Mazurkiewicz sets with no well-ordering of the reals

Mariam Beriashvili and Ralf Schindler

March 1, 2021

Abstract

There is a Mazurkiewicz set in the Cohen-Halpern-Levy model.

In [7], Stefan Mazurkiewicz presented the construction of a "pathological" subset M of the Euclidean plane $\mathbb{R}^2$ with the property that every line in the plane meets M in exactly two points (see e.g. [5, p. 212f.]). A set $M \subset \mathbb{R}^2$ is thus called a *Mazurkiewicz set* iff

$$\overline{\overline{M \cap \ell}} = 2$$

for every line ℓ in $\mathbb{R}^2$. Mazurkiewicz sets are also called "two-point sets."

The argument from [7] exploits (a fragment of) ZF plus the existence of a well-ordering of the reals. Mazurkiewicz sets may have a complicated descriptive set theoretical structure. E.g., in ZFC there is a Mazurkiewicz set which is nowhere dense and of Lebesgue measure zero and there is also a Mazurkiewicz set which is Lebesgue nonmeasurable and does not have the Baire property, cf. [3], see also [4]; moreover, no Mazurkiewicz set is F_σ, cf. [6], see also [5, p. 211f.].

It is not known if there is a Mazurkiewicz set which is analytic (or even Borel). Arnold Miller has shown in [8] that in Gödel's Constructible Universe L, there is a Mazurkiewicz set which is co-analytic.

Still in ZFC, there is a Mazurkiewicz set which is simultaneously a Hamel basis, cf. [4]. In joint work with Liuzhen Wu and Liang Yu, the present authors showed that there is a Hamel basis in the Cohen-Halpern-Lévy model (in which there is an infinite Dedekind-finite set of reals), see [1].

The current paper adds information about Mazurkiewicz sets in models of ZF with no well-ordering of the reals. Arnold Miller has shown that there exists a model of ZF with an infinite Dedekind-finite set of reals in which there is a Mazurkiewicz set, see [9, Theorem 5]. His model is the forcing extension of the Cohen-Halpern-Lévy model obtained by adding ω_1 Cohen reals. We shall prove here that there is a Mazurkiewicz set already in the Cohen-Halpern-Lévy model. In fact we shall present a general sufficient criterion for a Mazurkiewicz set to exist and show that it applies in the Cohen-Halpern-Levy model.

If x, y are reals, then we write $x \leq_T y$ to denote that x is Turing computable from y. If A is a set of reals, then we write

$$\text{comp}(A) = \{x \in \mathbb{R} : \exists x_1, \ldots, x_k \in A \, x \leq_T x_1 \oplus \ldots \oplus x_k\}$$

for the set of all reals which are Turing computable from finitely many elements of A. Trivially, $A \subset \text{comp}(A)$.

If x, $y \in \mathbb{R}^2$, $x \neq y$, then we write $\ell(x,y)$ for the unique line ℓ with $\{x,y\} \subset \ell$.

When talking about Turing reducibility, we shall often confuse elements of $\mathbb{R}^2$ with reals, i.e., elements of $\mathbb{R}$, e.g. by identifying (x,y) with $x \oplus y$, x, $y \in \mathbb{R}$.

The following is an immediate consequence of the proof of [2, Lemma 4.1].

Lemma 0.1 *Let $c \subset \mathbb{R}^2$ be a circle with center $(0,0)$ and radius $r > 0$. Let $x \in \mathbb{R}^2 \setminus c$, let ℓ, ℓ' be two lines, and assume that there are $y \in \ell \cap c$ and $z \in \ell' \cap c$ such such $y \neq z$ and x, y, z are collinear. Then*

$$r \leq_T x \oplus u \oplus v \oplus u' \oplus v'$$

for all $u \neq v$ and $u' \neq v'$ with $\ell = \ell(u,v)$ and $\ell' = \ell(u',v')$.

We may now formulate a sufficient criterion for a model of ZF to have a Mazurkiewicz set. This is a slight variant of [2, Theorem 4.2]. We are going to prove it by running the argument for [9, Theorem 5].

Theorem 0.2 (ZF) *Assume that there is some sequence $(A_i, r_i \colon i < \lambda)$ such that for all $i \leq j < \lambda$,*

(a) $A_i \subset A_j$,

(b) $\mathbb{R} = \bigcup_{k<\lambda} A_k$*,*

(c) r_i *is a real which is not in* $\mathrm{comp}(A_i)$*, and*

(d) $\mathrm{comp}(A_i \cup \{r_i\}) \subset A_{i+1}$*.*

There is then a Mazurkiewicz set.

Proof. For any $i < \lambda$, let $n(i)$ be the unique $n < \omega$ such that $i = \bar{\lambda} + n$, where $\bar{\lambda}$ is the largest limit ordinal $\leq i$. By replacing r_i by $r_i + n(i)$, we may and shall assume that $r_i > n(i)$ for all $i < \lambda$.

Let us recursively construct sets M_i, $i < \lambda$, such that for all $i \leq j < \lambda$,

(i) $M_i \subset M_j$,

(ii) $M_i \subset A_{i+1}$,

(iii) M_i doesn't have three distinct collinear points,

(iv) for all x, $y \in A_i$, $x \neq y$, $\overline{\overline{\ell(x,y) \cap M_i}} = 2$.

Fix $i < \lambda$ and suppose $(M_k \colon k < i)$ is already given. Write $M = \bigcup_{k<i} M_k$. Let c be the circle with center $(0,0)$ and radius r_i. Notice that (c) implies that $c \cap A_i = \emptyset$, so that by the inductive hypothesis (ii), $c \cap M = \emptyset$.

Let P denote the set of all $\{x,y\}$ such that x, $y \in A_i$, $x \neq y$, and $\overline{\overline{\ell(x,y) \cap c}} = 2$. Let $L = \{\ell(x,y) \colon \{x,y\} \in P\}$. For $\ell \in L$, let $\ell \cap c = \{a_\ell, b_\ell\}$, where $a_\ell <_{\mathrm{lex}} b_\ell$,[1] and let

$$m_\ell = \begin{cases} \emptyset & \text{if } \overline{\overline{\ell \cap M}} = 2 \\ \{a_\ell\} & \text{if } \overline{\overline{\ell \cap M}} = 1 \\ \{a_\ell, b_\ell\} & \text{if } \overline{\overline{\ell \cap M}} = 0 \end{cases}$$

Then let

$$M_i = M \cup \bigcup \{m_\ell \colon \ell \in L\}.$$

Let us check (i) through (iv). (i) and (iv) for M_i are both trivial. (ii) is true for M_i, as a_ℓ, $b_\ell \leq_T x \oplus y \oplus r_i \in \mathrm{comp}(A_i \cup \{r_i\}) \subset A_{i+1}$ for all $\{x,y\} \in P$ with $\ell = \ell(x,y)$. It then suffices to check (iii) for M_i.

Suppose that a, b, d are three distinct collinear points in M_i.

[1] Here, $<_{\mathrm{lex}}$ is the lexicographical order on $\mathbb{R}^2$ induced by the natural order on $\mathbb{R}$.

Case 1. a, b, $d \in M$.

This contradicts the inductive hypothesis (iii) for all $k < i$.

Case 2. a, $b \in M$, $d \in M_i \setminus M$.

Then $d \in m_\ell \subset c$ for some $\ell \in L$, say $\ell = \ell(x, y)$, $\{x, y\} \in P \subset A_i$. But then $d \leq_T a \oplus b \oplus x \oplus y$ which implies that $r_i \leq_T a \oplus b \oplus x \oplus y \subset \mathrm{comp}(A_i)$. This contradicts (c).

Case 3. $a \in M$, b, $d \in M_i \setminus M$.

Let $b \in m_\ell$, $d \in m_{\ell'}$, ℓ, $\ell' \in L$. We can't have that $\ell = \ell'$. Let $\{x, y\}$, $\{x', y'\} \in P$ be such that $\ell = \ell(x, y)$ and $\ell' = \ell(x', y')$. By Lemma 0.1,

$$r_i \leq_T a \oplus x \oplus y \oplus x' \oplus y' \in \mathrm{comp}(A_i).$$

Contradiction!

Case 4. a, b, $d \in M_i \setminus M$.

This contradicts $M_i \setminus M \subset c$.

This finishes the construction of M_i.

Notice now that our hypothesis that $r_i > n(i)$ for all $i < \lambda$ makes sure that $\bigcup\{M_i \colon i < \lambda\}$ will be a Mazurkiewicz set. □

We refer the reader to [1] for the definition of "the" Cohen Halpern-Levy model. This model is obtained by adding ω Cohen reals over L by focing with a finite support product of ω copies of Cohen forcing. Let A denote the (unordered) set of those Cohen reals. The associated Cohen-Halpern-Levy model is then the model $L(A)$, i.e., the smallest inner model of ZF which contains A as a set. Every real of $L(A)$ is in $L[a]$ for some $a \in [A]^{<\omega}$. See [1].

Corollary 0.3 *There is a Mazurkiewicz set in the Cohen-Halpern-Levy model.*

Proof. Let us work inside $L(A)$ and define a strictly increasing sequence $(\alpha_i \colon i < \omega_1)$ of countable ordinals as follows. Given $(\alpha_k \colon k < i)$, where $i < \omega_1$, write $\tilde{\alpha} = \sup(\{\alpha_k \colon k < i\})$ and let α_i be the least $\alpha > \tilde{\alpha}$ such that

(1) $L_\alpha[a] \models \mathsf{ZFC}^-$ for all $a \in [A]^{<\omega}$ and

(2) $(\mathbb{R} \cap L_\alpha) \setminus \bigcup\{L_{\tilde{\alpha}}[a] \colon a \in [A]^{<\omega}\} \neq \emptyset$.

Then let $A_i = \mathbb{R} \cap \bigcup\{L_{\alpha_i}[a] \colon a \in [A]^{<\omega}\}$, and let r_i be the $<_L$-least element of $(\mathbb{R} \cap L_{\alpha_{i+1}}) \setminus A_i$, where $<_L$ denotes the canonical well-order of L.

It is easy to see that $(A_i, r_i \colon i < \omega_1)$ satisfies the hypothesis of Theorem 0.2. □

References

[1] M. Beriashvili, R. Schindler, L. Wu, and L. Yu *Hamel bases and well-ordering the continuum*, Proc. Amer. Math. Soc. **146** (2018), pp. 3565-3573. https://ivv5hpp.uni-muenster.de/u/rds/specialsets.pdf

[2] Chad, B., Knight, R., and Suabedissen, R., *Set theoretic constructions of two-point sets*, Fund. Math. **203** (2009), pp. 179-189.

[3] Gelbaum, B.R., and Olmsted,J.M.H., *Counterexamples in analysis*, Holden-Day, San Francisco 1964.

[4] Kharazishvili, A. *Nonmeasurable sets and functions*, Elsevier, Amsterdam 2004.

[5] Kharazishvili, A. *Elements of combinatorial geometry. Part I*, Georgian Nat. Acad. of Sciences, Tbilisi 2016.

[6] Larman, D.G., *A problem of incidence*, J. London Math. Soc. **43** (1968), pp. 407-409.

[7] Mazurkiewicz, S., *Sur un ensemble plan qui a avec chaque droite deux et seulement deux points communs*, C. R. Varsovie, 7 (1914), 382-384

[8] Miller, A., *Infinite Combinatorics and Definability*, Annals of Pure and Applied Logic 41 (1989) 179-203, North-Holland

[9] Miller, A., *The axiom of choice and two-point sets in the plane*, preprint, available at https://www.math.wisc.edu/∼miller/res/two-pt.pdf