Math. Log. Quart. 45 (1999) 3, 361 – 369

Mathematical Logic
Quarterly

Strong Cardinals and Sets of Reals in $L_{\omega_1}(\mathbb{R})$

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Abstract. We generalize results of [3] and [1] to hyperprojective sets of reals, viz. to more than finitely many strong cardinals being involved. We show, for example, that if every set of reals in $L_{\omega_1}(\mathbb{R})$ is weakly homogeneously Souslin, then there is an inner model with an inaccessible limit of strong cardinals.

Mathematics Subject Classification: 03E35, 03E45, 03E55.

Keywords: Descriptive set theory, Large cardinals, Inner models.

1 Introduction and statement of results

It has turned out that there is a close connection between the two kinds of objects referred to in the title. For example, STEEL (unpublished) has observed recently that if V is a "minimal" inner model with a limit λ of strong cardinals, then in $V^{\mathrm{Col}(\lambda,\omega)}$ every projective subset of $\mathbb{R} \times \mathbb{R}$ can be uniformized by a function with a projective graph. By earlier (unpublished) work of WOODIN it is also the case in $V^{\mathrm{Col}(\lambda,\omega)}$ that every projective set of reals is Lebesgue measurable and has the property of Baire. Actually, [3] (*cum grano salis*) shows that an assumption being slightly weaker than the existence of ω many strong cardinals gives the exact consistency strength of these regularity properties of projective sets to hold simultaneously. (Cf. [3] for an exact statement of the results and also on background information.)

Projective subsets of $\mathbb{R}$ are those sets of reals appearing earliest in the hierarchy $L(\mathbb{R})$. Following the usual definition, we let

$$\begin{aligned} J_1(\mathbb{R}) &= V_{\omega+1}, \\ J_\lambda(\mathbb{R}) &= \textstyle\bigcup_{\gamma<\lambda} J_\gamma(\mathbb{R}) \quad \text{for } \lambda \text{ a limit ordinal,} \\ J_{\gamma+1}(\mathbb{R}) &= \mathrm{rud}(J_\gamma(\mathbb{R})), \end{aligned}$$

where rud denotes the rudimentary closure operator (cf. for example [7] for details). Then $\mathcal{P}(\mathbb{R}) \cap J_2(\mathbb{R})$ are precisely the projective subsets of $\mathbb{R}$.

1) The author gratefully acknowledges a fellowship from the Deutsche Forschungsgemeinschaft (DFG). The results in this paper were obtained while I was visiting the UC Berkeley. I do wish to say thanks to JOHN STEEL for his forceful guidiance.

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Here we are interested in certain properties of sets of reals in $J_\gamma(\mathbb{R})$ for $2 \le \gamma \le \omega_1$, like the above mentioned regularity properties. We generalize results of [3] and [1] to such sets of reals, viz. to more than finitely many strong cardinals being involved.

We use the core model theory of [8] and [3]. (In fact, for technical reasons explained in [3] we have to use the "Friedman-Jensen indexing" of extender sequences rather than the "Mitchell-Steel indexing". However, this won't be a serious issue here.) Our first theorem and features of its proof are the main technical tools used for showing the subsequent results.

Theorem 1.1. *Assume that there is no inner model with a Woodin cardinal, and that Ω is a measurable cardinal. Let K denote Steel's core model of height Ω. Let $1 \le \alpha \le \omega_1$. Suppose that $J^K_{\omega_1} \models$ "there are $< \omega\alpha$ many strong cardinals". Then the set of reals coding $K \cap HC = J^K_{\omega_1}$ is an element of $J_{1+\alpha}(\mathbb{R})$.*

For the case $\alpha = 1$ this follows from [3], which in turn generalizes a theorem of [2]. In fact, Theorem 1.1 is obtained by a straightforward generalization of arguments in [3]. One can show that Theorem 1.1 gives the best possible estimate of $K \cap HC$.

Hauser has shown in [1] that if the theory of $\mathbb{R}$ is frozen under all further forcing extensions, then there is an inner model with ω many strong cardinals. Due to the following result, we here get the ω many strong cardinals in K to be below ω_1, and also a generalization to hyperprojective levels.

Methods of Woodin (unpublished) show that the following is the best possible result. We write $\mathcal{M} \prec^{\mathbb{R}} \mathcal{N}$ to mean that $\mathcal{M} \models \varphi(x)$ iff $\mathcal{N} \models \varphi(x)$ for all $x \in \mathbb{R}^{\mathcal{M}}$ for transitive models $\mathcal{M} \subset \mathcal{N}$.

Theorem 1.2. *Assume that there is no inner model with a Woodin cardinal. Let $1 \le \alpha \le \omega_1$. Suppose that for all $\beta < \alpha$, for all partial orderings $P \in V$, and for all G being P-generic over V we have $J_{1+\beta}(\mathbb{R}^V) \prec^{\mathbb{R}} J_{1+\beta}(\mathbb{R}^{V[G]})$. Then there is an inner model K with $\omega\alpha$ many ordinals $\kappa < \omega_1$ such that $K \models$ "κ is a strong cardinal". In particular, if $\alpha = \omega_1$, then ω_1 is an inaccessible limit of strong cardinal in K.*

Weak homogenity implies generic absoluteness, so that as a corollary one obtains the following

Theorem 1.3. *Let $1 \le \alpha \le \omega_1$. Suppose that every set of reals in $J_{1+\alpha}(\mathbb{R})$ is weakly homogeneously Souslin. Then there is an inner model with $\omega\alpha$ many strong cardinals. If $\alpha = \omega_1$, then there is an inner model with an inaccessible limit of strong cardinals.*

It is, however, not known whether one can get a model in which every set of reals in $J_{\omega_1}(\mathbb{R})$ is weakly homogeneous starting from a model with an inaccessible limit of strong cardinals and a measurable above, say. (It is conjectured that this is not the case.) Anyway, Woodin has constructed a model in which every set of reals in $L_{\omega_1}(\mathbb{R})$ is weakly homogeneous starting from an assumption being considerably weaker than a Woodin cardinal.

Going further, Steel has shown in [8] that if every set of reals in $J_{\omega_1+1}(\mathbb{R})$ is weakly homogeneous, then there is an inner model with a Woodin cardinal.

Next, we have the following result giving information about consistency strength of global Lebesgue measurability. Except for the use of Ω in order to build K, methods of Woodin show that this is best possible.

Theorem 1.4. *Assume that there is no inner model with a Woodin cardinal, and that Ω is a measurable cardinal. Let K denote Steel's core model of height Ω. Let $1 \leq \alpha \leq \omega_1$. Suppose that for all partial orderings $P \in V_\Omega$ and for all G being P-generic over V we have that all sets of reals in $J_{1+\alpha}(\mathbb{R}^{V[G]})$ are Lebesgue measurable (in $V[G]$). Then there are $\omega\alpha$ many ordinals κ such that $K \models$ "κ is a strong cardinal". In particular, if $\alpha = \omega_1$, then ω_1 is an inaccessible limit of strong cardinals in K.*

Finally, our last theorem is obtained from Theorems 1.1, 1.2, and [9], and generalizes a main result of [3] (which in turn was obtained by exploiting an idea due to Hugh Woodin). Again, except for the use of Ω one can show that in fact this is best possible by lifting the arguments of Steel and Woodin mentioned in the first paragraph of this paper to higher levels of the hyperprojective hierarchy.

Theorem 1.5. *Assume that there is no inner model with a Woodin cardinal and that Ω is a measurable cardinal. Let K denote Steel's core model of height Ω. Let $1 \leq \alpha \leq \omega_1$. Suppose that inside $J_{1+\alpha}(\mathbb{R})$ every set of reals is Lebesgue measurable and has the property of Baire, and that every subset of $\mathbb{R} \times \mathbb{R}$ can be uniformized. Then $J^K_{\omega_1} \models$ "there are $\omega\alpha$ many strong cardinals". In particular, if $\alpha = \omega_1$, then inside K, $\omega_1 = \omega_1^V$ is an inaccessible limit of cardinals $\kappa < \omega_1$ such that $J^K_{\omega_1} \models$ "κ is a strong cardinal".*

We leave it to the reader's fantasy to derive more along these lines by using arguments of the present paper.

2 Proofs of the results

Well, the proof of Theorem 1.1 is the most tricky part. For this, we have to develop some core model theory, following [3]. Until the proof of Theorem 1.1, let us assume that Ω is a fixed measurable cardinal, that there is no inner model with a Woodin cardinal, and that K denotes Steel's core model of height Ω, as constructed in [8].

Let $\mathcal{M}$ be any premouse, and let $\kappa \leq \mathrm{On} \cap \mathcal{M}$. We denote by $d^{\mathcal{M}}(\kappa)$ the order type of all $\xi < \kappa$ such that $\mathcal{J}^{\mathcal{M}}_\kappa \models$ "ξ is a strong cardinal". Whereas in [3] we were only interested in the case $d^{\mathcal{M}}(\kappa) < \omega$, we will here have to weaken this to $d^{\mathcal{M}}(\kappa) < \kappa$.

For premice $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \trianglelefteq \mathcal{N}$ we say that $\mathcal{M}$ is a *δ-cutpoint in $\mathcal{N}$* if $\mathcal{M}$ is passive and has a largest cardinal, every cardinal in $\mathcal{M}$ remains a cardinal in $\mathcal{N}$, and whenever $F = E^{\mathcal{N}}_\nu \neq \emptyset$ is an extender with $\kappa = \mathrm{c.p.}(F) < \mathrm{On} \cap \mathcal{M}$ and $\nu \geq \mathrm{On} \cap \mathcal{M}$, then $d^{\mathcal{M}}(\kappa) < \min\{\delta, \kappa\}$ or else F is partial (i.e., there exists $\xi \in [\nu, \mathrm{On} \cap \mathcal{N}]$ such that $\varrho_\omega(\mathcal{J}^{\mathcal{N}}_\xi) \leq \kappa$).

A premouse $\mathcal{M}$ is called *δ-full* (with witness W) if $W \triangleright \mathcal{M}$ is a universal weasel such that $\mathcal{M}$ is a δ-cutpoint in W and every class-sized iterate W' of W with iteration map $i : W \longrightarrow W'$ has the definability property at all $\kappa < i(\mathrm{On} \cap \mathcal{M})$ with $d^{W'}(\kappa) < \min\{\delta, \kappa\}$ and κ is greater than the generators of all extenders used along the branch from W to W'. $\mathcal{M}$ is called *strongly δ-full* (with witness W) if W witnesses that $\mathcal{M}$ is δ-full and $\mathrm{On} \cap \mathcal{M}$ is a (successor) cardinal in W.

A premouse $\mathcal{M} \trianglerighteq \mathcal{N}$ is called a *δ-collapsing mouse for $\mathcal{N}$* if $\mathcal{M}$ is δ-full, $\mathcal{N}$ is a δ-cutpoint in $\mathcal{M}$, and $\varrho_\omega(\mathcal{M}) \leq \mathrm{On} \cap \mathcal{N}$. We remark that a premouse $\mathcal{N}$ is strongly δ-full just in case $\mathcal{N}$ is δ-full and for every δ-collapsing mouse $\mathcal{M}$ for $\mathcal{N}$ we have that $\varrho_\omega(\mathcal{M}) = \mathrm{On} \cap \mathcal{N}$.

A premouse $\mathcal{M}$ with top extender F and c.p.$(F) = \kappa$ is called a *δ-beaver* if $d^{\mathcal{M}}(\kappa) = \delta < \kappa$ and there is a universal weasel W witnessing that $\mathcal{J}^W_{\kappa^{+W}} = \mathcal{J}^{\mathcal{M}}_{\kappa^{+\mathcal{M}}}$ is δ-full and such that $\mathrm{Ult}(W, F)$ is fully iterable. A premouse $\mathcal{M}$ is called *internally δ-full* if for any iterate $\mathcal{M}^*$ of $\mathcal{M}$ with iteration map $i : \mathcal{M} \longrightarrow \mathcal{M}^*$, whenever there are $\overline{\delta} < \min\{\delta, \nu\}$ and F with critical point κ greater than the generators of all extenders used along the branch from $\mathcal{M}$ to $\mathcal{M}^*$ such that $(\mathcal{J}^{\mathcal{M}^*}_\nu, F)$ is a $\overline{\delta}$-beaver, then $E^{\mathcal{M}^*}_\nu = F$.

The following two lemmas are shown in [3] for the case $\delta < \omega$, but the proofs go thru virtually unchanged in the general case.

Lemma 2.1. *Let $\mathcal{M}$ be a countable premouse, and let δ be an ordinal. Then the following are equivalent:*

(1) *$\mathcal{M}$ is δ-full.*

(2) *$\mathcal{M}$ is internally δ-full with a largest cardinal λ, and for any $k < \omega$, $\mathcal{N}$ is k-iterable, $\mathcal{M} \trianglelefteq \mathcal{C}_{k+1}(\mathcal{N})$ (in fact, $\mathcal{M}$ is a δ-cutpoint in $\mathcal{C}_{k+1}(\mathcal{N})$), and $\varrho_{k+1}(\mathcal{N}) \geq \lambda$ whenever $\mathcal{N} \trianglelefteq \mathcal{M}$ is as follows:*

(a) *$\mathcal{M}$ is a δ-cutpoint in $\mathcal{N}$,*

(b) *$\mathcal{N}$ is k-iterable using extenders with critical point $\geq \lambda$ and index $\geq \mathrm{On} \cap \mathcal{M}$,*

(c) *for all k-iterates $\mathcal{N}^*$ of $\mathcal{N}$ as in (b), if $F = E^{\mathcal{N}^*}_\nu$ with $\nu \geq \mathrm{On} \cap \mathcal{M}$ has the critical point $\kappa < \lambda$, then $(\mathcal{J}^{\mathcal{N}^*}_\nu, F)$ is a $d^{\mathcal{M}}(\kappa)$-beaver (and redundantly, $d^{\mathcal{M}}(\kappa) < \min\{\delta, \kappa\}$).*

Lemma 2.2. *Let $\mathcal{M}$ be a countable premouse with top extender $F \neq \emptyset$ and with critical point κ, and let δ be an ordinal. Then the following are equivalent:*

(1) *$\mathcal{M}$ is a δ-beaver.*

(2) *$\mathcal{J}^{\mathcal{M}}_{\kappa^{+\mathcal{M}}}$ is internally δ-full, and for any $k < \omega$, $\varrho_{k+1}(\mathcal{N}) \geq \kappa^{+\mathcal{M}}$ and $\mathrm{Ult}(\mathcal{N}, F)$ is k-iterable whenever $\mathcal{N} \trianglelefteq \mathcal{J}^{\mathcal{M}}_{\kappa^{+\mathcal{M}}}$ is as follows:*

(a) *$\mathcal{J}^{\mathcal{M}}_{\kappa^{+\mathcal{M}}}$ is a δ-cutpoint in $\mathcal{N}$,*

(b) *$\mathcal{N}$ is k-iterable using extenders with critical point $\geq \kappa^{+\mathcal{M}}$,*

(c) *for all k-iterates $\mathcal{N}^*$ of $\mathcal{N}$ as in (b), if $G = E^{\mathcal{N}^*}_\nu$ with $\nu \geq \mathrm{On} \cap \mathcal{M}$ has the critical point $\mu \in \mathcal{M}$, then $(\mathcal{J}^{\mathcal{N}^*}_\nu, G)$ is a $d^{\mathcal{M}}(\mu)$-beaver (and redundantly, $d^{\mathcal{M}}(\mu) < \min\{\delta, \mu\}$).*

Given Lemmas 2.1 and 2.2, the proof of the next lemma is a routine exercise in constructibility theory. Actually, it is verified by exactly the same reasoning which establishes $(J_{\overline{\alpha}} : \overline{\alpha} \leq \alpha) \in J_{\alpha+1}$ from the uniformity of the J-hierarchy, say.

Lemma 2.3. *Let $\beta < \omega_1$ and $n < \omega$. Then the following sets of reals are elements of $J_{1+\beta+1}(\mathbb{R})$:*

(a) $A_{\omega\beta+n} = \{x : x$ *codes an $\omega\beta + n$-full premouse*$\}$,

(b) $B_{\omega\beta+n} = \{x : x$ *codes a strongly $\omega\beta + n$-full premouse*$\}$,

(c) $C_{\omega\beta+n} = \{x : x$ *codes an $\omega\beta + n$-collapsing mouse*$\}$,

(d) $D_{\omega\beta+n} = \{x : x$ *codes an $\omega\beta + n$-beaver*$\}$.

Proof. We use the well-known fact that for $k \leq \omega$, the assertion "x codes a 1-small k-iterable premouse" is uniformly $\Pi^1_2(x)$. This can be shown with [8, 2.4] and the concept of α-goodness (cf. [4, §5]; recall that we still assume that there is no inner model with a Woodin cardinal, so that all sufficiently iterable premice are 1-small, and K is universal for coiterable premice).

Well, it is easy to see that if $A_{\omega\beta+n}$ is in $J_{1+\beta+1}(\mathbb{R})$, then so are $C_{\omega\beta+n}$ and $B_{\omega\beta+n}$ (for the latter one uses the remark above after the definition of a collapsing mouse). We will thus restrict ourselves to showing that $\{A_{\omega\beta+n}, D_{\omega\beta+n}\} \subset J_{1+\beta+1}(\mathbb{R})$, given that $D_{\omega\overline{\beta}+\overline{n}} \in J_{1+\overline{\beta}+1}(\mathbb{R})$ for all $\omega\overline{\beta}+\overline{n} < \omega\beta+n$. In fact, by simultaneous induction on $(\beta, n) \in \omega_1 \times \omega$ (ordered lexicographically) we prove the following:

$(1)_{(\beta,n)}$ $\{A_{\omega\beta+n}, D_{\omega\beta+n}\} \subset J_{1+\beta+1}(\mathbb{R})$,

$(2)_{(\beta,n)}$ $(A_{\omega\overline{\beta}+\overline{n}} : \omega\overline{\beta}+\overline{n} < \omega\beta+n) \in J_{1+\beta+1}(\mathbb{R})$,

$(3)_{(\beta,n)}$ $(D_{\omega\overline{\beta}+\overline{n}} : \omega\overline{\beta}+\overline{n} < \omega\beta+n) \in J_{1+\beta+1}(\mathbb{R})$.

To commence, let $\beta = n = 0$. A premouse $\mathcal{M}$ is 0-full just in case there is a universal weasel $W \trianglerighteq \mathcal{M}$. By Lemma 2.1 and the fact that iterability is Π^1_2, it is easily seen that the set of all reals coding 0-full premice is a Π^1_3 subset of $\mathbb{R}$. (Actually, this is also a well-known fact.) This establishes $(1)_{(0,0)}$. Of course, $(2)_{(0,0)}$ and $(3)_{(0,0)}$ are vacuously true.

By Lemma 2.1 and straightforward first steps of our induction we then get that for $n < \omega$ the set of all reals coding n-full premice is a projective subset of $\mathbb{R}$. Using Lemma 2.2, the same holds for n-beavers if $n < \omega$. As the projective sets of reals are the ones in $J_2(\mathbb{R})$, we thus have $(1)_{(0,n)}$ for $n < \omega$. (This was already shown in [3].) As for $(2)_{(0,n)}$ and $(3)_{(0,n)}$, they follow from $(1)_{(0,\overline{n})}$ for all $\overline{n} < n$.

In fact, the set of codes of n-full premice is Π^1_{n+3} for $n < \omega$. In order to check this and also to handle the cases $\beta > 0$ and $n = 0$, we have to inspect a bit closerly what Lemma 2.1 combined with Lemma 2.2 buys us. Namely, from Lemma 2.1 we get that the set $A_{\omega\beta+n+1}$ is $\forall^{\mathbb{R}}\exists^{\mathbb{R}} D$, where $D \subset \mathbb{R}^3$ is arithmetical in $\mathbb{R} \setminus D_{\omega\beta+n}$, uniformly in all $\beta < \omega_1$ and $n < \omega$. Similarily, Lemma 2.2 gives that $D_{\omega\beta+n+1}$ is $\forall^{\mathbb{R}}\exists^{\mathbb{R}} D'$, where $D' \subset \mathbb{R}^3$ is arithmetical in $(\mathbb{R} \setminus A_{\omega\beta+n}) \times (\mathbb{R} \setminus D_{\omega\beta+n})$, uniformly in all $\beta < \omega_1$ and $n < \omega$. Notice that this immediately gives $(1)_{(\beta,n+1)}$ from $(1)_{(\beta,n)}$, and also $(2)_{(\beta,n+1)}$ viz. $(3)_{(\beta,n+1)}$ from $(2)_{(\beta,n)}$ viz. $(3)_{(\beta,n)}$ together with $(1)_{(\beta,n)}$, for any β, n.

We are hence left with the task of showing $(1)_{(\beta,0)}$, $(2)_{(\beta,0)}$, and $(3)_{(\beta,0)}$ from the inductive assumptions.

Well, we have that $(A_{\omega\overline{\beta}}+\overline{n} : \omega\overline{\beta}+\overline{n} < \omega\beta)$ is a recursive function with values in $J_{1+\beta}(\mathbb{R})$ (by $(1)_{(\overline{\beta},\overline{n})}$), moreover whose proper initial segements are in $J_{1+\beta}(\mathbb{R})$ as well (by $(2)_{(\overline{\beta},\overline{n})}$), and whose definition can hence be carried out over $J_{1+\beta}(\mathbb{R})$ (as $A_{\omega\overline{\beta}+\overline{n}+1}$ is uniformly $\forall^{\mathbb{R}}\exists^{\mathbb{R}} D$, where D is arithmetical in $\mathbb{R} \setminus D_{\omega\overline{\beta}+\overline{n}}$). But this implies $(2)_{(\beta,0)}$. Condition $(3)_{(\beta,0)}$ is established in much the same way. But now $(1)_{(\beta,0)}$ easily follows using Lemmas 2.1 and 2.2. □

Again, the proof in [3] of the next two theorems for the case $\gamma \leq \delta < \omega$ goes thru unchanged in the general case.

Theorem 2.4. *Assume that $\omega_1 = \omega_1^V$ is inaccessible in K, and suppose $\gamma < \omega_1$ to be such that $\mathcal{J}^K_{\omega_1} \models$ "γ is the order type of the strong cardinals". Let $\delta \geq \gamma$. Then for all premice $\overline{\mathcal{M}}$ we have that $\overline{\mathcal{M}} \triangleleft \mathcal{J}^K_{\omega_1}$ iff $\Phi_\delta(\overline{\mathcal{M}})$. Here, $\Phi_\delta(\overline{\mathcal{M}})$ denotes the following assertion: 'There is a countable strongly δ-full $\mathcal{M} \trianglerighteq \overline{\mathcal{M}}$ such that for all countable* *strongly δ-full $\mathcal{N}$, if $\mathcal{M}$, $\mathcal{N}$ coiterate to comparable premice $\mathcal{M}^*$, $\mathcal{N}^*$, respectively, such that $\mathcal{M}^* \trianglelefteq \mathcal{N}^*$, then*

(a) *$\mathcal{M}^*$ is a δ-cutpoint in $\mathcal{N}^*$,*

(b) *there is no drop along the main branches on either side, and*

(c) *if $i : \mathcal{M} \longrightarrow \mathcal{M}^*$ and $j : \mathcal{N} \longrightarrow \mathcal{N}^*$ denote the maps coming from the coiteration, then $i"\mathcal{M} \subset j"\mathcal{N}$'.*

The proof of Theorem 2.4 actually gives that regardless of the number of strong cardinals in $\mathcal{J}^K_{\omega_1}$ and regardless of whether ω_1 is inaccessible in K or not, $\Phi_\delta(\mathcal{J}^K_\kappa)$ holds as long as κ not greather than the δ^{th} strong cardinal in $\mathcal{J}^K_{\omega_1}$ (if there is one).

Theorem 2.5. *Assume that $\omega_1 = \omega_1^V$ is a successor in K, and suppose $\gamma < \omega_1$ to be such that $\mathcal{J}^K_{\omega_1} \models$ "γ is the order type of the strong cardinals". Let $\delta \geq \gamma$. Then there is some $\mathcal{N} \lhd \mathcal{J}^K_{\omega_1}$ such that for all premice $\overline{\mathcal{M}}$ we have that $\overline{\mathcal{M}} \lhd \mathcal{J}^K_{\omega_1}$ iff there is a δ-collapsing mouse $\mathcal{M}$ for $\mathcal{N}$ with $\overline{\mathcal{M}} \trianglelefteq \mathcal{M}$.*

We can now commence with proving the results of this paper.

Proof of Theorem 1.1. This is now a straightforward consequence of Theorems 2.4, 2.5, together with Lemma 2.3. □

Proof of Theorem 1.2. We may assume w.l.o.g. that there is no inner model with a Woodin cardinal. As in [1], say, Σ^1_3-absoluteness gives that V is closed under the †-operation, so that we can make sense of a global K by piecing together the core models built inside the various daggers. We now prove Theorem 1.2 by verifying the following two claims.

Notice that we may assume w.l.o.g. that $d^K(\kappa) < \kappa$ for all κ being measurable in $\mathcal{J}^K_{\omega_1}$.

Claim 1. *Let $\kappa < \omega_1$ be such that $d^K(\kappa) < \omega\alpha$ and $\mathcal{J}^K_{\omega_1} \models$ "κ is strong". Then $K \models$ "κ is strong", too.*

Proof. Suppose that $\kappa < \omega_1$ were a counterexample. We may then pick λ large enough such that $\mathcal{J}^K_\lambda \models$ "κ is not strong". Let $\omega\beta + n = d^K(\kappa) < \omega\alpha$, where $n < \omega$. Now any initial segment of K is γ-full for any γ, in any set generic extension of V (this follows from [8, 8.3]). Hence in $V^{\text{Col}(\lambda^+,\omega)}$ the following assertion is true:

There is a strongly $\omega\beta + n + 1$-full $\mathcal{N} \rhd \mathcal{J}^K_\kappa$ such that $\mathcal{N} \models$ "κ is not strong".

Let $z_0 \in \mathbb{R}^V$ be a code for $\mathcal{J}^K_\kappa$. Via coding, the displayed assertion is $\Sigma_\omega(J_{1+\beta}(\mathbb{R}))$ in z_0 by Lemma 2.3(b). It is true in $V^{\text{Col}(\lambda,\omega)}$, and hence in V as well by our absoluteness assumption. Let $\mathcal{N} \rhd \mathcal{J}^K_\kappa$, $\mathcal{N} \in V$, be a witness for the latter. Let $W \rhd \mathcal{N}$ witness that $\mathcal{N}$ is strongly $\omega\beta + n + 1$-full. We get an elementary embedding $\sigma : K \longrightarrow W$ from the coiteration of K with W as in the proof of [8, 8.10]. By $\mathcal{J}^K_\kappa = \mathcal{J}^W_\kappa$ and the fact that W witnesses that $\mathcal{N}$ is strongly $\omega\beta + n + 1$-full, we get that $\sigma \restriction \kappa + 1 = \text{id}$ (this follows by an argument as in the proof of [8, 5.1]). This implies that $\mathcal{J}^W_{\sigma(\omega_1)} \models$ "κ is strong", so a *fortiori* $\mathcal{J}^W_{\omega_1} \models$ "κ is strong". In turn, this implies that $\mathcal{N} = \mathcal{J}^W_{\text{On} \cap \mathcal{N}} \models$ "κ is strong", as On $\cap\, \mathcal{N}$ is a cardinal in W. Contradiction! □ (Claim 1)

Claim 2. *There are $\omega\alpha$ many $\kappa < \omega_1$ such that $\mathcal{J}^K_{\omega_1} \models$ "κ is strong".*

Proof. Suppose not. Let $\beta < \alpha$ and $n < \omega$ be such that $\omega\beta + n$ is the order type of the strong cardinals in $\mathcal{J}^K_{\omega_1}$.

Case 1. ω_1 is inaccessible in K.

Let $\Psi \equiv \Psi^{\omega\beta+n}_{\omega\beta+n+1}$ abbreviate the statement

for all M, M', if $\Phi_{\omega\beta+n+1}(M)$ and $\Phi_{\omega\beta+n+1}(M')$, then $M \trianglelefteq M'$ or $M \trianglelefteq M'$, and if $\mathcal{M} = \bigcup_{\Phi_{\omega\beta+n+1}(M)} M$, then $\mathrm{On} \cap \mathcal{M} = \omega_1$, and $\mathcal{M} \models \mathsf{ZFC}+$ "there are $\omega\beta + n$ many strong cardinals".

(Here, $\Phi_{\omega\beta+n+1}(M)$ is as in Theorem 2.4.) By Theorem 2.4, $\Phi_{\omega\beta+n+1}(M)$ holds iff $M \lhd \mathcal{J}^K_{\omega_1}$, so that Ψ holds (in V). However, via coding, Ψ is $\Sigma_\omega(J_{1+\beta}(\mathbb{R}))$ in a real coding $\omega\beta + n$ by Lemma 2.3(b). Hence Ψ holds in any set generic extension of V by our absoluteness assumption.

Let λ be any singular cardinal. By [5], $\lambda^{+K} = \lambda^+$. In the next paragraph let us work in $V[G]$, where G is $\mathrm{Col}(\lambda,\omega)$-generic over V.

Let $\mathcal{M}$ witness that Ψ holds. In particular, $\mathcal{M}$ is a premouse of height $\lambda^+ = \omega_1$. Let us first suppose that there is a (unique) $\kappa < \lambda$ such that $\mathcal{J}^K_{\lambda^+} \models$ "κ is strong" and $d^K(\kappa) = \omega\beta+n$. As cofinally proper initial segments of $\mathcal{M}$ are $\omega\beta+n+1$-full, an easy reflection argument gives that $\mathcal{M}$ itself is $\omega\beta+n+1$-full. Let $W \rhd \mathcal{M}$ witness that $\mathcal{M}$ is $\omega\beta+n+1$-full. We have that $\Phi_{\omega\beta+n+1}(\mathcal{J}^K_\kappa)$ by the remark after Theorem 2.4, and thus $\mathcal{J}^K_\kappa \lhd \mathcal{M} \lhd W$. Moreover, we get an elementary $\sigma : K \longrightarrow W$ from coiterating K with W as in the proof of Claim 1 with $\sigma \restriction \kappa + 1 = \mathrm{id}$. But on the other hand, κ is strong in $\mathcal{J}^K_{\lambda^+}$, but not in $\mathcal{M} = \mathcal{J}^W_{\lambda^+}$ by Ψ. Hence $\sigma \restriction \kappa + 1 \neq \mathrm{id}$. Contradiction!

Hence $\omega\beta+n$ is the order type of the strong cardinals in $\mathcal{J}^K_{\lambda^+}$, too, by Claim 1. But then we have $\Phi_{\beta+n+1}(\mathcal{J}^K_\xi)$ for all $\xi < \lambda^+$, again by the remark after Theorem 2.4, i.e., $\mathcal{M} = \mathcal{J}^K_{\lambda^+}$. But this implies $\mathcal{M} \models \neg\mathsf{ZFC}$, contradicting Ψ.

Case 2. ω_1 is a successor in K.

Then there is some $\mathcal{N} = \mathcal{J}^K_\xi \lhd \mathcal{J}^K_{\omega_1}$ such that $M \lhd \mathcal{J}^K_{\omega_1}$ iff there exists an $\omega\beta + n$-collapsing mouse $\mathcal{N}'$ for $\mathcal{N}$ such that $\mathcal{M} \trianglelefteq \mathcal{N}'$, by Theorem 2.5. Hence, via coding, the fact that $\omega_1 = \xi^{+K}$ is expressed by a $\Sigma_\omega(J_{1+\beta}(\mathbb{R}))$ statement in a parameter for a real coding $\mathcal{N}$ by Lemma 2.3(c). However, this statement becomes false in $V^{\mathrm{Col}(\omega_1,\omega)}$. Contradiction! □ (Claim 2) □

Notice that in the course of verifying Claim 2 in Case 1 we actually showed that $\mathcal{J}^K_{\omega_1} \models$ "there are more than $\omega\beta + n$ many strong cardinals" implies that $\Psi^{\omega\beta+n}_{\omega\beta+n+1}$ fails. This will be exploited during the proof of Theorem 1.5.

We also remark that looking at the previous proof a bit closerly gives that if the Σ_{n+4}-theory of $J_{1+\beta}(\mathbb{R})$ agrees with the Σ_{n+4}-theory of $J_{1+\beta}(\mathbb{R}^{V[G]})$ (with real parameters) whenever G is P-generic over V for some $P \in V$, then there are $\omega\beta + n$ strong cardinals smaller than ω_1 in K. This will be needed in the proof of Theorem 1.5.

Proof of Theorem 1.3. The existence of a weakly homogeneous tree trivially implies the existence of a measurable cardinal. Let Ω denote the least one.

Let $\beta < \alpha$. It suffices to show that for all partial orderings $P \in V_\Omega$ and for all G being P-generic over V we have $J_{1+\beta}(\mathbb{R}) \prec^{\mathbb{R}} J_{1+\beta}(\mathbb{R}^{V[G]})$, because then Theorem 1.2 straightforwardly gives Theorem 1.3.

But the desired absoluteness follows from the fact that every set of reals in $\Sigma_\omega(J_{1+\beta}(\mathbb{R}))$ is weakly homogeneous, exactly as in [8, p. 56f]. Alas, every such set is in $J_{1+\alpha}(\mathbb{R})$, and is thus weakly homogeneous by assumption. □

Proof of Theorem 1.4. Suppose Theorem 1.4 fails. Let $\omega\beta + n < \omega\alpha$ be the order type of the set of strong cardinals in K, where $n < \omega$. Assuming that for all singular cardinals (in V) $\lambda < \Omega$ there is some $\kappa < \lambda$ with $d^K(\kappa) = \omega\beta + n$ and $\mathcal{J}_\lambda^K \models$ "κ is strong", we may define $F(\lambda)$ to be the unique such. Hence F is regressive on the class of all singular cardinals below Ω, so that FODOR's Lemma gives us a stationary $A \subset \Omega$ with F being constant on A. But setting $\kappa = F(\lambda)$ for any $\lambda \in A$, we have that $d^K(\kappa) = \omega\beta + n$ as well as $K \models$ "κ is strong", contradicting the choice of $\omega\beta + n$ as the order type of the strong cardinals in K.

Hence we may choose a singular cardinal $\lambda < \Omega$ such that in $\mathcal{J}_\lambda^K$ there are exactly (in order type) $\omega\beta + n$ many strong cardinals. By [5], $\lambda^{+K} = \lambda^+$, and so $\lambda^+ = \omega_1^{V[G]}$ is a successor cardinal in K, where G is $\mathrm{Col}(\lambda, \omega)$-generic over V. Now work in $V[G]$ for the rest of this proof. Fix $f : \omega \longrightarrow \lambda$ bijective. Then $z_0 \in \mathbb{R}$ is a code for λ, where we set $(n, m) \in z_0$ iff $f(n) < f(m)$ (here we identify $\mathbb{R}$ with $\mathcal{P}(\omega \times \omega)$). For any $\xi < \omega_1$ such that $\xi \geq \lambda$, we my define $z_\xi \in \mathbb{R}$ by $(n, m) \in z_\xi$ iff $F \circ f(n) < F \circ f(m)$, where F is least (in the order of constructibility) in K with $F : \lambda \longrightarrow \xi$ bijective. But this F is also least in any large enough $\omega\beta + n$-collapsing mouse $\mathcal{M}$ for $\mathcal{J}_\lambda^K$ with $F : \lambda \longrightarrow \xi$ bijective, by Theorem 2.5. Hence by Lemma 2.3(c) the relation $\prec$ on $\mathbb{R} \times \mathbb{R}$ defined by $x \prec y$ iff there are $\xi < \xi'$ with $x = z_\xi$ and $y = z_{\xi'}$ is $\Sigma_\omega(J_{1+\beta}(\mathbb{R}))$ in z_0 and a code for a proper initial segment of $\mathcal{J}_{\omega_1}^K$. Notice that $\prec$ gives an uncountable sequence of distinct reals. Hence by [6] there is a set of reals in $\Sigma_\omega(J_{1+\beta}(\mathbb{R}))$ which is not Lebesgue measurable. However, this contradicts our assumption, which gives that every set of reals in $J_{1+\alpha}(\mathbb{R})$ is Lebesgue measurable (in $V[G]$). □

We remark that this proof in fact also shows that if in no forcing extension of V (for some forcing $P \in V_\Omega$) there is an uncountable sequence of distinct reals being in $J_{1+\alpha}(\mathbb{R})$, then there is an inner model with $\omega\alpha$ many strong cardinals. Again, except for the use of Ω this is best possible. WOODIN and STEEL (unpublished) independently of each other have both shown that if in no forcing extension of V (for some forcing $P \in V_\Omega$) there is an uncountable sequence of distinct reals being in $L(\mathbb{R})$, then $\mathrm{AD}^{L(\mathbb{R})}$ holds.

Proof of Theorem 1.5. Assuming the hypotheses of Theorem 1.5, a reasoning slightly generalizing [9] gives, for any $\beta < \omega_1$ and $n < \omega$ such that $\omega\beta + n < \omega\alpha$, a transitive model M_n^β of ZFC (of height ω_1) such that the following assertion holds: The Σ_n-theory of $J_{1+\beta}(\mathbb{R}^V)$ is the same as the Σ_n-theory of $J_{1+\beta}(\mathbb{R}^{M_n^\beta})$, which in turn is the same as the Σ_n-theory of $J_{1+\beta}(\mathbb{R}^{M_n^\beta[G]})$ whenever G is P-generic over M_n^β for some $P \in M_n^\beta$.

Well, using Σ_3^1-absoluteness as in the proof of Theorem 1.2, we can build $K^{M_n^\beta}$, and by the second remark following the proof of Theorem 1.2 we have that $K^{M_n^\beta} \models$ "there are at least $\omega\beta + n - 4$ strong cardinals".

Now let us suppose toward contradiction that there are $\omega\beta + n < \omega\alpha$ many strong cardinals in $\mathcal{J}_{\omega_1}^K$, where $n < \omega$. If ω_1 were a successor cardinal in K, we would get a contradiction exactly as in the proof of Theorem 1.4. We may hence assume w.l.o.g. that ω_1 is inaccessible in K. In particular, $\Psi_{\omega\beta+n+1}^{\omega\beta+n}$ is true, where this is the assertion from the proof of Theorem 1.2. (Notice that it is a lightface assertion.) Set $M = M_{n+100}^\beta$, so that K^M has at least $\omega\beta + n + 1$ many strong cardinals. However, by the choice of M we have that $\Psi^{\omega\beta+n}_{\omega\beta+n+1}$ is true in M. But this contradicts the first remark after the proof of Theorem 1.2. □

3 Open problems

(1) How can one remove the assumption on the existence of Ω from Theorems 1.3, 1.4, and 1.5? Recent work of STEEL and the author shows that this assumption may be weakened to "Ω is a Mahlo cardinal" if K is replaced by a smaller core model, but it would be highly desirable to remove Ω at all from the core model theory below an inaccessible limit of strong cardinals, say. Work of JENSEN as well as recent work of STEEL and the author in fact indicates that this should be possible.

(2) What is necessary to construct a model in which every projective set of reals is Lebesgue measurable and has the property of Baire, and every $\mathbf{\Pi}^1_{2n+1}$-relation admits a uniformizing function with a graph of prescribed complexity, for $n \geq 1$? [3] does not give full information about this. For example, in the case $n = 1$ if the uniformizing function is required to be $\mathbf{\Pi}^1_3$ (in the same parameter as the relation to be uniformized), then $\mathbf{\Delta}^1_2$-determinacy holds (cf. [8, Cor. 7.14]), whereas STEEL's proof shows that (still for $n = 1$) for the function to be $\mathbf{\Delta}^1_5$ an inaccessible κ with $V_\kappa \models$ "there is a strong cardinal" suffices.

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(Received: January 23, 1998)