : P \longrightarrow_* P' \text{ if and only if } \sigma : \langle |P|, F^n \rangle \longrightarrow_{\Sigma_0} \langle |P'|, F'^n \rangle \text{ for } n = 0, 1.$$

The nonexistence of a two headed Steel array follows from:

Lemma 5.6.8. *Let $\mathbb{N} = \langle N_i \mid i \leq \mu + 1 \rangle$ be a putative two headed Steel array. Let P be a countable bicephalus such that $\sigma : P \longrightarrow_* N_{\mu+1}$. Then P is countably normally iterable.*

We first show that this implies the nonexistence of a two headed Steel array $\mathbb{N}$. $N_{\mu+1}$ is pre-mouselike, since M_μ is mouselike. Hence P is pre-mouselike, since pre-mouselikeness is a Π_1 property. Hence P is solid. By lemma 5.6.7, P is ω_1+1 iterable. Hence, if $P = \langle |P|, \bar{F}^0, \bar{F}^1\rangle$, then $\bar{F}^0 = \bar{F}^1$. But $N_{\mu+1} = \langle |N_\mu|, F^0, F^1\rangle$ and $F^0 \neq F^1$. Hence we could easily choose P, σ such that $\bar{F}^0 \neq \bar{F}^1$. Contradiction!

We shall closely imitate the proof of Lemma 5.3.7 in order to prove lemma 5.6.8. Fix $\mathbb{N}$ and let $\sigma : P \longrightarrow_* N_{\mu+1}$, where P is countable. We again prove:

$(*)$ If I has length $\eta+1$, and we appoint ν_η, F_η such that $F_\eta \in \mathbb{F}_{\nu_\eta}$ and $\nu_\eta > \nu_i$ for all $i < \eta$, then letting $\gamma = T(\eta+1)$, we have:

- If P^*_η is a premouse then the n-ultrapower
$$\pi : P^*_\eta \longrightarrow^n_{F_\eta} P_{\eta+1} \text{ exists,}$$
where $n \leq \omega$ is maximal such that $\kappa_i < \rho^n_{P^*_\eta}$.
- If $P^*_\eta = P_\gamma$ is a bicephalus, then the Σ_0 ultrapower $\pi : P_\gamma \longrightarrow_F P_{\eta+1}$ exists.

$(**)$ If I has limit length, then I has a cofinal, well founded branch.

In a normal iteration of a bicephalus extenders are sometimes applied in a different way than in a normal iteration of a premouse. For this reason we must revise the definition of *realization*:

Definition 5.6.6. Let $\mathbb{N} = \langle N_i \mid i \leq \mu+1\rangle$ be a two headed putative Steel array. Let P be a countable bicephalus and let I be a countable normal iteration of P. By induction on $i < \mathrm{lh}(I)$ we define the set D_i of realizations of P_i in $\mathbb{N}$. Each element of D_i is a sequence:

$$\sigma = \langle\langle\sigma_j, \mu_j\rangle \mid j \leq_T i\rangle$$

such that $\sigma_j : P_j \to N_{\mu_j}$ for $j \leq_T i$. We inductively verify:

- $\mu_i \leq \mu_j$ for $j \leq i$
- If P_i is a bicephalus, then $\sigma_i : P_i \longrightarrow_* N_{\mu+1}$.
- If P_i is not a bicephalus, then $\mu_i \leq \mu$ and
$$\sigma_i : P_i \longrightarrow_{\Sigma_0^{(n)}} N_{\mu_i} \text{ whenever } \lambda_j < \rho^n_{P_i} \text{ for all } j < i$$
- If $(j, i]_T$ is drop free, then $\mu_j = \mu_i$ and $\sigma_j = \sigma_i \pi_{j,i}$.

We again define D_i by cases:

Case 1. $i = 0$. D_0 is the set of $\sigma = \{\langle\sigma_0, \mu_0\rangle\}$ such that $\mu_0 = \mu + 1$ and $\sigma_0 : P \longrightarrow_* N_{\mu+1}$.

Case 2. $i = j + 1$.

We again split into two cases:

Case 2.1. $j + 1$ is not a drop point. Thne $\sigma = \langle\langle\sigma_n, \mu_n\rangle \mid n \leq_T i\rangle \in D_i$ if and only if the following hold:

- $\sigma | n + 1 \in D_n$ if $n <_T i$.
- $\mu_i = \mu_n$ and $\sigma_n = \sigma_i \pi_{n,i}$ for $n <_T i$.
- If P_i is a bicephalus, then $\sigma_i : P_i \longrightarrow_* N_{\mu+1}$.
- If P_i is a premouse, then $\sigma_i : P_i \longrightarrow_{\Sigma_0^{(k)}} N_{\mu_i}$ whenever $\lambda_j < \rho^k_{P_i}$.

Case 2.2 is then exactly as before, as is Case 3.

As before we pick an injection n^* of $\mathrm{lh}(I)$ into ω. We then define $n(i)$ $(i < \mathrm{lh}(I))$ as before. We define "i survives at j" as before. We then define the relation R on D(where $D = \bigcup_{i<\mathrm{lh}(I)} D_i$) as before.

Definition 5.6.7. $\sigma' R \sigma$ if and only if there are i, j such that $i <_T j$, $\sigma' \in D_j$, $\sigma \in D_i$, $\sigma = \sigma' | i + 1$, and i does not survive at j.

As before, it turns out that, if R is ill founded then I has limit length and there us a cofinal well founded branch in I. As before, we assume that R is well founded. We can then literally take over the definition of 'enlargement', using the revised notion of realization. In fact, we can literally take over all the ensuing definitions and proofs in the proof of lemma 5.3.7, thus proving lemma 5.6.8.

This shows that at any point in the construction of a Steel array there is at most one possible application of Option 2, assuming that there is no inner model with a Woodin cardinal.

5.7 The model K^c

We continue to assume that there is no inner model with a Woodin cardinal. In the previous section we showed that there is a unique sequence $\mathbb{N} = \langle N_i \mid$

$i < \infty\rangle$ with the properties

- $\mathbb{N}$ is a robust Steel array.
- N_{i+1} is formed by Option 2 if possible; otherwise by Option 1.

Thus, the function $\langle N_i \mid i < \infty$ is defined recursively. In order to distinguish it from other Steel arrays, we may sometimes write:

$$\mathbb{N}^c = \langle N_i^c \mid i < \infty\rangle.$$

As we noted in §5.1 (following Lemma 5.2.5), the structure

$$N_\infty = \bigcup_{i<\infty} M_i||\tilde{\mu}_{i,\infty}$$

is a weasel and an inner model of ZFC^-. Since it is a weasel, we also denote by L^E or L^{E^c}. We set:

$$K_\alpha^c = K^c||\alpha =: \langle J_\alpha^{E^c}, E_\alpha^c\rangle$$

for limit ordinals α. Whenever α is a limit ordinal and $\tilde{\mu}_{i,\infty} < \alpha$ for $i < \alpha$, then $L_\alpha^E = N_\alpha$. Hence:

Lemma 5.7.1. *$\{\alpha \mid K_\alpha^c = N_\alpha^c\}$ is club in ∞.*

Using this we get:

Lemma 5.7.2. *K_α^c is fully β-iterable for all β.*

Proof. Let $\beta > \alpha$ such that $K_\beta^c = N^c$. Let $N_{\beta,\eta}$ denote the constructible extension of N_β of length $\beta+\omega\eta$. (Thus $N_{\beta,0} = N_\beta$. For $\eta > 0$, $N_{\beta,i} = L_{\beta+\omega i}^{E'}$, where $E' = E \cup \{\langle x, \beta\rangle \mid x \in E_\beta\}$ and $N_\beta = \langle L_\beta^E, E_\beta\rangle$.) There is a least η such that either $\rho_{N_{\beta,\eta}}^\omega < \beta$ or else $\rho_{N_{\beta,\eta}}^\omega = \beta$ and β is not Woodin in $N_{\beta,\eta}$. (Otherwise β would be Woodin in $N_{\beta,\infty}$.) But $N_{\beta,\eta}$ is then a restrained one small mouse. Moreover, by induction on $i \leq \eta$ we can prove: $N_{\beta,i} = N_{\beta+i}$ and $M_{\beta,i} = M_{\beta+i}$ for $i < \eta$. (In successor points in the induction we use that Option 2 is not available.) By §5.4 it follows that $N_{\beta+\eta}$ is uniquely normally iterable up to ∞. Hence $N_{\beta+\eta}$ is fully γ-iterable for all $\gamma < \infty$. Hence K_α^c is fully γ-iterable for $\gamma < \infty$, since $K_\alpha^c = N_{\beta+\eta}||\alpha$. QED(Lemma 5.7.2)

We now turn to the main result of this section, which says that K^c is universal in the sense that K^c "out iterates" any normally iterable mouse. We shall prove this using methods that we employed in the proof of the basic comparison lemma Lemma 3.5.1. However, we must apply them to a less wieldy situation. The precise statement we wish to prove is:

Theorem 5.7.3. *Let $\theta > 2^{2^\omega}$ be a regular cardinal such that $\alpha^\omega < \theta$ for all $\alpha < \theta$. Let $Q = K^c_\theta$. Let P be a premouse of height $< \theta$. Let S be a successful $\theta+1$ normal iteration strategy for P and S' a successful $\theta+1$ normal iteration strategy for Q. Coiterate P, Q using S, S'. Then the coiteration terminates below θ.*

Note that there are arbitrarily large θ with these properties. If $\alpha \geq 2^{2^\omega}$ is any cardinal, then $\theta = (\alpha^\omega)^+$ satisfies the condition. Before proving Theorem 5.7.3, we develop some methodology. By the condensation lemma for the Chang hierarchy (Lemma 5.3.1), it follows that:

Fact 0. $C^e_{\gamma,\theta} \prec_{\Sigma_1} C^e_{\gamma,\infty}$ for all e and all $\gamma < \theta$.

(We leave this to the reader.)

Our proof will make use of the condensation lemma for mouselike premice. However, we shall restrict ourselves to the application of the following weaker consequence:

Lemma 5.7.4. *Let N be a mouselike sound premouse. Let $\sigma\colon M \longrightarrow_{\Sigma_\omega} N$ with $\rho^\omega_M = \mathrm{crit}(\sigma)$ and $\sigma(\rho^\omega_M) = \rho^\omega_N$. Then $M \triangleleft N$.*

Proof. It follows easily that σ witnesses the phalanx $\langle N, M, \lambda\rangle$, where $\lambda = \rho^\omega_M$. M is sound above λ, since N is sound above $\sigma(\lambda)$. There are three possibilities. The first is that $M = \mathrm{core}(N)$. This is impossible, since $\rho^\omega_M < \rho^\omega_N$. Thus M is a proper segment of N unless the third possibility (c) arises. If λ is a cardinal in N, then (c) is excluded, since it would require that $\rho^\omega_{N||\gamma} < \lambda$ for a $\gamma \in N$ with $\gamma > \lambda$. If λ is not a cardinal in N, let κ be the largest $\kappa < \lambda$ which is a cardinal in N. (c) then requires:

$$\pi\colon N||\gamma \longrightarrow^*_F M \text{ where } F = E^N_\mu$$

for some $\mu \leq \gamma$ such that $\kappa = \mathrm{crit}(F)$ and $\lambda = \gamma^{+N||\gamma}$. Since $\rho^\omega_{N||\gamma} = \kappa$, we have $\rho^\omega_M = \kappa < \lambda$. Contradiction! Thus (c) fails. QED(Lemma 5.7.4)

We now introduce a concept which will be needed in the proof of Theorem 5.7.3 and will also play a large role in the next chapter, where we introduce the core model K.

Definition 5.7.1. Let $\theta > \omega$ be a regular cardinal. Let Q be a mouselike premouse of height θ. By the *stack over* Q ($\mathbb{S} = \mathbb{S}(Q)$) we mean the set of all mouselike premice N such that $Q \triangleleft N$, $Q \in N$, N is sound and $\rho^\omega_N = \Theta$.

Lemma 5.7.5. *Let $\mathbb{S} = \mathbb{S}(Q)$. Let $N, N' \in \mathbb{S}$. Then either $N \triangleleft N'$ or $N' \triangleleft N$.*

Proof. Let $\Omega > \theta$ be regular. Let $X \prec H_\Omega$ such that $N, N' \in X$ and $\bar{\theta} = X \cap \theta$ is an ordinal $< \theta$. (Such X clearly exists, since θ is regular.) Let $\sigma \colon \bar{H} \overset{\sim}{\longleftrightarrow} X$. Thus $\sigma \colon \bar{H} \prec H_\omega$. Let $\sigma(\bar{N}) = N, \sigma(\bar{N}') = N'$. Then $\bar{N} \triangleleft N$, where $\mathrm{ht}(\bar{N}) < \theta$. Hence $\bar{N} \triangleleft Q = N||\theta$. Similarly $\bar{N}' \triangleleft Q$. Hence $\bar{N} \triangleleft \bar{N}' \vee \bar{N}' \triangleleft \bar{N}$. Hence $N \triangleleft N' \vee N' \triangleleft N$. QED(Lemma 5.7.5)

It follows that the union:

$$S = S(Q) =: \bigcup \mathbb{S}(Q)$$

is a premouse extending Q.

We now again assume that $\theta > \omega$ is regular. Let Q be a premouse satisfying ZFC^- such that either $\mathrm{ht}(Q) = \theta$ or $\theta \in Q$. In either case $Q||\theta$ is a ZFC^- model, since θ, being a cardinal, cannot index and extender. We ask what happens when we apply a weakly amenable extender F pf length less than θ to Q. Since Q is a ZFC^- model, the Σ_0-ultrapower is the same as the $*$-ultrapower. We assume that F is a weakly amenable extender at $\kappa < \theta$ on Q and that $\pi \colon Q \longrightarrow_F Q'$ exists. Let $Q = J^E_\Omega$, $Q' = J^{E'}_{\Omega'}$. Then F has base $|J^E_\tau|$, where $\tau = \kappa^{+Q} < \theta$ and extension $\langle |J^{E'}_\nu|, \pi \restriction |J^E_\tau| \rangle$, where $\pi(\tau) = \nu$. Every element of $J^{E'}_\nu$ has the form $\pi(f)(\alpha)$, where $f \in J^E_\tau$ is a map defined on κ and $\alpha < \mathrm{lh}(F)$. The collection of such pair $\langle f, \alpha \rangle$ is a set in Q. In V, however, the function $\langle f, \alpha \rangle \mapsto \pi(f)(\alpha)$ maps this set onto ν. Hence $\nu < \theta$, since θ is regular in V.

We ask whether π takes θ to itself. If $\pi"\theta \subset \theta$ and $\mathrm{ht}(Q) = \theta$, it follows that $\mathrm{ht}(Q') = \theta$, since each ordinal element of Q' has the form $\pi(f)(\alpha)$, where $\alpha < \mathrm{lh}(F)$ and $f \colon \kappa \longrightarrow \mathsf{ON}$ in Q. Thus $\pi(f)(\alpha) < \pi(\gamma)$, where $\gamma = \mathrm{lub}\, f"\,\mathrm{lh}(F) < \theta$. If $\theta \in Q$, it follows for the same reason that $\pi(\theta) = \theta$. Since θ is regular in V we know that $Q||\theta$ is a ZFC^- model. If arbitrarily large $\alpha < \theta$ are cardinals in the sense of Q, then $Q||\theta$ is a full ZFC model. If not, there is a largest $\mu < \theta$ which is a cardinal in $Q||\theta$. Then:

Fact 1. If $Q||\theta$ models ZFC, then $\pi"\theta \subset \theta$.

Proof. Let $\eta \in Q'$. Each $\xi < \eta$ has the form: $\pi(f)(\alpha)$, where $\alpha < \mathrm{lh}(F)$ and $f \in Q$ such that $f \colon \kappa \longrightarrow \eta$. The set of such pairs is a set in $Q||\theta$, hence in V. Thus there is in V a map of this set onto η. Hence $\mathrm{card}(\eta) < \theta$. Hence $\eta < \theta$, since θ is regular. QED(Fact 1)

Fact 2. Let μ be the largest cardinal in $Q||\theta$. Set $\tilde{\mu} = \mathrm{lub}\, \pi"\mu$. Then $\tilde{\mu} < \theta$.

Proof. By a virtual repetition of the proof of Fact 1 we have: $\eta < \mu \longrightarrow \pi(\eta) < \theta$. But then $\tilde{\mu} < \theta$, since θ is regular. QED(Fact 2)

Fact 3. If μ is as in Fact 2 and $\kappa \neq \mathrm{cf}(\mu)$ in Q, then $\pi(\mu) = \tilde{\mu}$.

Proof. If $\kappa < \mathrm{cf}(\mu)$ in Q, then each $\xi < \pi(\mu)$ has the form $\pi(f)(\alpha)$ where $\alpha < \mathrm{lh}(F)$ and $f \in Q$ such that $f\colon \kappa \longrightarrow \mu$. But then $\pi(f)(\alpha) < \pi(\beta)$, where

$$\beta = \mathrm{lub}\{f(\xi) \mid \xi < \kappa\}.$$

Hence $\pi(\mu) = \bigcup \pi"\mu = \tilde{\mu}$. Now let $\kappa > \mathrm{cf}(\mu)$ in Q. Let $\gamma = \mathrm{cf}(\mu)$ in Q. There is $g \in Q$ such that $g\colon \gamma \longrightarrow \mu$ and $\mu = \mathrm{lub}\, g"\gamma$. But then $\pi(g)\colon \gamma \longrightarrow \pi(\mu)$ and $\pi(\mu) = \mathrm{lub}\,\pi(g)"\gamma = \tilde{\mu}$. QED(Fact 3)

Fact 4. If μ is as above and $\kappa \neq \mathrm{cf}(\mu)$, then $\pi"\theta \subset \theta$.

Proof. Let $\mu \leq \xi < \theta$. Then there is $g \in Q$ such that $g\colon \mu \xrightarrow{\text{onto}} \xi$. Hence $\pi(g)\colon \pi(\mu) \xrightarrow{\text{onto}} \pi(\xi)$, where $\pi(\xi) \geq \xi$. Hence $\pi(\xi) < \theta$, since $\theta > \pi(\mu)$ is regular. QED(Fact 4)

If, however, $\kappa = \mathrm{cf}(\mu)$ in Q, then things are very different:

Fact 5. If μ is as above and $\kappa = \mathrm{cf}(\mu)$ in Q, then $\pi(\mu) > \theta$. Moreover, $\tilde{\mu}$ is the largest cardinal in $Q'||\theta$.

Proof. Let $u \in Q||\theta$ such that $u \subset \{f \mid f\colon \kappa \longrightarrow \mu\}$ in Q. Then there is a $g \in Q$ such that $g\colon \kappa \longrightarrow \mu$ and

$$\pi(f)(\kappa) \neq \pi(g)(\kappa) \text{ for all } f \in u.$$

To see this, let $\langle f_i \mid i < \mu\rangle$ enumerate u in Q. Let $\langle \mu_i \mid i < \kappa\rangle$ be monotone such that $\mathrm{lub}\{\mu_i \mid i < \kappa\} = \mu$. Choose $g(i) \notin \{f_j(i) \mid j < \mu_i\}$ for $i < \kappa$. Then: $f_j(i) \neq g(i)$ for $j > \mu_i$. Hence $f(i) \neq g(i)$ for sufficiently large i, if $f \in u$. Hence:

$$\pi(g)(\kappa) \neq \pi(f)(\kappa) \text{ for } f \in u.$$

Using this, we see that there is a sequence $g_\xi(\xi < \beta)$ such that

$$\pi(g_\xi)(\kappa) \neq \pi(g_\zeta)(\kappa) \text{ for } \xi \neq \zeta.$$

Hence $\pi(\mu) \geq \theta$, since $\pi(g_\xi)(\kappa) < \pi(\mu)$ for $\xi < \theta$. θ is regular V, hence in Q', whereas

$$\mathrm{cf}(\pi(\mu)) = \pi(\kappa) < \theta \text{ in } Q'.$$

Hence $\pi(\mu) > \theta$. It remains to show that $\tilde{\mu}$ is the largest cardinal in $Q'||\theta$. Let $\tau > \tilde{\mu}$ be a cardinal in $Q'||\theta$. We derive a contradiction. Then $\mathbb{P}(\tilde{\mu}) \cap Q' \subset J_\tau^{E^{Q'}}$ by acceptability. Now suppose that $X, Y \in \mathbb{P}(\mu) \cap Q$ such that $X \neq Y$. Then X, Y have a point of difference $\xi < \mu$, i.e.:

$$\xi \in X \nleftrightarrow \xi \in Y.$$

But then $\pi(\xi) < \tilde{\mu}$ is apoint of difference of $\pi(X)$, $\pi(Y)$. Hence the map $X \mapsto \pi(X) \cap \tilde{\mu}$ injects $\mathbb{P}(\mu) \cap Q$ into $\mathbb{P}(\tilde{\mu}) \cap Q'$. This is a contradiction, since $\mathrm{card}(\mathbb{P}(\mu) \cap Q) = \theta$ and $\mathrm{card}(\mathbb{P}(\tilde{\mu}) \cap Q') \leq \tau < \theta$. QED(Fact 5)

We now turn to the proof of Theorem 5.7.3. Suppose not. Then P, Q have the coiteration $\langle I^P, I^Q \rangle$ of length $\theta + 1$, where:

$$I^P = \langle \langle P_i \rangle, \langle \nu_i \mid i \in A^P \rangle, \langle \pi^P_{i,j} \rangle, T^P \rangle,$$

$$I^Q = \langle \langle Q_i \rangle, \langle \nu_i \mid i \in A^Q \rangle, \langle \pi^Q_{i,j} \rangle, T^Q \rangle,$$

where $\langle \nu_i \mid i < \theta \rangle$ is the sequence of coiteration indices. (Hence $A^P \cup A^Q = \theta$.)

Let $\Omega > \mathrm{card}(H_\theta)$ be regular such that

$$S, S', I^P, I^Q, \mathbb{S} \in H_\Omega,$$

where S, S' are the iteration strategies for P, Q respectively and $\mathbb{S} = \mathbb{S}(Q)$ is the stack over Q. Pick $X \prec H_\Omega$ such that

- $\mathrm{card}(X) < \theta$
- $\bar{\theta} = X \cap \theta$ is transitive and $2^{2^\omega} < \bar{\theta}$
- $S, S', I^P, I^Q, \mathbb{S} \in X$.

This is possible by the regularity of θ. Let $\sigma \colon \hat{H} \stackrel{\sim}{\longleftrightarrow} X$ be the transitivization of X. Then $\sigma \colon \hat{H} \prec H_\Omega$, $\bar{\theta} = \mathrm{crit}(\sigma)$ and $\sigma(\bar{\theta}) = \theta$. Let:

$$\sigma(\bar{I}^P) = I^P, \sigma(\bar{I}^Q) = I^Q,$$

where:

$$\bar{I}^P = \langle \langle \bar{P}_i \rangle, \langle \bar{\nu}_i \rangle, \langle \bar{\pi}^P_{i,j} \rangle, \bar{T}^P \rangle$$
$$\bar{I}^Q = \langle \langle \bar{Q}_i \rangle, \langle \bar{\nu}_i \rangle, \langle \bar{\pi}^Q_{i,j} \rangle, \bar{T}^Q \rangle$$

Set: $H = \sigma^{-1}(H_\theta)$. Then $\sigma \upharpoonright H = \mathrm{id}$. Hence on both sides of the coiteration we have:

(1)(a) $i <_{\bar{T}} \bar{\theta} \longleftrightarrow i <_T \theta$ for $i < \bar{\theta}$

(b) $i <_{\bar{T}} j \longleftrightarrow i <_T j$ for $i, j < \bar{\theta}$

But then

(1)(c) $\bar{\theta} <_T \theta$,

since $\bar{\theta} < \theta$ is a limit point of the branch $\{i \mid iT\theta\}$. (*Note* This does *not* presuppose that there are cofinally many active points below θ. As we shall see, it is possible that there are no active points on the Q-side. Recall that if no $j > i$ is active, then $i <_T j$ and $\pi_{i,j} = \mathrm{id}$ for $j > i$.) We now specifically consider the P-side of the coiteration. Since $\mathrm{ht}(P) < \theta$, a straightforward induction on i shows:

(2) $\mathrm{ht}(P_i) < \theta$ for $i < \theta$.

(Hence $\nu_i \leq \mathrm{ht}(P_i) < \theta$.) Since $\sigma \restriction H = \mathrm{id}$ we have:

(3) $\bar{P}_i = P_i$ and $\bar{\pi}^P_{i,j} = \pi^P_{i,j}$ for $i \leq_T j < \theta$.

The branch $\{i \mid i <_{T^P} \theta\}$ has at most finitely many drop points. Hence the last drop point, if it exists, must lie below $\bar{\theta}$. Exactly as in the proof of Lemma 3.5.1 we then get:

(4) $\bar{P}_{\bar{\theta}}$, $\langle \bar{\pi}^P_{i,\bar{\theta}} \mid i <_{T^P} \bar{\theta} \rangle$ is the transitivised direct limit of:

$$\langle P_i \mid i < \bar{\theta} \rangle, \langle \pi^P_{i,j} \mid i \leq_{T^P} j <_{T^P} \bar{\theta} \rangle.$$

But then:

(5) $\bar{P}_{\bar{\theta}} = P_{\bar{\theta}}$, $\bar{\pi}^P_{i,\bar{\theta}} = \pi^P_{i,\bar{\theta}}$ for $i <_{T^P} \bar{\theta}$.

Hence:

(6) $\sigma \restriction P_{\bar{\theta}} = \pi^P_{\bar{\theta},\theta}$.

Proof. Let $x \in P_{\bar{\theta}}$. Then $x = \pi_{i,\bar{\theta}}(z)$ for some $i <_{T^P} \bar{\theta}$. Thus,

$$\sigma(x) = \sigma(\bar{\pi}_{i,\bar{\theta}}(z)) = \pi_{i,\theta}(z) = \pi_{\bar{\theta},\theta}(\pi_{i,\bar{\theta}}(z)) = \pi_{\bar{\theta},\theta}(x).$$

QED(6)

Exactly as in Lemma 3.5.1 we then get:

(7) Let i be least such that $\bar{\theta} <_{T^P} i <_{T^P} \theta$ and $P_i \neq P_{\bar{\theta}}$. Then $i = j + 1$ where j is active in I^P. Moreover:

$$E^{P_j}_{\nu_j}(X) = \{\alpha < \lambda_j \mid \alpha \in \sigma(X)\} \text{ for } X \in \mathbb{P}(\bar{\theta}) \cap P_{\bar{\theta}}.$$

We now turn to the Q-side. Here things are more complicated, since Q is not smaller than θ. It is therefore not clear that any $i < \theta$ is active on the I^Q-side. If, however, a truncation occurs on the main branch $\{\delta \mid \delta <_{T^Q} \theta\}$, then the Q_i's on this branch would be small from this point on. We could then repeat the proof of Lemma 3.5.1, obtaining a contradiction. Hence:

(8) The Q-side has no truncation on the main branch.

Proof(sketch). Suppose not. Then the last truncation point $i_0 + 1$ on the branch lies below $\bar{\theta}$. By induction on i it then follows that:

$$\text{ht}(Q_i) < \theta \text{ for } i_0 < i <_{T^Q} \bar{\theta}.$$

Hence:

$$\text{ht}(\bar{Q}_i) < \bar{\theta} \text{ for } i_0 < i <_{\bar{T}^Q} \bar{\theta}.$$

Since $\sigma \upharpoonright H = \text{id}$, we can repeat the proof of (2)-(7) on the Q-side, getting

- $\text{ht}(Q_i) < \theta$ for $i_0 < i <_{T^Q} \theta$
- $\bar{Q}_i = Q_i$ and $\bar{\pi}^Q_{i,j} = \pi^Q_{i,j}$ for $i_0 < i \leq_{T^Q} j <_{T^Q} \bar{\theta}$
- $\bar{Q}_{\bar{\theta}} = Q_{\bar{\theta}}$, $\bar{\pi}^Q_{i,\bar{\theta}} = \pi^Q_{i,\bar{\theta}}$ for $i_0 < i <_{T^Q} \bar{\theta}$
- $\sigma \upharpoonright Q_{\bar{\theta}} = \pi^Q_{\bar{\theta},\theta}$
- Let i' be least such that $\bar{\theta} <_{T^Q} i <_{T^Q} \theta$ and $Q_i \neq Q_{\bar{\theta}}$. Then $i' = j' + 1$ where j' is active in I^Q. Moreover:

$$E^{P_{j'}}_{\nu_{i'}}(X) = \{\alpha < \lambda_j \mid \alpha \in \sigma(X)\} \text{ for } X \in \mathbb{P}(\bar{\theta}) \cap Q_{\bar{\theta}}.$$

Hence, letting i, j be as in (7) we cannot have $j = j'$, since otherwise $E^{Q_j}_{\nu_i} = E^{P_j}_{\nu_i}$ and ν_i is not a point of difference. Repeating the rest of the proof of Lemma 3.5.1, we can then use the initial segment condition to show that $j \neq j'$ is also impossible. Contradiction! QED(8)

From this it follows by induction on j that:

(9) $\pi^Q_{i,j} \colon Q_i \longrightarrow_{\Sigma_\omega} Q_j$ cofinally for $i \leq_{T^Q} j <_{T^Q} \theta$.

However, it is *not* clear that $\pi^Q_{i,j}{}''\theta \subset \theta$ for $i < Tj <_t \theta$. We leave it to the reader to verify:

(10) Let $i \leq_T j <_T \theta$ such that $\pi_{i,j}{}''\theta \subset \theta$ in I^Q. Then:

- If $h \leq_T i$ and $\pi_{i,j}{}''\theta \subset \theta$, then $\pi_{h,j}{}''\theta \subset \theta$.
- If $j \leq_T h <_T \theta$ and $\pi_{j,h}{}''\theta \subset \theta$, then $\pi_{i,h}{}''\theta \subset \theta$.

- if $i \leq_T h \leq_T j$, then $\pi_{i,h}"\theta \subset \theta$ and $\pi_{h,j}"\theta \subset \theta$.

Call $j <_{T^Q} \theta$ is *tipping point* of I^Q if and only if there is $i <_{T^Q} j$ such that $\pi^Q_{i,j}"\theta \not\subseteq \theta$ and $\pi^Q_{i,h}"\theta \subset \theta$ for $i \leq_{T^Q} h <_{T^Q} \theta$.

(11) There are at most finitely many tipping points.

Proof. Suppose not. Let j_n be the n-th tipping point $(n < \omega)$. Then $\theta \in Q_{j_n}$ and $\pi_{j_n,j_{n+1}}(\theta) > \theta$. Hence:

$$\pi_{j_n,\theta}(\theta) = \pi_{j_{n+1},\theta}(\pi_{j_n,j_{n+1}}(\theta)) > \pi_{j_{n+1},\theta}(\theta).$$

Contradiction! QED(11)

(12) Every tipping point is a successor ordinal.

Proof. Suppose not. Let η be an exception. Then η is a limit ordinal and there is $i <_T \eta$ such that $\pi_{i,j}"\theta \subset \theta$ for $i <_T j <_T \eta$. Pick $\xi < \theta$ such that $\pi_{i,\eta}(\xi) \geq \theta$. Then:

$$\pi_{i,\eta}(\xi) = \bigcup\{\pi_{j,\eta}"\pi_{i,j}(\xi) \mid i \leq_T j <_T \eta\}.$$

Since $\eta < \theta$ and $\pi_{i,j}(\xi) < \theta$ for $i \leq_T j <_T \eta$, it follows that $\mathrm{card}(\pi_{i,\eta}(\xi)) < \theta$. But θ is a cardinal in V. Hence $\pi_{i,\eta}(\xi) < \theta$. Contradiction! QED(12)

(13) Let $i' + 1$ be a tipping point. Let $h = T(i+1)$ in I^Q. Then there is $\mu < \theta$ such that

- μ is the largest cardinal in $Q_h||\theta$
- $\kappa_i = \mathrm{cf}(\mu)$ in $Q_h||\theta$
- $\pi_{h,i+1}(\mu) > \theta$
- $\tilde{\mu} = \mathrm{lub}\,\pi_{h,i+1}"\mu$ is the largest cardinal in $Q_{i+1}||\theta$.

This follows by the application of Fact 1–Fact 5. We leave this to the reader.

(14) Let $\gamma = \sup\{i \mid i \text{ is a tipping point }\}$. Then

- $\pi^Q_{\gamma,i}"\theta \subset \theta$ (Hence $\pi_{\gamma,i} \colon Q_\gamma||\theta \longrightarrow_{\Sigma_0} Q_i||\theta$ cofinally for $\gamma \leq_T i <_T \theta$.
- If Q is a ZFC model, then $\gamma = 0$ and each Q_i is a ZFC model for $i < \theta$.
- If Q is not a ZFC model, then $Q_\gamma||\theta$ is not a ZFC model. (Hence $Q_\gamma||\theta$ has the largest cardinal in $Q_i||\theta$ for $\gamma \leq i \leq_{T^Q} \theta$.)

This follows from (12), again applying Fact 1–Fact 5. If γ is as in (14) it is clear that $\gamma < \bar{\theta}$. But then

(15) $\pi^Q_{\gamma,\theta}"\theta \subset \theta$.

Proof. Suppose not. Then there are arbitrarily large j such that $\gamma \leq_T j <_T \theta$ in I^Q and j is active in I^Q. (otherwise there would be j such that $\gamma <_T j <_T \theta$ and $Q_j = Q_\theta$. Hence $\pi_{j,\theta} = \mathrm{id}$. Hence:

$$\pi_{\gamma,\theta}"\theta = \pi_{\gamma,j}"\theta \subset \theta.$$

Contradiction!) Now let δ be least such that $\delta \geq \gamma$, $\delta <_{T^Q} \theta$, and $\pi_{\delta,\theta}"\theta \not\subseteq \theta$. Then there is a least $\mu < \theta$ such that $\pi_{\delta,\theta}(\mu) \geq \theta$. But then μ is a cardinal in Q_δ, since otherwise θ would not be a cardinal in Q_θ, hence not in V. Clearly $\delta, \mu < \bar{\theta}$., Set:

$$\mu_i = \pi^Q_{\delta,i}(\mu), Q'_i = Q_i|\mu_i = J^{E^{Q_i}}_{\mu_i} \text{ for } \delta \leq_T i \leq_T \theta \text{ in } I^Q.$$

and:

$$\pi'_{h,i} = \pi^Q_{h,i} \restriction Q'_h \text{ for } \gamma \leq_T h \leq_T i \text{ in } I^Q.$$

Set:

$$\langle \bar{Q}'_i \mid \gamma \leq_{\bar{T}} i \leq_{\bar{T}} \bar{\theta} \rangle = \sigma^{-1}(\langle Q'_i \mid \gamma \leq_T i \leq \theta \rangle)$$
$$\langle \bar{\pi}'_{h,i} \mid \gamma \leq_{\bar{T}} i \leq_{\bar{T}} \bar{\theta} \rangle = \sigma^{-1}(\langle \pi'_{h,i} \mid \gamma \leq_T h \leq_T i \leq_T \theta \rangle).$$

Since $\sigma \restriction H = \mathrm{id}$, we get:

$$\bar{Q}'_i = Q'_i, \bar{\pi}'_{i,j} = \pi'_{i,j} \text{ for } \gamma \leq_T i \leq_T j < \bar{\theta}.$$

But then exactly as before we get:

$$\bar{Q}'_{\bar{\theta}} = Q'_{\bar{\theta}}, \bar{\pi}'_{\bar{\theta},\theta} = \sigma \restriction Q'_\theta, \bar{\pi}'_{i,\bar{\theta}} = \pi'_{i,\bar{\theta}}$$

for $i <_T \bar{\theta}$. If i is least such that i is active in I^Q and $\bar{\theta} <_T i+1 <_T Q$ in I^Q, then it follows as before that:

$$\pi_{\delta i+1}(X) = \sigma(X) \cap \lambda_i \text{ for } X \in \mathbb{P}(\kappa_i) \cap Q_{\bar{\theta}}.$$

(Note that $X \in Q'_{\bar{\theta}}$, since $\mu_{\bar{\theta}}$ is a cardinal in $Q_{\bar{\theta}}$.)

Hence:

$$E^Q_{\nu_i}(X) = \{\alpha < \lambda_i \mid \alpha \in \sigma(X)\} \text{ for } X \in \mathbb{P}(\delta) \cap Q\delta.$$

We can the repeat the proof of Lemma 3.5.1, getting a contradiction. QED(15)

(16) Q is a ZFC model.

Proof. Suppose not. Let γ be as before. Then there is $\mu < \theta$ such that

$$Q_\gamma||\theta \models \mu \text{ is the largest cardinal.}$$

Hence:

$$Q_\theta||\theta \models \mu' \text{ is the largest cardinal,}$$

where $\mu' = \pi_{\gamma,\theta}(\mu)$. But $Q_\theta||\theta = P_\theta||\theta$ and:

$$\kappa_i \text{ is a cardinal in } P_\theta \text{ for active } i <_{T^Q} \theta.$$

Hence $Q_\theta||\theta$ is a ZFC model. Contradiction! QED(16)

Thus $\pi^Q_{i,j}``\theta \subset \theta$ and:

$$\pi^Q_{i,j} \colon Q_i \longrightarrow_{\Sigma_\omega} Q_j \text{ cofinally}$$

for $i \leq_{T^Q} j \leq_{T^Q} \theta$. Now let:

Def. $\tilde{Q} =: (J^E_{\bar{\theta}^+})^{P_{\bar{\theta}}}$

(17) $\tilde{Q} = (J^E_{\bar{\theta}^+})^{P_\theta}$.

Proof. Let i be least such that $\bar{\theta} <_T i+1 <_T \theta$ in I^P and i is active in I^P. Let $h = T(i+1)$. Then h is active I^P and $Ph = P_{\bar{\theta}}$. Moreover, $\tau_i = \bar{\theta}^{+P_{\bar{\theta}}}$. Hence: $(J^E_{\bar{\theta}^+})^{P_\theta} = (J^E_{\bar{\theta}^+})^{P_{i+1}} = \tilde{Q}$. QED(17)

But then:

(18) $\tilde{Q} = (J^E_{\bar{\theta}^+})^{Q_\theta} = (J^E_{\bar{\theta}^+})^{Q_{\bar{\theta}}}$.

Proof. $\tilde{Q} = (J^E_{\bar{\theta}^+})^{Q_\theta}$, since $Q_\theta||\theta = P_\theta||\theta$. We must show: $(J^E_{\bar{\theta}^+})^{Q_{\bar{\theta}}} = (J^E_{\bar{\theta}^+})^{Q_\theta}$. If $Q_{\bar{\theta}} = Q_\theta$, this is trivial. If not, there is a least i such that $\bar{\theta} <_T i+1 <_T \theta$ and i is active in I^Q. We can then repeat the proof of (17) on the Q side. QED(18)

We now set: **Def.** $Q' =: (J^E_{\bar{\theta}^+})^Q$, $\tilde{\pi} =: \pi_{0,\bar{\theta}} \upharpoonright Q'$.

Clearly $\tilde{\pi} \colon Q' \longrightarrow_{\Sigma_\omega} \tilde{Q}$. Moreover:

(19) $\tilde{\pi} \upharpoonright \bar{Q}_0 = \bar{\pi}_{0,\theta}$.

Proof. Let $x \in \bar{Q}_0 \subset H$. Since $\sigma \upharpoonright H\text{id}$, we have:

$$\begin{aligned} \tilde{\pi}_{0,\bar{\theta}}(x) = y &\iff \pi_{0,\theta}(x) = \sigma(y) = \pi_{\bar{\theta},\theta}(y) \\ &\iff \pi_{\bar{\theta},\theta}(x) = y. \end{aligned}$$

QED(19)

But then:

(20) $Q' \subset \hat{H}$

Proof. If $X \in \mathbb{P}(\bar{\theta}) \cap Q'$, then, letting $\tilde{X} = \tilde{\pi}(X)$ we have: $X = \bar{\pi}_{0,\bar{\theta}}"\tilde{X} \in \hat{H}$, since $\tilde{X}$, $\bar{\pi}_{0,bar\theta} \in \hat{H}$. Hence $X \in \hat{H}$. But each $x \in Q'$ os canonically coded by an $X \in \mathbb{P}(\bar{\theta} \cap Q'$. Since $\hat{H}$ is a ZFC^- model, x can be recovered from X in $\hat{H}$. Hence $x \in \hat{H}$. QED(20)

Hence:

(21) $Q' \in \hat{H}$.

Proof. Each $x \in Q'$ lies in a $Q'|\nu = \langle J_\nu^{E'}, F \rangle$ where $Q'||\nu$ has size $< \bar{\theta}$ in Q' and $\rho^\omega_{Q'||\nu} = \bar{\theta}$ and $Q'||\nu$ is mouselike. It follows easily that $Q'||\nu \in \bar{\mathbb{S}}$, where $\bar{\mathbb{S}} = \sigma^{-1}(\mathbb{S})$. Hence $Q'||\nu \lhd \bar{S}$, where $\sigma(\bar{S}) = S = \bigcup \mathbb{S}$. Hence $Q' \lhd \bar{S}$, where $\bar{S} \in \hat{H}$. Hence $Q' \in \hat{H}$. QED(21)

Now let: $Q'' = J_\nu^E = \sigma(Q')$. Then Q'' is a premouse extending $Q = \sigma(\bar{Q})$. Set:

$$F = \sigma \restriction (\mathbb{P}(\bar{\theta}) \cap Q')$$

Then F is an extender with base Q' and extension $\langle Q'', \sigma \restriction Q' \rangle$. The length of F is $\theta = \sigma(\bar{\theta})$. Then F is a full extender. F is weakly amenable since $\mathbb{P}(\bar{\theta}) \cap Q' = \mathbb{P}`(\bar{\theta}) \cap Q''$. Then the structure $\langle J_\nu^E, F \rangle$ satisfies all conditions for being an active premouse *except* the initial segment condition. We can remedy this by shortening F. Since θ is regular, there is a least λ such that $\mathrm{ht}(Q') < \lambda < \theta$ and:

$$\sigma(f)(\alpha) < \lambda \text{ whenever } \alpha < \lambda, f \in Q', \text{ and } f \colon \bar{\theta} \longrightarrow \theta.$$

Set $F^* = F|\lambda$. Then F^* is a full extender with base Q' and extenstion $\langle Q^*, \sigma^* \rangle$, where $\sigma^* \colon Q' \longrightarrow_F Q^*$. Let $Q^* = J_{\nu^*}^{E^*}$. Each $x \in J_{\nu^*}^{E^*}$ has the form: $\sigma^*(f)(\alpha)$, where $\alpha < \lambda$ and $f \in Q'$ such that $f \colon \bar{\theta} \longrightarrow Q'$. Hence we can define $\tilde{\sigma} \colon Q^* \longrightarrow_{\Sigma_0} Q''$ by:

$$\tilde{\sigma}(\sigma^*(f)(\alpha) = \sigma(f)(\alpha)$$

for all such α, f. Then $\lambda = \mathrm{crit}(\tilde{\sigma})$ and $\tilde{\sigma}(\lambda) = \theta$. Moreover $\tilde{\sigma}\sigma^* = \sigma \restriction Q'$. Hence:

(22) $Q^*||\lambda = Q''||\lambda = J_\lambda^E$ where $Q'' = J_\tau^E$.

However, we can improve this to:

(23) $Q^* = Q''||\nu^*$.

Proof. Let $\alpha < \nu^*$, Then $\alpha \in \sigma^*(Q'||\eta)$ for an $\eta \in Q'$ such that $\rho^\omega_{Q'||\eta} = \bar{\theta}$. Hence $\sigma^*(Q'||\eta) = Q^*||\sigma^*(\eta)$ where $\rho^\omega_{Q^*||\sigma^*(\eta)} = \lambda$. Moreover:

$$\tilde{\sigma}(Q^*||\sigma(\eta)) = \sigma(Q'||\eta) = Q''||\sigma(\eta),$$

where $\rho^{\omega}_{Q'||\sigma(\eta)} = \theta$. By the condensation Lemma 5.7.4 it follows that: $Q^*||\sigma^*(\eta) = Q''||\sigma^*(\eta)$. QED(23)

However, we can then conclude:

(24) $E^{Q''}_{\nu^*} = \emptyset$.

Proof. Suppose not. Then $Q''|\nu^*$ is a sound premouse and $\rho^{\omega}_{Q''||\nu^*} \leq \lambda$, since λ is the largest cardinal in $J^{E^{Q''}}_{\nu^*}$. But λ is, in fact, a cardinal in Q''. Hence $\rho^{\omega}_{Q''||\nu^*} = \lambda$. But then $Q||\nu^*$ is not 1-small by Lemma 3.8.9. Hence $Q = Q''||\theta$ is not 1-small. Contradiction! QED(24)

By this and Lemma 5.2.8, we then conclude:

(25) F^* is not robust in $\langle J^E_{\nu^*}, F^*\rangle$.

Proof. Suppose not. $Q = N_\theta = M_\theta$ in the Steel array which constructs K^c, since $\theta > \omega$ is regular in V. But λ is a cardinal in Q and:

$$J^{E^Q}_{\nu^*} \models \lambda \text{ is the largest cardinal.}$$

Hence ν^* is cardinally absolute in Q. Since $E^Q_{\nu^*} = \emptyset$. We conclude by Lemma 5.2.8 that $J^E_{\nu^*} = M_i$ for an $i < \theta$ such that N_{i+1} is formed by Option 1. But if F^* were robust in $\langle J^E_{\nu^*}, F^*\rangle$, we would be obligated to use option 2. Contradiction! QED(25)

We now produce the ultimate contradiction by proving:

(26) F^* is robust in $\langle J^E_{\nu^*}, F^*\rangle$.

Proof. The condition 'F^* is robust in $\langle J^E_{\nu^*}, F^*\rangle$' can be reformulated as follows: let $g\colon \omega \longrightarrow \lambda$ and let $X = \langle X_i \mid i \in \omega\rangle$ map ω into $\mathbb{P}(\bar{\theta}) \cap Q''$. Set:

$$D = \{\langle i_1, \cdots, i_n, j\rangle \mid i_1, \cdots, i_n, j < \omega \wedge \prec g(i_1), \cdots, g(i_n) \succ \in F^*(X_j)\}$$

$$A = \{\langle a_1, \cdots, a_n, \varphi\rangle \mid \varphi \text{ is a } \Sigma_1 \text{ formula } \wedge a_1, \cdots, a_n \subset \omega \wedge C^E_{c,\infty} \models \varphi[g\text{"}a_1, \cdots, g\text{"}a_n]\}$$

where $c = \text{lub}\, g\text{"}\omega$. Then there is $\bar{g}\colon \omega \longrightarrow \bar{\theta}$ such that

(a) For all $i_1, \cdots, i_n < \omega$ and $j < \omega$:

$$\prec \bar{g}(i_1), \cdots, \bar{g}(i_n) \succ \in X_j \longleftrightarrow \langle i_1, \cdots, i_n, j\rangle \in D.$$

(b) For all $a_1, \cdots, a_n \subset \omega$ and all Σ_1 formula φ:

$$C^E_{\bar{c},\bar{\theta}} \models \varphi[\bar{g}\text{"}a_1, \cdots, \bar{g}\text{"}a_n] \longleftrightarrow \langle a_1, \cdots, a_n, \varphi\rangle \in A$$

where $\bar{c} = \text{lub}\, \bar{g}\text{"}\omega$.

(We leave it to the reader to verify this formulation.) We first note that A, D are subsets of $\mathbb{P}(H_{\omega_1})^{n+1}$. But then H contain an enumeration of all such subsets by 2^{2^ω}, since H_θ does. Hence $\sigma(A) = A$, $\sigma(D) = D$, since $\sigma \upharpoonright H = \mathrm{id}$.

The existence statement that there is $\bar{g}\colon \omega \longrightarrow \bar{\theta}$ satisfying (a), (b) is a statement about $X, \theta, Q' = J_\tau^{E'}, A, D$ holding $\hat{H}$. Hence it suffices to show that the same statement holds of $\sigma(X) = \langle F(X_i) \mid i \in \omega\rangle, \sigma(\bar{\theta}) = \theta, \sigma(Q^*) = Q'', A = \sigma(A), D = \sigma(D)$ in H_Ω. This is, in fact trivial if we take $\bar{g}$ as being our original g. Then $g\colon \omega \longrightarrow \theta$ and:

(a') For all $i_1, \cdots, i_n < \omega$ and $j < \omega$:

$$\prec g(i_1), \cdots, g(i_n) \succ \in X_j \longleftrightarrow \langle i_1, \cdots, i_n, mj\rangle \in D.$$

(b') For all $a_1, \cdots, a_n \subset \omega$ and all Σ_1 formula φ:

$$C_{c,\theta}^{E''} \models \varphi[g\text{"}a_1, \cdots, g\text{"}a_n] \longleftrightarrow \langle a_1, \cdots, a_n, \varphi\rangle \in A$$

(a') holds since $F^*(X_j) = \lambda \cap F(X_j)$. (b') holds because $C_{c,\theta}^{E''} \prec_{\Sigma_1} C_{c,\infty}^{E''}$. QED(26)

This completes the proof of Theorem 5.7.3.

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[2] Handwritten notes
[3] Handwritten notes