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Vitali sets and well-orderings of $\mathbb{R}$

According to [PP, p. 433], it was S. Feferman who realized that the model H which is studied in [PP] (and which is refered to as the "Cohen-Halpern-Lévy model) has a Vitali set, whereas there is no well-ordering of $\mathbb{R}$ in that model as had been shown before by P. Cohen. In the present note, we aim to bring this result back to memory.

Before doing so, let us note that the following still seems to be open, cf. [PP, p. 433].

Question. Assume that there is a Hamel basis, by which we just mean that $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$, has a basis. Must there be a well-ordering of the reals? Must there be a Sierpiński set? A Luzin set? A Bernstein set?

Let G be $\mathbb{C}(\omega)$-generic over L, and let $(c_n : n<\omega)$ be the sequence of Cohen reals added. Let

$$H = \mathrm{HOD}^{L[G]}_{\{c_n : n<\omega\} \cup \{\{c_n : n<\omega\}\}}.$$

By Cohen, $H \models ZF +$ "there is no well-ordering of the reals."

Suppose $L[G]$ were to define a well-ordering of the reals from the parameters $\vec{\alpha}$, $c_0, \ldots, c_{k-1}$, $\{c_n : n<\omega\}$. For notational simplicity, let's assume $k=0$. (As $L[G]$ is $\mathbb{C}(\omega)$-generic over $L[c_0, \ldots, c_{k-1}]$, the argument to follow is virtually the same in the general case.)

Let $m > k$. For some γ and some formula φ,

$$c_m(k) = n \quad \text{iff} \quad L[G] \models \varphi(k, n, \vec{\alpha}, \gamma, \{c_n : n<\omega\})$$

$$\text{iff} \quad \exists p \in G \;\; p \Vdash^{\mathbb{C}(\omega)}_{L} \varphi(\check{k}, \check{n}, \vec{\check{\alpha}}, \check{\gamma}, \tau), \qquad (*)$$

where τ is a canonical name for the set $\{c_n : n<\omega\}$.

We claim that (*) is equivalent to

$$1\!\!1 \Vdash^{\mathbb{C}(\omega)}_{L} \varphi(\check{k}, \check{n}, \vec{\check{\alpha}}, \check{\gamma}, \tau).$$

If $s \in \mathbb{C}(\omega)$, $s \Vdash^{\mathbb{C}(\omega)}_{L} \neg\varphi(\check{k}, \check{n}, \vec{\check{\alpha}}, \check{\gamma}, \tau)$, then we may easily find automorphisms $\pi, \pi' \in L$ and $q \in \mathbb{C}(\omega)$ such that

$q \leq \pi(p)$, $q \leq \pi'(s)$.

Let H be $\mathbb{C}(\omega)$-generic over L. Then $\pi^{-1}{}''H$ and $\pi'^{-1}{}''H$ are $\mathbb{C}(\omega)$-generic over L, $p \in \pi^{-1}{}''H$, $s \in \pi'^{-1}{}''H$, and $L[H] = L[\pi^{-1}{}''H] = L[\pi'^{-1}{}''H]$.

We may in fact let $\pi = \mathrm{id}$ and π' is given by a permutation e of ω such that

$e'' \operatorname{supp}(s) \cap \operatorname{supp}(p) = \emptyset$, and

$\operatorname{dom}(\pi'(r)(\ell)) = \operatorname{dom}(r(e(\ell)))$ and

$\pi'(r)(\ell)(\ell') = r(e(\ell))(\ell')$ for $\ell, \ell' < \omega$.

Then $\tau^H = \tau^{\pi^{-1}{}''H} = \tau^{\pi'^{-1}{}''H}$, which gives a contradiction.

But then $c_m \in L$, contradiction!

The argument in fact shows that $\{c_n : n < \omega\}$ cannot be enumerated in H.

<u>Claim 1.</u> (Feferman) $H \models$ "There is a Vitali set."

Prf.: Let us work inside H.

For each $x \in {}^\omega\omega$ there is a unique finite set $C \subset \{c_n : n<\omega\}$ such that

- $x \in L[C]$, and
- if $C' \subset \{c_n : n<\omega\}$ is finite with $x \in L[C']$, then $C \subset C'$.

To see this, notice first that the argument on pp. 2–3 above shows that for all $x \in {}^\omega\omega$ there is some $C \subset \{c_n : n<\omega\}$ finite with $x \in L[C]$. Now let $C, C' \subset \{c_n : n<\omega\}$ both be finite with $x \in L[C] \cap L[C']$. Write $D = C \cap C'$. $C \setminus D$ is $\mathbb{C}$-generic over $L[D]$, and $C' \setminus D$ is $\mathbb{C}$-generic over $L[D, C\setminus D]$, so that $x \in L[D]$. (Cf. [Sch, p. 123, Problem # 6.12]).

The desired C is hence the intersection of all finite $C' \subset \{c_n : n<\omega\}$ with $x \in L[C']$.

Let us write $C(x)$ for this unique finite $C \subset \{c_n : n<\omega\}$.

Notice that $C(x)$ only depends on the constructibility degree of x, i.e., if $L[x] = L[y]$, then $C(x) = C(y)$. Let us write $[x] = \{y : L[x] = L[y]\}$ for the constructibility degree of x, and $\mathcal{C} = \{[x] : x \in {}^\omega\omega\}$. For $d \in \mathcal{C}$, we also write $C(d)$ for $C(x)$ for some / all $x \in d$.

For $d \in \mathcal{C}$, let $\vec{c}(d)$ be the enumeration of $C(d)$ in lexicographic order. We may then associate to each $d \in \mathcal{C}$ the canonical well-ordering of ${}^\omega\omega \cap L[\vec{c}(d)]$. Let us write $<_d$ for this well-ordering of ${}^\omega\omega \cap L[\vec{c}(d)]$.

We may now easily define a Vitali set: For each $[x]_{E_0} = \{y : y \, E_0 \, x\}$, let us pick the $<_d$ - least y such that $y \, E_0 \, x$, where $d = [x]$.

(This is well-defined, as $[x]$ of course only depends on the E_0-degree of x.)

$\dashv$ (Claim 1)

<u>Claim 2.</u> ([PP]) $H \models$ "There is a Luzin set".

Prf.: Again, let us work inside H.

We have

(*) Let $A = \bigcup \{A_n : n < \omega\} \subset {}^\omega\omega$ be such that each A_n is countable. Then A is countable.

This is because for each n we may ~~pick~~ let $C(n)$ the unique finite $C \subset \{c_n : n < \omega\}$ which is $\subset$-minimal such that an enumeration of A_n exists inside $L[C]$; letting $\vec{c}$ be the enumeration of $C(n)$ according to $<_{lex}$, we may then let $e_n : \omega \to A_n$ be the $<_{L[\vec{c}]}$-least enumeration of A_n inside $L[\vec{c}]$.

Then $(n,m) \mapsto e_n(m)$ is onto A. Hence (*).

This shows that $\Lambda \subset {}^\omega\omega$ is Luzin iff $\Lambda \setminus \mathcal{O}$ is at most countable for every $\mathcal{O}$ which is a (ctble.) union of open intervals with rational end-points, and Λ is uncountable.

Let $\Lambda \in L$ be such that Λ is Luzin inside L. We claim that Λ is Luzin in H.

Let $((p_i, q_i) : i < \omega)$ be an enumeration of all open intervals with rational end-points.

Let $X \subset \omega$, $X \in H$, be such that $\bigcup_{i \in X} (p_i, q_i)$ is dense. Suppose that $\Lambda \setminus \bigcup_{i \in X} (p_i, q_i)$ would not be countable. Let τ be a name for X, and let $p \in \mathbb{C}(\omega)$ be such that

$$p \Vdash^{\mathbb{C}(\omega)}_{L} \text{“} \check{\Lambda} \setminus \bigcup_{i \in \tau} (p_i, q_i) \text{ is not countable.”}$$

As $\mathbb{C}(\omega)$ is countable, there is <u>one</u> $q \leq p$ such that for uncountably many $x \in \Lambda$,

$$q \Vdash^{\mathbb{C}(\omega)}_{L} \text{“} \check{x} \notin \bigcup_{i \in \tau} (p_i, q_i) \text{”}.$$

Let $U \subset \Lambda$ be the set of such x.

Let $\mathcal{O} = \bigcup \{ (p_i, q_i) : \exists r \leq q \; r \Vdash^{\mathbb{C}(\omega)}_{L} \check{i} \in \tau \}$.

Clearly, $\mathcal{O} \supset \bigcup_{i \in X} (p_i, q_i)$ and is hence dense and open. $\mathcal{O} \in L$, so $\Lambda \setminus \mathcal{O}$ must be countable. But $U \subset \Lambda \setminus \mathcal{O}$ and U is uncountable. Contradiction! $\dashv$

<u>Claim</u> 3. $H \models$ "There is a Sierpiński set."

Prf.: Let $S \in L$ be any Sierpiński set in L. Suppose that S is not Sierpiński in ~~L~~ H, so that there is some null set $N \in H$ with $S \cap N$ not being countable. Let $p \in \mathbb{C}(\omega)$ and τ be a name for N such that

$$p \Vdash^{\mathbb{C}(\omega)}_{L} \text{"} \check{S} \cap \tau \text{ is not countable, and } \tau \text{ is null."}$$

For any $q \leq p$, let

$$A_q = \{ x \in \mathbb{R} \cap L : q \Vdash^{\mathbb{C}(\omega)}_{L} \check{x} \in \tau \}.$$

If G_q is $\mathbb{C}(\omega)$-generic with $q \in G_q$, then $A_q \subset \tau^{G_q}$ and A_q is null in $L[G_q]$.

We may have picked N as a G_δ-set, say $N = \bigcap_n \mathcal{O}_n$, where each $\mathcal{O}_n$ is the union of open intervals with $\mu(\mathcal{O}_n) \leq \frac{1}{n+1}$. For $n<\omega$ let $\mathring{\mathcal{O}}_n$ be a name for $\mathcal{O}_n$, and let

$$\sigma = \{ ((n,i)^\vee, q) : q \Vdash (p_i, q_i) \subset \mathring{\mathcal{O}}_n \}.$$

The relation $q \Vdash \check{x} \in \tau$ can then be written as $\forall n \exists i \, [((n,i)^\vee, q) \in \sigma \wedge p_i < x < q_i]$ which is arithmetic in σ. The fact that A_q is null may then be written as a Σ^1_2 statement. As A_q is null in $L[G_q]$, it must be null in L. But now $A = \bigcup_{q \leq p} A_q$ is, in L, a countable union of null sets, hence null in L.

Therefore, $S \cap A$ is countable. On the other hand, $\tau^G \subset A$, and $S \cap \tau^G$ is uncountable in $L[G]$. Contradiction! $\dashv$

Claim 4. $H \models$ "There is a Bernstein set".

Pf.: We again work in H. Let

$$B = \{b \in {}^\omega\omega : \exists \text{ even } n<\omega\, [\, 2^n < \overline{\overline{C(b)}} \leq 2^{n+1}]\, \},$$

and $$B' = \{b \in {}^\omega\omega : \exists \text{ odd } n<\omega\, [\, 2^n < \overline{\overline{C(b)}} \leq 2^{n+1}]\, \},$$

where $C(-)$ is as on p. 5.

Let $P = [T]$, where T is a perfect tree on ω. Let $\overline{\overline{C(T)}} < 2^n$, where n is even; let $c_{i_1}, \ldots, c_{i_{\overline{\overline{C(T)}}}}$ be such that $T \in L[c_{i_1}, \ldots, c_{i_{\overline{\overline{C(T)}}}}]$, $i_n \neq i_m$ for $n \neq m$, and pick $c_{i_{\overline{\overline{C(T)}}+1}}, \ldots, c_{i_{2^{n+1}}}$ such that still $i_n \neq i_m$ for $n \neq m$. Write $c = \bigoplus_{n=1}^{2^{n+1}} c_{i_n}$. It is easy to find some $b \in {}^\omega\omega$ such that $L[T \oplus b] = L[c]$, and $b \in [T]$.

As $b \in L[c]$, $C(b) \subset \{c_1, \ldots, c_{i_{2^{n+1}}}\}$. If $\overline{\overline{C(b)}} \leq 2^n$, then $T \oplus b \in L[C(T) \cup C(b)]$, and $\overline{\overline{C(T) \cup C(b)}} < 2^{n+1}$, which contradicts $C(T \oplus b) = C(c)$ and $\overline{\overline{C(c)}} = 2^{n+1}$. Therefore, $\overline{\overline{C(b)}} > 2^n$, and of course $\overline{\overline{C(b)}} \leq 2^{n+1}$.

Hence $b \in P \cap B$. We have shown that

$B \cap P \neq \emptyset$ for every perfect P.

Virtually the same argument shows that $B' \cap P \neq \emptyset$ for every perfect P.

We have shown that B is a Bernstein set. $\dashv$

We do not know if H has a Hamel basis. Cf. [PP, p. 433].

[PP] D. Pincus, K. Prikry, Luzin sets and well-orderings of the continuum, Proc. Amer. Math. Soc. 49 (1975), pp. 429—435.

[Sch] R. Schindler, Set theory. Exploring independence and truth, Springer-Verlag 2014.