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Realizing a model of "there is a cofinal ω_1-sequence of E_0-degrees but no well-ordering of the reals" as a symmetric model

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We wish to thank L. Halbeisen for his interest in seeing the model described in [BeS1] and [BeS2] being represented as a symmetric model.

Let us call φ a <u>good permutation</u> iff there is some $n<\omega$ such that $\varphi : {}^n\omega \to {}^n\omega$ is a permutation of ${}^n\omega$ and if $k \leq n$, $s, t \in {}^n\omega$, and $s\restriction k = t\restriction k$, then $\varphi(s)\restriction k = \varphi(t)\restriction k$. Every good permutation φ induces an automorphism, call it π^φ, of $\mathbb{C}$: If $\varphi : {}^n\omega \to {}^n\omega$, then

$$\pi^\varphi(p) = \begin{cases} \varphi(s)\restriction lh(p) & \text{for some/all } s \text{ with } s \supset p \text{ if } lh(p) \leq n \\ \varphi(p\restriction n)^\frown p\restriction[n, lh(p)) & \text{o.w.} \end{cases}$$

Let $\vec{\varphi} = (\varphi_i : i<\omega_1)$ be such that every φ_i is a good permutation and $\varphi_i = id$ for all but finitely many $i<\omega_1$. We then write $\pi^{\vec{\varphi}}$ for the automorphism of $\mathbb{C}(\omega_1)$

which is defined by

$$\pi^{\vec{\varphi}}(p)(i) = \pi^{\varphi_i}(p(i)).$$

Notice that $\pi^{\vec{\varphi}}(p)(i) = p(i)$ for all but finitely many $i<\omega_1$.

Let us write G for the group of all automorphisms of $\mathbb{C}(\omega_1)$ of the form $\pi^{\vec{\varphi}}$, where $\vec{\varphi} = (\varphi_i : i<\omega_1)$ is a sequence of good permutations and $\varphi_i = \mathrm{id}$ for all but finitely many $i<\omega_1$.

For each countable $I \subset \omega_1$, let H^I be the collection of all $\pi^{\vec{\varphi}} \in G$ such that $\varphi_i = \mathrm{id}$ (and hence $\pi^{\varphi_i} = \mathrm{id}$ and $\pi^{\vec{\varphi}}(p)(i) = p(i)$) for all $i \in I$. We write $\mathcal{F}$ for the collection of all $H \lhd G$ such that $H^I \lhd H$ for some countable $I \subset \omega_1$. Notice that $H^I \cap H^{I'} = H^{I \cup I'}$. It is easy to verify then that $\mathcal{F}$ is a filter, cf. [Sch, p. 126, Problem 6.22]. *)

Now let g be $\mathbb{C}(\omega_1)$-generic over L, and let $N \subset L[g]$ be the collection of

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*) [Sch] R. Schindler, Set theory, Springer-Verlag 2014.

all τ^g, where τ is symmetric, i.e.,

$$\mathrm{sym}_G(\tau) = \{\pi \in G : \pi(\tau) = \tau\} \in \mathcal{F}.$$

It is easy to see that every real in $L[g]$ has a symmetric name: Let $x \in {}^\omega\omega \cap L[g]$, say $x = \sigma^g$. Let $\tau = \{((n,m)^\vee, p) : p \in A_n \wedge p \Vdash \sigma(\check{n}) = \check{m}\}$, where A_n is a maximal antichain of conditions deciding $\sigma(\check{n})$, $n<\omega$. Let $I \subset \omega_1$ be the collection of all $i<\omega_1$ such that for some $n<\omega$ and some $p \in A_n$, $p(i) \neq \emptyset$. As $\mathbb{C}(\omega_1)$ has the c.c.c., I is at most countable, and if $\pi \in H^I$, then $\pi(\tau) = \tau$. But $\tau^g = \sigma^g = x$. We have shown that ${}^\omega\omega \cap L[g] \subset N$.

We claim that if $(c_i : i<\omega_1)$ is the sequence of Cohen reals given by g, then $(c_i : i<\omega_1) \notin N$, and also that N does not have a well-ordering of its reals. Both of these facts follow immediately from:

(*) If $A \subset OR$, $A \in N$, then $A \in L[x]$ for some $x \in {}^\omega\omega \cap N = {}^\omega\omega \cap L[g]$.

To verify $(*)$, let $A = \tau^{g}$, where τ is symmetric, say $H^{I} \subset \mathrm{sym}_{G}(\tau)$, $I \subset \omega_1$ being countable. Then $\xi \in A$ iff $\exists p \in g \; p \Vdash \check{\xi} \in \tau$. However, $p \Vdash \check{\xi} \in \tau$ is equivalent to $\pi(p) \Vdash \check{\xi} \in \tau$ for every $\pi \in H^{I}$, so that in fact

$p \Vdash \check{\xi} \in \tau$ is equivalent with $\bar{p} \Vdash \check{\xi} \in \tau$, where $\bar{p}(i) = p(i)$ for $i \in I$ and $\bar{p}(i) = \emptyset$ otherwise. But this means that A can be computed inside $L[g \restriction I]$. As $g \restriction I$ may be coded by a real in $L[g \restriction I]$, $(*)$ is shown.

Writing again $(c_i : i < \omega_1)$ for the sequence of Cohen reals given by g, let us set $\bar{d}_i = [c_i]_{E_0} = \{ c \in {}^{\omega}\omega : \exists n_0 \, \forall n \geq n_0 \;\; c(n) = c_i(n) \}$ for $i < \omega_1$. Let us verify that $(\bar{d}_i : i < \omega_1)$ has a symmetric name. Well, $\{ (p(i)^{\vee}, p) : p \in \mathbb{C}(\omega_1) \}$ is the canonical name for the strict initial segments of c_i, and for each good permutation φ, $\{ (\pi^{\varphi}(p(i))^{\vee}, p) : p \in \mathbb{C}(\omega_1) \}$ is the canonical name for the strict initial segments of a typical real which is E_0-equivalent to c_i. Then

$$\tau_i = \{ (\{ (\pi\varphi(p(i)))^{\vee}, p) : p \in \mathbb{C}(\omega_1)\}, \mathbb{1}_{\mathbb{C}(\omega_1)}) : \varphi \text{ a good permutation} \}$$

is the canonical name for $\bar{d}_i$. It is very easy to verify that $\pi(\tau_i) = \tau_i$ for every $\pi \in G$, i.e., $\mathrm{sym}_G(\tau_i) = G$: if $\pi\vec{\varphi} \in G$, then $\{ \pi\varphi^{(i)} \circ \pi\varphi : \varphi \text{ a good permutation} \} = \{ \pi\varphi : \varphi \text{ a good permutation} \}$. Hence every τ_i is symmetric, and $\{ ((i, \tau_i), \mathbb{1}_{\mathbb{C}(\omega_1)}) : i < \omega_1 \}$ is a symmetric name for $\vec{d} = (\bar{d}_i : i < \omega_1)$. [We have abused the notation by writing (i, τ_i) for the canonical ~~pair~~ name for the pair of i and the interpretation of τ_i.]

Now let the reals z_i, $i < \omega_1$, be defined as in [BeS1]. Let us repeat this definition for the convenience of the reader.

For $i < \omega_1$, let $\beta(i)$ be the least $\beta > \max(\omega, i)$ such that $L_\beta \models$ "i is countable," and let $e_i : \omega \leftrightarrow L_i$ be the $L_{\beta(i)}$-least bijection, $e_i \in L_{\beta(i)}$. Let $E_i \subset \omega^2$ be s.t. $(\omega; E_i) \overset{e_i}{\cong} (L_i; \in)$, and let g_i be the set of all $n<\omega$ s.t. there are $j<i$ and $k,m<\omega$ with $e_i(n)=(j,k,m)$ and $p(j)(k)=m$ for some $p\in g$. I.e., E_i is a canonical code for L_i, and g_i codes $g\upharpoonright i$ relative to E_i. $z_i \in {}^{\omega}\omega$ is defined by

$$z_i(2\ell) = \begin{cases} 1 & \text{iff } (\ell)_0 \; E_i \; (\ell)_1 \\ 0 & \text{iff } (\ell)_0 \; \not{E}_i \; (\ell)_1 \end{cases}$$

$$z_i(2\ell+1) = \begin{cases} 1 & \text{iff } \ell \in g_i \\ 0 & \text{iff } \ell \notin g_i \end{cases},$$

where $((\ell)_0, (\ell)_1) = e(\ell)$ for some canonical $e:\omega \leftrightarrow \omega\times\omega$. (Cf. [BeS1, pp. 1 f.].) As on p.4 of [BeS1], let us write $d_i = [z_i]_{E_0}$ for the E_0-equivalence class of z_i, $i<\omega_1$.

(**) For all $x \in {}^{\omega}\omega \cap N = {}^{\omega}\omega \cap L[g]$ there is some $i<\omega_1$ such that $x \in L[z]$ for all $z \in d_i$, in fact $x \leq_T z$ for all $z \in d_i$. [Here, $\leq_T$ denotes Turing reducibility. We may assume w.l.o.g. that e.g. $e_i(2n)=n$ for all $n<\omega$ and all $i<\omega_1$.]

The proof of $(**)$ is trivial.

By $(*)$, $(z_i : i<\omega_1) \notin N$. A simple variat of the proof that $\vec{\bar{d}} = (\bar{d}_i : i<\omega_1) \in N$ shows that $\vec{d} = (d_i : i<\omega_1) \in N$, though: Notice that for $i<\omega_1$, $z_i \upharpoonright \mathrm{Even} \in L$; a canonical name for $z_i \upharpoonright \mathrm{Odd}$ is then computable in a trivial way from the (check name of) $z_i \upharpoonright \mathrm{Even}$ and the canonical name for $g \upharpoonright i$; ~~this~~ a canonical name δ_i for $d_i = [z_i]_{E_0}$ may then be obtained in a fashion as the name for $\bar{d}_i$ was obtained on pp. 4f. above. We will have that $\mathrm{sym}_G(\delta_i) = G$. The point is that every $\pi\vec{\varphi} \in G$ is non-trivial only at finitely many $j<\omega_1$, i.e., if $\vec{\varphi} = (\varphi_j : j<\omega_1)$, then $\varphi_j = \mathrm{id}$ for all but finitely many $j<\omega_1$, so that $\pi\vec{\varphi}$ will move $z_i \upharpoonright \mathrm{Odd}$ to an E_0-equivalent variant thereof.

The proofs of Claim 5, 6 of [BeS1] and of the lemmas in [BeS2] go thru as before.

[BeS1] M. Beriashvili, R. Schindler, Bernstein sets don't give Vitali sets.

[BeS2] M. Beriashvili, R. Schindler, Luzin and Sierpiński sets in N.