Suslin Cardinals and Scales From AD

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Basic Facts

We work throughout in $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}$.

Our goal is to give a complete as possible picture of the structure of the Suslin cardinals and scaled pointclasses just assuming AD. The methods are "descriptive set theoretic."

The methods here just use AD although AD^+ is needed to get that the supremum of the Suslin cardinals is a Suslin cardinal.

Will use work of Cabal in particular work of Kechris, Steel, and Woodin, and in particular a method of Martin for analyzing the next Suslin cardinal.

Definition

$A \subseteq \omega^\omega$ is κ-Suslin if there is a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ with

$$A = p[T] = \{x \colon \exists f \in \kappa^\omega \, \forall n \, (x \restriction n, f \restriction n) \in T\}.$$

We let $S(\kappa)$ denote the pointclass of κ-Suslin sets.
$S(<\kappa) = \bigcup_{\lambda<\kappa} S(\lambda)$.

Definition

κ is a Suslin cardinal if $S(\kappa) - S(<\kappa) \neq \emptyset$.

Suslin representations are closely related to scales.

Definition

An α-semiscale on $A \subseteq \omega^\omega$ is a sequence of norms $\varphi_n \colon A \to \alpha$ such that if $\{x_m\}_{m \in \omega} \subseteq A$, $x_m \to x \in \omega^\omega$, and for each n, $\varphi_n(x_m)$ is eventually equal to some $\lambda_n \in \mathrm{On}$, then $x \in A$.

We have the following easy and basic fact:

Fact

$A = p[T]$ for a tree T on $\omega \times \alpha$ iff A admits an α-semiscale.

If $A = p[T]$, let $\varphi_n(x) =$ the nth coordinate of the leftmost branch of T_x. $\vec{\varphi} = \vec{\varphi}_T$ is the semiscale from T.

If $\{\varphi_n\}$ is a semiscale on A, define T by $(s, \vec{\alpha}) \in T$ iff $\exists x \in A\ \exists n\, [s = x \restriction n \wedge \vec{\alpha} = (\varphi_0(x), \dots, \varphi_{n-1}(x))]$. $T = T_{\vec{\varphi}}$ is the tree from $\vec{\varphi}$.

To get scales, we add the lower semicontinuity property:

Definition

$\vec{\varphi}$ is a scale on A if it is a semiscale and whenever $x_m \to x$ and $\varphi_n(x_m) \to \lambda_n$ then $\varphi_n(x) \leq \lambda_n$.

If $A = p[T]$ and we let

$$\varphi'_n = |\langle \varphi_0(x), \dots, \varphi_n(x) \rangle|_{\text{lex}},$$

then $\vec{\varphi}'$ is a scale on A (with norms into α^n).

- If $\vec{\varphi}$ is a scale on A, then $\vec{\varphi}_{T_{\vec{\varphi}}} = \vec{\varphi}$.
- If $T = T_{\vec{\varphi}}$ is the tree of a scale $\vec{\varphi}$, then $T_{\vec{\varphi}_T} = T_{\vec{\varphi}_{T_{\vec{\varphi}}}} = T_{\vec{\varphi}} = T$.

Not every tree T, however, is the tree of a scale. For a general tree T we just have $T_{\vec{\varphi}_T} \subseteq T$ and these trees have the same projection (if $x \in p[T]$ and $\vec{\beta}$ is the left-most branch of T_x then $(x, \vec{\beta}) \in [T_{\varphi_T}]$).

- For any tree T we have $\varphi_T = \varphi_{T_{\varphi_T}}$.

In particular, both maps $T \mapsto T_{\varphi_T}$ and $\varphi \mapsto \varphi_{T_\varphi}$ are idempotent.

The scale $\vec{\varphi}'$ corresponding to a semiscale $\vec{\varphi}$ may be on a slightly bigger ordinal, but we have the following.

Fact

For all cardinals κ, the following are equivalent.

1. *A is κ-Suslin.*
2. *A admits a κ-semiscale.*
3. *A admits a κ-scale.*
4. *A admits a κ-very good scale.*

proof of (1) $\Rightarrow$ (3): Let $A = p[T]$ where T is a tree on $\omega \times \kappa$.

Case 1. $\mathrm{cof}(\kappa) > \omega$.

For $x \in p[T]$, let $\psi_0(x) =$ least $\alpha < \kappa$ such that $T_x \restriction \alpha$ is illfounded. Let

$$\psi_n(x) = |(\psi_0(x), \varphi_0^{\psi_0(x)}(x), \ldots, \varphi_{n-1}^{\psi_0(x)}(x))|_{\psi_0(x)}$$

where φ_n^α is the semiscale corresponding to $T \restriction \alpha$ and $|(\alpha, \beta_0, \ldots, \beta_{n-1})|_\gamma$ denotes the rank of $(\beta_0, \ldots, \beta_{n-1})$ in lexicographic ordering on $(\gamma + 1)^{n+1}$.

Case 2. $\mathrm{cof}(\kappa) = \omega$.

Let $\kappa = \sup_n \kappa_n$.

Can easily get a T' with $p[T'] = p[T] = A$ and such that if $(s, \vec{\alpha}) \in T'$ then $\alpha_i < \kappa_i$. Let $\vec{\psi}$ be scale from T', where ψ_n uses lex ordering on κ_n^{n+1}.

Lemma

Let κ be a Suslin cardinal.

1. *There is a strictly increasing sequence $\{A_\alpha\}_{\alpha<\kappa}$ of κ-Suslin sets.*
2. *If cof$(\kappa) > \omega$ then we may take the A_α to be $< \kappa$ Suslin.*

Proof. Suppose first $\mathrm{cof}(\kappa) = \omega$. Let $B \in S(\kappa) - S(<\kappa)$ and let $\vec{\varphi}$ be a scale on B with $|\varphi_n| = \lambda_n < \kappa$. We must have $\sup_n \lambda_n = \kappa$. Let $A = \{x : x' \in B\}$ where $x'(i) = x(i+1)$. Define $\vec{\psi}$ on A by: $\psi_0(x) = \varphi_{x(0)}(x')$ and $\psi_{i+1}(x) = \psi_i(x')$. $\vec{\psi}$ is a κ-scale on A and $|\psi_0| = \kappa$. Let $A_\alpha = \{x \in A : \psi_0(x) \leq \alpha\}$.

Suppose next that $\mathrm{cof}(\kappa) > \omega$. Let A and $\vec{\psi}$ be as above so $|\psi_0| = \kappa$. For $\alpha < \beta < \kappa$ let

$$A_{\alpha,\beta} = \{x\colon \forall i\; \psi_i(x) < \beta\} \cup \{x\colon \forall i\; \psi_i(x) \leq \beta \wedge \psi_0(x) \leq \alpha\}.$$

Each $A_{\alpha,\beta}$ is κ-Suslin. Order the indices by reverse lexicographic ordering.

For each $\alpha < \kappa$, there is an $x \in A$ with $\psi_0(x) = \alpha$. Let $\beta = \sup_i \psi_i(x)$. Then

$$x \in A_{\alpha,\beta} - \bigcup_{(\alpha',\beta')<(\alpha,\beta)} A_{\alpha',\beta'}.$$

Thus we get an increasing κ-length subsequence of κ-Suslin sets.

We have the following immediate fact.

Lemma

For any κ, $S(\kappa)$ is closed under $\exists^{\omega^\omega}$, $\wedge_\omega$, $\vee_\omega$.

We also have:

Theorem (Kechris)

For any Suslin cardinal κ, $S(\kappa)$ is non-selfdual.

Recall a $\boldsymbol{\Gamma}$-norm on A is a norm $\varphi\colon A \to \mathrm{On}$ such that the following relations are in $\boldsymbol{\Gamma}$:

$$x <^* y \leftrightarrow (x \in A) \wedge (y \notin A \vee (y \in A \wedge \varphi(x) < \varphi(y)))$$
$$x \leq^* y \leftrightarrow (x \in A) \wedge (y \notin A \vee (y \in A \wedge \varphi(x) \leq \varphi(y)))$$

A $\boldsymbol{\Gamma}$-scale on A is a scale $\{\varphi_n\}$ with each φ_n a $\boldsymbol{\Gamma}$-norm.

Definition

pwo($\boldsymbol{\Gamma}$) if every $A \in \boldsymbol{\Gamma}$ admits a $\boldsymbol{\Gamma}$-norm. scale($\boldsymbol{\Gamma}$) if every $A \in \boldsymbol{\Gamma}$ admits a $\boldsymbol{\Gamma}$-scale.

Note that if φ is a $\boldsymbol{\Gamma}$-norm on the $\boldsymbol{\Gamma}$ set A, then each

$$\begin{aligned} A^{\varphi}_{\leq\alpha} &= \{x \in A \colon \varphi(x) \leq \alpha\} \in \boldsymbol{\Delta} \\ A^{\varphi}_{<\alpha} &= \{x \in A \colon \varphi(x) < \alpha\} \in \boldsymbol{\Delta} \end{aligned}$$

[If $\varphi(x_0) = \alpha$ then $x \in A^{\varphi}_{\leq\alpha} \leftrightarrow x \leq^* x_0 \leftrightarrow \neg(x_0 <^* x)$, and likewise for $A^{\varphi}_{<\alpha}$.]

So if $\boldsymbol{\Delta}$ is closed under $\wedge$, $\vee$ (e.g., if $\boldsymbol{\Gamma}$ is closed under $\wedge$, $\vee$) then each $A^{\varphi}_{\alpha} = \{x \colon \varphi(x) = \alpha\} \in \boldsymbol{\Delta}$.

Basic Facts: Pointclasses

Recall a pointclass $\boldsymbol{\Gamma}$ is a collection $\boldsymbol{\Gamma} \subseteq \mathcal{P}(\omega^\omega)$ closed under Wadge reduction (continuous preimages).

We say $\boldsymbol{\Gamma}$ is selfdual if $\boldsymbol{\Gamma} = \check{\boldsymbol{\Gamma}} = \{\omega^\omega - A : A \in \boldsymbol{\Gamma}\}$.

Wadge's Lemma: For any A, B either $A \leq_w B$ or $B \leq_w \omega^\omega - A$.

This gives a strict linear order on the Wadge degrees, the equivalence classes of pairs $(A, \omega^\omega - A)$ under Wadge reduction.

Martin-Monk Theorem: The Wadge degrees are wellfounded.

We let $o(A)$ denote the Wadge rank of $[(A, \omega^\omega - A)]$. We let $o(\boldsymbol{\Gamma}) = \sup\{o(A) : A \in \boldsymbol{\Gamma}\}$.

A is selfdual if $A \equiv_w \omega^\omega - A$.

The pointclasses are essentially the initial segment of the Wadge degrees. Nonselfdual pointclasses correspond to nonselfdual degrees (a selfdual pointclass $\boldsymbol{\Delta}$ can occur in one of 4 ways).

Fact (Steel, VanWesep)

The selfdual and the nonselfdual Wadge degrees alternate. At limit ordinals of cofinality ω there is a selfdual degree and at ordinals of uncountable cofinality a nonselfdual degree.

- *If $o(A)$ is a limit of cofinality ω, A is the degree of a countable join of sets of lower degree.*
- *If A is selfdual and $o(A)$ a successor, then A is the degree of the join of the two sets in the previous nonselfdual pair.*

Definition

sep($\boldsymbol{\Gamma}$) if for every $A, B \in \boldsymbol{\Gamma}$ with $A \cap B = \emptyset$ there is a $C \in \boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ with $A \subseteq C \subseteq \omega^\omega - B$.

Theorem

For every nonselfdual $\boldsymbol{\Gamma}$, exactly one of sep($\boldsymbol{\Gamma}$), sep($\check{\boldsymbol{\Gamma}}$) *holds.*

Fact

For any nonselfdual $\boldsymbol{\Gamma}$ we have pwo($\boldsymbol{\Gamma}$) $\Rightarrow \neg$sep($\boldsymbol{\Gamma}$) *(and so* sep($\check{\boldsymbol{\Gamma}}$)*).*

So, at most one side can have the pwo property.

We are mainly interested in Levý classes:

Definition

A nonselfdual pointclass $\boldsymbol{\Gamma}$ is a Levý pointclass if it is closed under $\exists^{\omega^\omega}$ or $\forall^{\omega^\omega}$.

Remark

A nonselfdual pointclass closed under $\exists^{\omega^\omega}$ is closed under $\vee^\omega$. Likewise, closure under $\forall^\omega$ implies closure under $\wedge^\omega$.

Let $\boldsymbol{\Sigma}^1_\alpha$ $(\alpha < \Theta)$ enumerate the Levý classes closed under $\exists^{\omega^\omega}$, and $\boldsymbol{\Pi}^1_\alpha$ their duals.

The Levý classes fall into projective-like hierarchies analyzed as follows.

Definition

For $\boldsymbol{\Gamma}$ a pointclass, $\delta(\boldsymbol{\Gamma})$ is the supremum of the lengths of the $\boldsymbol{\Gamma}$ prewellorderings of ω^ω.

Fact

If Λ is a selfdual pointclass closed under $\exists^{\omega^\omega}$ and $\wedge$, then $o(\boldsymbol{\Delta}) = \delta(\boldsymbol{\Delta})$.

Remark

In this case $o(\Lambda)$ is also the supremum of the lengths of the Λ wellfounded relations.

Definition (Steel)

For $\mathbf{\Gamma}$ a pointclass, let

$$\Lambda(\mathbf{\Gamma}) = \bigcup \{\Lambda \subseteq \mathbf{\Gamma} \colon \Lambda \text{ is selfdual, closed under } \exists^{\omega^\omega}, \wedge\}.$$

So, $\Lambda(\mathbf{\Gamma})$ is selfdual, closed under $\exists^{\omega^\omega}$, $\wedge$, and $\lambda = o(\Lambda(\mathbf{\Gamma}))$ is a limit ordinal.

Type 1: If $\operatorname{cof}(\lambda) = \omega$ and $o(B) = \lambda$, then B is selfdual and $\exists^{\omega^\omega} B$ is equal to $\bigcup_\omega \Lambda$. Let $\mathbf{\Sigma}^\lambda_0 = \bigcup_\omega \Lambda$, and then define $\mathbf{\Sigma}^\lambda_n$, $\mathbf{\Pi}^\lambda_n$ over this as usual.

Types 2 and 3: If $\mathrm{cof}(\lambda) > \omega$ and $o(B) = \lambda$, then $(B, \omega^\omega - B)$ is a nonselfdual pair. We assume these classes are not closed under quantifiers in these cases. Let $\boldsymbol{\Sigma}^\lambda_{-1}$ be the side with $\mathrm{sep}(\boldsymbol{\Sigma}^\lambda_{-1})$, and apply quantifiers to get the $\boldsymbol{\Sigma}^\lambda_n$, $\boldsymbol{\Pi}^\lambda_n$ as usual.

[Type 3 means $\boldsymbol{\Sigma}^\lambda_{-1}$ is closed under $\wedge$, type 2 if it is not.]

Type 4: If $\mathrm{cof}(\lambda) > \omega$ and both the B and $\omega^\omega - B$ sides are closed under quantifiers, let $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\lambda)$ be the side such that $\mathrm{sep}(\check{\boldsymbol{\Gamma}}(\lambda))$. Let $\boldsymbol{\Sigma}^\lambda_1 = \boldsymbol{\Gamma} \vee \check{\boldsymbol{\Gamma}}$, $\boldsymbol{\Pi}^\lambda_1 = \boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}}$. Apply quantifiers to get the $\boldsymbol{\Sigma}^\lambda_n$, $\boldsymbol{\Pi}^\lambda_n$.

We call the classes $\boldsymbol{\Pi}^\lambda_{-1}$ the Steel Pointclasses.

The $\boldsymbol{\Sigma}^\lambda_n$, $\boldsymbol{\Pi}^\lambda_n$ for $n \geq -1$ enumerate all of the Levý classes.

A quick proof

Quick proof that the Steel class $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^{\lambda}_{-1}$ is closed under $\forall^{\omega^\omega}$:

Recall sep($\check{\boldsymbol{\Gamma}}$), we show $\exists^{\omega^\omega}\check{\boldsymbol{\Gamma}} \subseteq \check{\boldsymbol{\Gamma}}$.

$\exists^{\omega^\omega}\check{\boldsymbol{\Gamma}}$ is easily a pointclass, so if this fails then $\boldsymbol{\Gamma} \subseteq \exists^{\omega^\omega}\check{\boldsymbol{\Gamma}}$.

Let $A \in \boldsymbol{\Gamma} - \boldsymbol{\Delta}$ and write $A = p[B]$ with $B \in \check{\boldsymbol{\Gamma}}$. Let $C = (\omega^\omega - A) \times \omega^\omega$, so $C \in \check{\boldsymbol{\Gamma}}$. Let $D \in \boldsymbol{\Delta}$ separate B, C. Then $A = p[D] \in \boldsymbol{\Delta}$, a contradiction.

Remark

All of the Levý classes are closed under $\wedge$, $\vee$ except for the Steel classes in type 2 and the $\boldsymbol{\Sigma}^{\lambda}_0$, $\boldsymbol{\Pi}^{\lambda}_0$ in type 4.

All of the Levý classes are closed under $\wedge^{\omega}$, $\vee^{\omega}$ except for the above and $\boldsymbol{\Sigma}^{\lambda}_0$, $\boldsymbol{\Pi}^{\lambda}_0$ in type 1.

Prewellordering Property

Type 1: Let $A = \bigcup_n A_n$, each $A_n \in \boldsymbol{\Delta}$. For $x \in A$ let $\varphi(x) = \mu n\,(x \in A_n)$. Then $x <^* y \leftrightarrow \exists n\,(x \in A_n \wedge y \notin A_n)$. Likewise for $\leq^*$. Then propagate by periodicity.

Types 2 and 3: We show the Steel class $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^{\lambda}_{-1}$ has the prewellordering property. Let ρ be least such that $\boldsymbol{\Delta}$ is not closed under ρ-unions (AD^+). ρ is regular. $\rho \leq \lambda$ as otherwise there is a $\boldsymbol{\Delta}$ prewellordering of length λ, a contradiction.

In fact, $\rho = \mathrm{cof}(\lambda)$: If $\rho < \lambda$ and $\{A_\alpha\}_{\alpha<\rho} \subseteq \boldsymbol{\Delta}$ is increasing with union $A \notin \boldsymbol{\Delta}$, then we must have $\sup_{\alpha<\rho} o(A_\alpha) = \lambda$. Otherwise let $B \in \Lambda$ with $o(B) > o(A_\alpha)$ for all α. By the coding lemma there is a $\boldsymbol{\Delta}$ set C of reals z each of which reduces some A_α to B, and every A_α is reduced to B by some $z \in C$. Then $x \in A \leftrightarrow \exists z \in C\,(z(x) \in B)$. So, $A \in \boldsymbol{\Delta}$, a contradiction.

If $\rho > \mathrm{cof}(\lambda)$ then for some $\beta < \lambda$, for cofinally many $\alpha < \rho$ $o(A_\alpha) < \beta$. This is a contradiction as above.

Following Steel, let

$$\mathbf{\Gamma}^* = \{\bigcup_{\alpha<\rho} A_\alpha \colon \forall \alpha\ (A_\alpha \in \mathbf{\Delta}) \wedge \{A_\alpha\}_{\alpha<\rho} \text{ is } \mathbf{\Sigma}^1_1 \text{ bounded } \}.$$

Claim

$\mathbf{\Gamma} = \mathbf{\Gamma}^*$.

Proof If $\bigcup_\rho \mathbf{\Delta} \subseteq \check{\mathbf{\Gamma}}$ (and so equal), then $\mathrm{pwo}(\check{\mathbf{\Gamma}})$, a contradiction. So, $\bigcup_\rho \mathbf{\Delta} \subseteq \mathbf{\Gamma}$. Note that $\bigcup_\rho \mathbf{\Delta}$ is closed under $\exists^{\omega^\omega}$. So, $\bigcup_\rho \mathbf{\Delta} \supseteq \exists^{\omega^\omega} \mathbf{\Gamma} = \mathbf{\Sigma}^\lambda_0$.

Next, $\mathbf{\Gamma} \subseteq \mathbf{\Gamma}^*$. For let $A \in \mathbf{\Gamma}$, and let Let C be the set of codes of $\mathbf{\Delta}^1_1$ subsets of A.

More precisely, let $S \subseteq \omega^\omega \times \omega^\omega$ be $\boldsymbol{\Sigma}^1_1$. Say $S = p[F]$ where F is closed.

Then,

$$\begin{aligned} x \in C &\leftrightarrow S_x \subseteq A \\ &\leftrightarrow \forall y \, (y \in S_x \rightarrow y \in A) \\ &\leftrightarrow \forall y, z \, ((y, z) \in F \rightarrow y \in A) \end{aligned}$$

So, $C \in \boldsymbol{\Gamma}$ ($\boldsymbol{\Gamma}$ is closed under unions with open sets).

Write $C = \bigcup_{\alpha<\rho} C_\alpha$, an increasing union with each $C_\alpha \in \boldsymbol{\Delta}$.

Let $A_\alpha(x) \leftrightarrow \exists y \, (C_\alpha(y) \wedge S_y(x))$, so $A_\alpha \in \boldsymbol{\Delta}$. This works.

We show that $\bigcup_\rho \boldsymbol{\Delta} \subseteq \exists^{\omega^\omega} \boldsymbol{\Gamma}$. Let $A = \bigcup_{\alpha<\rho} A_\alpha$ with each $A_\alpha \in \boldsymbol{\Delta}$. Fix a $B \in \boldsymbol{\Gamma} - \boldsymbol{\Delta}$ and write it as a $\boldsymbol{\Sigma}^1_1$ bounded union: $B = \bigcup_{\alpha<\rho} B_\alpha$.

By $\boldsymbol{\Sigma}^1_1$ boundedness, II has a winning strategy σ such that if $x \in B$ then $\sigma(x)$ is a $\boldsymbol{\Gamma}$ code for some A_α where $\alpha > \varphi(x) = \mu\beta\ (x \in B_\beta)$.

Then $x \in A \leftrightarrow \exists y\ (y \in B \wedge (\sigma(y))(x) \in U)$, where $U \in \boldsymbol{\Gamma} - \boldsymbol{\Delta}$.

So, $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}^* \subseteq \exists^{\omega^\omega} \boldsymbol{\Gamma}$.

Finally, $\mathbf{\Gamma}^*$ is closed under $\forall^{\omega^\omega}$:

If $A \subseteq \omega^\omega \times \omega^\omega$ is in $\mathbf{\Gamma}^*$, $A = \bigcup_{\alpha<\rho} A_\alpha$ (a $\mathbf{\Sigma}^1_1$ bounded union), then $B = \forall^{\omega^\omega} A = \bigcup_{\alpha<\rho} B_\alpha$ where $B_\alpha = \forall^{\omega^\omega} A_\alpha$. This is also a $\mathbf{\Sigma}^1_1$ bounded union.

It now follows that $\mathbf{\Gamma} = \mathbf{\Gamma}^*$ and $\bigcup_\rho \mathbf{\Delta} = \mathbf{\Sigma}^\lambda_0$.

Exactly same arguments work in type 4 except $\mathbf{\Gamma} = \mathbf{\Gamma}^* = \exists^{\omega^\omega} \mathbf{\Gamma}$.

We now easily have pwo($\boldsymbol{\Pi}^{\lambda}_{-1}$). For let $A \in \boldsymbol{\Pi}^{\lambda}_{-1}$,

$$A = \bigcup_{\alpha<\rho} A_\alpha$$

an increasing $\boldsymbol{\Sigma}^1_1$-bounded union, where $A_\alpha \in \boldsymbol{\Delta}$.

For $x \in A$ let $\varphi(x) = \mu\alpha \, (x \in A_\alpha)$.

This is $\boldsymbol{\Pi}^{\lambda}_{-1}$-norm on A. For example

$$x <^*_\varphi y \leftrightarrow \exists \alpha < \rho \, (x \in A_\alpha \wedge y \notin A_\alpha).$$

This is easily a $\boldsymbol{\Sigma}^1_1$-bounded union.

Type 4: We have $\mathbf{\Gamma} = \mathbf{\Gamma}^* = \exists^{\omega^\omega} \mathbf{\Gamma} = \bigcup_\lambda \mathbf{\Delta}$. Note here that $\rho = \lambda$: if $\rho = \mathrm{cof}(\lambda) < \lambda$ then there is a $\mathbf{\Delta}$ pwo of length ρ. Let $A \in \mathbf{\Gamma} - \mathbf{\Delta}$ and write $A = \bigcup_{\alpha<\rho} A_\alpha$, $A_\alpha \in \mathbf{\Delta}$. By coding lemma there in a $\mathbf{\Delta}$ set of codes C which reduce the A_α to a $\check{\mathbf{\Gamma}}$ set. This computes $A \in \check{\mathbf{\Gamma}}$, a contradiction.

$$x \in A \leftrightarrow \exists \sigma \, (\sigma \in C \wedge \sigma(x) \in U)$$

where $U \in \check{\mathbf{\Gamma}} - \mathbf{\Delta}$.

We use here that $\mathbf{\Sigma}^\lambda_{-1}$ is closed under $\wedge$. More generally, if $\mathbf{\Sigma}^\lambda_{-1}$ is closed under intersections with $\mathbf{\Delta}$, then λ is regular.

Note in this case that

$$\begin{aligned}\boldsymbol{\Gamma} &= \boldsymbol{\Sigma}^1_1\text{-bounded unions of } \boldsymbol{\Delta} \text{ sets}\\ &= \boldsymbol{\Delta}\text{-bounded unions of } \boldsymbol{\Delta} \text{ sets (usual boundedness argument)}\\ &= \bigcup_{\lambda} \boldsymbol{\Delta} \text{ (since } \boldsymbol{\Gamma} = \exists^{\omega^\omega} \boldsymbol{\Gamma})\end{aligned}$$

Recall $\boldsymbol{\Pi}^\lambda_1 = \boldsymbol{\Gamma}(\lambda) \wedge \check{\boldsymbol{\Gamma}}(\lambda)$. We show pwo($\boldsymbol{\Pi}^\lambda_1$).

Let $A \in \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ (where $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\lambda)$.

Say $A = B \cap C$ where $B \in \boldsymbol{\Gamma}$, $C \in \check{\boldsymbol{\Gamma}}$.

Write $B = \bigcup_{\alpha<\lambda} B_\alpha$, an increasing $\boldsymbol{\Delta}$-bounded union.

Write $D = \omega^\omega - C = \bigcup_{\alpha<\lambda} D_\alpha$, an increasing $\boldsymbol{\Delta}$-bounded union.

For $x \in A = B \cap (\omega^\omega - D)$, let $\varphi(x) = \mu\alpha\ (x \in B_\alpha)$.

Then

$$x <^*_\varphi y \leftrightarrow (x \in C) \wedge \exists \alpha < \lambda\ \exists \beta \leq \alpha\ (x \in B_\beta) \wedge (y \notin B_\beta \vee y \in D_\alpha).$$

This is an intersection of a $\check{\boldsymbol{\Gamma}}$ set ($\{(x, y) \colon x \in C\}$) with a $\boldsymbol{\Gamma}$ set (a λ-union of $\boldsymbol{\Delta}$ sets). Similarly for $\leq^*_\varphi$.

Useful Fact

Lemma (Martin)

Let $\boldsymbol{\Gamma}$ be nonselfdual, closed under $\forall^{\omega^\omega}$, $\vee$, and assume pwo($\boldsymbol{\Gamma}$)*. Then $\boldsymbol{\Delta}$ is closed under $< \delta(\boldsymbol{\Delta})$ unions and intersections.*

Proof.

Otherwise by the coding lemma $\check{\boldsymbol{\Gamma}} = \bigcup_\kappa \boldsymbol{\Delta}$ for some least $\kappa < \delta$. But then pwo($\check{\boldsymbol{\Gamma}}$), a contradiction. □

Wellordered Unions

Theorem

Suppose $\boldsymbol{\Gamma}$ *is nonselfdual,* pwo($\boldsymbol{\Gamma}$)*, and* $\exists^{\omega^\omega}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$*. Then* $\boldsymbol{\Gamma}$ *is closed under wellordered unions.*

Case I: $\boldsymbol{\Gamma}$ closed under $\wedge^\omega$, $\vee^\omega$ but not $\forall^{\omega^\omega}$.

Let

$$\delta_1 = \sup\{|\prec| \colon \prec \in \boldsymbol{\Gamma} \ \wedge \ (\prec \text{ is wellfounded })\}$$

$$\delta_2 = \sup\{|\prec| \colon \prec \in \exists^{\omega^\omega}\check{\boldsymbol{\Gamma}} \ \wedge \ (\prec \text{ is wellfounded })\}$$

Let ρ be least so that some $\bigcup_{\alpha<\rho} A_\alpha \notin \boldsymbol{\Gamma}$, with each $A_\alpha \in \boldsymbol{\Gamma}$. So, ρ is regular, uncountable. Easily $\delta_1 < \delta_2$ (use closure of $\boldsymbol{\Gamma}$ under $\wedge^\omega$). We must have $\bigcup_\rho \boldsymbol{\Gamma} \supseteq \exists^{\omega^\omega} \check{\boldsymbol{\Gamma}}$. We must have $\rho \geq \delta_2$ as otherwise there is a least $\rho' \leq \rho$ such that $\boldsymbol{\Delta}_1 = \boldsymbol{\Delta}(\exists^{\omega^\omega} \check{\boldsymbol{\Gamma}})$ is not closed under wellordered unions, and by the coding lemma $\bigcup_{\rho'} \boldsymbol{\Delta}_1 = \exists^{\omega^\omega} \check{\boldsymbol{\Gamma}}$ which shows pwo($\exists^{\omega^\omega} \check{\boldsymbol{\Gamma}}$), a contradiction to periodicity (which shows pwo($\forall^{\omega^\omega} \boldsymbol{\Gamma}$)). So, $\delta_1 < \delta_2 \leq \rho$.

Let $\prec \in \exists^{\omega^\omega} \check{\boldsymbol{\Gamma}}$ be wellfounded of length $< \delta_1$, and write $\prec = \bigcup_{\alpha<\rho} A_\alpha$, each $A_\alpha \in \boldsymbol{\Gamma}$. For $x \in \mathrm{dom}(\prec)$, let $\zeta(x) =$ the eventual rank of x in A_α (for α large enough). This is an order-preserving map from $\prec$ to δ_1, a contradiction.

Case II: $\boldsymbol{\Gamma}$ is closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$.

So, $\boldsymbol{\Gamma}$ is closed under $\wedge^\omega$, $\vee^\omega$.

By pwo($\boldsymbol{\Gamma}$), clearly $\check{\boldsymbol{\Gamma}}$ is also not closed under wellordered unions. Let ρ_1 be least so $\boldsymbol{\Gamma}$ is not closed under ρ-unions. Let ρ_2 be same for $\check{\boldsymbol{\Gamma}}$. ρ_1, ρ_2 are regular.

Then $\bigcup_{\rho_1} \boldsymbol{\Gamma} \supseteq \check{\boldsymbol{\Gamma}}$. So argument from before shows every $A \in \check{\boldsymbol{\Gamma}}$ is a $\boldsymbol{\Sigma}^1_1$-bounded union of length ρ_1 of $\boldsymbol{\Gamma}$ sets. Let φ be corresponding norm on A.

Play game: I plays x, II plays y, z, and II wins iff
$(x \in A) \to (y \text{ codes } A_\alpha) \wedge (z \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta)$ for some $\alpha \geq \varphi(x)$.

By $\boldsymbol{\Sigma}^1_1$-boundedness II has a winning strategy τ.

Define a relation $\prec$ by $x \prec y$ iff $(x, y \in A) \wedge (\tau(y)_1 \notin A_{\tau(x)_0})$.

Then $\prec$ is a $\check{\boldsymbol{\Gamma}}$ wellfounded relation of length ρ_2 (since ρ_2 is regular). By the coding lemma, $\rho_2 > \rho_1$.

A symmetrical argument shows $\rho_1 > \rho_2$, a contradiction.

Case III: $\boldsymbol{\Gamma}$ not closed under $\forall^\omega$ or $\wedge^\omega$.

We must have that λ is of Type 1, so $\mathrm{cof}(\lambda) = \omega$.

Also, $\boldsymbol{\Gamma} = \boldsymbol{\Sigma}_0^\lambda = \bigcup_\omega \Lambda$.

Let ρ be least such that $\bigcup_\rho \boldsymbol{\Gamma} \not\subseteq \boldsymbol{\Gamma}$. So, ρ is regular. By coding lemma, $\rho > \lambda$.

By Wadge, $\bigcup_\rho \boldsymbol{\Gamma} \supseteq \boldsymbol{\Sigma}_1^\lambda$. Let $A \in \check{\boldsymbol{\Gamma}} - \boldsymbol{\Gamma}$ and write $A = \bigcup_{\alpha<\rho} A_\alpha$ with each $A_\alpha \in \boldsymbol{\Gamma}$.

As before, let S be universal $\boldsymbol{\Sigma}^1_1$ and let $C(z) \leftrightarrow \forall x\, (S_z(x) \to x \in A)$. So, $C \in \boldsymbol{\Pi}^\lambda_1$. Write $C = \bigcup_{\alpha<\rho} C_\alpha$ with $C_\alpha \in \boldsymbol{\Gamma}$. Let $A'_\alpha(x) \leftrightarrow \exists z \in C_\alpha\, (S_z(x))$, so $A'_\alpha \in \boldsymbol{\Gamma}$ and A is the $\boldsymbol{\Sigma}^1_1$ bounded union of the A'_α. So, we may assume the $\{A_\alpha\}_{\alpha<\rho}$ is a $\boldsymbol{\Sigma}^1_1$ union.

Let U be universal $\boldsymbol{\Sigma}^\lambda_0$ set.

Play the game where I plays x, II plays y, z and II wins iff

$$(x \in A) \to \exists \alpha < \rho\, (U_y = A_\alpha \wedge z \in A_\alpha - \bigcup_{\beta<\alpha} A_\beta)$$

By boundedness, II has a winning strategy τ. This gives a $\boldsymbol{\Pi}^{\lambda}_0$ wellfounded relation (in fact prewellordering) of length ρ:

$$x_1 \prec x_2 \leftrightarrow (x_1, x_2 \in A) \wedge (\tau(x_2)_1 \notin U_{\tau(x_1)_0})$$

By the coding lemma we then have $\bigcup_\rho \boldsymbol{\Gamma} \subseteq \boldsymbol{\Sigma}^{\lambda}_1$, and so $\bigcup_\rho \boldsymbol{\Gamma} = \boldsymbol{\Sigma}^{\lambda}_1$.

Let $\rho' \leq \rho$ be least such that $\bigcup_{\rho'} \boldsymbol{\Delta}^{\lambda}_1 \not\subseteq \boldsymbol{\Delta}^{\lambda}_1$. By the coding lemma and the above, $\bigcup_{\rho'} \boldsymbol{\Delta}^{\lambda}_1 \subseteq \boldsymbol{\Sigma}^{\lambda}_1$ and so $\bigcup_{\rho'} \boldsymbol{\Delta}^{\lambda}_1 = \boldsymbol{\Sigma}^{\lambda}_1$. This show pwo($\boldsymbol{\Sigma}^{\lambda}_1$), a contradiction.

Lengths of Wellordered Unions

Theorem (Chuang)

Let $\boldsymbol{\Gamma}$ be nonselfdual, closed under $\forall^{\omega^\omega}$ and $\vee$. Then there is no strictly increasing or decreasing sequence of sets in $\boldsymbol{\Gamma}$ of length $(\delta(\boldsymbol{\Gamma})^+)$.

Proof. Let $\delta = \delta(\boldsymbol{\Gamma})$ be the supremum of the lengths of the $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ prewellorderings.

Let $\{A_\alpha\}_{\alpha<\delta^+}$ be a strictly increasing sequence of $\boldsymbol{\Gamma}$ sets. We may assume $\bigcup_{\beta<\alpha} A_\beta \subsetneq A_\alpha$.

Note that $\mathrm{pwo}(\exists^{\omega^\omega}\boldsymbol{\Gamma})$ and $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ is closed under $\wedge^\omega$, $\vee^\omega$, so is closed under wellordered unions.

Case I: $\boldsymbol{\Gamma}$ not closed under $\exists^{\omega^\omega}$.

Let φ be the norm corresponding to $\{A_\alpha\}_{\alpha<\delta^+}$.

Let $x \prec y \leftrightarrow (x, y \in A) \wedge (\varphi(x) < \varphi(y))$, so $\prec \in \exists^{\omega^\omega}\boldsymbol{\Gamma}$.

Let U be universal for $\boldsymbol{\Gamma}$.

Define

$$C(x, y, z) \leftrightarrow \exists \alpha < \delta^+ \, [(U_x = A_\alpha) \wedge (y, z \in A) \wedge (\varphi(y) = \alpha) \wedge (\varphi(z) = \alpha + 1)]$$

By coding lemma, let $S \subseteq C$ be in $\exists^{\omega^\omega} \boldsymbol{\Gamma}$ and such that for all $\alpha < \delta^+$ there is an $(x, y, z) \in S$ with $U_x = A_\alpha$.

Every set in $\boldsymbol{\Gamma}$ is a δ union of sets in $\boldsymbol{\Delta}$, so S is a δ-union of sets in $\exists^{\omega^\omega} \boldsymbol{\Delta} \subseteq \check{\boldsymbol{\Gamma}}$. Say $S = \bigcup_{\beta<\delta} S_\beta$, where $S_\beta \in \check{\boldsymbol{\Gamma}}$.

Note that every $\check{\boldsymbol{\Gamma}}$ wellfounded relation has length $< \delta$ (otherwise, by the coding lemma, there is a $\check{\boldsymbol{\Gamma}}$ unbounded subset of a $\boldsymbol{\Gamma}$ complete norm, a contradiction; we use here the closure of $\check{\boldsymbol{\Gamma}}$ under $\wedge$).

Consider the following prewellordering on S_β:

$$
\begin{aligned}
(x_1, y_1, z_1) \leq_\beta (x_2, y_2, z_2) &\leftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge \varphi(y_1) \leq \varphi(y_2) \\
&\leftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge y_1 \in U_{x_2} \\
&\leftrightarrow (x_1, y_1, z_1), (x_2, y_2, z_2) \in S_\beta \wedge z_2 \notin U_{x_1}
\end{aligned}
$$

So, $\leq_\beta$ in the intersection of $S_\beta \times S_\beta$ with a $\boldsymbol{\Gamma}$ or a $\check{\boldsymbol{\Gamma}}$ set. So, $\leq \beta \in \check{\boldsymbol{\Gamma}}$.

This gives a one-to-one map $\alpha \mapsto (\beta, \gamma)$ of δ^+ into $\delta \times \delta$, a contradiction: For $\alpha < \delta^+$, let $\beta < \delta$ be least such that some $(x, y, z) \in S_\beta$ has $\varphi(y) = \alpha$, and let $\gamma = |(x, y, z)|_{\leq_\beta}$.

Case II: $\boldsymbol{\Gamma}$ closed under $\exists^{\omega^\omega}$.

Let $\{A_\alpha\}_{\alpha<\delta^+}$ and $\prec$ be as before. Then $\prec \in \exists^{\omega^\omega}(\boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}})$ (this class is closed under wellordered unions).

Use coding lemma to get $S \in \exists^{\omega^\omega}(\boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}})$ as before. Note that $\exists^{\omega^\omega}(\boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}}) = \bigcup_\delta \check{\boldsymbol{\Gamma}}$. So write $S = \bigcup_{\beta<\delta} S_\beta$ with each $S_\beta \in \check{\boldsymbol{\Gamma}}$.

Every $\check{\boldsymbol{\Gamma}}$ wellfounded relation has length $< \delta$ as before, and this gives a contradiction as before.

Remark

In case II we can avoid appeal to closure theorem by considering cases as to whether there is a $\boldsymbol{\Gamma}$ wellfounded relation of length δ^+.

Question

Can we remove the assumption that $\boldsymbol{\Gamma}$ is closed under $\vee$ in Chuang's theorem?

Remark

There are two cases where $\forall^{\omega^\omega}\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}$, pwo($\boldsymbol{\Gamma}$), and $\boldsymbol{\Gamma}$ is not closed under $\vee$. Namely, at the base of a Type 2 hierarchy ($\boldsymbol{\Gamma}$ is the Steel class $\boldsymbol{\Pi}^{\lambda}_{-1}$), and for the class $\boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}}$ where $\boldsymbol{\Gamma}$ is closed under quantifiers.

In the second case similar arguments show that there is no strictly increasing or decreasing sequence of $\boldsymbol{\Gamma} \wedge \check{\boldsymbol{\Gamma}}$ sets of length λ^+ ($\lambda = o(\Lambda)$, $\Lambda = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$).

For the Steel pointclass we don't know.

Two Background Results

We will need the following two fundamental results.

Theorem (Steel, Woodin)

(AD) *The Suslin cardinals are closed below their supremum.*
(AD^+) *The Suslin cardinal are closed.*

Theorem (Martin, Woodin)

Every tree T on $\omega \times \kappa$, where κ is less than the supremum of the Suslin cardinals, is weakly homogeneous.

Type 1 Case

Suppose κ is a limit Suslin cardinal (i.e., κ is a limit of Suslin cardinals).

First consider the case $\mathrm{cof}(\kappa) = \omega$. Let $\Lambda = S(< \kappa)$. So, Λ is closed under quantifiers, negation. Let $\lambda = o(\Lambda)$.

Claim. $\lambda = \kappa$.

Proof. If $\lambda > \kappa$ then there is a Λ prewellordering of length κ. By the coding lemma, $S(\kappa) \in \Lambda$, a contradiction. So, $\lambda \leq \kappa$. To show $\lambda \geq \kappa$ it suffices to show that there are Λ prewellorderings of length $\kappa' < \kappa$, where κ is a Suslin cardinal.

There is a κ' increasing sequence $\{B_\beta\}_{\beta<\kappa'}$ of sets in $S(\kappa') \subseteq \Lambda$. We can find a pointclass $\Gamma' \subseteq \Lambda$ with $\exists^{\omega^\omega} \Gamma' = \Gamma'$, $\mathrm{pwo}(\Gamma')$. By closure theorem this gives a Γ' prewellordering of length κ.

Since $S(\kappa)$ is closed under $\exists^{\omega^\omega}$ and $\wedge^\omega$, $\vee^\omega$, $S(\kappa) \supseteq \mathbf{\Sigma}^\kappa_1$. By coding lemma, $S(\kappa) \subseteq \mathbf{\Sigma}^\kappa_1$ (there is a prewellordering of length λ in $\mathbf{\Sigma}^\kappa_0$). So, $S(\kappa) = \mathbf{\Sigma}^\kappa_1$.

We show scale($\mathbf{\Sigma}^\kappa_0$).

Let $\kappa = \sup_n \kappa_n$, each κ_n a Suslin cardinal. Let $A \in \mathbf{\Sigma}^\kappa_0$, say $A = \bigcup_n A_n$ where $A_n \in S(\kappa_n)$. Let $A_n = p[T_n]$ and let ψ^n be the corresponding scale. For $x \in A$ let $\varphi_0(x) = \mu n \ (x \in A_n)$, and

$$\varphi_{i+1}(x) = \langle \varphi_0(x), \psi_i^{\varphi_0(x)}(x) \rangle.$$

This is easily a $\mathbf{\Sigma}^\kappa_0$ scale. Then propagate by periodicity.

So, scale($\boldsymbol{\Pi}^{\lambda}_1$), scale($\boldsymbol{\Sigma}^{\lambda}_2$), scale($\boldsymbol{\Pi}^{\lambda}_3$), $\dots$.

By Kunen-Martin, a $\boldsymbol{\Pi}^{\lambda}_1$ scale has norms of length $\leq \lambda^+$ (all initial segments are in $\boldsymbol{\Delta}^{\lambda}_1 \subseteq S(\lambda) = \boldsymbol{\Sigma}^{\lambda}_1$).

By the coding lemma, there are $\boldsymbol{\Sigma}^{\lambda}_1$ prewellorderings of length any $\alpha < \lambda^+$. So, λ^+ is the supremum of the lengths of the $\boldsymbol{\Sigma}^{\lambda}_1$ wellfounded relations and so λ^+ is regular. The $\boldsymbol{\Pi}^{\lambda}_1$-norms on a $\boldsymbol{\Pi}^{\lambda}_1$-complete set must have length λ^+ (otherwise we violate boundedness of $\boldsymbol{\Sigma}^{\lambda}_1$ sets in the norm).

So, λ^+ is a regular Suslin cardinal. From the coding lemma, $S(\lambda^+) = \boldsymbol{\Sigma}^{\lambda}_2$.

Quick Summary of Projective Arguments

The remaining arguments are just as in the projective hierarchy.

A $\mathbf{\Pi}^{\lambda}_{2n+3}$ norm on a $\mathbf{\Pi}^{\lambda}_{2n+3}$ set has length $\delta^{\lambda}_{2n+3} = \delta(\mathbf{\Delta}^{\lambda}_{2n+3})$. Such a set cannot be α-Suslin for $\alpha < \delta^{\lambda}_{2n+3}$ (coding lemma). So, δ^{λ}_{2n+3} is a Suslin cardinal and easily $S(\delta^{\lambda}_{2n+3}) = \mathbf{\Sigma}^{\lambda}_{2n+4}$. From coding lemma and closure of $\mathbf{\Pi}^{\lambda}_{2n+3}$ under $\vee$ we have that δ^{λ}_{2n+3} is regular.

A $\mathbf{\Pi}^{\lambda}_{2n+2}$ set has a scale with norms in $\mathbf{\Delta}^{\lambda}_{2n+3}$ and so of length $< \delta^{\lambda}_{2n+3}$. So, the set is λ_{2n+3}-Suslin for some

$$\delta_{2n+1} < \lambda_{2n+3} < \delta_{2n+3}.$$

Clearly $S(\lambda_{2n+3}) \supseteq \mathbf{\Sigma}^{\lambda}_{2n+3}$, and we have equality.

Claim (Kechris)

$\delta_{2n+3} = (\lambda_{2n+3})^+$ and $\mathrm{cof}(\lambda_{2n+3}) = \omega$.

Proof. By Kunen-Martin every $\mathbf{\Sigma}^{\lambda}_{2n+3}$ wellfounded relation has length $\leq (\lambda_{2n+3})^+$, so $\delta_{2n+3} \leq (\lambda_{2n+3})^+$, and so $\delta_{2n+3} = (\lambda_{2n+3})^+$.

If $\mathrm{cof}(\lambda_{2n+3}) > \omega$, then $\mathbf{\Sigma}^{\lambda}_{2n+3} = \bigcup_{\lambda_{2n+3}} S(< \lambda_{2n+3})$ (using the coding lemma for $\supseteq$). So, $\mathbf{\Sigma}^{\lambda}_{2n+3} = \bigcup_{\rho} \mathbf{\Delta}^{\lambda}_{2n+3}$ where $\rho \leq \lambda_{2n+3}$ is least so that $\mathbf{\Delta}^{\lambda}_{2n+3}$ is not closed under ρ-unions.

Then we have $\mathrm{pwo}(\mathbf{\Sigma}^{\lambda}_{2n+3})$, a contradiction.

Type 2, 3 cases

Suppose now κ is a limit Suslin cardinal with $\mathrm{cof}(\kappa) > \omega$. Let $\Lambda = S(<\kappa)$. We assume we are in Type 2 or 3, that is, the Steel class $\boldsymbol{\Pi}^{\lambda}_{-1}$ is not closed under $\exists^{\omega^\omega}$.

As above we have $\lambda = \kappa$.

$\boldsymbol{\Sigma}^{\kappa}_{0}$ is closed under wellordered unions, so $S(\kappa) = \bigcup_\kappa S(<\kappa) \subseteq \boldsymbol{\Sigma}^{\kappa}_{0}$.

So, either $S(\kappa) = \boldsymbol{\Sigma}^{\kappa}_{0}$ or $S(\kappa) = \boldsymbol{\Sigma}^{\kappa}_{-1}$. We show the latter case does not occur.

Let $\mathbf{\Gamma}$ be the collection of $A = p[T]$ where T is a homogeneous tree on κ. $\mathbf{\Gamma}$ is easily a pointclass.

We have $S(\kappa) = \exists^{\omega^\omega} \mathbf{\Gamma}$ as every tree on $\omega \times \kappa$ is weakly homogeneous (we are assuming κ is not the largest Suslin cardinal). So, $\mathbf{\Gamma} \not\subseteq \mathbf{\Lambda}$.

It suffices to show the following:

Claim

$\forall^{\omega^\omega} \mathbf{\Gamma} \subseteq S(\kappa)$.

Proof. Let $A(x) \leftrightarrow \forall y\, B(x, y)$, where $B = p[T]$, T a homogeneous tree on $\omega \times \omega \times \kappa$.

For $x \in A$ consider the game:

I plays out y, II plays out $\vec{\alpha}$, and II wins iff $(x, y, \vec{\alpha}) \in [T]$.

This is a closed game for II, and II has a winning strategy by the homogeneity of T.

So define T' on $\omega \times \kappa$ by:
$(s, \vec{\alpha}) \in T'$ iff for all $u = u_m \in \omega^{<\omega}$ with $m < (s)$ we have

$$(s \restriction \mathrm{lh}(u), u, (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{\mathrm{lh}(u)}})) \in T$$

where i_k is such that $u_{i_k} = u \restriction k$.

Remark

It follows that $\boldsymbol{\Gamma} = \boldsymbol{\Pi}^{\lambda}_{-1}$. This gives another characterization of the Steel pointclasses as the projections of κ-homogeneously Suslin trees.

Remark

One can also rule out the case $S(\kappa) = \boldsymbol{\Sigma}^{\lambda}_{-1}$ by using Woodin's proof of the closure of the Suslin cardinals which shows that there is a κ prewellordering in $S(\kappa)$ (and so $S(\kappa) \not\subseteq \boldsymbol{\Sigma}^{\lambda}_{-1}$).

So, $S(\kappa) = \exists^{\omega^\omega} \boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}$ the Steel pointclass,

First we show scale$(S(\kappa))$. Since cof$(\kappa) > \omega$,
$S(\kappa) = \bigcup_\kappa S(< \kappa) = \bigcup_\kappa \Lambda$.

Let $A \in S(\kappa)$, $A = p[T]$, T a tree on $\omega \times \kappa$.

For $x \in A$, let $\varphi_0(x) = \mu\alpha\ (x \in p[T \restriction \alpha])$. Let

$$\varphi_{n+1}(x) = \langle \varphi_0(x), \psi_0^{\varphi_0(x)}(x), \dots, \psi_n^{\varphi_0(x)}(x) \rangle,$$

where ψ^α is the scale from $T \restriction \alpha$. Each $\varphi_n \in \bigcup_\kappa \Lambda = S(\kappa)$.

We now show scale($\mathbf{\Gamma}$) where $\mathbf{\Gamma} = \mathbf{\Pi}^{\lambda}_{-1}$ is the Steel pointclass.

Let $\rho \leq \kappa$ be least such that $\bigcup_\rho \mathbf{\Delta} \not\subseteq \mathbf{\Delta}$ (so actually $\rho = \mathrm{cof}(\kappa)$). Recall $\mathbf{\Gamma} = \mathbf{\Sigma}^1_1$-bounded ρ-unions of $\mathbf{\Delta}$ sets.

Let $A \in \mathbf{\Gamma}$. Let U be universal $\mathbf{\Sigma}^1_1$. Let $C = \{y \colon U_y \subseteq A\}$. Then, as before, $C \in \mathbf{\Gamma}$. Let $C = p[T]$, T a tree on $\omega \times \kappa$. Let S be a tree on $(\omega)^3$ with $U(y, x) \leftrightarrow (x, y) \in p[S]$.

Define the tree V on $(\omega)^3 \times \kappa$ by:

$$(s, t, u, \vec{\alpha}) \in S \leftrightarrow (s, t, u) \in S \wedge (t, \vec{\alpha}) \in V.$$

Clearly $A = p[V]$.

Identify V with a tree V' on $\omega \times \kappa$ by reverse lexicographic order on $\omega \times \omega \times \kappa$ (i.e., order by ordinal first).

Modifying V' slightly to V'' we may assume that for $(s, \vec{\alpha}) \in V''$ we have $\alpha_0 > \max\{\alpha_1, \dots, \alpha_k\}$.

Let $\{\varphi_i\}$ be the scale corresponding to V''.

Each φ_i maps into κ. We show that $<_i^*$, $\leq_i^*$ are in $\boldsymbol{\Gamma}$.

For example, $<_i^* = \bigcup_{\alpha<\rho} B_\alpha$ where:

$$(x,y)\in B_\alpha \leftrightarrow \exists \beta\leq f(\alpha)\,[((x,y)\in A\wedge \varphi_i(x)=\beta)\wedge\neg(y\in A\wedge\varphi_i(y)\leq\beta)$$

where $f\colon \rho\to\kappa$ is cofinal.

To see this is a $\mathbf{\Sigma}^1_1$-bounded union, let $E\subseteq <_i^*$ be $\mathbf{\Sigma}^1_1$. Let $E'=\{x\colon \exists y\,(x,y)\in E\}$, so $E'\in\mathbf{\Sigma}^1_1$ and $E'\subseteq A$.

Let $E'=U_y$. Fix $\vec{\alpha}$ such that $(y,\vec{\alpha})\in[T]$. This gives an ordinal $\gamma<\kappa$ such that $E'\subseteq p[V''\restriction\gamma]$.

This shows E' is bounded in the $\{B_\alpha\}$ union.

We propagate scales from $\boldsymbol{\Sigma}^\lambda_0$ by periodicity to $\boldsymbol{\Pi}^\lambda_1$, $\boldsymbol{\Sigma}^\lambda_2$, $\boldsymbol{\Pi}^\lambda_3, \dots$.

As before, all the δ^λ_{2n+1} are regular Suslin cardinals, and $S(\delta^\lambda_{2n+1}) = \boldsymbol{\Sigma}^\lambda_{2n+2}$.

Also as before, $\delta^\lambda_{2n+1} = (\lambda_{2n+1})^+$ where $\mathrm{cof}(\lambda_{2n+1}) = \omega$. Also, $S(\lambda_{2n+1}) = \boldsymbol{\Sigma}^\lambda_{2n+1}$.

Consider now limit Suslin κ with $\Lambda = S(<\kappa)$ of type 4.

We previously showed $\lambda = o(\Lambda) = \kappa$. Recall λ is regular in this case. Previous arguments also showed scale($\boldsymbol{\Gamma}$), where $\boldsymbol{\Gamma}$ is at the base (so pwo($\boldsymbol{\Gamma}$), $\boldsymbol{\Gamma}$ closed under quantifiers).

However, we cannot propagate scales by periodicity.

Question

How do we get the next Suslin cardinal?

We will present a method of Martin for constructing the next Suslin cardinal.

The Largest Suslin Cardinal

Assume there is a largest Suslin cardinal κ.

Claim

κ is a regular limit Suslin cardinal. $\boldsymbol{\Gamma} = S(\kappa)$ and scale($\boldsymbol{\Gamma}$). Also, $S(\kappa)$ is closed under quantifiers.

Proof. $S(\kappa)$ is closed under $\forall^{\omega^\omega}$ as otherwise $\forall^{\omega^\omega} S(\kappa)$ would be a larger pointclass admitting scales (by periodicity). So, $S(\kappa)$ is closed under quantifiers.

So, $\boldsymbol{\Delta} = \boldsymbol{\Gamma} \cap \check{\boldsymbol{\Gamma}}$ is closed under quantifiers, $\wedge$, $\vee$. Let $\lambda = o(\boldsymbol{\Delta})$.

We must have that λ is regular as otherwise $\boldsymbol{\Gamma}$ would not be closed under $\wedge^\omega$, $\vee^\omega$.

Suppose $\lambda < \kappa$. There is a κ increasing sequence $\{A_\alpha\}_{\alpha<\kappa}$ of sets in $S(\kappa)$. Let $\boldsymbol{\Gamma} = S(\kappa)$ or $\boldsymbol{\Gamma} = \check{S}(\kappa)$, with pwo($\boldsymbol{\Gamma}$). This gives either an increasing or a decreasing κ-length sequence of sets in $\boldsymbol{\Gamma}$, contradicting Chuang's theorem.

So, $\kappa = \lambda$. In particular, κ is regular. Since cof$(\kappa) > \omega$, $S(\kappa) = \bigcup_\kappa \boldsymbol{\Delta}$.

κ must be a limit of Suslin cardinals as if there were a largest Suslin $\kappa' < \kappa$ then $S(\kappa) = \bigcup_\kappa S(\kappa')$, a contradiction as we can find a $\boldsymbol{\Gamma}_0 \supseteq S(\kappa')$ closed under wellordered unions.
Previous arguments show scale$(S(\kappa))$.

The Envelope

Let $\mathbf{\Gamma}$ be a pointclass, and $\kappa \in \mathrm{On}$. We define the $\mathbf{\Gamma}, \kappa$ envelope as follows.

Definition (Martin)

Let $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$, each $A_\alpha \subseteq \omega^\omega$. Then $\overline{\mathcal{A}}$ is the set of $A \subseteq \omega^\omega$ such that for all countable $S \subseteq \omega^\omega$, there is an $\alpha < \kappa$ such that $S \cap A = S \cap A_\alpha$.

We let

$$\Lambda(\mathbf{\Gamma}, \kappa) = \{\overline{\mathcal{A}} \colon \mathcal{A} \subseteq \mathbf{\Gamma} \wedge \|\mathcal{A}\| \leq \kappa\}.$$

Lemma

Let $\boldsymbol{\Gamma}$ be nonselfdual, closed under $\forall^{\omega^\omega}$, and $\mathrm{pwo}(\boldsymbol{\Gamma})$ *(if $\boldsymbol{\Delta}$ not closed under $\exists^{\omega^\omega}$ the assume* $\mathrm{scale}(\exists^{\omega^\omega}\boldsymbol{\Gamma}$ *with norms into κ). Then $\Lambda(\boldsymbol{\Delta},\kappa)=\Lambda(\boldsymbol{\Gamma},\kappa)=\Lambda(\exists^{\omega^\omega}\boldsymbol{\Gamma},\kappa)$ where $\kappa=\delta(\boldsymbol{\Delta})$.*

Proof. There is an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ prewellordering (W,φ) of length κ such that for all $\alpha<\kappa$, $\{x\in W\colon \varphi(x)=\alpha\}\in\boldsymbol{\Delta}$.

Let $\{A_\alpha\}_{\alpha<\kappa}$ be given with $A_\alpha\in\exists^{\omega^\omega}\boldsymbol{\Gamma}$.
By coding lemma, $C=\{(x,y)\colon x\in W\wedge y\in A_{\varphi(x)}\}\in\exists^{\omega^\omega}\boldsymbol{\Gamma}$. Recall $\exists^{\omega^\omega}\boldsymbol{\Gamma}=\bigcup_\rho\boldsymbol{\Delta}$ where $\rho=\mathrm{cof}(\kappa)$. Write $C=\bigcup_{\beta<\rho}C_\beta$ with $C_\beta\in\boldsymbol{\Delta}$.

For $\alpha<\kappa$, $\beta<\rho$, let

$$y\in A_{\alpha,\beta}\leftrightarrow\exists x\,((x\in W\wedge\varphi(x)=\alpha)\wedge(x,y)\in C_\beta).$$

This works if $\exists^{\omega^\omega}\boldsymbol{\Delta}=\boldsymbol{\Delta}$.

Assume now $\boldsymbol{\Gamma} = \forall^{\omega^\omega} \boldsymbol{\Delta}$ (sketch).

Let φ_n be an $\exists^{\omega^\omega}$-scale on an $\exists^{\omega^\omega} \boldsymbol{\Gamma}$ set W. Let C be as above using norm φ_0 (we may assume $\varphi_)$ is onto κ).

For $\alpha < \kappa$ let $\beta(\alpha) < \kappa$ be the least reliable ordinal $> \alpha$.

By Becker-Kechris there is uniformly in $\alpha < \kappa$ and $z \in \omega^\omega$ a closed game $G_\alpha(z)$ for II, where I plays ordinals $< \beta(\alpha)$, II plays ordinals $< \kappa$, such that II has a winning strategy iff $z \in A_\alpha$.

Let $G_{\alpha,\gamma}(z)$ be the game where II's moves are restricted to ordinals $< \gamma$.

Let $A_{\alpha,\gamma,\delta}$ be the set of z such that I doesn't win $G_{\alpha,\gamma}$ with rank less than δ. Each $A_{\alpha,\gamma,\delta} \in \boldsymbol{\Delta}$ by the closure of $\boldsymbol{\Delta}$ under $< \kappa$ unions and intersections.

The next lemma is a "universality" property.

Lemma

Same hypotheses as previous lemma. then there is a single $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with each $A_\alpha \in \boldsymbol{\Delta}$ such that every set in $\Lambda(\boldsymbol{\Gamma}, \kappa)$ is Wadge reducible to a set in $\overline{\mathcal{A}}$.

Proof. Let (W, φ) again be an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ pwo on an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ complete set W. Let U_1, U_2 be a universal $\boldsymbol{\Gamma}$ sets for subsets of ω^ω and $(\omega^\omega)^2$.

Define

$$(x, y) \in A_\alpha \leftrightarrow \exists z, u\, [(z \in W \wedge \varphi(z) = \alpha) \wedge U_2(x, z, u) \wedge U_1(u, y)].$$

Each $A_\alpha \in \boldsymbol{\Gamma}$, Given any $\mathcal{A}' = \{A'_\alpha\}_{\alpha<\kappa}$ with $A'_\alpha \in \boldsymbol{\Gamma}$, by the coding lemma there is an x_0 such that $A'_\alpha = (A_\alpha)_{x_0}$ for all $\alpha < \kappa$.

Let $A' \in \overline{\mathcal{A}'}$.

For each Turing degree d let $\beta(d) < \kappa$ be least such that $A' \cap d = A'_{\beta(d)} \cap d = (A_{\beta(d)})_{x_0}$.

Define $A \subseteq \omega^\omega \times \omega^\omega$ by

$$(x, y) \in A \leftrightarrow \forall^* d \, [(x, y) \in A_{\beta(d)}].$$

Then $A \in \overline{\mathcal{A}}$ and $A' = A_{x_0}$ so $A' \leq_W A$.

Corollary

Under same hypotheses, $\Lambda(\boldsymbol{\Gamma}, \kappa)$ is closed under $\wedge$, $\vee$, $\neg$.

This follows immediately using $\Lambda(\boldsymbol{\Gamma}, \kappa) = \Lambda(\boldsymbol{\Delta}, \kappa)$.

The next lemma says that if $\boldsymbol{\Gamma}$ is closed under quantifiers then so is $\Lambda(\boldsymbol{\Gamma}, \kappa)$.

Lemma

Suppose $\boldsymbol{\Gamma}$ is nonselfdual, closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$, and $\mathrm{pwo}(\boldsymbol{\Gamma})$. *Let $\kappa = o(\boldsymbol{\Delta})$. Then $\Lambda(\boldsymbol{\Gamma}, \kappa)$ is closed under $\exists^{\omega^\omega}$, $\forall^{\omega^\omega}$.*

Proof.

Let $A \subseteq \omega^\omega \times \omega^\omega$ be in $\overline{\mathcal{A}}$ with $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$, and each $A_\alpha \in \boldsymbol{\Delta}$. Let $B(x) \leftrightarrow \exists y\, A(x, y)$.

View every $z \in \omega^\omega$ as coding a countable set $s_z \subseteq \omega^\omega$ and a function $f_z \colon s_z \to \{0,1\}$. For every degree d let $\alpha_z(d) < \kappa$ be least (if it exists) such that

$$\forall x \in s_z\, [f_z(x) = 1 \leftrightarrow \exists y \leq d\, (x,y) \in A_{\alpha_z(d)}].$$

Let $C = \{z \colon \forall^* d\, (\alpha_z(d) \text{ is defined })\}$. Then $C \in \boldsymbol{\Gamma}$.

Let φ be the norm on C corresponding to the pwo

$$z_1 \preceq z_2 \leftrightarrow \forall^* d\, [\alpha_{z_1}(d) \leq \alpha_{z_2}(d)].$$

This is a $\boldsymbol{\Gamma}$ pwo on C and so has length κ.

For $z \in C$ define $B_{\varphi(z)}$ by

$$z \in B_{\varphi(z)} \leftrightarrow \forall^* d\, [\exists y \leq d\, ((x,y) \in A_{\alpha_z(d)})].$$

Coding Measures

Let $\boldsymbol{\Gamma}$ and $\kappa = o(\boldsymbol{\Delta})$ be as above. Fix an $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ norm (W, φ) of length κ (with each $W_\alpha \in \boldsymbol{\Delta}$).

Let U be universal $\exists^{\omega^\omega}\boldsymbol{\Gamma}$. For $z \in \omega^\omega$ let $B_z = \{\alpha < \kappa \colon \exists x \in W\ (\varphi(x) = \alpha\}$. By the coding lemma, every subset of κ is of the form B_z.

Let μ be a measure on κ. The code set is defined by

$$C_\mu = \{z \colon \mu(B_z) = 1\}.$$

Lemma

Let Γ, κ be as above. Then $A \in \Lambda(\Gamma, \kappa)$ iff there is a measure μ on κ such that $A \leq_W C_\mu$.

Proof. ($\Leftarrow$) Let μ be a measure on κ. Define $\{A_\alpha\}_{\alpha<\kappa}$ by $z \in A_\alpha \leftrightarrow \alpha \in B_z$. Then $C_\mu \in \overline{\mathcal{A}}$ (by countable additivity of the measure).

($\Rightarrow$) Let $A \in \overline{\mathcal{A}}$, $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$. Define $f \colon \mathcal{D} \to \alpha$ by $f(d) = \mu\alpha < \kappa$ $(A \cap d = A_\alpha \cap d)$. Let $\mu = f(\nu)$ where ν is the Martin measure on the degrees.

Let $R(x, y) \leftrightarrow (x \in W \wedge y \in A_{\varphi(x)}$, so $R \in \exists^{\omega^{\omega}} \boldsymbol{\Gamma}$ by the coding lemma.

By *s-m-n* theorem there is a continuous function h such that $U_{h(y)} = \{x \in W \colon y \in A_{\varphi(x)}\}$. That is, $B_{h(y)} = \{\alpha < \kappa \colon y \in A_\alpha\}$.

Then

$$\begin{aligned} y \in A &\leftrightarrow \forall^*_\nu d \, (y \in A_{f(d)}) \\ &\leftrightarrow \forall^*_\mu \alpha \, (y \in A_\alpha) \\ &\leftrightarrow h(y) \in C_\mu. \end{aligned}$$

Upper bound for the next (semi) scale

Theorem

Let $\boldsymbol{\Gamma}$ be nonselfdual, closed under $\forall^{\omega^\omega}$, and $\mathrm{pwo}(\boldsymbol{\Gamma})$. Assume every $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ set admits a $\exists^{\omega^\omega}\boldsymbol{\Gamma}$ scale with norms into $\kappa = \delta(\boldsymbol{\Delta})$. Assume also that there is a Suslin cardinal greater than κ. Then every set in $\forall^{\omega^\omega}\check{\boldsymbol{\Gamma}}$ admits a semi-scale with all norms in $\Lambda(\boldsymbol{\Gamma}, \kappa)$.

Proof. In all cases, $\mathrm{cof}(\kappa) > \omega$. [This follows by usual boundedness arguments if $\boldsymbol{\Gamma}$ is closed under $\vee$. $\boldsymbol{\Gamma}$ cannot be at the base of a type 1 hierarchy. The remaining problematic case is where $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 \wedge \check{\boldsymbol{\Gamma}}_)$ where $\boldsymbol{\Gamma}_0$ is closed under quantifiers. This case cannot occur as otherwise $\check{\boldsymbol{\Gamma}}_0$ would admit scales with a fixed projective class over $\boldsymbol{\Gamma}_0$, contradicting the next result.]

Let T be a tree on $\omega \times \kappa$ with $A = p[T]$ a complete $\exists^{\omega^\omega} \boldsymbol{\Gamma}$ set. Since $\mathrm{cof}(\kappa) > \omega$ we may assume that for all $(s, \vec{\alpha}) \in T$ that $\alpha_0 > \max\{\alpha_1, \ldots, \alpha_k\}$.

T is weakly homogeneous (there is a larger Suslin cardinal.

The homogeneous tree construction produces a tree T' with $p[T'] = B = \omega^\omega - A$, so $B \in \forall^{\omega^\omega} \check{\boldsymbol{\Gamma}}$.

For $x \in B$, let $f \colon T_x \to \kappa$ be the rank function (using KB order on T; view T as a homogeneous tree on $\omega \times \omega \times \kappa$ with measures $\mu_{s,t}$).

Let

$$\varphi_n(x) = [f \restriction T_{x\restriction \mathrm{lh}(t_n),t_n}]_{\mu_{x\restriction \mathrm{lh}(t_n),t_n}}.$$

Then $\vec{\varphi}$ is a semiscale on B. We show the norms are in $\Lambda(\boldsymbol{\Gamma},\kappa)$.

We have (assume $x \restriction \mathrm{lh}(t_n) = y \restriction \mathrm{lh}(t_n) = s_n$)

$$x <_n^* y \leftrightarrow \forall_{\mu_{s_n t_n}} \vec{\alpha}\ [\exists \beta < \kappa\ (|(s_n,\vec{\alpha})|_{T_x} \leq \beta \wedge \neg |(s_n,\vec{\alpha})|_{T_y} \leq \beta)].$$

The relation

$$\begin{aligned} R(\vec{w},x,y) \leftrightarrow (\vec{w} \in W) \wedge \exists \vec{\alpha},\beta < \kappa\ [(\varphi(\vec{w}) = \vec{\alpha}) \\ \wedge (|(s_n,\vec{\alpha})|_{T_x} \leq \beta \wedge \neg |(s_n,\vec{\alpha})|_{T_y} \leq \beta)]. \end{aligned}$$

is in $\exists^{\omega^\omega}\boldsymbol{\Gamma}$. This shows $<_n^* \in \Lambda(\boldsymbol{\Gamma},\kappa)$.

Remark

It is not clear if we can get a scale whose norms are in $\Lambda(\boldsymbol{\Gamma}, \kappa)$. The norms from the Suslin representation from the homogeneous tree construction are of the form $\psi_n(x) = [f' \restriction x \restriction \mathrm{lh}(t_n), t_n]_{\mu_{x \restriction \mathrm{lh}(t_n), t_n}}$ where f' is the minimal "almost everywhere rank function." We don't know if these norms are in $\Lambda(\boldsymbol{\Gamma}, \kappa)$.

If this is true then we can fill the remaining gap in the final analysis.

Lower bound for next scale

Lemma

Let $\boldsymbol{\Gamma}$ be nonselfdual, closed under $\forall^{\omega^\omega}$, and $\mathrm{pwo}(\boldsymbol{\Gamma})$. Let A be $\forall^{\omega^\omega}\check{\boldsymbol{\Gamma}}$-complete. Then A does not admit a scale all of whose norms are Wadge reducible to some $B \in \Lambda(\boldsymbol{\Gamma}, \kappa)$.

Proof. Fix $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$ with $A_\alpha \in \boldsymbol{\Delta}$ which is universal for $\Lambda(\boldsymbol{\Gamma}, \kappa)$.

Define $D \subseteq \omega^\omega \times \omega^\omega$ by:

$$D(x,y) \leftrightarrow \exists r \leq_T x\ \forall^*_\nu d\ \exists \alpha < \kappa\ \forall n,m \in \omega$$
$$[y(n) = m \leftrightarrow \exists z \leq_T d\ \forall w \leq_T d\ r(n,m,x,z(w),w) \in A_\alpha].$$

$D \in \forall^{\omega^\omega} \exists^{\omega^\omega} \boldsymbol{\Gamma}$, and all sections of D are countable.

Consider $D^c \in \exists^{\omega^\omega} \forall^{\omega^\omega} \check{\boldsymbol{\Gamma}}$.

Since we are assuming every $\forall^{\omega^\omega} \check{\boldsymbol{\Gamma}}$ set admits a scale with norms reducible to B, D^c has a uniformizing function f of the form

$$f(x)(n) = m \leftrightarrow \reflectbox{G} \langle z, w \rangle \; r(n, m, x, z, w) \in B$$

for some continuous $r \colon \omega^2 \times (\omega^\omega)^3 \to \omega^\omega$.

Then

$$f(x)(n) = m \leftrightarrow \forall^*_\nu d\, \exists z \leq_T d\, \forall w \leq_T d\, r(n, m, x, z(w), w) \in B.$$

Let $x \geq_T r$ and $y = f(x)$, so $\neg D(x, y)$.

For $d \in \mathcal{D}$, let $\alpha(d) < \kappa$ be such that $A_{\alpha(d)} \cap d = B \cap d$.

Then

$$y(n) = m \leftrightarrow \forall^*_\nu d\, \exists z \leq_T d\, \forall w \leq_T d\, r(n, m, x, z(w), w) \in A_{\alpha(d)}$$

which shows $D(x, y)$, a contradiction.

Remark

A semi-scale can be converted to a scale within the next projective class over a class containing the norms of a semiscale. So, if $\boldsymbol{\Gamma}$ (and hence Λ) is closed under quantifiers, then Λ is the least class containing the norms of a semiscale on a $\check{\boldsymbol{\Gamma}}$ complete set.

Question

In this case do we get a scale on $\check{\boldsymbol{\Gamma}}$ with norms in Λ?

Corollary

If $\boldsymbol{\Gamma}$ is nonselfdual, closed under quantifiers, and scale($\boldsymbol{\Gamma}$), then $\lambda = o(\Lambda)$ is the next Suslin cardinal after $\kappa = o(\boldsymbol{\Delta})$ (assuming κ is not the largest Suslin cardinal).

Lemma

Suppose $\boldsymbol{\Gamma}$ *is nonselfdual, closed under quantifiers, and* scale$(\boldsymbol{\Gamma})$ *(and* $\kappa = o(\boldsymbol{\Delta})$ *is not the largest Suslin cardinal). Then every set in* Λ *is* λ*-Suslin.*

Proof. Let $B \in \Lambda$ and $A \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$. Let $A = p[T]$ with T a tree on $\omega \times \lambda$. Let $\{\psi_n\}$ be the scale on A from T. Fix nonselfdual $\boldsymbol{\Gamma}_0 \subsetneq \boldsymbol{\Gamma}_1 \subseteq \Lambda$ with $B, \omega^\omega - B \in \boldsymbol{\Gamma}_0$ and pwo$(\boldsymbol{\Gamma}_1)$, $\exists^{\omega^\omega}\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_1$. Fix n so that $|\psi_n| >$ sup of the lengths of the $\boldsymbol{\Gamma}_1$ prewellorderings.

For $\alpha < |\psi_n|$, let $A_\alpha = \{x \in A : \psi_n(x) \leq \alpha\}$. We cannot have $A_\alpha \in \boldsymbol{\Gamma}_0$ for all α, so $B \leq_W A_\alpha$ for some α.

But A_α and hence B is λ-Suslin.

Assume $\boldsymbol{\Gamma} = S(\kappa)$ is closed under quantifiers, $\kappa = o(\boldsymbol{\Delta})$ is not the largest Suslin cardinal, and $\Lambda = \Lambda(\boldsymbol{\Gamma}, \kappa)$. Let $\lambda = o(\Lambda)$. So, $\mathrm{cof}(\lambda) = \omega$.

Let $\boldsymbol{\Sigma}_0 = \boldsymbol{\Sigma}_0^\lambda = \bigcup_\omega S(< \kappa)$, etc. Recall $\mathrm{pwo}(\boldsymbol{\Sigma}_0)$, $\mathrm{pwo}(\boldsymbol{\Pi}_1)$, etc.

Lemma

$S(\lambda) = \boldsymbol{\Sigma}_1$.

Proof. $S(\lambda)$ is closed under $\wedge^\omega$, $\vee^\omega$, and $\exists^{\omega^\omega}$ so $\boldsymbol{\Sigma}_1 \subseteq S(\lambda)$. From the coding lemma, $S(\lambda) \subseteq \boldsymbol{\Sigma}_1$.

Lemma

$\delta_1 = \delta(\Delta_1) = \lambda^+$ and λ^+ is regular.

Proof. δ_1 is the supremum of the lengths of the $\mathbf{\Sigma}_1$ wellfounded relations, and so is regular. Since $\mathbf{\Delta}_1 = S(\lambda)$ we have $\delta_1 = \lambda^+$ by Kunen-Martin and the coding lemma.

Lemma

Let $B \in \Lambda$, $\rho < \lambda$, and $\mathcal{B} = \{B_\beta\}_{\beta<\rho}$ be such that $B_\beta \leq_W B$ for each β. Then $\overline{\mathcal{B}} \subseteq \Lambda$.

Proof.
For $\rho_0, \rho_1 < \lambda$ let Λ_{ρ_0,ρ_1} be the union of all $\overline{\mathcal{B}}$ where $\mathcal{B} = \{B_\beta\}_{\beta<\rho_0}$ and $o(B_\beta) < \rho_1$.

If the lemma fails, we first show that some Λ_{ρ_0,ρ_1} contains $\boldsymbol{\Sigma}_2$.

Each Λ_{ρ_0,ρ_1} is selfdual and is closed under $\exists^\omega$. So if lemma fails then $\Lambda_{\rho_0,\rho_1} \supseteq \boldsymbol{\Pi}_1$ for some ρ_0, ρ_1.

Now we run the previous argument for the closure of Λ under quantifiers. Let $B \in \overline{\mathcal{B}}$ where $B = \{B_\beta\}_{\beta<\rho_0}$ and let $A = \exists^{\omega^\omega} B$.

View every real z as coding a countable set s_z and a function $f_z \colon s_z \to \{0,1\}$. Define

$$z \in C \leftrightarrow \forall^* d\, \exists \alpha < \rho_0\, \forall x \in s_z\, (f_z(x) = 1 \leftrightarrow \exists y \leq_T d\, (x,y) \in B_\alpha).$$

Easily $C \in \Lambda$. Let $\prec$ be the prewellordering on C as before. So, $\prec \in \Lambda$ and so $\rho_0' = |\prec| < \lambda$.

Define for $\alpha = |z|$ $(z \in C)$

$$x \in A'_\alpha \leftrightarrow \forall^* d\, \exists y \leq_T d\, (x,y) \in B_{\alpha_z(d)}.$$

where $\alpha_z(d)$ is the least $\alpha < \rho_0$ satisfying the above. Then $A \in \overline{\{A'_\alpha\}}_{\alpha<\rho_0'}$.

So, we may assume $\Lambda_{\rho_0,\rho_1} \supseteq \boldsymbol{\Sigma}_2$. Fix $A \in \overline{\mathcal{A}}$, $\mathcal{A} = \{A_\alpha\}_{\alpha<\rho_0}$ with A a complete $\boldsymbol{\Sigma}_2$ set.

Now we run the previous non-uniformization argument.

Define

$$D(x,y) \leftrightarrow \exists \tau \leq_T x \; \forall^* d \; \exists \alpha < \rho_0 \; \forall m,n \in \omega \\ [y(m) = n \leftrightarrow \exists z \leq_T d \; \tau(x,z,m,n) \in A_\alpha].$$

Easily $D \in \Lambda$ and so is λ-Suslin. So D^c admits a $\boldsymbol{\Sigma}_2$ uniformization, and thus a uniformization f reducible to A, say by τ. For $x \geq_T \tau$ we have $D(x, f(x))$, a contradiction.

Lemma

λ is closed under ultrapowers.

Proof. Let μ be a measure on $\alpha < \lambda$ and $\beta < \lambda$. Fix a prewellordering $\prec$ of length $> \max\{\alpha, \beta\}$ and let $\boldsymbol{\Gamma}_0 \subsetneq \Lambda$ be nonselfdual containing $\prec$ and closed under $\exists^{\omega^\omega}$, $\wedge$, $\vee$. Let U be universal $\boldsymbol{\Gamma}_0$.

Then $C_\mu \in \Lambda(\boldsymbol{\Gamma}_0, \alpha) \subseteq \Lambda$ as witnessed by the sequence $A_\alpha = \{x : \exists y \; |y|_\prec = \alpha \wedge U(x, y)\}$ as before,

From this and the coding lemma we easily have that $j_\mu(\beta) < \lambda$.

Lemma

$\delta_1 = \lambda^+$ is closed under ultrapowers.

Proof. Follows immediately from previous lemma and $\mathrm{cof}(\lambda) = \omega$.

Lemma

δ_1 is a Suslin cardinal, $S(\delta_1) = \boldsymbol{\Sigma}_2$, and $\mathrm{scale}(\boldsymbol{\Sigma}_2)$.

Proof. Let $A = p[T]$ be a complete $\boldsymbol{\Sigma}_1$ set, where T is a tree on $\omega \times \lambda$. T is weakly homogeneous, and the homogeneous tree construction gives a tree T' with $\omega^\omega - A = p[T']$. and T' is a tree on $\omega \times \sup_\mu j_\mu(\delta_1)$ where *mu* ranges over measures on λ. So, T' is on $\omega \times \delta_1$.

This show δ_1 is a Suslin cardinal and $\boldsymbol{\Sigma}_2 \subseteq S(\delta_1)$. Since $\mathrm{pwo}(\boldsymbol{\Pi}_1)$, the coding lemma gives $S(\delta_1) \subseteq \boldsymbol{\Sigma}_2$.

Since δ_1 is regular and $\boldsymbol{\Sigma}_2$ is closed under wellordered unions, previous arguments give scale($\boldsymbol{\Sigma}_2$).

Remark

We can show that $\boldsymbol{\Delta}_1$ (and $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Pi}_1$) is closed under measure quantification by measures on λ. Using this can show that every $\boldsymbol{\Pi}_1$ set admits a semi-scale whose norms are $\boldsymbol{\Pi}_1$.

Question

Do we have scale($\boldsymbol{\Sigma}_0$)? Do we have scale($\boldsymbol{\Pi}_1$)?

Definition

A tree T on $\omega \times \kappa$ is strongly homogeneous if there are measures μ_s on T_s such that:

- $\vec{\mu}$ witnesses the homogeneity of T.
- There are μ_s measure one sets A_s such that for all x with T_x wellfounded, The ranking function f on $T_x \restriction \vec{A}_s$ has minimal values for $[f_s]_{\mu_s}$ where f_s is the function on T_s induced by f.

Fact

If every κ-homogeneously Suslin set is κ-strongly homogeneously Suslin (where $\kappa = o(\mathbf{\Delta})$ in the type 4 case) then every set in Λ admits a scale all of whose norms are in Λ.

Question

Is every κ-homogeneously Suslin set a κ-strongly homogeneously Suslin set?

Let $\boldsymbol{\Gamma}$ be nonselfdual, closed under quantifiers, $\boldsymbol{\Gamma} = S(\kappa)$ where $\kappa = o(\boldsymbol{\Delta})$. Let $A \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$ and let $A = p[T]$ where T is a tree on $\omega \times \kappa$.

Definition (Steel)

$\mathrm{Env}(\boldsymbol{\Gamma})$ is the set of $A \subseteq \omega^\omega$ such that for some $z_0 \in \omega^\omega$, for any countable set of reals z containing z_0 we have $A \cap z \in L(T, z)$.

$\mathrm{Env}'(\boldsymbol{\Gamma})$ is the set of $A \subseteq \omega^\omega$ such that for some $z_0 \in \omega^\omega$, for any countable set of reals z containing z_0 we have $A \cap z$ is definable in $L(T, z)$ from finitely many ordinals, T, z.

Remark

We can also consider the variations $\tilde{\text{Env}}$, $\tilde{\text{Env}}'$ which are defined like Env and Env′ except we replace "for all countable z containing z_0" with "for all $d \geq_T z_0$."

Clearly $\text{Env} \subseteq \tilde{\text{Env}}$ and $\text{Env}' \subseteq \tilde{\text{Env}}'$.

Theorem

For $\boldsymbol{\Gamma}$ as above, $\Lambda(\boldsymbol{\Gamma}, \kappa) = Env(\boldsymbol{\Gamma}) = Env'(\boldsymbol{\Gamma}) = \tilde{Env}(\boldsymbol{\Gamma})$.

Proof. Clearly $\text{Env}' \subseteq \text{Env} \subseteq \tilde{\text{Env}}$ and $\tilde{\text{Env}}' \subseteq \tilde{\text{Env}}$.

We first show that $\Lambda(\boldsymbol{\Gamma}, \kappa) \subseteq \mathrm{Env}'(\boldsymbol{\Gamma})$.

Let $A \in \overline{\mathcal{A}}$ where $\mathcal{A} = \{A_\alpha\}_{\alpha<\kappa}$, each $A_\alpha \in \boldsymbol{\Gamma}$.

Let $W = p[T] \in \boldsymbol{\Gamma} - \check{\boldsymbol{\Gamma}}$ and $\varphi = \varphi_0$ a $\boldsymbol{\Gamma}$-norm on W.

From the coding lemma, the relation

$$R(x, y) \leftrightarrow (x \in W) \wedge (y \in A_{\varphi(x)})$$

is in $\boldsymbol{\Gamma}$. Let z_0 be a $\boldsymbol{\Gamma}$ code for R. Let z be a countable set containing z_0. We claim that $A \cap z$ is ordinal definable in $L(T, z)$ from T, z and z_0.

Let $\alpha < \kappa$ be such that $A \cap z = A_\alpha \cap z$. It suffices to show that $A_\alpha \cap z$ is definable in $L(T, z)$ from T, z, α, and z_0. I

This follows as in Becker-Kechris. For $y \in z$, consider the game $G_{\alpha,T,z}(y)$:

I $\quad x', \vec{\beta}$

II $\quad x, \vec{\gamma}, \vec{\delta}$

Rules: $(x' \restriction n, \vec{\beta} \restriction n) \in T$, $(x \restriction n, \vec{\gamma} \restriction n) \in T$, $\beta_0 = \gamma_0 = \alpha$, $(z_0 \restriction n, x \restriction n, y \restriction n, \vec{\delta} \restriction n) \in T$ (we assume W is universal $\boldsymbol{\Gamma}$ set).

In addition, II must make the additional moves as in Becker-Kechris to prove that $\varphi(x') \leq \varphi(x)$.

These games are (uniformly) closed for II and are in $L(T, z)$ (for z containing z_0).

Winning the game is absolute to $L(T, z)$.

$$\begin{aligned} y \in A_\alpha &\leftrightarrow \exists x\, [(x \in W \wedge \varphi(x) = \alpha) \wedge W(z_0, x, y)] \\ &\leftrightarrow \text{II has a winning strategy in } G_{\alpha,T,z}. \end{aligned}$$

So, $z \cap A = z \cap A_\alpha$ is definable from α and z_9 in $L(T, z)$.

We next show that $\tilde{\mathrm{Env}} \subseteq \tilde{\mathrm{Env}}'$. Let $A \in \tilde{\mathrm{Env}}$. So, $\forall^* d\ (A \cap d \in L(T, d))$.

Consider the following game:

I x

II y, z

II wins iff $y \geq_T x$, $z \leq_T y$ and the following holds:

Let $d = \{y' \colon y' \leq_T y\}$. Let α, φ be the least ordinal and formula such that $A \cap d$ is definable in $L(T, d)$ by the formula φ using parameters α, T, d, and some $w \in d$. Then $A \cap d$ is definable in $L(T, d)$ using φ and the parameters α, T, d, and z.

If I had a winning strategy σ, let $y \geq_T \sigma$ be such that $A \cap d \in L(T, d)$ (d as above). Let α,φ be the least ordinal and formula such that $A \cap d$ is definable in $L(T, d)$ from α, T, d, and some $w \in d$. II can then play y, w to defeat σ.

Fix a strategy τ for II. Then for any $d \geq_T \tau$, $A \cap d$ is definable in $L(T, d)$ from an ordinal, T, d, and τ.

Namely, let α, φ be least such that $A \cap d$ is definable in $L(S, d)$ from α, S, d and some $w \in d$ by φ. Then $u \in A \cap d$ iff $L(S, d) \models \exists x \in d\ \exists v\ (\varphi(\alpha, S, d, \tau(x)_1) \wedge u \in v)$.

Finally, we show that $\tilde{\mathrm{Env}}'(\boldsymbol{\Gamma}) \subseteq \Lambda(\boldsymbol{\Gamma}, \kappa)$.

As before, view every z as coding a countable set s_z and an $f_z \colon z \to \{0, 1\}$.

For degree d, let $\alpha_z(d) = \langle \alpha, \beta, \gamma, \varphi \rangle < \kappa$ be least, if it exists, such that $\beta, \gamma < \alpha$ and for all $u \in s_z$ we have

$$f_z(u) = 1 \ \leftrightarrow L_\alpha(T \cap \beta, d) \models \varphi(\gamma, T \cap \beta, d, u)$$

Let $C = \{z \colon \forall^* z\ \alpha_z(d) \text{ is defined}\}$. (we suppress the fixed real parameter). As before, $C \in \boldsymbol{\Gamma}$ and the natural norm ψ on C has length κ.

For $z \in C$ with $\psi(z) = \alpha$ define A_α by:

$$x \in A_\alpha \leftrightarrow \forall^* d\ \exists \alpha, \beta, \gamma, \varphi < \kappa\ [\alpha_z(d) = \langle \alpha, \beta, \gamma, \varphi \rangle \\ \wedge L_\alpha(T \cap \beta, d) \models \varphi(\gamma, T \cap \beta, d, x)].$$

Then $A \in \overline{\{A_\alpha\}}_{\alpha < \kappa}$.