B1

of $HOD_{\sigma_n \cup \{\sigma_n\}}$.
Then there is a filter G_A on $\mathbb{P}$ s.t.
* G_A is HOD_s - generic
* $HOD_{\{s,A\}} = HOD_s[G_A]$

Proof Let $H = HOD_s$. Know: $H = HOD_H^{HOD_{\sigma_n \cup \{\sigma_n\}}}$.
Working in $HOD_{\sigma_n \cup \{\sigma_n\}}$.
Let $\mathbb{P}$ be the Vopěnka algebra for adding a subset of κ^{σ_n} to HOD_H.
$(\mathbb{P}, \leq) \cong (\{ e \subseteq \mathcal{P}(\kappa^{\sigma_n}) \mid e \text{ is } OD_H \}, \subseteq)$ via π
$G_A = \{ c \in \mathbb{P} \mid A \in \pi(c) \}$
(1), (2) are then standard facts about Vopěnka algebra. ☒

Define
$\mathbb{P}$ = the set of all pairs (s, F) such that
- $s \in T$ and $F : T \to V$
- $F(\emptyset) = \mathcal{P}_{\omega_1}(\kappa_0^{\cdot})$
- $\forall \langle \sigma_0 \ldots \sigma_n \rangle \in T : \mu_{n+1}(F(\langle \sigma_0 \ldots \sigma_n \rangle)) = 1$

Ordering
$\langle s_0, F_0 \rangle \leq \langle s_1, F_1 \rangle$ iff
- $s_0 \supseteq s_1$
- $F_0 \subseteq F_1$
- $\forall i \in dom(s_0) - dom(s_1) \quad s_0(i) \in F_1(s_0 \upharpoonright i)$

Lemma (Prikry property) $\mathbb{P}$ has the Prikry property, i.e. if z is a countable set of terms, $(s_0, F_0) \in \mathbb{P}$ and φ is a formula then there is some H s.t. $(s_0, H) \in \mathbb{P}$ and $(s_0, H) \Vdash \varphi[\tau]$ for all $\tau \in z$.

Proof Exercise: there is a DC-free proof of the lemma.

B2

Now let G be $\mathbb{P}$-generic$/V$. Let
$S_G = \bigcup\{s \mid (\exists F)(s, F) \in G\} = \langle \sigma_i \mid i \in \omega \rangle$
We will use the Prikry property to show:

Lemma $(\forall i < \omega)$ $\mathcal{P}(\Theta_i) \cap \mathrm{HOD}^V_{S_G \restriction (i+1)} = \mathcal{P}(\Theta_i) \cap \mathrm{HOD}^{(V[G], V)}_{\{S_G\}}$

Rem If lemma holds: then
$\mathrm{HOD}^{(V[G], V)}_{\{S_G\}} \models \exists$ infinitely many Woodin cardinals
This is because $(\forall i < \omega)$: $\mathrm{HOD}^V_{S_G \restriction (i+1)} \models \Theta_i$ is Woodin.

Proof of the lemma
$\subseteq$ Holds because we use V as a predicate
$\supseteq$ If not: There are:

- Formula $\varphi(x_1, x_2, x_3)$,
- $\xi \in \mathrm{On}$
- $m > i$
- $(S_G \restriction m, F) \in G$

s.t.
$(S_G^m, F) \Vdash \{\beta < \Theta \mid (V[\dot{G}], V) \models \varphi[\beta, \xi, \dot{S}_G] \notin \mathrm{HOD}^V_{\dot{S}_G \restriction (i+1)}$

By Prikry property there are densely many condition of the form $(S_G \restriction m, H)$ that decide the statement
"$(V[G], V) \models \varphi[\beta, \xi, \dot{S}_G]$"
so wma $(S_G \restriction m, H) \in G$. This means:
$\{\beta < \Theta_i \mid (S_G \restriction m, H) \Vdash (V[G], V) \models \varphi[\beta, \xi, \dot{S}_G]\} \in \mathrm{HOD}^V_{S_G \restriction m}$.
But then this set is in $\mathrm{HOD}^V_{\dot{S}_G \restriction (i+1)}$ by our above arrangements. Contradiction ☒.

Let $N = \mathrm{HOD}^{(V[G], V)}_{\{s_G\}}$ where G is $\mathbb{P}$-generic $/V$ and $s_G = \langle \sigma_i \mid i \in \omega \rangle$. So $N \models ZFC$ and $\omega_1^V = \sup \theta_i$.

<u>Lemma</u> V is a derived model of N. More precisely: There is a $Col(\omega, <\omega_1^V)$-generic $/N$ filter K s.t. $V = L(\mathrm{Hom}^*_K, \mathbb{R}^*_K)$.

<u>Proof</u> Let $N_i = \mathrm{HOD}^V_{s_G \restriction (i+1)}$, $\theta_i = \theta^{\mathrm{HOD}^V_{s \restriction (i+1)}}$.

We know:

$\mathcal{P}(\theta_i) \cap N_j = \mathcal{P}(\theta_i) \cap N_j = \mathcal{P}(\theta_i) \cap N$ for $j \geq i$.

(In V^2) There is a filter K that is $Col(\omega, <\omega_1^V)$-generic$/N$ s.t. $\mathbb{R}^V = \mathbb{R}^*_K$. This is because each $x \in \mathbb{R}^V \cap s_G(i)$ then x can be absorbed by a Vopěnka algebra of size $<\omega_1^V$, namely a Vopěnka algebra for N.

Now to see that $\mathcal{P}(\mathbb{R})^V = \mathrm{Hom}^*_K$. Enough to see $\mathcal{P}(\mathbb{R})^V \subseteq \mathrm{Hom}^*_K$, otherwise we get a sharp for V in the generic extension of V.

Let $B \in \mathcal{P}(\mathbb{R})^V$. B is Suslin co-Suslin

Martin's theorem ($AD + DC_{\mathbb{R}}$) B is homogeneously Suslin.

Then we can code the homogeneity system by / by the next Suslin cardinal a countable sequence of ordinals that is bounded below θ. (Measures are OD by Kunen.) So we can get trees T, U s.t. $p[T] = B = \neg p[U]$ and T, U are OD from that sequence f. Now since f is bounded below θ: there is i s.t. $s_G(i) \geq f$ and $s_G(i) \cap On \in N$ & there is g generic over N_i s.t. the collapse of f is in $N_i[g]$. Since the corresponding collapse $\pi \in N_i[g]$ We have $f \in N_i[g]$. So for all $j \geq i$: $N_j[g]$ can decode f to recover the trees T and U, so

$p[T] \cap \mathbb{R}^{N_j[g]} = B \cap \mathbb{R}^{N_j[g]} \cap \neg p[U] \cap \mathbb{R}^{N_j[g]}$. This shows $B \in \mathrm{Hom}^*_K$ $\boxtimes$.

~~[illegible]~~ Next goal: let φ be a Σ_1-formula
Let φ be a Σ_1-fmla and $V \models \varphi[\mathbb{R}]$. WTS: $M_{\mathcal{P}(\mathbb{R})} \models \varphi(\mathbb{R})$.

<u>Lemma</u> There is $A \in \mathrm{Hom}^N_{<\omega_1^V}$ s.t. $L(A, \mathbb{R}^N) \models \varphi[\mathbb{R}^N]$.
<u>Proof</u> Let γ be least s.t. $L_\gamma(\mathcal{P}(\mathbb{R})) \models \varphi[\mathbb{R}]$ and there is a sequence $\langle \alpha_i \mid i<\omega \rangle$ s.t. $\Theta = \sup_{i \in \omega} \Theta_{\alpha_i}$ and $\langle \alpha_i \mid i \in \omega \rangle$ is definable in $L_\gamma(\mathcal{P}(\mathbb{R}))$ from a set of reals and no ordinal parameters. Let $j : (N, \in) \to (M, E)$ be a stationary tower map induced by a $\mathbb{Q}^N_{<\omega_1^V}$-generic $/N$. We have:
(1) $\mathrm{crit}(j) = \omega_1^N$ and $j(\omega_1^N) = \omega_1^V$ (2) $\mathbb{R}^{(M,E)} = \mathbb{R}^V$
(3) $j(\mathrm{Hom}^N_{<\omega_1^V}) \supseteq \mathcal{P}(\mathbb{R})^V = \mathrm{Hom}^*_\kappa$
(4) $j(A) = A^*$ all $A \in \mathrm{Hom}^N_{<\omega_1^V}$
(5) $\gamma \in \mathrm{wfp}((M, E))$

<u>Case 1</u> Suppose $j(\mathrm{Hom}^N_{<\omega_1^V}) \supsetneq \mathcal{P}(\mathbb{R})^V$, so there is some $A \in j(\mathrm{Hom}^N_{<\omega_1^V}) - \mathcal{P}(\mathbb{R})$. Since $(M,E) \models (L_\gamma(A, \mathbb{R}^{(M,E)}) \models \varphi(\mathbb{R}^{(M,E)}))$ (because φ is Σ_1) by elementarity of j we have $A \in \mathrm{Hom}^N_{<\omega_1^V}$ s.t. $L(A, \mathbb{R}^N) \models \varphi[\mathbb{R}^N]$
<u>Case 2</u> $j(\mathrm{Hom}^N_{<\omega_1^V}) = \mathrm{Hom}^*_\kappa = \mathcal{P}(\mathbb{R}^V)$
We can pick γ s.t. $\gamma \in \mathrm{rng}(j)$. Then $L_\gamma(\mathcal{P}(\mathbb{R})) \in \mathrm{rng}(j)$. Hence there is some sequence $\langle \alpha_i \mid i \in \omega \rangle \in \mathrm{rng}(j)$ s.t. $\Theta = \sup_{i \in \omega} \Theta_{\alpha_i}$. Why: We know such a sequence is definable in some $B \in \mathcal{P}(\mathbb{R})$ without ordinals. Now $B = C^*$ for some $C \in N[g]$ where g is $<\omega_1^V$ generic over N. By replacing N by $N[g]$ if necessary we can assume $C \in N$ and $C^* = B$. So $B = C^* = j(C) \in \mathrm{rng}(j)$, hence $\langle \alpha_i \mid i \in \omega \rangle \in \mathrm{rng}(j)$. Say $j(\langle \beta_i \mid i \in \omega \rangle) = \langle \alpha_i \mid i \in \omega \rangle$.
From $\langle \beta_i \mid i \in \omega \rangle$ we choose a sequence $\langle B^*_i \mid i \in \omega \rangle$ cofinal in in $\mathrm{Hom}^N_{<\omega_1^V}$. This is a contradiction,

as we can code $\langle B_i \mid i \in \omega \rangle$ by a $D \in \mathrm{Hom}^V_{<\omega_1}$, $B_i \leq_W D$ all i. But $\langle B_i \mid i \in \omega \rangle$ is cofinal in $\mathrm{Hom}^V_{<\omega_1^V}$. ↯ ⊠

Now since $L(A, \mathbb{R}^N) \models \varphi[\mathbb{R}^N]$ we $j \to L(A^*, \mathbb{R}^V) \models \varphi[\mathbb{R}^V]$; here $A^* \in \mathcal{P}(\mathbb{R})^V$.
So $M_{P(\mathbb{R})} \models \varphi[\mathbb{R}^V]$.

<u>CASE 3</u> No largest Suslin cardinal + $cf(\Theta) > \omega$ + + Θ singular.

Since $cf(\Theta) > \omega$ we have DC by Solovay.
Since every regular $< \Theta$ is measurable: Let μ be a measure on $rng(g) \cap cof(\omega)$ where $g : cf(\Theta) \to \Theta$ cofinal increasing. For each α s.t. $\Theta_\alpha < \Theta$, $cf(\alpha) = \omega$ let $I_\alpha = \{A \subseteq \Theta_\alpha \mid \sup(A) < \Theta_\alpha\}$

$$\Rightarrow \begin{cases} \mathrm{HOD}_{I_\alpha} \models AD^+ + AD_{\mathbb{R}} \\ \Theta_\alpha = \Theta^{\mathrm{HOD}_{I_\alpha}} \\ \forall X \in \mathrm{HOD}_{I_\alpha} : \Theta^{\mathrm{HOD}_{I_\alpha}} = \text{limit of Woodins in } \mathrm{HOD}_X \end{cases}$$

Our N will be a ZFC model s.t.
ω_1^V = limit of limits of Woodins in N

Let μ_α be a supercompact measure on $\mathcal{P}_{\omega_1}(I_\alpha)$.

<u>Lemma</u> For each α s.t. $cf(\alpha) = \omega$, $\Theta_\alpha < \Theta$ there is a measure 1 many $\sigma \in \mathbb{P}_{\omega_1}(I_\alpha)$ s.t.

- $\mathrm{HOD}_{\sigma \cup \{\sigma\}} \models AD^+ + AD_{\mathbb{R}}$
- σ has transitive collapse $= \{A \subseteq \Theta \mid \sup A < \Theta\}$ as computed in $\mathrm{HOD}_{\sigma \cup \{\sigma\}}$

Define

T_0 = the set of all $\langle \sigma_0, \dots \sigma_n \rangle$ s.t. $\forall i \leq n$:

- $(\exists \alpha_i)(\,cf(\alpha_i) = \omega \;\&\; \theta_{\alpha_i} < \theta)$
- $\theta_{\alpha_i} = \sup\{\gamma \mid \gamma \in \sigma_i\}$
- $\sigma_i \in \mathcal{P}_{\omega_1}(I_{\alpha_i})$
- $HOD_{\sigma_i \cup \{\sigma_i\}} \models AD^+ + AD_{\mathbb{R}}$
- σ_i collapses to $\{A \subseteq \theta \mid \sup(A) < \theta\}$ in $HOD_{\sigma_i \cup \{\sigma_i\}}$

T = the set of all $s = \langle \sigma_0 \dots \sigma_n \rangle$ s.t.

- $s \in T_0$
- $\mathcal{P}(\mathbb{R})^{HOD_s} = \mathcal{P}(\mathbb{R})^{HOD}$
- $(\forall i \leq n)$
  - $\alpha_i < \alpha_{i+1}$
  - $\sigma_k \subseteq \sigma_i$ and $\sigma_k \in HOD_{\sigma_i \cup \{\sigma_i\}}$ all $k \leq i$
  - σ_k countable in $HOD_{\sigma_i \cup \{\sigma_i\}}$ all $k < i$
  - $\mathcal{P}(\theta_i) \cap HOD_{s \restriction (i+1)} = \mathcal{P}(\theta_i) \cap HOD_s$

    where $\theta_i = \Theta^{HOD_{\sigma_i \cup \{\sigma_i\}}}$

Now define Prikry forcing:

$\mathbb{P}$ = the set of all pairs (s, F) such that $s \in T$,

$F : T \longrightarrow V$ and

$(\forall t \in T)$ $t^\frown\langle \sigma \rangle \in T$ for all $\sigma \in F(t)$ and

$\forall^*_\mu \alpha \; \forall^*_{\mu_\alpha} \sigma \in \mathcal{P}_{\omega_1}(I_\alpha)$ $\sigma \in F(t)$

<u>Ordering</u>:

$\langle s_0, F_0 \rangle \leq \langle s_1, F_1 \rangle$ iff $s_0 \supseteq s_1$ and

- $(\forall i \in dom(s_0) - dom(s_1))$ $s_0(i) \in F_1(s_1 \restriction i)$
- $F_0 \subseteq F_1$

22.7.2010 14:00 Steve Jackson

The largest Suslin cardinal.

Assume there is a largest Suslin cardinal κ.

Claim. · κ is a ~~regular~~ limit Suslin cardinal.
· $\Gamma = S(\kappa)$ and scale(Γ)
· $S(\kappa)$ is closed under quantifiers.

The Envelope

Let Γ be a pointclass and $\kappa \in On$. We define $\underline{\Gamma, \kappa}$-envelope as follows

Definition (Martin) Let $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ each $A_\alpha \subseteq \mathbb{R}$. Then $\bar{\mathcal{A}}$ = the set of all $A \in \mathcal{P}(\mathbb{R})$ such that for all countable $S \subseteq \mathbb{R}$ there is an $\alpha < \kappa$ s.t. $S \cap A = S \cap A_\alpha$.
We let

$$\Lambda(\Gamma, \kappa) = \{\bar{\mathcal{A}} \mid \mathcal{A} \subseteq \Gamma \ \& \ \text{card}(\mathcal{A}) \leq \kappa\}$$

Lemma Let Γ be nonselfdual, closed under $\forall^{\mathbb{R}}$ and pwo(Γ) (if Δ not closed under $\exists$ & assume scale($\exists^{\mathbb{R}}\Gamma$ with norms into κ)). Then

$$\Lambda(\Delta, \kappa) = \Lambda(\Gamma, \kappa) = \Lambda(\exists^{\mathbb{R}}\Gamma, \kappa) \text{ where } \kappa = \delta(\Delta).$$

Lemma Assumptions of the previous lemma. Then there is a single $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa \rangle$ with each $A_\alpha \in \Delta$ s.t. every set in $\Lambda(\Gamma, \kappa)$ is Wadge reducible to a set in $\bar{\mathcal{A}}$.

<u>Corollary</u> Under same hypotheses: $\Lambda(\Gamma,\kappa)$ is closed under $\wedge, \vee, \neg$.
Why: Because $\Lambda(\Gamma,\kappa) = \Lambda(\Delta,\kappa)$

<u>Lemma</u> Suppose Γ is nonselfdual closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$ and pwo(Γ). Let $\kappa = o(\Delta)$. Then $\Lambda(\Gamma,\kappa)$ is closed under $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$.

Coding measures

Let Γ and $\kappa = o(\Delta)$ be as above. Fix an $\exists^{\mathbb{R}}\Gamma$ norm (W,φ) of length κ (with each $W_\alpha \in \Delta$.)

Let U be a universal $\exists^{\mathbb{R}}\Gamma$-set. For $z \in {}^\omega\omega$ let $B_z = \{\alpha < \kappa \mid (\exists x \in W)(\varphi(x) = \alpha)\}$. By the Coding lemma every subset of κ is of the form B_z. For a measure μ on κ:

$$C_\mu = \{z \mid \mu(B_z) = 1\}$$

<u>Lemma</u> Γ, κ as above. Then $A \in \Lambda(\Gamma,\kappa)$ iff there is a measure on κ s.t. $A \leq_w C_\mu$.

Upper bound for the next semiscale

<u>Theorem</u> Γ nonselfdual, closed under $\forall^{\mathbb{R}}$ and pwo(Γ). Assume every $\exists^{\mathbb{R}}\Gamma$ set admits a $\exists^{\mathbb{R}}\Gamma$ scale with norms into $\kappa = \delta(\Delta)$. Assume also that there is a Suslin cardinal greater than κ. Then every set in $\forall^{\mathbb{R}}\check{\Gamma}$ admits a semiscale into κ.

Rem It is not clear if we can get a scale whose norms are in $\Lambda(\Gamma,\kappa)$:

Question Can we find a homogeneous tree T on $\omega\times\kappa$ a countable family A_s of μ_s of measure 1 sets s.t.

$$\forall x\left[T_x \text{ is wf} \Rightarrow [f^{\text{rank}}_{x\restriction s}]_{\mu_s} = \text{leftmost branch of scale} = [f^i_x \restriction B_s]_{\mu_s}\right]$$

Lower bound for the next scale

Lemma Γ nonselfdual, closed under $\forall^{\mathbb{R}}$ and pwo(Γ). Let A be $\forall^{\mathbb{R}}\check{\Gamma}$-complete. Then A does not admit a scale all of whose norms are Wadge reducible to some $B\in\Lambda(\Gamma,\kappa)$.

Proof Idea: this is the "largest countable Γ" argument.

Remark A semiscale can be converted to a scale within the next projective class

Lemma Suppose Γ is nonselfdual, closed under quantifiers and scale(Γ) (and $\kappa=o(\Delta)$ is not the largest Suslin cardinal). Then every set in Γ is λ-Suslin.

Assume $\Gamma=S(\kappa)$ is closed under quantifiers, $\kappa=o(\Delta)$ is not the largest Suslin cardinal and $\Lambda=\Lambda(\Gamma,\kappa)$. Let $\lambda=o(\Lambda)$. So $\mathrm{cf}(\lambda)=\omega$.

Let $\Sigma_0=\Sigma_0^\lambda=\bigcup_\omega S(<\kappa)$ etc. Recall pwo(Σ_0), pwo(Π_1) etc.

<u>Lemma</u> $S(\lambda) = \Sigma_1$.

<u>Lemma</u> $\delta_1 = \delta_1(\Delta_1) = \lambda^+$ and λ^+ is regular.

<u>Lemma</u> Let $B \in \Lambda$, $\rho < \lambda$ and $\mathbb{B} = \{B_\beta \mid \beta < \rho\}$ be s.t. $B_\beta \subseteq_W B$ for each β. Then $\overline{\mathbb{B}} \leq \Lambda$.

22.7.2010 16:45 DISCUSSION: Nam Trang

Continuation of the lecture in the morning.

$\mathbb{P}$ has the Prikry property:
Let G be V-generic for $\mathbb{P}$, $s_G = \bigcup\{s \mid (\exists F)((s,F)\in G\}$,
$N = \mathrm{HOD}^{(V[G],V)}_{\{s_G\}} \models ZFC + \omega_1^V =$ limit of limit of Woodins.
Let $s_G = \langle \sigma_i \mid i\in\omega\rangle$.

<u>Lemma</u> (a) $(\forall i\in\omega)\ \mathcal{P}(\theta_i)\cap \mathrm{HOD}^V_{s_G\restriction(i+1)} = \mathcal{P}(\theta_i)\cap \mathrm{HOD}^{(V[G],V)}_{\{s_G\}}$
where $\theta_i := \theta^{\mathrm{HOD}_{\sigma_i\cup\{\sigma_i\}}}$ (in $\mathrm{HOD}_{\sigma_i\cup\{\sigma_i\}} \models AD_{\mathbb{R}}$)

(b) Vopěnka holds. $\forall i\in\omega\ \forall A\subseteq\theta_i$ bounded $\exists \mathbb{P}$, $\|\mathbb{P}\|<\theta_i$ in $\mathrm{HOD}^V_{s_G\restriction(i+1)}$ and
$\mathrm{HOD}_{\{s_G\restriction(i+1), A\}} = \mathrm{HOD}_{s_G\restriction(i+1)}[G_A]$.
HOD

Induced filter from A.

(c) θ_i is a limit of Woodins in $\mathrm{HOD}^{(V[G],V)}_{\{s_G\}}$

To show (c): Note that $\mathrm{HOD}^V_{\sigma_i\cup\{\sigma_i\}} \models AD_{\mathbb{R}}$ so $\mathrm{HOD}^V \models \theta_i$ is a limit of Woodins
and $\mathcal{P}(\theta_i)\cap \mathrm{HOD}^{(V[G],V)}_{\{s_G\}} = \mathcal{P}(\theta_i)\cap \mathrm{HOD}^V_{s_G\restriction(i+1)}$

Fix G $\mathbb{P}$-generic$/V$ and $s_G = \langle\sigma_i \mid i\in\omega\rangle$.
$\forall x\in\mathbb{R}$ $N[x] \models ZFC + \omega_1 =$ limit of limit of Woodins
and $V = D(N[x]\$, \omega_1)$

<u>Def</u> (Woodin) Assume δ is a limit of Woodins.
$\mathrm{Hom}_{<\delta}$ is <u>weakly sealed</u> if the following holds.
1) If $\kappa<\delta$ is Woodin and $G\subseteq \mathbb{Q}_{<\kappa}$ is generic $/V$
let $j: V \to \mathrm{Ult}(V,G)$ be the generic map. Then $j(\mathrm{Hom}_{<\delta}) = \mathrm{Hom}^{V[G]}_{<\delta}$

2) 1) holds in $V[H]$ for any H that is $<\delta$ generic.

Main
Lemma Exactly one of the following holds:

(1) $\exists x \in \mathbb{R}$ s.t. $A \in \mathrm{Hom}^{N[x]}_{<\omega_1^V}$ s.t. $L(A, \mathbb{R}^{N[x]}) \models \varphi[\mathbb{R}^{N[x]}]$
(We are assuming $V \models \varphi[\mathbb{R}^V]$ when φ is Σ_1)

(2) $\mathrm{Hom}^N_{<\omega_1^V}$ is weakly sealed.

Proof Assume $V \models \varphi[\mathbb{R}^V]$ where φ is Σ_1. Let γ be large enough s.t. $L_\gamma(\mathcal{P}(\mathbb{R})) \models \varphi[\mathbb{R}^V]$. For $x \in \mathbb{R}^V$ let $j_x : (N[x], \in) \to (M_x, E_x)$ induced by a $\mathbb{Q}^{N[x]}_{<\omega_1^V}$-generic s.t.

① $\mathrm{cr}(j_x) = \omega_1^{N[x]}$ and $j(\omega_1^{N[x]}) = \omega_1^V$

② $\mathbb{R}^{(M_x, E_x)} = \mathbb{R}^V$

③ $\mathrm{Hom}^* = \mathcal{P}(\mathbb{R})^V \subseteq j_x(\mathrm{Hom}^{N[x]}_{<\omega_1^V})$

④ $\forall A \in \mathrm{Hom}^{N[x]}_{<\omega_1^V}$: $j_x(A) = A^*$

⑤ For every successor Woodin cardinal $\kappa < \omega_1^V$ in $N[x]$ there is an $N[x]$-generic $H \subseteq \mathbb{Q}^{N[x]}_{<\kappa}$ inducing
$j_H : N[x] \to \mathrm{Ult}(N[x], H)$ and
$k_H : \mathrm{Ult}(N[x], H) \to (M_x, E_x)$ so that
$j_x = k_H \circ j_H$

Case 1 $\mathcal{P}(\mathbb{R})^V \subsetneq j_x(\mathrm{Hom}^{N[x]}_{<\omega_1^V})$ for some $x \in \mathbb{R}^V$.
Already done

Case 2 $\mathcal{P}(\mathbb{R})^V = j_x(\mathrm{Hom}^{N[x]}_{<\omega_1^V})$ all $x \in \mathbb{R}^V$.
We have:
$j_H(\mathrm{Hom}^{N[x]}_{<\omega_1^V}) = \mathrm{Hom}^{N[x][H]}_{<\omega_1^V}$ (Takes a little argument)
(Note: We don't get weakly sealed this way as $\mathbb{Q}_{<\kappa}$ are not weakly homogeneous)
(2) Holds by varying the embedding j_x to include any given condition.

This ~~proves~~ gives (1) in the statement of the Main Lemma. ☒ (ML)

Now: if (2) holds then B13

<u>Lemma</u> $\mathrm{Hom}^N_{<\omega_1^V} = L(\mathrm{Hom}^N_{<\omega_1^V}) \cap \mathcal{P}(\mathbb{R})$

Assuming this lemma: Then $L(\mathrm{Hom}^N_{<\omega_1^V})$ is a counterexample to the theorem in the sense that $L(\mathrm{Hom}^N_{<\omega_1^V}) \models AD^+ + \varphi[\mathbb{R}^N]$ but for no $A \in \mathrm{Hom}^N_{<\omega_1^V}$ $L(A, \mathbb{R}^N) \models \varphi[\mathbb{R}^N]$. & $\Theta^{L(\mathrm{Hom}^N_{<\omega_1^V})} < \Theta^V$.
By repeating this we get an infinite descending sequence of ordinals.

<u>Proof of the lemma</u>

<u>Sublemma</u> If $\mathbb{P} \in V^N_{\omega_1^V}$, $G \subseteq \mathbb{P}$ generic/N then in $N[G]$ there is an elementary embedding
$$j_G : L(\mathrm{Hom}^N_{<\omega_1^V}) \to L(\mathrm{Hom}^{N[G]}_{<\omega_1^V})$$
s.t.
$$j_G(\mathrm{Hom}^N_{<\omega_1^V}) = \mathrm{Hom}^{N[G]}_{<\omega_1^V}$$

Assuming the sublemma, we ~~get~~ prove now the lemma:
If the lemma fails, let α be least s.t.
$\mathrm{Hom}^N_{<\omega_1^V} \neq L_\alpha(\mathrm{Hom}^N_{<\omega_1^V}) \cap \mathcal{P}(\mathbb{R})$
Take A is definable without ordinal parameters s.t. ~~A∉~~ $A \in L(\mathrm{Hom}^N_{<\omega_1^V}) \cap \mathcal{P}(\mathbb{R}) - \mathrm{Hom}^N_{<\omega_1^V}$.
(from a pair of trees (T, S) representing a $\mathrm{Hom}^N_{<\omega_1^V}$-set by a fml φ.)
Then use the tree production lemma.
The hypo of the TPL holds for φ. We get $A \in \mathrm{Hom}^N_{<\omega_1^V}$.
☒

<u>Proof of Sublemma</u> Let $\kappa < \omega_1^V$ is a limit of Woodins in N δ_i, $i \in \omega$ s.t. ~~|P|~~ $|\mathbb{P}| < \kappa$. Let $\delta_\omega > \kappa$ be a Woodin.

Find $G_\omega \subseteq \mathbb{P}^N_{<\delta_\omega}$ that is generic over N s.t. $G_i = G_\omega \cap \mathbb{P}^N_{<\delta_i}$ is N-generic for $\mathbb{P}^N_{<\delta_i}$. Let $\sigma = \bigcup_{i\in\omega} \mathbb{R}^{N[G_i]}$. There are

$j_i : N \to_{G_i} M_i$ be the generic embeddings.

Let M^* be the direct limit. This is embeddable into M_ω, hence well-founded. We have

$$j_i(\mathrm{Hom}^N_{<\omega_1^V}) = \mathrm{Hom}^{N[G_i]}_{<\omega_1^V}.$$

Let $j : N \to M^*$ be the direct limit map.
We get $j^*(\mathrm{Hom}^N_{<\omega_1^V}) = \mathrm{Hom}^{N(\sigma)}_{<\omega_1}$.

Let $N[G](\tau)$ be the symmetric extension of $N[G]$ for $Col(\omega, <\kappa)$ s.t. $N(\sigma) = N[G](\tau)$. We have
$j^* : L(\mathrm{Hom}^N_{<\omega_1^V}) \to L(\mathrm{Hom}^{N(\sigma)}_{<\omega_1^V})$ and

$$j^*(\mathrm{Hom}^N_{<\omega_1^V}) = \mathrm{Hom}^{N(\sigma)}_{<\omega_1^V}$$

Also: $j^* : L(\mathrm{Hom}^{N[G]}_{<\omega_1^V}) \to L(\mathrm{Hom}^{N(\sigma)}_{<\omega_1^V})$
and
$j^*(\mathrm{Hom}^{N[G]}_{<\omega_1^V}) = \mathrm{Hom}^{N(\sigma)}_{<\omega_1^V}$. (for some different j^*)

Now use fixpoints + Use trees to show that the two maps move sets of reals correctly. Then this can be used to embed $L(\mathrm{Hom}^N_{<\omega_1^V}) \to L(\mathrm{Hom}^{N[G]}_{<\omega_1^V})$

We assumed $M_1^\# \leq_T x$,
G generic over $L[x]$ for $Col(\omega, <\kappa)$, $\kappa = 1^{st}$ inacc of $L[x]$
In $L[x,G]$ defined a DLS $\mathcal{F}$:
Indices: (N,s) where N is M_1-like, $\delta^N < \omega_1$, $s \in On^{<\omega}$
N is strongly s-iterable:
Given a good stack $(\mathcal{T}_0 \dots \mathcal{T}_n)$ on N
(Each $\mathcal{T}_i$ maximal or else has a last model without dropping on the main branch.) $\mathcal{T}_{i+1}$ on the last model of $\mathcal{T}_i$ or on $L(M(\mathcal{T}_i))$ (if $\mathcal{T}_i$ maximal).
Let $\mathcal{T}_i$ be on N_i: We demand that there are $b_0 \dots b_n$ s.t.
$$i_{b_k}(\mathrm{type}(s^- \cup \delta^{N_{k-1}})^{N_{k-1}|\max(s)}) = $$
$$= \mathrm{type}(s^- \cup \delta^{\mathcal{T}_k}(\mathcal{T}_k))^{M_{b_k}^{\mathcal{T}_k}|\max(s)}$$
We then define strongly s-iterable as before.
→ Need this revision in order to get absoluteness.

(N,s) indexes $H_s^N = \mathrm{Hull}^{N|\max(s)}(\gamma_s^N \cup s^-)$
$(N,s) \leq^* (P,t)$ iff there is a good stack on N with last model P and $s \subseteq t$.
$\pi_{(N,s)(P,t)} = i_{b_k} \circ \dots \circ i_{b_0} \restriction H_s^N$ for any such good stack.

$M_\infty = $ dir lim
$M_\infty^+ = $ the dir lim of all iterates of M_1 by its canonical strategy Σ_{M_1} in $HC^{L[x,G]}$
We have $\pi: M_\infty \longrightarrow M_\infty^+$ and $\pi \restriction (\delta_\infty + 1) = id$.
For any $s \in On$ we let $s^* = \pi_{(N,s),\infty}(s)$. Then the map $s \mapsto s^*$ is OD in $L[x,G]$.

<u>Claim</u> $\delta_\infty = \kappa^{+L[x,G]}$ $\left(= \omega_2^{L[x,G]} = \Theta^{L(\mathbb{R})^{L[x,G]}}\right)$

Proof $\delta_\infty \le \kappa^{+L[x,G]}$: Take $\xi < \delta_\infty$. Say $\pi_{(N,s),\infty}(\bar\xi) = \xi$ for $\bar\xi < \gamma^N_s$. The DLS of all (P,s) s.t. $(N,s) \le^* (P,s)$ gives us a map from $HC^{L[x,G]}$ onto $\sup \pi_{(N,s),\infty}[\gamma^N_s]$ in $L[x,G]$.

To see $\kappa^{+L[x,G]} \le \delta_\infty$. Pick $\alpha < \kappa^{+L[x,G]} = \kappa^{+L[x]}$. Let $s \in (C_\kappa^{V})^{<\omega}$ and τ a term s.t. $(\forall \xi < \alpha)(\exists \bar\beta < \kappa)\ \xi = \tau^{L[x]}[\bar\beta, s]$ $\tau^{L_{\max(s)}}$

Let $\eta < \alpha$. Have $\eta = \tau^{L_{\max(s)}[x]}[\bar\beta, s^-]$ some $\bar\beta \in \kappa^{<\omega}$.

Let N be $s \cup \{\eta\}$- strongly iterable s.t. $\bar\beta < 1^{st}$ measurable in N, and x being $\mathbb{B}^N_{\delta^N}$-generic $/N$ (Extender algebra)

$(\omega_1^{L[x, N_1|\delta^N]} < \omega_1^{L[x,G]})$

Let $\rho^{(N,s)}_{\tau,\bar\beta} = \{\xi \mid (\exists p \in \mathbb{B}^N_\delta)\ p \Vdash \tau^{L_{\max(s)}[\dot{x}_g]}[\bar\beta, s^-] = \check\xi\}$

Note $\mathrm{otp}(\rho^N_{\tau,\bar\beta}) < \delta^N$ (δ-c.c.).

$$\rho^{(N,s)} = \bigcup \{\rho_{\sigma,\bar\alpha} \mid \sigma, \bar\alpha < 1^{st} \text{ inaccessible of } N\}$$

$\mathrm{otp}(\rho^{(N,s)}) < \delta^N$. Let

η = the ν-th element of $\rho^{(N,s)}$

$\nu < \gamma^N_s$.

Let $\nu^\infty_\eta = \pi_{(N,s),\infty}(\nu)$

Show (a) ν^∞_s does not depend on (N,s)

(b) $\eta < \xi < \alpha \Rightarrow \nu^\infty_\eta < \nu^\infty_\xi$

Proof: Exercise.

Let $\Lambda_\infty = \Sigma_{M_1} \restriction$ trees on M_∞ in $M_\infty | \kappa_\infty$ (finite stacks)

$\kappa_\infty = \kappa^* =$ the least inacc $> \delta_\infty$ of M_∞.

Claim $\Lambda_\infty \in \mathrm{HOD}^{L[x,G]}$

Proof Given $\mathcal{T}$ normal on M_∞, every $\mathcal{T} \restriction \lambda$ short:

If $\mathcal{T}$ short : $\Lambda_\infty(\mathcal{T})$ = the unique b s.t. $Q(\mathcal{T}) \trianglelefteq M_b^{\mathcal{T}}$.
If $\mathcal{T}$ maximal: Note for $s \in On^{<\omega}$
$M_\infty \models$ I am s^*-iterable for good stacks in $M_\infty | \kappa_\infty$
Why: Pick $(N, s) \in \mathcal{F}$ [s.t. x is $\mathbb{B}^N_{\delta N}$-generic/N.] $s' - = s$
Then $N | \kappa \subseteq HC^{L[x,G]}$. So
$N | \max(s') \models$ I am s-iterable
$H^N_{s'} \models$ same, $\pi_{(N,s'),\infty} : H^N_{s'} \xrightarrow{\Sigma_0} M_\infty$. So
$M_\infty | s'^* \models$ I am s^*-iterable.

More precisely:
$M_\infty \models$ I am s^*-iterable in $L[y, H]$ where (y, H) is
$Col(\omega, \delta_\infty) \times Col(\omega, <\kappa_\infty)$ for ....

For each s^* pick a branch (in V) b_{s^*} which witnesses s^*-iterability for $\mathcal{T}$. (Is cofinal and $i_b(\text{type}^{M_\infty|\max(s^*)}(s^* - \cup \delta_\infty)) = \text{type}^{L[M(\mathcal{T})]|\max(s^*)}(s^* - \cup \delta(\mathcal{T}))$)
Let $b = \Sigma_{M_1}(\mathcal{T})$. Then $L(M(\mathcal{T})) = M_b$ and $\delta(\mathcal{T}) = i_b(\delta_\infty)$
Then

$$b = \lim_{s^*} b_{s^*}$$

Because $\gamma^{M_\infty}_{s^*}$ are cofinal in δ_∞.
b is independent of how b_{s^*} were generically chosen in $M_\infty^{Col(\omega, |\mathcal{T}|)}$. Hence $b \in HOD^{L[x,G]}$. So $\Lambda_\infty \in HOD^{L[x,G]}$.

<u>Claim</u> $HOD^{L[x,G]} \subseteq L[M_\infty, \Lambda_\infty]$ (Hence $=$)
<u>Proof</u> We can find an $A \subseteq \delta_\infty = \kappa^{+L[x,G]}$ s.t.
(1) $HOD^{L[x,G]} = L[A]$
(2) A is definable without parameters over $L[x,G]$
(Use Vopěnka.)

Let $\xi \in A$ iff $L[x, G] \models \varphi[\xi]$. Let B18

M^*_∞ = dir lim of $\mathcal{F}^{L[y,H]}$ where

y, H is M_∞-generic ($Col(\omega, \delta_\infty) \times Col(\omega, <\kappa_\infty)$)

Claim $M^*_\infty = \lim$ "$\mathcal{F}^{M_\infty}$" $= \mathcal{F}^{L[y,H]} \cap M_\infty$

Proof Given $(N, s) \in \mathcal{F}^{L[y,H]}$: In M_∞ we have $(P, s) \in \mathcal{F}^{M_\infty}$. Let $\tau^{y \times H \restriction \xi}$ ① $= N | \delta^N$. Let $g = y \times H$. Let

$p \Vdash \tau$ is M_1-like and "$(L[\tau], s) \in \mathcal{F}^{L[y,H]}$"

For $q \leq p$ let $g_q = (g \restriction Col(-) - dom(q)) \cup q$

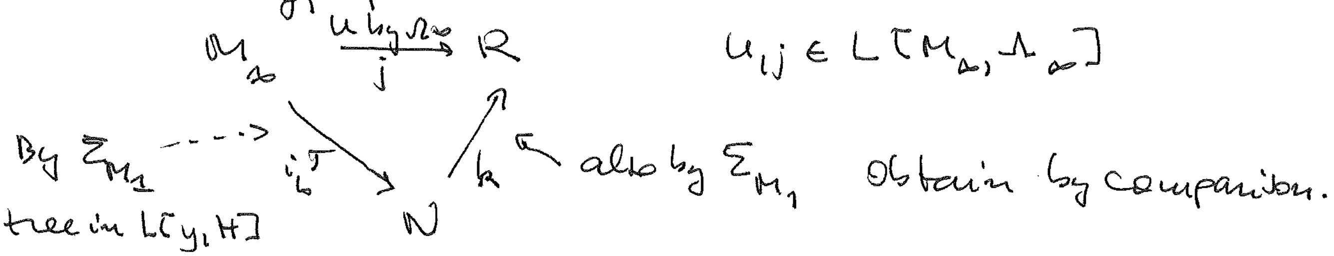
$p \in g_q$. Let $N_q = \tau^{g_q} \in L[y, H]$.

$(N_q, s) \in \mathcal{F}^{L[y,H]}$ → there are only cbly many

Now compare all N_q simultaneously and also with P. The coiteration terminates at R. So: $P \to R$, $N_q \to R$

$(N_q, s) \leq^* (R, s) \in M_\infty$ by symmetry.

Similarly, if we have



then we can find b in $L[M_\infty, \Lambda_\infty][y, H]$:

Use a tree searching for k, b s.t. $j = k \circ i$.

b is unique making the diagram commutative, because it moves types of indiscernibles correctly.

(This needs some elaboration.)

So $\mathcal{F}^{M_\infty}$ is dense in $\mathcal{F}^{L[y,H]}$

$\lim \mathcal{F}^{M_\infty} = M^*_\infty$

Let $i: M_\infty \to M^*_\infty$ be the map given by Λ_∞.

① $\xi < \kappa_\infty$ → need this since we only want countably many N_q's.

~~Let~~ Claim For $\xi < \delta_\infty$:

$$\xi \in A \iff M_\infty \models \left(\Vdash^{Col(\omega,\delta_\infty)\times Col(\omega,<\kappa_\infty)} \quad \right)$$

$$\iff M_\infty \models \left(\Vdash^{Col \times Col} L[y,H] \models \varphi[i(\xi)] \right)$$

Proof. Fix ξ. Let $\xi = \pi_{(N,\xi),\infty}(\bar\xi)$ with $s \in (u^V)^{<\omega}$ and ~~ξ~~ $\bar\xi < \gamma^N_s$. Point: can choose N s.t. x is $\mathbb{B}^N_{\delta N}$-generic$/N$, hence $L[x,G]$ is a $Col(\omega,\delta^N)\times Col(\omega,<\kappa)$-generic ~~Fat~~ extension of N.

Then ~~By~~

$$\xi \in A \iff L[x,G] \models \varphi[\xi]$$

$$\iff N \models \left(\Vdash^{Col\times Col} \underbrace{L[y,H] \models \varphi[\pi_{(N,s),\infty}(\bar\xi)]} \right)$$

$\psi(s,\bar\xi)$ ~~...~~ Can be expressed in N so we can write it as ψ

N.b. if $\pi: N \to M_\infty$ is an iteration map via Σ_{M_1} then $\pi \restriction \gamma^N_s = \pi_{(N,s),\infty} \restriction \gamma^N_s$ so they agree on $\bar\xi$. Hence:

$$\iff M_\infty \models \left(\Vdash^{Col\times Col} L[y,H] \models \psi(s,\xi) \right)$$

$$\iff M_\infty \models \left(\Vdash^{Col\times Col} L[y,H] \models \varphi \text{ ~~...~~ } [i(\xi)] \right)$$

as $i(\xi) = \pi_{(M_\infty,s^*),\infty}(\xi)$ ⊠

<u>Exercise</u> $i \restriction \eta$ $(\eta < \delta_\infty) \in M_\infty$. Hence $V^{HOD^{L[x,G]}}_{\delta_\infty} = M_\infty | \delta_\infty$

<u>Thm</u> (Woodin) (PD) For a cone of x: $HOD^{L[x,G]} \models \omega_2^{L[x,G]}$ is Woodin

Hence $L(M_\infty, \Lambda_\infty) \models \delta_\infty$ is Woodin.

<u>Exercise</u> $\delta_\infty^{+M_\infty} < \delta_\infty^{+L[M_\infty,\Lambda_\infty]}$; $|\kappa|^{L[M_\infty,\Lambda_\infty]} = \delta_\infty$.

We can use this to show:

Let $\Lambda = \Sigma_{M_1} \upharpoonright (\text{trees in } M_1 | \nu)$ where ν = the first inaccessible $> \delta_{M_1}$ in M_1.

Then ~~$L[M_1, \Lambda]$~~ $V_{\delta^{M_1}}^{L[M_1,\Lambda]} = M_1 \| \delta^{M_1}$ and $L[M_1, \Lambda] \models \delta^{M_1}$ is Woodin.

Sketch: Let M_∞ = the dir lim of $\mathcal{F}^{M_1}$ where $\mathcal{F}^{M_1}$ = the DLS for M_1 up to ν

M_∞^* = dir lim of $\mathcal{F}^{M_\infty}$

$$M_1 \xrightarrow[i]{} M_\infty \xrightarrow[i(i)]{} M_\infty^*$$

(Note: Adding Λ_∞ to M_∞ does not add bounded subsets of δ_∞)

$i(i)$ maps $L[\Lambda_\infty, M_\infty]$ to $L[\Lambda_\infty^*, M_\infty^*]$

use this to show:

$$\text{Hull}^{L[M_\infty, \Lambda_\infty]}(\text{rng}(i)) \cong L[M_1, \Lambda]$$

Point: Definitions are allowed to act on Λ_∞.
$i(i)$ preserves Λ_∞-definitions.

23.7.2010 14:00 Steve Jackson

Lemma Let $B \in \lambda$, $\rho < \lambda$ and $\mathcal{B} = (B_\beta \mid \beta < \rho)$ be s.t. $B_\beta \leq_W B$ for each β. Then $\overline{\mathcal{B}} \leq \lambda$

Lemma λ is closed under ultrapowers

Lemma $\delta_1 = \lambda^+$ is closed under ultrapowers.

Lemma δ_1 is a Suslin cardinal, $S(\delta_1) = \Sigma_2$ and scale(Σ_2)

Rem We can show that Δ_1 (and Σ_1, Π_1) is closed under measure quantification by measures on λ. Using this one can show that every Π_1 set admits a semi-scale with norms in Π_1

Question Do we have scale(Σ_0), scale(Π_1)?

Definition A tree on $\omega \times \kappa$ is strongly homogeneous if there are measures μ_s on T_s s.t.

- $\vec{\mu}$ witnesses the homogeneity of T
- there are measures μ_s on A_s s.t. for all x with T_x wellfounded, the ranking function $T_x \restriction \vec{A}_s$ has minimal values $[f]_{\mu_s}$ where f_s is the function on T_s induced by f

Fact If every κ-hs is strongly κ-hs then we can fill the gap ~~with~~ above where we have only semiscale instead of a scale.

Γ nonselfdual, closed under quantifiers, $\Gamma = S(\kappa)$ where $\kappa = o(\Delta)$. Let $A \in \Gamma - \check{\Gamma}$ and let $A = p[T]$ where T is on $\omega \times \kappa$.

<u>Definition</u> (Steel) $Env(\Gamma)$ is the set of all $A \subseteq \omega^\omega$ s.t. for some $z_0 \in \omega^\omega$, for any countable set of reals z containing z_0 we have $A \cap z \in L(T, z)$

$Env'(\Gamma)$ = the set of all $A \subseteq {}^\omega\omega$ s.t. for some $z_0 \in {}^\omega\omega$: for any countable set of reals z containing z_0 we have $A \cap z$ is definable in $L(T, z)$ from finitely many ordinals, T and z.

<u>Remark</u> We can consider the variations $\widetilde{Env}$, $\widetilde{Env}'$ where we consider "$d \geq_T z_0$" instead of "z containing z_0". Clearly $Env \subseteq \widetilde{Env}$, $Env' \subseteq \widetilde{Env}'$

<u>Theorem</u> For Γ as above: $\mathcal{L}(\Gamma, \kappa) = Env(\Gamma) = Env'(\Gamma) = \widetilde{Env}(\Gamma)$

26.7.2010 9:30 John Steel

Analyse ~~L(R)~~ $HOD^{L(\mathbb{R})}$ on the assumption: $M_\omega^\#$ exists. Let Σ_0 = the unique IS of $M_\omega^\#$
Actually, it is ~~important~~ possible to do it under weaker $AD^{L(\mathbb{R})}$. Let

M_∞ = dil lim all ctbl Σ_0-iterates of M_ω via trees on ~~M_ω~~ $M_\omega | \delta_0^{M_\omega}$, so that there is no drop on the main branch.

Recall: $M_\omega = \text{Hull}^{M_\omega}(\Gamma)$ whenever $\Gamma \subseteq On$ proper class
So M_ω is sound. This soundness can be used to show that the system of iterates is directed.

$\Lambda_\infty = \Sigma_0 \restriction$ trees in $M_\infty \restriction \lambda_\infty$ based on $M_\infty | \delta_0^{M_\infty}$
(where $\lambda_\infty = \sup_{i \in \omega} \delta_i^{M_\infty}$)

Then: $HOD^{L(\mathbb{R})} = L(M_\infty, \Lambda_\infty)$

Approximate via a DLS defined over $L(\mathbb{R})$.

Def $WG(M, \omega)$ is:

I	$\mathcal{T}_0$		$\mathcal{T}_1$
II		b_0	b_1

($\mathcal{T}_i$ on $M_{b_{i-1}}^{\mathcal{T}_{i-1}}$ where $M_{b_{-1}}^{\mathcal{T}_{-1}} = M$; $\mathcal{T}_i$ normal)

II wins iff $\lim_i M_{b_i}^{\mathcal{T}_i}$ exists and is w.f.

"II has a winning strategy in $WG(M, \omega)$" ~~...~~ is $\partial^{\mathbb{R}}\Pi_1^1 = \Sigma_1(L(\mathbb{R}))$
II has a ws $\Rightarrow$ II has a w.s. in $L(\mathbb{R})$.

Fact If M, N are $\partial^{\mathbb{R}}\Pi_1^1$ iterable project to ω are sound and ω-small then $M \trianglelefteq N$ or $N \trianglelefteq M$.

So the <u>Mouse-set-conjecture holds in L(ℝ)</u> :

In L(ℝ), TFAE for a countable transitive a and $b \subseteq a$:

(1) $b \in OD(a \cup \{a\})$

(2) b is $C_{\Sigma^2_1(a)}$

(3) b is in some ω_1-iterable mouse over a

(4) b is in some ω-small, ω_1-iterable mouse over a.

<u>Rem</u> AD ⟹ every ~~every~~ ω_1-iterable mouse is (ω_1+1) iterable.

The proof (1) ⟺ (2) is just an abstract computation.

(3) ⟹ (1) : Define b from its stage constructed in any M over a

(1) ⟹ (4) : this is the "correctness" of M_ω. Enough to show (ETS) : $b \in M_\omega(a)$. But then $b \in M_\omega(a) \| \omega_1^{M_\omega(a)}$ and this is iterable in L(ℝ): the iteration strategy: $T \mapsto$ the unique cof b s.t. M_b is $WG(M,\omega)$-iterable

To $b \in M_\omega(a)$: iterate $M_\omega(a) \to \cdots M_i \to N$ via $\Sigma_0^{M_\omega(a)}$ so that for some G generic / $\mathrm{col}(\omega, <\lambda^N)$: $\mathbb{R}^*_G = \mathbb{R}^V$.

So b is $OD(a \cup \{a\})^{D(N,\lambda^N)} = L(\mathbb{R})$. So $b \in N$, so $b \in M_{\omega_1}(a)$.

<u>Def</u> A premouse M is full iff $(\forall \gamma < On^M)(\forall b \subseteq M\|\gamma)$
$b \in OD(M\|\gamma, \{M\|\gamma\}) \Rightarrow b \in M$

So: $M_\omega \| \lambda^{M_\omega}$ and its iterates are full.

<u>Def</u> A premouse M is k-suitable iff there are $\delta_0 < \cdots < \delta_k$ Woodins s.t. $M \models \delta_i$'s are the unique Woodins <u>and</u>
$On = \delta_k^{++}$ ~~and~~ <u>and</u> ZFC$^-$
and M is full and ω-small.
(To be safe, add the requirement: no $M|\gamma$, $\gamma < On$ has this property.)

Paper: Woodin's analysis of $HOD^{L(\mathbb{R})}$.

We write $k = k(M)$ (k as above)

Crucial Definition Let $A \subseteq \mathbb{R}$, M be a countable premouse, $M \models ZFC - \{Powerset\}$ and $M \models \delta^+$ exists, let τ be a $Col(\omega,\delta)$ term. Then τ captures A over M iff for every g $Col(\omega,\delta)$-generic/M
$$\tau^g = A \cap M[g]$$

Example Let $A \subseteq \mathbb{R}$ be $OD^{L(\mathbb{R})}$, $\delta = \delta_k^{M_\omega}$. Then there is $\tau \in M_\omega$ s.t. τ captures A.
Exercise using $\mathbb{R}$ genericity iterations.

For τ a term, δ as above let
$$\tau^* = \{(p,\sigma) \mid p \in Col(\omega,\delta) \ \& \ \sigma \subseteq Col(\omega,\delta) \times \{\check{n} \mid n \in \omega\} \ \& \ p \Vdash \sigma \in \tau\}$$

Assuming $\Vdash \sigma \subseteq \check{\mathbb{R}}$: (we assume such terms always satisfy this)
- $\tau = \tau^*$
- $\tau^* = \tau^{**}$

Definition τ is invariant iff for all g, h generic for $Col(\omega,\delta)$: $M[g] = M[h] \Rightarrow \tau^g = \tau^h$ (M-definable)

For invariant τ, σ TFAE
(1) $\sigma^* = \tau^*$
(2) $\sigma^g = \tau^g$ on all $M[g]$, g $Col(\omega,\delta)$ generic/M
(3) — " — some — " —
Pf: Exercise

$\tau^* =$ the ~~~~ unique standard invariant term capturing A over M, if exists

We write: $\tau^* =$ ~~~~ $\tau^M_{A,\delta}$

Rem extended Let M be k-suitable, $A \in OD^{L(\mathbb{R})}$, $\delta \in M_\omega(M)$ then τ^M_{A,δ_n} exists for all $k \leq n$.

Definition Let $A_i \subseteq \mathbb{R}$ be OD, $\vec{A} = \langle A_0, \ldots, A_n \rangle$. Let M be k-suitable then Σ is an $\vec{A}$-iteration strategy for M iff Σ is a winning strategy in $WG(M, \omega)$ for II s.t. if $M \xrightarrow{\pi} N$ is an iteration map via Σ then

(1) N is k-suitable

(2) $\pi(\tau^M_{A_i, \delta_j^M}) = \tau^N_{A_i, \delta_j^N}$ for all i, j ~~~~ ($j = k$ is enough)

M is $\vec{A}$-iterable iff π has such a strategy.

Lemma If $\vec{A} \in (OD^{L(\mathbb{R})})^{<\omega}$ then for any Σ_0-iterate N of M_ω there is a Σ_0-iterate P of N s.t. for all $k < \omega$ Σ_0 is an $\vec{A}$ ~~~~ IS for $P \| \delta_k^{+P}$.

Proof (by picture.) Assume $N = N_0 \xrightarrow{\pi_0} N_1 \xrightarrow{\pi_1} N_2 \to \cdots$ and π_i moves $\tau^{N_i}_{A,\delta_k}$ incorrectly. Iterate each N_i to N_i^* to make $D(N_i, x_i^{N_i,*}) \equiv L(\mathbb{R})$ in a way that makes the diagram commute and that we have embeddings π_i^* (take sort).

$$
\begin{array}{ccccccc}
N_0^* & \xrightarrow{\pi_0^*} & N_1^* & \xrightarrow{\pi_1^*} & N_2^* & \text{---} & N_\omega^* \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
N_0 & \to & N_1 & \to & N_2 & \to \cdots & N_\omega
\end{array}
$$

The map-composition: bottom row + right column is an iteration map, so N_ω^* is w.f. In each $D(N_i^*, x^{N_i^*})$: A is OD, say from α. Then for sufficiently large i, $\pi_i^*(\alpha)$ is fixed, ow N_ω^* would be illfounded. From that point on, the terms are moved correctly. ⊠

<u>Def</u> For M, N k-suitable $\pi : M \to N$ is an A-iteration map iff π arises from a play according to an A-iteration strategy.

<u>Def</u> M is strongly A-iterable iff whenever $\pi : M \to N$, $\sigma : M \to N$ are A-IMs then $\pi \restriction H^M_A = \sigma \restriction H^M_A$. (Here M is k-suitable, $A \in OD^{<\omega} \cap \mathcal{P}(\mathbb{R})$)

Here: for P k-suitable over $A \in OD^{<\omega} \cap \mathcal{P}(\mathbb{R})$, $P = L(P \cup \sigma(P))$

$\gamma_{(P,A)} = \sup\{\xi < \delta_0^P \mid \xi$ is definable over P from ~~parameters~~ ~~the~~ $\{\tau^P_{A,\delta_k}\}\}$

Similarly for $\vec{A} = \{A_0 \ldots A_k\}$

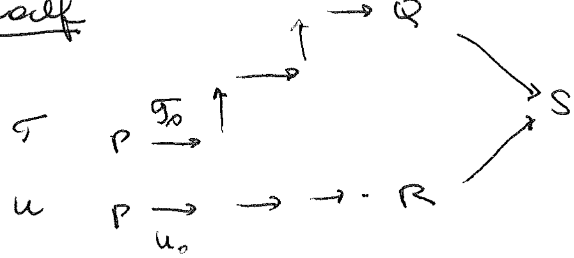
$\gamma_{(P,\vec{A})} < \delta_0^P$

$$H_{(P,\vec{A})} = \mathrm{Hull}^P(\gamma_{P,\vec{A}} \cup \{\tau^P_{A_0,\delta_k}, \ldots \tau^P_{A_i,\delta_k}\})$$

$$H_{P,\vec{A}} \cap \delta_0^P = \gamma_{(P,\vec{A})}$$

<u>Lemma</u> Let N be a Σ_0-iterate of M_ω s.t. Σ_0 is an $\vec{A}$-IS for $P = N \mid \delta_k^{++P}$. Then P is strongly $\vec{A}$ iterable.

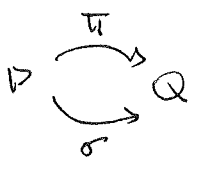
<u>Proof</u>



u_i below δ_0

This is like the M_1 argument before – check this

Also appeal to the D-J property of Σ_0 :



two Σ_0-iterates : $\pi \restriction H^P_{\vec{A}} = \sigma \restriction H^P_{\vec{A}}$ using DJ-property of Σ_0.

Let

$I^* = \{(N, \vec{A}) \mid \exists k\ N$ is k-suitable and N is strongly A-iterable $\}$

$(N, \vec{A}) \leq (P, \vec{B})$ iff there is an $\vec{A}$ IM $\pi : N \to (P|k(N)^{++P})$ ~~and~~ $k(N) \leq k(P)$ and $\vec{A}$ is an initial segment of $\vec{B}$.

$P_{(N,\vec{A}),(P,\vec{B})} : H^N_{\vec{A}} \to H^{P|(k(N)^{++P})}_{\vec{B}}$

is the common value of all k-iteration maps.

$\mathcal{F}$ = the corresponding DLS,

$\mathcal{F}$ is definable over $L(\mathbb{R})$

Claim: M_∞ = dir lim of $\mathcal{F}$

Def $\Phi_k = Th^{L(\mathbb{R})}(\alpha_0, .. \alpha_k)$ where α_i are $\mathbb{R}$-indiscernibles coded as set of reals.

So: Φ_k is $OD^{L(\mathbb{R})}$

Lemma Suppose $B \subseteq \mathbb{R}$ is $OD^{L(\mathbb{R})}$ and A is $OD^{L(\mathbb{R})}$ and $A \leq_W B$. Then ~~for any~~ there are densely many $(N, \vec{C}) \in I$ s.t. $A, B \in \vec{C}$ and

$\tau^N_{A, \delta^N_k} \in H^N_B$

and hence

if $\pi : H^N_A \longrightarrow H^P_{\vec{B}}$ is $\pi_{(N,B),(P,\vec{B})}$ then

$\pi(\tau^N_{A,\delta_k}) = \tau^P_{A,\delta_k}$.

Proof Choose any $(N, \vec{C})$ s.t. $A, B \in \vec{C}$ and Σ_0 is a $\vec{C}$-~~SM~~ IS for $N^* \triangleright N$, N^* a Σ_0-iterate of M_ω.

Also make sure x is $\mathbb{B}^N_{\delta^N_0}$-generic/$N$ where $A \leq_W B$ via x.

For τ a standard invariant term in $Col(\omega, \delta_k)$ where $k = k(N)$ pick an $r_\tau \in \mathbb{B}^N_{\delta^N_0}$ s.t.

$s_\tau \Vdash^{q}_{\mathbb{B}^N_{\delta_0^N}} \quad \Vdash^{h}_{Col(\omega,\delta_k)} \quad \tau = \dot{x}_g^{-1}(\tau_B^k)$

↑ Wadge reduction

if s_τ exists.

Here $k = g * h$ as $col(\omega, \delta_k)$-generic by rearrangement of generics. Then

$\tau \neq \sigma \Rightarrow s_\tau \perp s_\sigma$

Since there are $< \delta_0^N$ such s_τ's and τ's (s_τ determines τ)

So $< \delta_B^N$ many. So all $\tau \in H^N_B$. So $\tau_A \in H^N_B$. ⊠

<u>Corollary</u> dirlim $\mathcal{F}$ = lim of all $H_{(N, \Phi_k)}$ s.t. $N = P \restriction \delta_j^{++}$ for P a Σ_0 iterate of M_ω.

To show this limit is $M_\infty | \lambda_\infty$:

<u>Lemma</u> Let N^* be a Σ_0-iterate of $M^\#_\omega$ and $N = N^* | \delta_k^{++N^*}$. Let

$S_j = Th^{M_\omega(N)}(\alpha_0 \ldots \alpha_j \cup N | \delta_k)$

(very large cardinals.)

Then

(1) $\forall j \exists p \; S_j \in H^N(\delta_k^{++} \cup \tau^N_{\Phi_p, \delta_k})$

(2) $\forall j \exists p \; \tau^N_{\Phi_p, \delta_k}$ is in S_j.

<u>Proof</u> (2) is easy — use the theory $Th^{D(M_\omega(N), \lambda)}(\alpha_0 .. \alpha_p)$ to figure out Φ_p.

(1) Given j take $p = j+5$. ~~Idea Priky force premise~~

Idea: Over $L(\mathbb{R})$ Priky force a premouse whose derived model is $L(\mathbb{R})$.

For a countable transitive well-founded self well-ordered add a Turing degree above a

letting

T = tree of a scale on universal Σ^2_1 set in $L(\mathbb{R})$

In $L[T, d]$: take all o-suitable Q s.t. Q is ϕ-iterable and $Q \leq_T d$, Q over a.

$\overset{d}{Q}(a)$ = result of comparing all of them and making all $z \leq_T d$ generic / Q.

They can be compared in $L[T, d]$.

Given $d_0 < d_1 < \dots < d_n$

$Q_0 = Q^{d_0}_{\emptyset}$

$\vdots$

$Q_{i+1} = Q^{d_{i+1}}_{Q_i}$

Let $\langle d_i \mid i \in \omega \rangle$ be Prikry. Can show for

$Q_\infty = \bigcup_i Q_i$

$L[Q_\infty] \cap P(\gamma) \subseteq Q_i$ for any $\gamma < o(Q_i)$.

(all Q_i are OD-full)

Moreover: There is an iterate of M_ω via Σ_0 s.t. it is of the form $Q^{\vec{d}}_\infty$ some Prikry-generic $\vec{d}$.

Can then define S_j from Φ_{j+5} using the Prikry forcing.

The rest is similar to the M_1-argument ⊠.

26.7.2010 14:00 Ralf Schindler

Core model induction in $L(\mathbb{R})$

Def Let $\gamma \geq \aleph_1$ be a cardinal and $A \in H_\gamma$. A model operator over A on H_γ is a partial function $F: H_\gamma \to V$

$$\mathcal{M} = (|\mathcal{M}|, \in, A, E, B, S) \longmapsto F(\mathcal{M}) = \mathcal{N}$$

($|\mathcal{M}|$: transitive rud closed $\in A$; E: extender sequence; B: something else; S: stratification)

where

$$\mathcal{N} = (|\mathcal{N}|, \in, A, \tilde{E}, B')$$ such that

($|\mathcal{N}|$: transitive rud closed)

- $\mathcal{N}$ is an "end-extension" of $\mathcal{M}^{\#}$, $\mathcal{M} \in |\mathcal{N}|$
- $F(\mathcal{M}) = \mathrm{Hull}_{\Sigma_1}^{F(\mathcal{M})}(|\mathcal{M}| \cup \{|\mathcal{M}|\})$
- [For no $\alpha \in [\mathrm{On} \cap \mathcal{M}, F(\mathcal{M}) \cap \mathrm{On})$: $\rho_\omega(F(\mathcal{M})|\alpha) < \mathcal{M} \cap \mathrm{On}$]

Examples • $\mathrm{rud}_{\vec{E}} = F$
- $F = M_n^{\#}$

$$F(\mathcal{M}) = \begin{cases} \text{the least iterable mouse } P \text{ with crit pts} > \\ \mathrm{ht}(\mathcal{M}) \text{ s.t. either } \rho_P^\omega < \mathrm{ht}(\mathcal{M}) \text{ or else} \\ P \text{ has } n \text{ Woodins + Sharp and is not sound.} \end{cases}$$

- F feeding in info about Σ_1 an IS for $\mathcal{N}$ coded by A.

Def (Iteration strategies + Condensation relative to F). Let $(\mathcal{M}, \bar{\mathcal{M}})$ be given.* Suppose $\pi: \bar{\mathcal{M}} \to F(\mathcal{M})$ is either Σ_0 cofinal or Σ_2. Then $\bar{\mathcal{M}} = F(\pi^{-1}(\mathcal{M}))$.

* F condenses well

<u>Exercise</u> Assume $F : H_\nu \to H_\nu$ is a model operator which condenses well. Let $\kappa > \nu$. Then there is at most one extension $\tilde{F}$ of F, $\tilde{F} : H_\kappa \to H_\kappa$ s.t. $\tilde{F}$ also condenses well.

<u>Def</u> Let $F : H_\nu \to H_\nu$ be an MO. A model $\mathcal{M} = (|\mathcal{M}|, \in, A, E, B, S)$ is a potential premouse iff There is ~~&~~ $\vec{\mathcal{M}} = (\mathcal{M}_i \mid i < \theta)$ a sequence of models; write $\mathcal{M}_\theta = \mathcal{M}$, satisfying

- $\mathcal{M}_{i+1} = (F(\mathcal{M}_i), \vec{\mathcal{M}} \restriction (i+1))$
- E is a coherent extender sequence.

~~D~~ $\mathcal{M}$ is a <u>premouse</u> iff all proper initial segments are sound.

<u>Def</u> $K^{c,F}(P)$ - construction. This is like an ordinary K^{c} construction with the exception that the step ~~$\to$~~ $M_\xi \mapsto \mathcal{J}^{E^{M_\xi}}_{\alpha+1, M_\xi}$ do $M_\xi \mapsto F(M_\xi)$.

<u>Example</u> Don't add any extenders, $\nu = \infty$. Then $K^{c,F}(P) = L^F(P)$. Point: If F condenses well then $L^F(P) \models GCH$ etc. (P countable & in $L^F(P)$)

As usual: Countable substructures of models N_ξ from the $K^{c,F}(P)$ construction are ω_1-iterable in this sense: if $\mathcal{T}$ is a countable tree on W ~~with last model M_ξ~~ and $\sigma : W \to N_\xi$ then $\mathcal{T}$ has a last model embeddable into some $W_{\bar\xi}$, $\bar\xi \leq \xi$ or else there is ~~or else there $\bar\xi \leq \xi$.~~ is a maximal branch b s.t. $\mathcal{M}^{\mathcal{T}}_b$ is embeddable into $W_{\bar\xi}$

Def A premouse $\mathcal{M}$ is F-small iff $M||\kappa \models$ No Woodins where $\kappa = cr(E_\alpha^{\mathcal{M}})$ some α.

$M_1^F(\mathcal{M})$ = the least (ω_1+1)-iterable premouse
↳ Above $\mathcal{M}$ and not sound above $\mathcal{M}$.

Def Assume $\mathcal{T}$ is an IT on an F-pm which does not have a definable Woodin card. We say that $\mathcal{T}$ is guided by L^F iff $\forall \lambda < lh(\mathcal{T})$: $[0, \lambda)_{\mathcal{T}}$ = the unique cofinal branch b for $\mathcal{T}\restriction\lambda$ s.t. for some $Q \trianglelefteq \mathcal{M}_b^{\mathcal{T}\restriction\lambda}$ s.t. Q either projects below $\delta(\mathcal{T}\restriction\lambda)$ or else $\delta(\mathcal{T}\restriction\lambda)$ is not definably Woodin over Q (briefly Q kills Woodinness of $\delta(\mathcal{T}\restriction\lambda)$) and $Q \triangleleft L^F(\mathcal{M}(\mathcal{T}\restriction\lambda))$

Plan: $K^{c,F}(P)$ is fully iterable via the strategy of producing trees which are guided by L^F.

Theorem ($K^{c,F}$ existence dichotomy). For simplicity assume Ω is a measurable cardinal.
Let F be a model operator on $H_\Omega = V_\Omega$. Let $K^{c,F}(P)$ be the result of the $K^{c,F}(P)$-construction inside V_Ω.
Let Σ be the partial strategy of producing IT's which are guided by L^F. Then:

① If Σ produces a model with a Woodin, i.e. there is a tree $\mathcal{T}$ of limit length on $K^{c,F}(P)$ guided by L^F s.t. $L^F(\mathcal{M}(\mathcal{T})) \models \delta(\mathcal{T})$ is Woodin then $K^{c,F}(P)$ reaches $M_1^F(P)$ + $M_1^F(P)$ is iterable.

② If hypo ① fails then $K^{c,F}(P)$ is $\Omega+1$ iterable

If ② applies, isolate $K^F(P)$ and use it to get a contradiction from the favorite background hypothesis.

Proof

This is like the proof in the classical case where F = rud +
+ uses that F condenses well.

Rem There are more "local" versions of the K^F-existence dichotomy
(For instance:)

Applications Show PD from various hypotheses

Theorems $\neg\Box_\kappa \Rightarrow V$ is closed under $M_n^\#$ (suitable κ)

Theorem There ~~is a s~~ are ω pairs of successor cardinals with the tree property with sup δ s.t. $2^{\aleph_0} < \delta$. Then H_δ is closed under $M_n^\#$

Theorem Let κ be singular, $cf(\kappa) > \omega$. Suppose $\{\alpha < \kappa \mid 2^\alpha = \alpha^+\}$ is stat costat. Then H_κ is closed under $M_n^\#$.

Theorem Suppose CH + there is a (homogeneous) presaturated ideal on ω_1. Then PD holds. (i.e. H_{ω_1} is closed under all $M_n^\#$)

Theorem (Woodin) There is ω_1-dense ideal on ω_1. Then PD.

26.7.2020 4:45 pm Paul Larson – Discussion – 1 –

Theorem 9.40

Suppose $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass, $V = L(\Gamma, \mathbb{R})$, $AD_{\mathbb{R}}$ + "Θ regular".
Let $G_0 \subseteq \mathbb{P}_{max}$ be $L(\Gamma, \mathbb{R})$ be generic and let $H_0 \subseteq Col(\omega_3, \mathcal{P}(\mathbb{R}))$ (Here $\mathcal{P}(\mathbb{R})$ is essentially H_{ω_3}) be $L(\Gamma, \mathbb{R})[G_0]$ – generic. Then
$L(\Gamma, \mathbb{R})[G_0][H_0] \models ZFC + MM^{++}(c)$

$MM^{++}(c)$ is : • MM for posets of size $2^{\aleph_0}$ plus
• For any collection $(\tau_\alpha \mid \alpha < \omega_1)$ of $\mathbb{P}$-names for stationary subsets of ω_1, each τ_α^G is stat.

Def $\mathbb{P}_{max}$ is the set of $\langle (M, I), a \rangle$ s.t.
- M is a countable transitive model of ZFC + MA_{ω_1}
- I is a precipitous ideal on ω_1^M in M
- (M, I) is iterable by repeated application of generic ultrapowers by I.
- $a \in \mathcal{P}(\omega_1)^M$ and $\exists x \in \mathcal{P}(\omega)^M$ s.t. $\omega_1^{L[x,a]} = \omega_1^M$.

Ordering:
$\langle (M, I), a \rangle \le \langle (N, J), b \rangle$
if – $\langle (N, J), b \rangle \in H(\omega_1)^M$
– $\exists j : (N, J) \to (N^*, J^*)$ in M s.t. $j(b) = a$ and $J^* = I \cap N^*$
(so j is an iteration map of length ω_1^M)
Note: j is uniquely determined by j(b).

Facts

ⓐ If $G \subseteq \mathbb{P}_{max}$ is a filter let
$A_G = \bigcup \{ a \mid \langle (M, I), a \rangle \in G \}$
For all $p \in G$, $p = \langle (M, I), a \rangle$ there is unique $j_p : (M, I) \longrightarrow (M^*, I^*)$ s.t. $j_p(a) = A_G$.

ⓑ ~~B~~ let $\mathcal{P}(\omega_1)_G = \bigcup \{ j_P(\mathcal{P}(\omega_1)^M) \mid P = \langle (M, I), a \rangle \in G \}$

ⓒ $\mathbb{P}_{max} \in L(\mathbb{R})$.

<u>Theorem 9.33/35</u> Suppose that $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass and $L(\Gamma, \mathbb{R}) \models AD^+$. Let $G \subseteq \mathbb{P}_{max}$ be ~~L(R)~~ $L(\Gamma, \mathbb{R})$-generic. Then in $L(\Gamma, \mathbb{R})[G]$:

① $\mathcal{P}(\omega_1)_G = \mathcal{P}(\omega_1) \quad \in L(\mathbb{R})[G]$

② $L(\mathbb{R})[G] \models c = \aleph_2$

③ $\forall A \in \mathcal{P}(\mathbb{R}) \cap L(\Gamma, \mathbb{R}) : L(A, \mathbb{R})[G] \models ZFC$

④ $\forall A \in \mathcal{P}(\omega_1) - L(\mathbb{R}) : G \in L(\mathbb{R})[A]$.

<u>Proof of T 9.40</u>$^{(*)}$ $L(\Gamma, \mathbb{R})[G_0] \models \omega_2\text{-}DC$ so ETS $L(\Gamma, \mathbb{R})[G_0] \models MM^{++}(c)$

Let $\tau_{\mathbb{P}}, \tau_{\mathbb{D}}, \tau_{\mathcal{S}}$ be $\mathbb{P}_{max}$ names for:

$\tau_{\mathbb{P}}$ a poset on ω_2 preserving stationary subsets of ω_1

$\tau_{\mathbb{D}}$ an ω_1-sequence of dense subsets of $\tau_{\mathbb{P}}$

$\tau_{\mathcal{S}}$ an ω_1-sequence of $\tau_{\mathbb{P}}$-names for stat subsets of ω_1.

Fix a coding of elements of $H(\omega_2)$ by reals

- first code elements of $H(\omega_2)$ by subsets of ω_1.
- then: since each subset of ω_1 is in $L[x]$ for some $x \subseteq \omega$ code this by $x^{\#}$ and the relevant term.

Letting $B_{\mathbb{P}}, B_{\mathbb{D}}, B_{\mathcal{S}}$ be the set of codes for elements of $\tau_{\mathbb{P}}, \tau_{\mathbb{D}}, \tau_{\mathcal{S}}$ we have that for any TCM of ZFC and closed under ~~#s~~ daggers for reals:

$B_{\mathbb{P}} \cap M$ decodes as a $\mathbb{P}_{max}$-name for a p.o. on a subset of ω_2^M.

$B_{\mathbb{D}} \cap M$ decodes as an ...

Let T_0 be a tree on $\omega^3 \times On$ s.t.

$p[T_0] = B_{\mathbb{P}} \times B_{\mathbb{D}} \times B_{S}$ and $p[T_1]$ = its complement.

This is possible due to AD^+: it implies reflection to Hom*/Suslin.

If $j: M \to M^*$ where $M \models ZFC$ transitive and $T_0, T_1 \in M$ then $p[T_i] \subseteq p[j(T_i)]$ $i = 0, 1$. (M^*)

<u>Theorem 9.38</u> Assume $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ is a pointclass and $L(\Gamma, \mathbb{R}) \models AD^+$. Then $\forall X \subseteq On$ in $L(\Gamma, \mathbb{R})$ $\exists Y \subseteq On$ in $L(\Gamma, \mathbb{R})$ s.t.

① $X \in L[Y]$

② $\forall$ countable $t \subseteq \omega_1$ $\exists N \models ZFC$ proper class model s.t. $L[Y, t] \subseteq N$ and

- $L[Y, t] \cap V_\kappa = N \cap V_\kappa$ for the least strongly inaccessible (in $L[Y,t]$)
- $\exists \delta \leq \omega_1^V$ s.t. δ is Woodin in N.

<u>Proof later</u>

- Let S be this set ↑ for T_0, T_1
- Let μ be the club measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$

  normality: if $f: \mathcal{P}_{\omega_1}(\mathbb{R}) \to \mathcal{P}(\omega_1)$ is such that $f(\sigma) \subseteq \sigma$ for $\sigma \neq \emptyset$ then $\exists x \in \mathbb{R}$ s.t. $\{\sigma \mid x \in f(\sigma)\} \in \mu$.

Take $\prod_{\sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})} L(S, \sigma) / \mu = L(S^*, \mathbb{R})$

Let T_0^*, T_1^* be the images of T_0, T_1 under the up map. Then $p[T_0^*] = p[T_0]$ $(= B_{\mathbb{P}} \times B_{\mathbb{D}} \times B_S)$ and $p[T_1^*] = p[T_1]$. So $L(S^*, \mathbb{R}) \models p[T_0^*]$ decodes as....

So: $\exists \sigma \in \mathcal{P}_{\omega_1}(\mathbb{R})$ $L(S, \sigma)$ also thinks this.

Force over $L(S, \sigma)$ with $\mathbb{P}_{max}^{L(S,\sigma)}$; call this generic g. Then $L(S, \sigma) \models ZFC$, etc let t be an enumeration of σ in $L(S, \sigma)[g]$. Then $L(S, \sigma)[g] = L(S, t)$.

Let N be as for $L(S,t)$ (T 9.30). Let $\mathbb{P}$ be the realization of $\tau_{\mathbb{P}}$ by g. $N \models \mathbb{P}$ preserves stat subsets of ω_1. Let $h \subseteq \mathbb{P}$ be N-generic.

Let δ be Woodin in N. Let K be $N[h]$ generic for $Col(\omega_1, <\delta)$. Force over $N[h][K]$ with ccc forcing to get MA_{ω_1}; call this extension N^1. Let λ be the least strongly inaccessible of N^1. Then $\langle (N^1_\lambda, NS^{N^1}_{\omega_1}), A_g \rangle \in \mathbb{P}_{max}$ and $= p_0$ is above all $\langle (M, I), a \rangle$ for all $\langle (M,I), a\rangle$ in g. Let $p_0 \in G \subseteq \mathbb{P}_{max}$, $L(\Gamma, \mathbb{R})$-generic. Then $j_{p_0}(\mathbb{P}) \subseteq \tau_{\mathbb{P}}^G$ $\quad j_{p_0}(K) =$ filter in $\tau_{\mathbb{P}}^G$

$$j_{p_0} : (N^1_\lambda, NS^{N^1}_{\omega_1}) \longrightarrow (N^*, J^*)$$

$$J^* = NS_{\omega_1} \cap N^*$$

<u>Theorem 9.36</u> Assume Γ is a pointclass, $L(\Gamma, \mathbb{R}) \models AD^+ + \Theta$ reg, $G \subseteq \mathbb{P}_{max}$ is $L(\Gamma, \mathbb{R})$-generic. Then $L(\Gamma, \mathbb{R}) \models \omega_2$-DC.

<u>Proof</u> It suffices to prove ω_2-DC_Γ. Suppose $R \subseteq \Gamma \times \Gamma$. Work in $L(\mathbb{R}, \Gamma)[G]$. Find $\kappa < \Theta$ s.t. all ω_1-sequences from $R \restriction w(\kappa)$ have extensions in $w(\kappa) = \{A \in \Gamma \mid w(A) < \kappa\}$.
For all $\kappa < \Theta$: $|w(\kappa)^{\aleph_1}| < \Theta$. Why:

- $c = \mathcal{P}(\omega_1) = \aleph_2$
- $\exists B \subseteq \mathbb{R}$ coding $R \restriction w(\kappa)$, $\exists A \subseteq \mathbb{R}$ coding a $\mathbb{P}_{max}$ name for B.

So in $L(A, \mathbb{R})[G]$ can find an ω_2-sequence through R.