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Luzin and Sierpiński sets in N

Let N be the model as being defined on p. 5 of [BeS]. We aim to verify that there are Luzin and Sierpiński sets in N.

<u>Definition.</u> $L \subset {}^\omega\omega$ is called <u>Luzin</u> iff L is uncountable and $\overline{\overline{L \cap M}} \leq \aleph_0$ for every meager set. $S \subset {}^\omega\omega$ is called <u>Sierpiński</u> iff S is uncountable and $\overline{\overline{S \cap N}} \leq \aleph_0$ for every null set.

In what follows, we shall feel free using the notations introduced in [BeS].

For each $\alpha < \omega_1$, let $\kappa(\alpha) < \omega_1$ be the least κ such that $L_\kappa[z] \models ZFC^-$ for all/some $z \in d_\alpha$.

<u>Lemma</u>. $N \models$ "There is a Sierpiński set."

Proof: Let us define a normal function $f : \omega_1 \longrightarrow \omega_1$ as follows, working entirely inside N.

For $\alpha < \omega_1$, let $\mathcal{N}_\alpha$ be the collection of all G_δ null sets which have a real code in $L_{k(\alpha)}[z]$ for some/all $z \in d_\alpha$. Notice that $\mathcal{N}_\alpha$ is countable, so that $\bigcup \mathcal{N}_\alpha$ is a null set for all $\alpha < \omega_1$.

Given $\alpha < \omega_1$, let $f(\alpha)$ be the least $\beta > \alpha$ such that there is some $x \in {}^\omega\omega$ such that for all/some $z \in d_\beta$, and f.a./some $\bar{z} \in d_\alpha$:

$$(*) \qquad x \in L_{k(\beta)}[z] \setminus \left(L_{k(\alpha)}[\bar{z}] \cup \bigcup \mathcal{N}_\alpha \right)$$

For limit λ, let $f(\lambda) = \sup_{\alpha < \lambda} f(\alpha)$.

We then let S be the collection of reals x s.t. $(*)$ holds ~~with~~ for some

$\alpha < \omega_1$, with β being $f(\alpha)$, i.e.,

$$S = \{ x \in {}^\omega\omega : \exists \alpha < \omega_1 \, [x \in L_{\kappa(f(\alpha))}[z] \setminus (L_{\kappa(\alpha)}[\bar{z}] \cup \bigcup \mathcal{N}_\alpha) \text{ for some/all } z \in d_{f(\alpha)} \text{ and } \bar{z} \in d_\alpha] \}.$$

It is easy to see that S is a Sierpiński set. $\dashv$

Virtually the same proof shows:

<u>Lemma.</u> $N \models$ "There is a Luzin set."

Proof: As the proof of the previous lemma, replacing $\mathcal{N}_\alpha$ with the collection of all meager sets which have a real code in $L_{\kappa(\alpha)}[z]$, some/all $z \in d_\alpha$. $\dashv$

For the record, let us also state:

<u>Lemma.</u> $N \models$ "There is no Hamel basis."

This immediately follows from [BeS]

together with the following.

Recall that a <u>Hamel basis</u> is a basis for $\mathbb{R}$ construed as a vector space over $\mathbb{Q}$.

<u>Lemma</u> (folklore). In ZF, if there is a Hamel basis, then there is a Vitali set.

Proof: Fix a Hamel basis B. For each x, there is a unique finite $b_x \subset B$ of least size such that $[x]_{E_0} \subset \langle b_x \rangle$. Using a well-ordering of the finite sequences of rationals, we may then for each $x \in {}^\omega\omega$ pick $y \in [x]_{E_0}$ such that if $y = \sum \vec{r}\, b_x$, $\vec{r} \in {}^{<\omega}\mathbb{Q}$, then $\vec{r}$ is the least $\vec{r}'$ s.t. $\sum \vec{r}' b_x \in [x]_{E_0}$. This gives a Vitali set. ⊣

We showed there are Bernstein, Luzin and Sierpiński sets in N, but no Vitali sets and no Hamel basis.

[BeS] Beriashvili, Schindler, Bernstein sets don't give Vitali sets.