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Mazurkiewicz sets in the Cohen-Halpern-Levy model

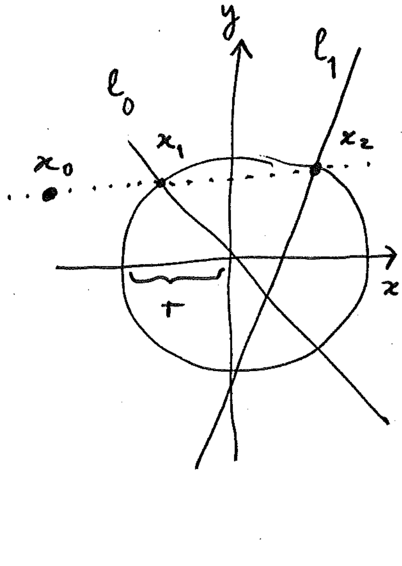
If $x, y \in \mathbb{R}^2$, $x \neq y$, then we write $\ell(x,y)$ for the (straight) line given by x, y.

A set $M \subset \mathbb{R}^2$ is called a <u>Mazurkiewicz set</u> (or, <u>2-point set</u>) iff $\overline{\overline{M \cap \ell}} = 2$ for every line $\ell = \ell(x,y)$, some $x, y \in \mathbb{R}^2$, $x \neq y$.

ZFC (or, ZF + there is a well-order of $\mathbb{R}$) proves the existence of a Mazurkiewicz set.

We will combine arguments from [CKS] and [M] to show that in the Cohen-Halpern-Levy model there is a Mazurkiewicz set.

We first steal the following statement from [CKS], see the proof of Lemma 4.1.



<u>Lemma 1.</u> Let ℓ_0, ℓ_1 two lines, $\ell_0 \neq \ell_1$, and let $x_0 \in \mathbb{R}^2 \setminus (\ell_0 \cup \ell_1)$. Let $x_1 \in \ell_0$, $x_2 \in \ell_1$, $\|x_1\| = \|x_2\|$, $x_1 \neq x_2$, and suppose that x_0, x_1, x_2 are collinear. Then $\|x_1\| = r$ is the solution of a

polynomial in a way that τ is computable from x_0, y_0, y_1, z_0, z_1 for all $y_0 \neq y_1$, $z_0 \neq z_1$ with $\ell_0 = \ell(y_0, y_1)$, $\ell_1 = \ell(z_0, z_1)$.

Let $(c_n : n < \omega)$ be a sequence of mutually generic Cohen reals over L, and write $A = \{c_n : n < \omega\}$. Then $L(A)$ is the associated Cohen–Halpern–Levy model, see e.g. [BSWY].

$\mathbb{R} \cap L(A) = \bigcup \{\mathbb{R} \cap L[a] : a \in [A]^{<\omega}\}$.

For an ordinal α we write $\mathbb{R}_\alpha = \bigcup \{\mathbb{R} \cap L_\alpha[a] : a \in [A]^{<\omega}\}$. If α is countable, then $\mathbb{R}_\alpha \subset L_\alpha[(c_n : n < \omega)]$, so $(\mathbb{R} \cap L) \setminus \mathbb{R}_\alpha \neq \emptyset$.

It is thus easy to construct in $L(A)$ a sequence $(\alpha_i, x_i : i < \omega_1)$ s.t. $(\alpha_i : i < \omega_1)$ is a strictly increasing and continuous sequence of countable ordinals and

$$x_i \in (\mathbb{R} \cap L_{\alpha_{i+2}}) \setminus \mathbb{R}_{\alpha_{i+1}}$$

for every $i < \omega_1$.

We are now going to prove the following, using the argument from [M], pp. 5-6, cf. also Theorem 4.2 of [CKS]:

Theorem 1. There is a Mazurkiewicz set in the Cohen-Halpern-Levy model $L(A)$.

Proof: In what follows, we sometimes identify reals with points in $\mathbb{R}^2$. We will recursively construct $(M_i : i<\omega_1)$ s.t. $M_i \subset M_j$ for $i \leq j$ and each M_i is a partial Mazurkiewicz set in the sense that if $x, y \in \mathbb{R}_{\alpha_{i+1}}$ and $\ell(x,y)$ has distance $\leq n(i)$ to $(0,0)$, then $\overline{\overline{\ell(x,y) \cap M_i}} = 2$, and M_i doesn't have 3 distinct collinear points; also, $M_i \subset \mathbb{R}_{\alpha_{i+2}}$. Here, $n(i)$ is the unique $n<\omega$ s.t. $i = \lambda + n$ for $\lambda =$ the largest limit ordinal $\leq i$.

$\bigcup\{M_i : i<\omega_1\}$ will then be a Mazurkiewicz set.

Fix $i<\omega$, and suppose all M_j, $j<i$ have been constructed.

Let $L = \{ \ell(x,y) : x,y \in \mathbb{R}_{\alpha_{i+1}}, \ \ell(x,y) \cap C_{n(i)} \neq \emptyset \}$, when $C_{n(i)}$ is the circle around $(0,0)$ with radius $n(i)$.

For all $\ell \in L$ pick $a_\ell \subset \ell$, $\overline{\overline{a_\ell}} \leq 2$ s.t.
$\overline{\overline{\ell \cap \left(\bigcup_{j<i} M_j \cup a_\ell \right)}} = 2$, $a_\ell \subset C_{n(i)+x_i}$ = the circle with center $(0,0)$ and radius $n(i)+x_i$.

Set $M_i = \bigcup_{j<i} M_j \cup \bigcup \{ a_\ell : \ell \in L \}$.

We claim that M_i is as desired. We have to see that M_i doesn't have 3 collinear points. Deny, and let x, y, z be a counterexample.

The cases $\{x,y,z\} \subset \bigcup_{j<i} M_j$, $\{x,y,z\} \subset M_i \setminus \bigcup_{j<i} M_j$, and $x \in M_i \setminus \bigcup_{j<i} M_j$, $\{y,z\} \subset \bigcup_{j<i} M_j$ are trivially ruled out. $\{x,y\} \subset M_i \setminus \bigcup_{j<i} M_j$, $z \in \bigcup_{j<i} M_j$ contradicts Lemma 1:

Let $x \in a_\ell$, $y \in a_{\ell'}$. We cannot have $\ell = \ell'$. If $\ell = \ell(x_1, x_2)$, $\ell' = \ell(y_1, y_2)$, then by Lemma 1, $n(i) + x_i$ would be computable from $\mathbb{Z}$, $x_1, x_2, y_1, y_2 \in \mathbb{R}_{\alpha_{i+1}}$, so $x_i \in \mathbb{R}_{\alpha_{i+1}}$ $\lightning$ $\dashv$

[CKS] Chad, Knight, Suabedissen, "Set theoretic constructions of two-point-sets" Fund. Math.

[M] A. Miller, "The axiom of choice and two-point sets in the plane."

[BSWY] Beriashvili, Schindler, Wu, Yu, "Hamel bases and well-ordering the continuum," Proc. Americ. Math. Soc.

Appendix to: "Mazurkiewicz sets in the Cohen-Halpern-Levy model"

The following states an abstract version of what we did above.

<u>Theorem 2.</u> Assume ZF. Suppose that there is $(R_i, r_i : i<\alpha)$ such that $R_i \subset R_j$, $i \le j$, $\mathbb{R} = \bigcup_{i<\alpha} R_i$, for all $x_1, \dots, x_k \in R_i$, r_i is not computable from $x_1, \dots, x_k$, and if y is computable from $r_i, x_1, \dots, x_k$, $x_1, \dots, x_k \in R_i$, then $y \subset R_{i+1}$. Then there is a Mazurkiewicz set.

Proof: Construct $(M_i : i<\alpha)$ s.t. $M_i \subset M_j$, $i \le j$, no M_i has 3 collinear points, $\overline{\overline{\ell(x,y) \cap M_i}} = 2$ for all $x, y \in R_i$, $M_i \subset R_{i+1}$.

Fix i, and suppose M_j, $j<i$, are given. Write $M = \bigcup_{j<i} M_j$. For all $x, y \in R_i$ s.t. $\ell = \ell(x,y)$ intersects the circle C_{r_i} around $(0,0)$ with

radius r_i in 2 points we pick

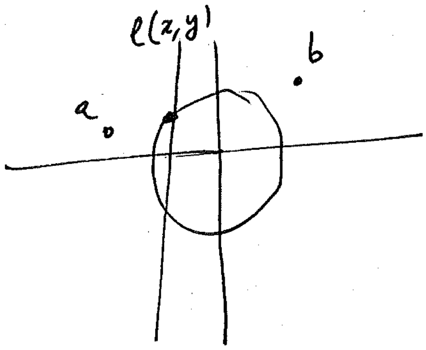
$a_\ell \in c \cap \ell$ s.t. $\overline{\overline{\ell \cap (M \cup a_\ell)}} = 2$.

Let $M_i = M \cup \bigcup \{ a_\ell : \ell = \ell(x,y),$ some $x, y \in R_i \}$.

We only need to see M_i doesn't have 3 collinear points.

<u>1st case.</u> M_i has 3 coll. old points. no!

<u>2nd case.</u> M_i has 2 old points, one new point which are coll.



The new point is then computable from elements of R_i, so that r_i would become computable from elements of R_i. ⚡

<u>3rd case.</u> M_i has 1 old point, 2 new points which are coll.

This contradicts Lemma 1.

<u>4th case.</u> M_i has 3 new points which are coll. no!