Working in $N_x^*[h]$ iterate the picture to make $\mathbb{R}^*$ generic

$$
\begin{tikzcd}
N_0^0 \arrow[r] \arrow[dr] & N_1^0 & \cdots \\
 & Q_0 \arrow[u] & \\
N_0^1 \arrow[r] \arrow[dr] & N_1^1 & \cdots \\
 & Q_0^1 \arrow[u] & \\
\vdots & & \\
N_0^\omega \arrow[r] \arrow[dr] & N_1^\omega & \cdots \\
 & Q_0^\omega \arrow[u] &
\end{tikzcd}
$$

$B \in D(N_i^\omega)$ the new derived models $D(Q_i^\omega)$ for all i. At least, it is possible to arrange this. By setting it up so that N_x^* thinks all of its interpretations of B are s.t. $L(B,\mathbb{R}) \models AD^+$.

So far we have: For every $B \in \mathbb{B}$ there is a hod pair (P,Σ) s.t. Σ has BC and ~~Σ~~ is FPR and strongly respects B.

<u>Note</u>: given $\langle B_i \mid i \in \omega \rangle$ we can get a hod pair (P,Σ) s.t. Σ is FPR, has BC and strongly respects $\bigoplus_{i \le k} B_i$ all k. Do this by comparing (P_i,Σ_i)'s.

<u>Backtrack</u> Take $\alpha = -1$, i.e. $HOD|\Theta_0$.
$\mathbb{B} = \{ A \subseteq \mathbb{R} \mid A \text{ is OD} \}$.
Let $\langle A_i \mid i \in \omega \rangle$ be a semiscale on Π^2_1. By the above, get (P,Σ) s.t. P is a hod prem, $\lambda^P = 0$, Σ is FPR and has BC and strongly respects $\bigoplus_{i \le k} A_i$ all k.

<u>Then</u> $\langle A_i \mid i \in \omega \rangle$ guides Σ.

<u>Proof</u> Given $\mathcal{T}$ on P. We have: if $\Sigma(\mathcal{T}) = b$ then $(M_b^{\mathcal{T}})$ $\forall i$ $i_b^{\mathcal{T}}(\tau^P_{A_i}) = \tau^{M_b^{\mathcal{T}}}_{A_i}$ but $\sup_{i \in \omega} \gamma^{\mathcal{T}}_{A_i} = \delta(\mathcal{T})$.

<u>Why</u>: If not then letting $S = \mathrm{Hull}^{M_b^\sigma}_{\Sigma_1}(\{\pi^{M_b^\sigma}_{A_i} \mid i\in\omega\}\cup\gamma)$ after collapse with $i: S \to M_b^\sigma$. Then $\mathrm{cr}(\pi)=\gamma$ and $S \models \gamma$ is Woodin. Also S is full because $\langle A_i \mid i\in\omega\rangle$ is a semiscale on Π^2_1. So $Lp(S|\gamma) \models \gamma$ is Woodin. Hence $M_b^\sigma \models \gamma$ is Woodin. ↯.

We can now show $\delta^{M_\infty(P,\Sigma)} = \Theta_0$

<u>Proof</u> $\leq$ comes from the covering system $\mathcal{F}$

$\geq$ because otherwise Σ would be ~~...~~ $<\Theta_0$-Suslin so $w(\Sigma)<\Theta_0$. But we know $w(\Sigma) \geq w(A_i)$ all i.

Why is $\Sigma < \Theta_0$ Suslin:

$$\begin{tikzcd} & M_\infty \lhd \mathrm{HOD}|\Theta_0 \\ P \arrow[ur, "\pi"] \arrow[dr, "\sigma"'] & \\ & \cdot \arrow[uu, "\sigma"'] \end{tikzcd}$$

<u>See</u>: Scales in $K(\mathbb{R})$

<u>John says</u> Enough to look at the tree of ~~all~~ all suitable attempts and is significantly simpler (since we are merely looking for a Suslin representation and not for a scale).

Back to Θ_α and $\Theta_{\alpha+1}$ for $\alpha \geq 0$.

We have: (P,Σ) with Σ FPR+BC

$M_\infty(P^-,\Sigma_{P^-})|\Theta_\alpha = V^{\mathrm{HOD}}_{\Theta_\alpha}$. Also $w(\Sigma_{P^-}) = \Theta_\alpha$. For all $B \in \mathcal{B}$: some tail of (P,Σ) strongly respects B.

Consider $N(P,\Sigma)$ = dir lim of all Σ-iterates of P <u>above</u> P^-.

This is like $\Theta=\Theta_0$ case. Show by the same argument that $N_\infty(P,\Sigma) = V^{\mathrm{HOD}_{\Sigma_{P^-}}}_{\Theta_{\alpha+1}}$ and get $\langle A_i \mid i\in\omega\rangle$ semiscale on $\Pi^2_1(\Sigma_{P^-}) \subseteq OD_{\Sigma_{P^-}}$. ~~Say~~

~~Suppose~~ $\sup_{i<\omega} \gamma^{N_\infty(P,\Sigma)}_{A_i,\infty} = \Theta_{\alpha+1}$

For $\gamma < \Theta_{\alpha+1}$ let

$B^{\gamma} = \{((Q,\Lambda), x, (y,z))$ s.t.

1. (Q,Λ) is a hod pair with FPR + BC and $M_\infty(Q,\Lambda) = V_{\Theta_\alpha}$
2. x codes Q
3. y codes R_y s.t. (R_y,Λ) is a suitable pair
4. z is in the least $OD_{\{\Lambda\}}$-set C s.t. (R_y,Λ) is C-iterable in the sense of getting HOD_Λ and $\pi_{R_y,\infty}(\gamma_C^{R_y}) \geq \gamma$

Let $\langle \gamma_i \mid i\in\omega \rangle$ be a sequence s.t.

$\sup_{i\in\omega} \gamma_i = \Theta_{\alpha+1}$

Let $B_i = B^{\gamma_i}$

Let (P,Σ) be a hod pair with FPR + BC,

$M_\infty(P^-, \Sigma_{P^-}) = V^{HOD}_{\Theta_\alpha}$, Σ strongly respects $\bigoplus_{i\in k} B_i$ all k.

Claim Σ is guided by $\langle B_i \mid i\in\omega \rangle$

Proof ETS: If $\mathcal{T}$ is on P and b is the branch, $Q = M_b^{\mathcal{T}}$

then $\sup_{i\in\omega} \gamma_{B_i}^{Q,\Sigma_{Q^-}} = \delta^Q$.

[In $HOD_{\Sigma_{Q^-}}$-limit, the $\gamma_{B_i,\infty}$'s are cofinal in $\Theta_{\alpha+1}$ and (Q, Σ_{Q^-}) is a suitable pair which is $\bigoplus_{i\in k} B_i$ iterable $\forall k$, $i_{Q,\infty}(\gamma_{B_i}^{Q}) > \gamma_i$. Let us backup.]

Suppose $\gamma = \sup \gamma_{B_i}^{Q,\Sigma_{Q^-}}$. Let $H = \mathrm{Hull}^{Q}_{\Sigma_1}(\gamma \cup \{\tau_{B_i}^{Q,\Sigma_{Q^-}} \mid i\in\omega\})$.

If there is π s.t. then we are done, because H is full by our requirement on Σ and $H \models \gamma$ Woodin

$$\begin{tikzcd} & Q \\ P \arrow[ur, "\vec{\sigma}"] \arrow[dr, "\pi"'] & \\ & H \arrow[uu, "\sigma"'] \end{tikzcd}$$

So $Q \models \gamma$ Woodin, ↯.

π exists if $P = \cancel{\ldots} \bigcup_{i \in \omega} H_{B_i}^{P, \Sigma_{P^-}}$ To get such a P just consider a countable version of HOD $^{\Sigma_{P^-}}$:[Look for $N_\infty^{L[T,d]}$ where T is the tree for $\Sigma_1^2(\mathrm{Code}(\Sigma_{P^-}))$ on a cone of d. - No!])

$\Rightarrow$ Let Γ be a good pointclass, $w(\Gamma) > \Theta_{\alpha+1}$. Work in some N_x^* that can see everything we want. Then let Φ be the N_x^* sets of Wadge rank $< \Theta_{\alpha+1}$.

$\mathcal{N} = \mathcal{W}_\infty(P, \Sigma_P)^{(L[\Phi, \mathbb{R}])^{N_x^*}}$. $\square$

14:00

Remark

Step 1 $V_{\Theta_0}^{HOD}$

Step 2 $V_{\Theta_1}^{HOD}$

⋮

Step ω $V_{\Theta_\omega}^{HOD}$ If $\Theta_{\omega+1}$ exists then we can get (P, Σ) s.t. $\lambda^P = \omega$, Σ is FPR+BC $\forall n$ $M_\infty(P(n), \Sigma_{P(n)}) | \Theta_n = V_{\Theta_n}^{HOD}$

Compare all of them.

(If $\Theta_{\omega+1}$ does not exist take just the union of $V_{\Theta_n}^{HOD}$

This generalizes to arbitrary cof ω-step.

Now assume Θ_{ω_1+1} exists.

Step ω_1 Let $\Gamma > w(\Theta_{\omega_1})$. Let $A \subseteq \mathbb{R}$ with $w(A) = \omega_1$.

B = the set of all σ s.t. $\sigma^{-1}[A] = \langle P, \Sigma \rangle$, Σ FPR+BC, $M_\infty(P, \Sigma) = V_{\Theta_\alpha}^{HOD}$ all $\alpha < \omega_1$.

Get some N_x^* capturing A, B, then do hod pair construction inside N_x^*. We will get (P, Σ) s.t.

- Σ is FPR + BC
- λ^P = the least measurable in P
- $(\forall \alpha < \lambda^P)\ \ M_\infty(P(\alpha), \Sigma_{P(\alpha)}) | \Theta_\alpha = V_{\Theta_\alpha}^{HOD}$)

What about $M_\infty(P, \Sigma)$.

$B(P, \Sigma) = \{ Q \mid Q \triangleleft_{hod} R$ where R is a Σ-iterate of $P \}$

$\leq^*$ in $B(P, \Sigma)$:

$Q \leq^* R \iff (\exists \gamma \leq \lambda^R)(R(\gamma) \in I(Q, \Sigma_Q))$ and

$i^\Sigma_{QR} : Q \to R(\gamma)$ the corresponding embedding

Let $M_\infty^-(P, \Sigma) = \text{dir lim}(B(P, \Sigma), \leq^*)$ under $i^\Sigma_{Q,R}$.

Exercise $M_\infty^-(P, \Sigma) = M_\infty(P, \Sigma) | \delta^{M_\infty(P, \Sigma)}$

It follows: $M_\infty^-(P, \Sigma) | \Theta_{\lambda^P} = V^{HOD}_{\Theta_{\lambda^P}}$

Claim $M_\infty^-(P, \Sigma) = V^{HOD}_{\Theta_{\omega_1}}$.

Proof (Sketch) Let $\alpha < \omega_1$. By induction:

$M_\infty^-(P, \Sigma) | \Theta_\alpha = V^{HOD}_{\Theta_\alpha}$ for $\alpha < \omega_1$.

Want this for $\alpha + 1$. Let $R \in B(P, \Sigma)$ be s.t.

$M_\infty(R, \Sigma_R) = M_\infty^-(P, \Sigma)(\alpha + 1)$

WTS: $M_\infty(R, \Sigma_R) | \Theta_{\alpha+1} = V^{HOD}_{\Theta_{\alpha+1}}$.

For this: Need that the HYPO holds for (R, Σ_R).

So, let $R^* \in I(R, \Sigma)$ be s.t. $R \trianglelefteq_{hod} R^*$.

Then use the following fact about derived models:

Notation: $D^*(P, \Sigma) = D(\text{Ult}(P, \Sigma)(\lambda^P), \Sigma)$ where

μ is the order 0 measure on $cf(\lambda^P)$.

One can show:

(1) $\quad D^*(P,\Sigma) = \{A \mid w(A) < \Theta_{\lambda^P}\}$

Using this,

$P \models$ In my M_1 derived model, $\forall \alpha < \lambda^P$ $(P(\alpha), \Sigma_{P(\alpha)})$ is "good" (i.e. good for computing HOD)

$\rightsquigarrow$

$R^* \models$ —"—

Then

$D^*(R^*, \Sigma_{R^*}) = \{A \mid w(A) < \Theta_{\lambda^{R^*}}\} \Rightarrow (R, \Sigma_R)$ is good

An idea of the proof of (1): Interpretation of strategies to generic extensions.

Given a hod pair (P,Σ) with BC+FPR and $\alpha < \lambda^P$ we can find $T, S \in P$ s.t. (T,S) interpret $\Sigma_{P(\alpha)}$ in generic extensions, i.e.

$$\left.\begin{array}{l} p[T]^{P[g]} = \Sigma_{P(\alpha)} \restriction \mathbb{R}^{P[g]} \\ p[S]^{P[g]} = \ldots \end{array}\right\} (2)$$

So for all $\alpha < \lambda^P$: $\Sigma_{P(\alpha)} \in D^*(P,\Sigma)$. ~~and~~ For any $A \in D^*(P,\Sigma)$ $\exists \alpha < \lambda^P$ s.t. $A \leq_W Code(\Sigma_{P(\alpha)})$

i.e. $(T,S) \in P[g]$ that gives A. Let $\delta_\beta < \lambda^P$ be s.t. δ_β Woodin in $P[g]$. Let T^*, S^* be the $\delta_{\beta+1}$ versions of T,S. Then show that $\Sigma_{P(\beta+1)}$ gives A by the following: Iterate to absorb reals $x \in A \Leftrightarrow$ x is generic of $P(\beta+1)$ by $\Sigma_{P(\beta+1)}$ giving R s.t. $x \in p[i^{\Sigma}_{P(\beta+1),R}(T^*)]$

Also: $\forall \alpha < \lambda^P$ $w(Code(\Sigma_{P(\alpha)})) < \Theta_{\lambda^P}$ So

$D^*(P,\Sigma) = \{A \mid w(A) < \Theta_{\lambda^P}\}$

Exercise Show

$D^*(P,\Sigma) \models \Sigma_{P(\alpha+1)}$ is not OD from $\Sigma_{P(\alpha)}$.

An idea toward (2). $\leadsto$ Unable to record.

THE PROOF OF MSC

WTS: $x \in OD_y \Leftrightarrow \exists y$-mouse M s.t. $x \in M$.

Assumptions: $\neg(AD_{\mathbb{R}} + \Theta$ regular$)$

Assume not:

Suppose: Γ is the largest s.t.

$L_\alpha(\Gamma, \mathbb{R}) \models MC$ & $L_{\alpha+1}(\Gamma, \mathbb{R}) \models \neg MC$

Then there is (P, Σ) s.t.

Let M be the largest initial of V where MC holds.

$\cancel{M = L_\alpha(\Gamma, \mathbb{R})}$ Let $\Gamma = \mathcal{P}(\mathbb{R})^M$ where $M = L_\alpha(\Gamma, \mathbb{R})$

and $L_{\alpha+1}(\Gamma, \mathbb{R}) \models \neg MC$

Fix $x \in OD^{L_{\alpha+1}(\Gamma, \mathbb{R})}$ x is not in any y-mouse.

Then there is (P, Σ) s.t.

Either λ^P is limit and

1. $\forall Q \in B(P, \Sigma): \Sigma_Q \in \Gamma$
2. Σ is Γ-FPR and has BC
3. $L(\Sigma, \mathbb{R}) \models x$ is OD
4. Σ has good properties one needs to compute HOD^M

this is equivalent to full FPR. this is part of the minimality assumption.

or else λ^P is a successor ordinal and

1. $\Sigma_P \in \Gamma$
2. Σ is Γ-FPR and has BC
3. $L(\Sigma, \mathbb{R}) \models x$ is OD
4. As above

By assumption: $\Gamma \lhd (\Sigma^2_1)^{L(\Sigma, \mathbb{R})}$. Some have a good $\Gamma^* > \Gamma$ s.t. $(P, \Sigma) \in \Delta_{\Gamma^*}$.

We have: For any Σ-iterate Q of P: x is in Σ_Q-mouse M_Q

Why: $L(\Sigma, \mathbb{R}) = L(\Sigma_\alpha, \mathbb{R})$ by MC in ~~L~~ $L(\Sigma, \mathbb{R})$.

We can assume:

$A = \{(Q, M_Q) \mid Q \in I(P, \Sigma)\} \in P^*$

Let N_x^* be for Γ^* s.t. $(N_x^*, \delta_x, \Sigma_x)$ Suslin captures (P, Σ), A. Let $N = L[E]^{N_x^* | \delta_x}$. ETS: $x \in N$.

For the above, it is enough to prove:

<u>Lemma</u>: There is $Q \in I(P, \Sigma) \cap N$ s.t. $\Sigma_Q \restriction N \in L[N]$.

<u>Why</u>: Let $N^* = (L[E, \Sigma_Q][Q])^N$. By universality $M_Q \trianglelefteq N^*$. But $x \in M_Q$ so $x \in N^* \subseteq N$.

<u>Proof of the Lemma</u> By induction on $\overbrace{B(P, \Sigma)}^{I(P,\Sigma) \cup} \cap N_x^* | \delta_x$.

<u>Step 1</u> N captures a tail of $(P(0), \Sigma_{P(0)})$

<u>Proof</u> By what we did before: the least strong cardinal of N is a limit of Woodin cardinals.

[Let η be one of them st. η is a successor Woodin and a cutpoint]

Let η be a Woodin in N that is not a limit of points ξ with $\hat{Lp}(N|\xi) \models \xi$ is Woodin.

Then $L[E]^{N|\gamma}$ is an iterate of $P(0)$ above the largest
↑above
ξ as above. We want to construct a direct limit of all iterates of $P(0)$ in "N at κ" (where κ is the least strong) (this is like the construction of $M_{(12)}$ in $L[x]$). Working in N define the following system.

$I^* = \{S \mid$ • $S \in N|\kappa$, $S = L[E]$ is a suitable mouse
(S has 1-Wdn ω- and etc)
• S is ~~full~~ (i.e. as certified by $L[E]$-construction of N)
• for some cutpoint ξ: $S \in H_{\xi+\omega+1, N}$ and H^N_ξ is generic for the extender algebra of $S\}$

Let

I = the set of all (S, A) s.t. $S \in I^*$,
$A \in D(N, \kappa)$ and $D(N,\kappa) \models S$ is A-iterable
↑new derived model

Let $\mathcal{F} = \{H^S_A \mid (S, A) \in I\}$

$R^* = \operatorname{dirlim}(\mathcal{F}, \leq^*)$ under the A-iteration maps
($\leq^*$ as before)

We showed $R^* = (V_\theta^{HOD})^{D(N,\kappa)}$.

! Change the definition of I ! ⟶ NEXT PAGE

I = the set of all (S, τ) s.t.

- τ is a term for a set of reals in $D(N, \kappa)$ s.t. $\Vdash S$ is τ iterable

$\mathcal{F}^{\in N} = \{ H^S_\tau \mid (S, \tau) \in I \}$

R^* = dir lim of $\mathcal{F}$ under the partial iteration maps

Let $R = Lp_\omega(R^*)$. $R \in N$ as N is full

Claim R is an iterate of $P(0)$

Proof Let $g \subseteq Col(\omega, \kappa)$. In $N^*_x[g]$ construct $\langle \mathcal{T}_i \mid i \in \omega \rangle$ s.t.

$P(0) \xrightarrow{\mathcal{T}_0} \xrightarrow{\mathcal{T}_1} \longrightarrow \cdots$ R = dir lim along $\langle \mathcal{T}_i \rangle_i$.

Fix $\langle \eta_i \mid i \in \omega \rangle$ in $N^*_x[g]$ s.t. η_i is a cutpoint of N, $Lp(N|\eta_i) \models \eta_i$ is Woodin and $\eta_i \nearrow \kappa$.

η_i is not a limit of Lp-Woodins. Let

$\xi_i = \sup\{\beta \mid \beta < \eta_i \text{ \& } \beta \text{ is } Lp\text{-Woodin}\}$

$P_i = L[E]^{N|\xi_i}$. We have P_i iterates to P_{i+1} (by universality). Let $\mathcal{T}_i$ be the corresponding tree.

and ~~$\mathcal{T}$~~ $\mathcal{T}_{-1}$ the tree from $P(0)$ to P_0.

Let $\pi_i :$ ~~P~~ $P_i \to P_{i+1}$ and P_ω = dir lim P_i under π_i. Then $P_\omega = R$:

- $\Sigma_{P(0)}$ is guided by some $\vec{A} = \langle A_i \mid i \in \omega \rangle$
- $A_i \in D(N, \kappa)$
- $\pi_i = \bigcup_{k \in \omega} i_{(P_i, \bigoplus_{j \leq k} A_j)(P_{i+1}, \bigoplus_{j \leq k} A_j)}$

Claim $\Sigma_R \restriction N \in L[N]$

Proof Given $\mathcal{T}$ on N: if $\mathcal{T}$ is short then $N \models \mathcal{T}$ is short (by universality) so N finds the branch as the unique branch with the Q-structure.

Now what if $\mathcal{T}$ is maximal; let E be an extender with $\operatorname{cr}(E) = \kappa$. Strength of $E \stackrel{\text{def}}{=} \nu_E >$ $>\operatorname{rank}(\mathcal{T})$. So $\mathcal{T} \in V^{\operatorname{Ult}(N,E)}_{\nu_{i(E)}}$. Let $Q = L_{P_\omega}(M(\mathcal{T}))$.

$$R \longrightarrow J_E(R)$$
$$\xrightarrow{\ \sigma\ }$$

~~Let ... Ult(N,E) be~~

Claim $\Sigma(\mathcal{T}) = b \iff \exists \sigma : Q \to J_E(R)$ s.t. $j_E = \sigma \circ i_b^{\mathcal{T}}$.

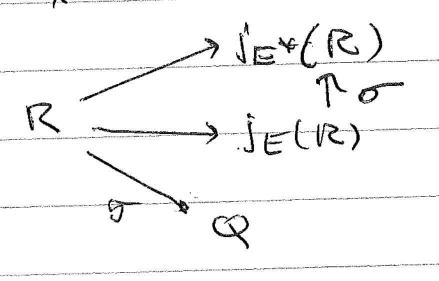
Proof Note: $J_E(R)$ is an iterate of Q.

We have Q is $\oplus$ iterable from the point of view of $\operatorname{Ult}^{i<n}(N,E)$ and ~~...~~

$$\pi_i : i_{(Q, j_E(\oplus_{k<i} A_k))} \restriction H^Q_{j_E(\oplus_{k<i} A_k)} \longrightarrow O \triangleleft J_E(R)$$

Let E^* be the background certificate of E

$$N^*_x \longrightarrow \operatorname{Ult}(N^*_x, E^*)$$



Let $\pi_i^* : H^Q_{j(E^*)(\oplus_{k<i} A_k)} \to J_{E^*}(R)$

Now $\pi^* = \bigcup_{i<\omega} \pi_i^*$ then $\pi^* : Q \longrightarrow j_{E^*}(R)$ is the iteration embedding.

So now $j_{E^*} = \pi^* \circ i_b^{\mathcal{T}}$ (pullback by σ) so $j_E = \pi \circ i_b^{\mathcal{T}}$.

$\boxtimes$ CASE 1

28.2010. 5:00 DISCUSSION: TREVOR WILSON

We prove ① + ② + ③ $\Rightarrow$ ~~AD~~ ~~L~~ $(\exists A \subseteq \mathbb{R}) \models AD^+ + \Theta = \Theta_1$

Here:

① $\mathbb{P}_{max}$ axiom $(*)$, in fact is consequence:
I_{NS} (i.e. NS ideal on ω_1) is ω_2-saturated and quasihomogeneous.

Quasihomogeneity: If $X \subseteq \mathcal{P}(\omega_1)$ and $X \in OD_{\mathbb{R}}$ then if there is a stationary-costationary set in X then every stationary-costationary $A \subseteq \omega_1$ is $\equiv_{NS}$ to some $A' \in X$.

② Stationarily many amenably closed hulls $X \prec H_{\omega_3}$: Let $H \cong X$, H transitive. If $A \subseteq \omega_2^H$ is amenable to H then $A \in H$.

③ Every $f: \omega_2 \to \omega_2$ is bounded by a canonical function on a stationary set. ~~That is~~ (This is implied by J_{NS} weakly presaturated. $J_{NS} = I_{NS}^{\omega_2} \upharpoonright cof(\omega)$).

First: ① + ② $\Rightarrow AD^{K(\mathbb{R})}$

① $\Rightarrow$ $j: V \to M[G]$ with $j(\omega_1^V) = \omega_2^V$ and $V[G] \models {}^{\omega}M \subseteq M$
$\forall x \in \mathbb{R}$ $j \upharpoonright HOD_x$ is amenable to V. (What j says about $OD_{\mathbb{R}}$-sets is independent of generics)

② $\Rightarrow$ mouse / strategy reflection ~~on~~ at ω_2 (extend from ~~the~~ $\mathcal{P}(\omega_1) \to \mathcal{P}(\omega_2)$) (Reference: Steel: PFA $\Rightarrow AD^{L(\mathbb{R})}$.)

② $\Rightarrow$ $H \cong X \prec V_\eta$ (η large) is AC (amenably closed) and $X \cap \omega_2 \in \omega_2$, $|X| = \omega_1$ then $A \in (H_{\omega_3})^H$ then $Lp(A) \in H$.

To get $AD^{K(\mathbb{R})}$. Using j extend mouse reflection at ω_1. Important point: $j(K) \in V$ for local $K = K^{M_n^\#(\mathbb{R})}$.

~~How to get past $K(\mathbb{R})$:~~

Required for $\Theta > \Theta_0 = \Theta^{K(\mathbb{R})}$.

- $\mathrm{cof}(\Theta_0) = \omega$
- strategy with condensation at Θ_0

Let $H = \mathrm{HOD}^{K(\mathbb{R})} | \Theta_0$. $\bar{H}, \bar{\Theta}_0$ = collapses in AC hull $\in HC^M$

<u>Claim</u> $\bar{H}^+ = Lp_\omega(\bar{H})^{j(K(\mathbb{R}))}$ is suitable in $j(K(\mathbb{R}))$.

<u>Proof</u> Let T be a tree for $(\Sigma^2_1)^{j(K(\mathbb{R}))}$. $T \in V$.

$Lp(\overline{K(\mathbb{R})}) = Lp_2(\bar{\mathbb{R}}) = Lp(Lp(\bar{\mathbb{R}})) = Lp_2(\bar{\mathbb{R}})^{j(K(\mathbb{R}))}$

But $Lp_2(\bar{\mathbb{R}})^{j(K(\mathbb{R}))} \models \bar{\Theta}_0$ is the only Woodin HOD

<u>Claim</u> $\mathrm{cf}(\Theta_0) = \omega$ (Under our hypo: $2^{\aleph_0} = \aleph_2$)

<u>Proof</u> $\mathrm{cof}(\Theta_0) \neq \omega_1, \omega_2$. Otherwise $\mathrm{cof}(\bar{\Theta}_0) = \omega_1$ by amenable closure. Let $\gamma = \sup j[\bar{\Theta}_0] < j(\bar{\Theta}_0)$.

Let E ~~...~~ $= E_j | \gamma$.

$$L[T, \bar{H}] \xrightarrow{i} L[i(T), i(\bar{H})] \xrightarrow{\alpha \geq \gamma} L[j(T), j(\bar{H})]$$

$Q = Q(i(\bar{H})) \in \mathrm{Ult}^E$. ↯. ☒

To see $\Theta_0 < \omega_3$: Let $h : \omega_2 \xrightarrow{bi} \mathbb{R}$. Define $f : \omega_2 \to \omega_2$

$f(\alpha) = \mathrm{On} \cap Lp(V_\omega \cup h[\alpha])$

Take $g : \omega_2 \xrightarrow{bi} \beta < \omega_3$ s.t.

$f(\alpha) \leq \mathrm{otp}(g[\alpha])$ for stat many α.

Take $Y \prec V_\gamma$. Let $\pi : X \cong Y$ with X transitive.

Here $|Y| = \omega_1$,

$\alpha = Y \cap \omega_2 \in \omega_2$ s.t. $f(\alpha) \leq \mathrm{otp}(g[\alpha])$

$\pi^{-1}(\Theta_0) \leq \mathrm{On} \cap Lp(V_\omega \cup h[\alpha]) \leq \mathrm{otp}(g''\alpha) = \pi^{-1}(\beta)$. ☒

Take $A \subseteq OD^{K(\mathbb{R})} \cap P(\mathbb{R})$ countable, W-cofinal, assume $A \in X$ (By AC hulls).

In $K(\mathbb{R})$ we have $\mathcal{F}$, $\mathcal{I}$ giving $M_\infty | \delta_\infty = \mathrm{HOD}^{K(\mathbb{R})} | \Theta_0$.

Collapse $\bar{\mathcal{F}}, \bar{\mathcal{I}}, \bar{M}_\infty$ (HC^M).

In $j(K(\mathbb{R}))$ let $((N_i, A_i) \mid i \in \omega)$ be increasing cofinal in $\bar{\mathcal{I}}$. Let $M^*_\infty = \text{quasilimit}_{i<\omega} N_i$ WRT all $A \in OD^{j(K(\mathbb{R}))}$.

Quasilimit: maps are only defined on reasonable hulls

$$\bar{M}_\infty \cong H^{M^*_\infty}_{j[(OD^{K(\mathbb{R})})^{<\omega}]}$$

M^*_∞ is suitable and ~~A-iterable~~ strongly $\bar{A}$-quasi-iterable for $\vec{A} \in \overline{(OD^{K(\mathbb{R})})^{<\omega}}$

$\bar{M}_\infty = M^*_\infty = \bar{H}^+$ $\quad (\bar{H}^+ = Lp_\omega(\mathrm{HOD}^{K(\mathbb{R})} | \Theta_0)^{j(K(\mathbb{R}))})$

One can check: For every quasi-iterate W of $\bar{M}_\infty$:

$W = H^W_{j[A^{<\omega}]}$ (δ-c.c. for extender algebra)

So $j[A]$ guides a strategy Σ for $\bar{M}_\infty$.

<u>Lemma</u> Σ has <u>weak condensation</u>. that is: If $e : \bar{M}_\infty \longrightarrow S$ factors into a Σ-iteration map ~~then~~ $\bar{M}_\infty \to W$ then S is suitable.

(We assume S is countable in M.)

$$\begin{tikzcd} & S \arrow[d] & \\ \bar{M}_\infty \arrow[ur] \arrow[r, "\Sigma"'] \arrow[rr, bend right, "\pi"'] & W \arrow[r] & j(\bar{M}_\infty) \end{tikzcd}$$

Show: $\pi = j \restriction \bar{M}_\infty$ (Exercise)

(This is because the elements of the direct limit system are countable, so they do not move.) So e factors into j.

$$\begin{tikzcd} & L[e(T), S] \arrow[d] \\ L[T, \bar{M}_\infty] \arrow[ur, "e^*"] \arrow[r, "j"'] & L[j(T), j(\bar{M}_\infty)] \end{tikzcd}$$

As before, S is full.

By elementarity of j: $\exists N^*$ suitable in $K(\mathbb{R})$ s.t. A guides a strategy Σ for N^* with weak condensation + + Dodd-Jensen property.

<u>Claim</u> A tail of Σ has branch condensation.

<u>Proof</u> Otherwise iterate

$$\begin{tikzcd} N_\alpha \arrow[r, "\Sigma"] & N_{\alpha+1} \arrow[r, "i_b"', "\sigma"] \arrow[dr, "i_c"', "\sigma, \Sigma"] \arrow[rrr, bend left, "\Sigma"] & M_b^T \arrow[r, "t"] \arrow[d] & W \arrow[d, "\Sigma"] \\ & & M_c^\sigma \arrow[r, "\Sigma"] & N_{\alpha+2} \end{tikzcd}$$

this is an instance of a "bad" sequence and can be obtained for all $\alpha < \omega_1$:

$$\begin{tikzcd} S: N_\alpha \arrow[r, "j_\alpha"] & N_{\alpha+1} \arrow[r, "j_{\alpha+1}"] \arrow[dr, "i_{c_\alpha}"'] \arrow[ur, "i_{b_\alpha}"] & N_{\alpha+2} \\ & \cdot \arrow[ur] & \cdot \arrow[u] \end{tikzcd}$$

N_α is suitable.

Assume: S is in a $\mathbb{P}_{max}$ extension of ~~$HOD_{\mathbb{R}}$~~. $K(\mathbb{R})$.

$(\ast)$ $\Rightarrow$ $\forall X \subseteq \omega_1$ $X \in L(\mathbb{R})$ or X is in a $\mathbb{P}_{max}$ extension of $HOD_{\mathbb{R}}$)

Take $P \triangleleft K(\mathbb{R})$ least s.t. $\Vdash^{P}_{\mathbb{P}_{max}}$ "$\exists$ bad sequence of length ω_1"

Take U a universal $(\Sigma^2_1)^P$ set. Let $\mathcal{B} \subseteq P$ be a sjs, $U \in \mathcal{B}$, T a tree for $\oplus\mathcal{B}$ given by $\mathcal{B}$.
In $p^{\mathbb{P}_{max}}$ let $P \cong X \prec H(\omega_2)$, X is U-iterable, ~~phase~~ P has a bad sequence of length ω_1^P. Let x code P. In $L[T, x]$ iterate p. Get a bad sequence $(N_\alpha, b_\alpha, c_\alpha, j_{\alpha\beta}\ (\alpha < \beta < \omega_1^{L[T,x]})$. Let $\gamma = \omega_1^{L[T,x]}$ and $N_\gamma = \operatorname{dir\,lim}_{j_{\alpha,\beta}} N_\alpha \quad (\alpha \leq \beta < \gamma)$ (suitable.)

$L[T, x]$ knows sequence of $\tau^{N_\gamma}_{B_i}$'s. $\exists \alpha_0 < \gamma$ s.t. $\{\tau^{N_\gamma}_{B_i} \mid i \in \omega\} \subseteq \mathrm{rng}(j_{\alpha\beta})$. By term condensation: $\forall \alpha > \alpha_0$ $j_{\alpha,\alpha+1}$ maps terms correctly (since it comes from a collapse) $\lightning$.

Show $AD^{L^\Sigma(\mathbb{R})}$ $\quad$ ($L^\Sigma(\mathbb{R}) = L(\Sigma, \mathbb{R})$)

By quasihomogeneity get $j(T') \in V$, T' a tree for $\oplus \mathcal{B}$. $\mathcal{B}$ a sjs at ϑ_0 guiding Σ. In $V^{Col(\omega,\omega_1)}$ use T' to identify good branches (since $j(T') \in V$). ~~The~~ Continue as for $AD^{L(\mathbb{R})}$ in Steel-Zoble: Determinancy from strong reflection.

3.8.2010 9:30 GRIGOR SARGSYAN

RECALL (P,Σ) captured by N_x^* where $N = L[E]^{V_x^*|\delta_x}$.
$\exists Q \in I(P,\Sigma) \cap N$ s.t. $\Sigma_Q \restriction N \in L[N]$.

By induction on $I(P,\Sigma) \cup B(P,\Sigma)$ we showed capturing for $(P(0), \Sigma_{P(0)})$, $\exists Q \in N \cap I(P(0), \Sigma_{P(0)})$ s.t. $\Sigma_Q \in L[N]$.

This is the general case

Suppose $Q \in I(P,\Sigma) \cup B(P,\Sigma)$ & λ^Q is a successor &
$\exists S \in I(Q^-, \Sigma_{Q^-}) \cap N$ s.t. $\Sigma_S \restriction N \in L[N]$.

Want: Find $R \in I(Q, N_Q) \cap N$ s.t. $\Sigma_R \restriction N \in L[N]$. Let
$N^* = (L[E, \Sigma_S \restriction N])^N$.

As before, the least strong of N^* is a limit of Woodins.
Let λ be this strong. Repeat the construction from above. Let $R^* = \mathrm{HOD}^{D(N^*,\lambda)}|\theta$. ~~Let $R^* = \theta_0^{\mathrm{HOD}}$~~
$R = L_{P_\omega}^{\Sigma(R^*)^-}(R^*)$. $R \in N^*$ because of universality.
Then use extenders with cr. pt. λ to get $\Sigma_R \restriction N^* \in L[N^*]$.
So: Given $\mathcal{T}$ on R let E be on N^* an extender with $\mathrm{cr}(E) = \lambda$, $\mathrm{lh}(E) > \mathrm{rank}(\mathcal{T})$. Let b be the branch of $\mathcal{T} \iff \exists \sigma$ s.t. $j_E \restriction R = \sigma \circ i_b^{\mathcal{T}}$. $R \xrightarrow[\mathcal{T},b]{} \cdot \xrightarrow{\sigma} \cdot$, $R \xrightarrow{j_E(R)}$.

Need $\Sigma_R \restriction N \in L[N]$

Let $\mathcal{T}$ be a tree in N. Let $\alpha = \mathrm{rank}(\mathcal{T})$. Let
$N^{**} = (L[E, \Sigma_R])^{N^*}$. Use extender with cr pt $> \alpha$
Then let λ^* be the least strong of N^{**}. We have R^{**}, the version of R in N^{**}. Also, $\mathcal{T}$ is generic over N^{**} at the least Woodin.
Let π be the iteration map from R to R^{**}, $\pi \in N^*$. Then, by the same proof as before,

look for b s.t. $(\exists\sigma)(\pi=\sigma\circ i^{\mathcal{T}}_b)$
$$\begin{tikzcd} & R^{**} \\ R \arrow[ur, "\pi"] \arrow[dr, "q_{,b}"'] & \\ & Q \arrow[uu, "e"'] \end{tikzcd}$$
This finishes the general successor case.

<u>Limit case</u> (General) Suppose $Q\in(I(P,\Sigma)\cup B(P,\Sigma))\cap N_x^*|\delta_x$ and $\forall R\in B(Q,\Sigma_Q)\cap N_x^*|\delta_x$ $\exists S\in I(R,\Sigma_R)\cap N$ s.t. $\Sigma_S\restriction N\notin L[N]$.

<u>NTS</u> $\exists R\in I(Q,\Sigma_Q)\cap N$ s.t. $\Sigma_R\restriction N\in L[N]$

Assume Q is countable (otherwise we have to do the same we did in the general case above.)

Let κ be the least strong of N that reflects the set of strong cardinals.

<u>Lemma</u> If $S\in B(Q,\Sigma_Q)\cap N_x^*|\kappa$ then S is "captured" below κ. This means:

$\exists R\in I(S,\Sigma_S)\cap N|\kappa$ s.t. $\Sigma_R\restriction N\in L[N]$

<u>Proof</u> Suppose not. Let $R\in I(S,\Sigma_S)\cap N$. $\Sigma_R\restriction N\in L[N]$. Let $\kappa_1>\operatorname{rank}(R)$ a strong cardinal of N. Let E an extender on N s.t. $\operatorname{cr}(E)=\kappa$, $\operatorname{lh}(E)>\kappa$, E reflects the set of strongs. Then κ_1 is strong in $\operatorname{Ult}(N,E)$. Moreover: because $V^N_{\kappa_1+1}\subseteq\operatorname{Ult}(N,E)$, $\Sigma_R\restriction\in V^N_{\kappa_1}\in\operatorname{Ult}(N,E)$ (by capturing). Now let Λ be the strategy of R in $\operatorname{Ult}(N,E)$ we get by stretching $\Sigma_R\restriction V^N_{\kappa_1}$ using the extenders with cr pts κ in $\operatorname{Ult}(N,E)$.
NTS: If E_1,E_2 on $\operatorname{Ult}(N,E)$ and $\operatorname{cr}(E_1)=\kappa=\operatorname{cr}(E_2)$ then $j_{E_1}(\Sigma_R\restriction V^N_{\kappa_1})\restriction\lambda_1=j_{E_2}(\Sigma_R\restriction V^N_{\kappa_1})\restriction\lambda_1$ where $\lambda_1=\operatorname{lh}(E_1)\le\operatorname{lh}(E_2)$. To see hull condensation (Possibly branch condensation?)

<u>Claim</u> $\Lambda = \Sigma_R \restriction Ult(N,E)$

<u>Proof</u> Let E^* be the background certificate of E. Let

$$\begin{tikzcd}
N^*_x \arrow[r] & Ult(N^*_x, E^*) \\
N \arrow[u] \arrow[r] \arrow[dr] & j_{E^*}(N) \\
 & Ult(N,E) \arrow[u, "\sigma"']
\end{tikzcd}$$

$\sigma(\Lambda)$ = the strategy in $j_{E^*}(N)$ of $\sigma(R)=R$ given by stretching $\Sigma_R \restriction V^N_{\kappa_1}$ using extenders with $cr = \kappa$.

Note: $V^N_{\kappa_1} = V^{j_{E^*}(N)}_{\kappa_1}$, $\sigma \restriction V^N_{\kappa_1} = id$

ETS: $\sigma(\Lambda) = \Sigma_R \restriction j_{E^*}(N)$ and then we hull cond.
However, the equality is clear since the extenders on $j_{E^*}(N)$ do move Σ to itself, as they are actual extenders. $\boxtimes$

Now working in N, form the following limit. Let $\mathcal{F}$ = the set of all (S,Λ) s.t.

- (S,Λ) is a hod pair
- Λ is FPR + BC
- $S \in V^N_\kappa$

Here "FPR" means that it is certified by background constructions. Let

$M_\infty = \text{dirlim}\ (\mathcal{F}, \leq^*)$ under the iteration maps. ← the natural relation

$M_\infty \in H^N_{\kappa^+}$. Let

$\Lambda^* = \bigoplus_{\alpha < \lambda^{M_\infty}} \Lambda_{M_\infty(\alpha)}$ where

$\Lambda_{M_\infty(\alpha)}$ = the common IS coming from some (S,Λ) s.t. S iterates to $M_\infty(\alpha)$ via Λ.

let $R^* = Lp_{\omega}^{\Lambda^*}(M_\infty)$

<u>Claim</u> For some $\alpha \le \lambda^{R^*}$, $R^*(\alpha)$ is a Σ_Q-iterate of Q.

<u>Proof</u> M = dir lim of all Σ_Q-iterates of Q that are in $V_\kappa^{N_x^*}$.

Then $M \trianglelefteq_{hod} R^*$. Why: let

$\mathcal{F}^* = \{ S \mid S \in B(Q, \Sigma_Q) \cap V_\kappa^{N_x^*} \}$

let M^* = dir lim of $\mathcal{F}^*$. Then $M | \delta^M = M^*$.

But $M^* \triangleleft R^*$ because everything in $B(Q, \Sigma_Q) \cap V_\kappa^{N_x^*}$ is captured below κ in N.

Let $\alpha \le \lambda^{R^*}$ be s.t. $R^*(\alpha)$ an iterate of Q.

let $R = R^*(\alpha)$. WTS: $\Sigma_R \restriction N \in L[N]$

<u>Case 1</u> $cf(\lambda^R)$ is not measurable.

<u>Pf</u> Easy: $\Sigma_R = \bigoplus_{\alpha < \lambda^R} \Sigma_{R(\alpha)}$ and $\Sigma_{R(\alpha)} \restriction N \in L[N]$ $\forall \alpha < \lambda^R$

<u>Case 2</u> $cf(\lambda^R)$ is measurable.

<u>Notation</u> Given $S \in B(Q, \Sigma_Q) \cap N_x^* | \delta_x$ we let $(R^S, \Phi^S) \in L[N]$ be the captured tail of S.

(i.e. $R^S \in I(S, \Sigma_S)$, $\Phi^S = \Sigma_{R^S} \restriction N$)

$$\begin{tikzcd} & & j_E(R) \\ ? & R \arrow[ur] \arrow[dr, "\mathcal{T}_b"'] & \\ & & M_b^{\vec{\mathcal{T}}} \arrow[uu, "\sigma"'] \end{tikzcd}$$

$\mathcal{T}$ $\quad -\kappa^+ \quad -\kappa$

The challenge is to guess the actual embedding $M_b^{\vec{\mathcal{T}}} \longrightarrow j_E(R)$

let $\mathcal{E}$ = the set of all extenders on the E-sequence with cut pt κ that reflect the set of all strategies.

(†) Working in N define Λ a strategy for R as follows, given $\vec{\mathcal{T}}$ on R with essential components

$\vec{\mathcal{T}} = \langle M_\alpha, M_\alpha^*, \vec{\mathcal{T}}_\alpha, i_{\alpha\beta} \mid \alpha,\beta \le \eta \rangle$ is via Λ iff

$\exists (R_\alpha, \Lambda_\alpha)$, $\cancel{\langle \pi_\alpha \mid \alpha \le \eta \rangle}$ s.t.

① $(R_\alpha, \Lambda_\alpha)$ is a hod pair

② if $E \in \mathcal{E}$ s.t. there is a strong cardinal between $(\mathrm{rank}(R_\alpha), lh(E))$

$\Lambda_\alpha \restriction Ult(N,E) \in Ult(N,E)$

③ if λ is a strong cardinal s.t. $\lambda > \mathrm{rank}(\mathcal{T}), \mathrm{rank}(R_\alpha)$

$\forall E \in \mathcal{E}$ with $lh(E) > \lambda$ $\exists (\pi^E_\alpha \mid \alpha \le \eta)$ s.t.

- $\pi_0^E = j_E \restriction R = M_0$
- $\pi_\alpha^E : M_\alpha \to j_E(R)$
- $\pi_\alpha^E = \pi_\beta^E \circ i_{\alpha\beta}$

$$\begin{tikzcd} & j_E(R) & \\ M_\alpha \arrow[ur, "\pi_\alpha"] \arrow[rr, "i_{\alpha\beta}"] & & M_\beta \arrow[ul, "\pi_\beta"'] \end{tikzcd}$$

- R_α iterates via Λ_α to $\pi_\alpha(M_\alpha^*)$
- $\mathcal{T}_\alpha$ is according to $\Lambda^{\pi_\alpha^E}_{\alpha, \pi_\alpha(M_\alpha^*)}$ ($\vec{\mathcal{T}}_\alpha$ is a tree on M_α^*)

④ Moreover: if $\vec{\mathcal{U}}$ is a stack on M_α^*, $\alpha < \eta$, $E \in \mathcal{E}$ as in ③ and $\langle \pi_\xi^E \mid \xi \le \eta \rangle$ are as in ③ also λ a strong cardinal as in ③ then

$b = \Lambda^{\pi_\alpha^E}_{\alpha, \pi_\alpha^E(M_\alpha^*)}(\vec{\mathcal{U}}) \leftrightarrow b$ is the unique branch s.t. $\exists \sigma : M_b^{\vec{\mathcal{U}}} \to j_E(R)$ s.t. $\pi_\alpha = \sigma \circ i_b^{\vec{\mathcal{U}}}$.

<u>Claim</u> $\Lambda = \Sigma_R \restriction N$ → also $\vec{\mathcal{T}}$ up to the last branch is according to Σ.

<u>Proof</u> Suppose $\vec{\mathcal{T}}$ is according to Λ, $\mathcal{T} = \langle M_\alpha, M_\alpha^*, \vec{\mathcal{T}}_\alpha, i_{\alpha\beta} \rangle_{\alpha\beta}$

WTS: $\mathcal{T}$ is according to Σ.

Fix $\langle \langle R_\alpha, \Lambda_\alpha \rangle \mid \alpha < \eta \rangle$ for $\vec{\mathcal{T}}$. [Let λ be a strong cardinal $> \mathrm{rank}\{R^{M_\alpha^*} \mid \alpha \le \eta\}$. Let $E \in \mathcal{E}$ with $lh(E) > \lambda$. We have: $\Phi^{M_\alpha^*} = \Sigma_{M_\alpha^*} \restriction N$, $\Phi^{M_\alpha^*} \restriction Ult(N,E)$]

DELETE

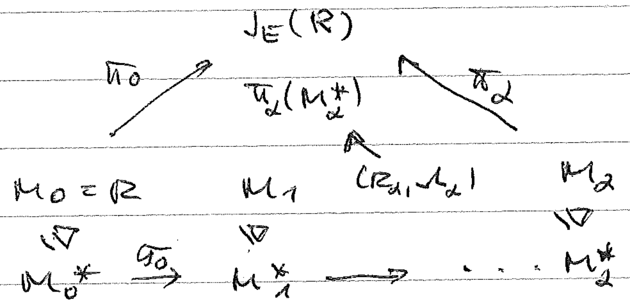
[Let $\langle \pi_\alpha \mid \alpha \leq \eta \rangle = \langle \pi_\alpha^E \mid \alpha \leq \eta \rangle$ and let E^* be the background certificate of E. Let $\tau : \mathrm{Ult}(N, E) \to \mathrm{Ult}(N, E^*)$.

DELETE

Want to define a strategy for M_α^* on N. Let $\lambda > \mathrm{rank}\{R^{M_\xi^*} \mid \xi < \eta\}$, λ strong. Given $\vec{u}$ on M_α^* let Ψ_α be a strategy for M_α^* be defined as:

$\Psi_\alpha(\vec{u}) = b \iff \forall E \in \mathcal{E}$ with $lh(E) > \lambda$

$$\cancel{\Lambda_\alpha} \quad (\Lambda_\alpha)^{\pi_\alpha^E}_{\pi_\alpha^E(M_\alpha^*)}(\vec{u}) = b \quad , \alpha \leq \eta$$



<u>Claim</u> $\Psi_\alpha = \Sigma_{M_\alpha^*} \restriction N$

<u>Proof</u> Induction on α.

<u>$\alpha = 0$</u> ~~Let~~ Given $\vec{u}$ on M_0^* ~~we~~ WTS

$\Psi_0(\vec{u}) = \Sigma_{M_0^*}(\vec{u})$ (But $\Sigma_{M_0^*} \restriction N \in L[N]$)

Let $E \in \mathcal{E}$: $\Sigma_{M_0^*} \restriction \mathrm{Ult}(N, E) \in \mathrm{Ult}(N, E)$

so by the above picture for $\alpha = 0$:

(R_0, Λ_0) and $(M_0^*, \Sigma_{M_0^*})$ compare to a common place before going to $\pi_0(M_0^*)$.

So $\Sigma^{\pi_0}_{\pi_0(M^*)} = \Sigma_{M_0^*}$ by hull condensation.

then

$$(\Lambda_0)_{\pi_0(M_0^*)} = \Sigma_{\pi_0(M_0^*)}$$

<u>Case $\alpha+1$</u> Suppose $\Psi_\alpha = \Sigma_{M^*_\alpha}$.
WTS: $\Psi_{\alpha+1} = \Sigma_{M^*_{\alpha+1}}$.

<u>The problem</u> If $\pi^E_\alpha : M_\alpha \to j_E(R)$ is an iteration embedding via Ψ_α then the proof for $\alpha=0$ works here.

$\boxed{\text{Things got messed up now. Go back to (†) and start again}}$

Define a strategy Λ for R in N. Given
$\vec{\mathcal{T}} = (\vec{\mathcal{T}_\alpha}, M_\alpha, M^*_\alpha, i_{\alpha\beta})_{\alpha,\beta}$
a tree according to Λ iff $\exists$ $(\langle (R_\alpha, \Lambda_\alpha) \mid \alpha \leq \eta \rangle, \langle \Psi_\alpha \mid \alpha \leq \eta \rangle$ s.t.

1. $(R_\alpha, \Lambda_\alpha)$ are hod pairs
2. $(M^*_\alpha, \Psi_\alpha)$ is a hod pair s.t. $\vec{\mathcal{T}_\alpha}$ is according to Ψ_α
3. Ψ_β extends Ψ_α for $\beta > \alpha$
4. For every $E \in \mathcal{E}$ s.t. $\exists$ strong cardinal $\lambda \in (\mathrm{rank}\{R_\alpha (\alpha \leq \eta\}), \mathrm{lh}(E)]$ there is a sequence $\langle \pi^E_\alpha \mid \alpha \leq \eta \rangle$ s.t.
   - (a) $\pi^E_\alpha : M^{\vec{\mathcal{T}}}_\alpha \to j_E(R)$
   - (b) $\pi^E_\alpha = \pi^E_\beta \circ i_{\alpha\beta}$
   - (c) $\pi_\alpha \restriction M^*_\alpha$ is the iteration map by Ψ_α
   - (d) R_α iterates to $\pi_\alpha(M^*_\alpha)$ via Λ_α
   - (e) $(\Lambda_\alpha)^{\pi^E_\alpha}_{\pi_\alpha(M^*_\alpha)} = \Psi_\alpha$.
5. For every $\overbrace{\alpha \leq \eta}^{\vec{u} \text{ on } M^*_\alpha}$ and E as in (4) s.t. $\mathrm{lh}(E) > \mathrm{rank}(\vec{u})$
   $\Psi_\alpha(\vec{u}) = b \leftrightarrow b$ is unique s.t. $\exists \sigma$ $\pi_\alpha \restriction M^*_\alpha = \sigma \circ i^{\vec{\mathcal{T}}}_b$

<u>Claim</u> $\Lambda = \Sigma$

<u>Proof</u> Let $\vec{\mathcal{T}}$ be according to ~~the~~ Λ and $\vec{\mathcal{T}}$ without its last branch be according to Σ

Let $\langle (R_\alpha, \Lambda_\alpha) \mid \alpha < \eta \rangle$, $\langle \Psi_\alpha \mid \alpha \leq \eta \rangle$ be as in the def of $\vec{\mathcal{T}}$

<u>Claim</u> $\Psi_\alpha = \Sigma_{M^*_\alpha} \upharpoonright N$

<u>Proof</u> By induction on α.

<u>$\alpha = 0$</u> Trivial

<u>$\alpha+1$</u> We have $\Sigma_{M^*_\alpha} \upharpoonright N = \Psi_\alpha$. Let $\vec{\mathcal{U}}$ be a tree of limit length according to both $\Psi_{\alpha+1}$ and $\Sigma_{M^*_{\alpha+1}}$. WTS $\Psi_{\alpha+1}(\vec{\mathcal{U}}) = \Sigma_{M^*_{\alpha+1}}(\vec{\mathcal{U}})$. Let $E \in \mathcal{E}$ be s.t. there is a strong cardinal in the interval $(\mathrm{rank}(\{R^{M^*_\xi}_\xi \mid \xi \leq \eta\}), lh(E))$.

DELETE [Then $\forall \xi \leq \alpha$ $\pi^E_\xi : M_\xi \to j_E(R)$ is the embedding]

<u>Claim</u> $\pi^E_{\alpha+1} \upharpoonright M^*_{\alpha+1} : M^*_{\alpha+1} \to \pi_{\alpha+1}(M^*_{\alpha+1})$ is the iteration embedding according to $\Sigma_{M^*_{\alpha+1}}$

<u>Proof</u> Let $x \in M_{\alpha+1}$. Then $x = i_{\alpha,\alpha+1}(f)(a)$ where a is a generator of $\vec{\mathcal{T}}$.

So $\pi_{\alpha+1}(x) = \pi_{\alpha+1}(i_{\alpha,\alpha+1}(f))(\pi_{\alpha+1}(a)) = \pi_\alpha(f)\,\pi_{\alpha+1}(a)$

But $\pi_{\alpha+1}(a)$ comes from the iteration embedding according to $(\Psi_\alpha)_{i_{\alpha,\alpha+1}(M^*_\alpha)}$. By the IH: $(\Psi_\alpha)_{i_{\alpha,\alpha+1}(M^*_\alpha)} = \Sigma_{i_{\alpha,\alpha+1}(M^*_\alpha)}$. $\boxtimes$

Now we have $R^{M^*_{\alpha+1}} \in \mathrm{Ult}(N, E)$,

$$\underbrace{\Psi^{M^*_{\alpha+1}} \upharpoonright \mathrm{Ult}(N,E)}_{= \Sigma_{M^*_{\alpha+1}} \upharpoonright \mathrm{Ult}(N,E)} \in \mathrm{Ult}(N,E)$$

$$\begin{tikzcd}
 & j_E(R) & & \\
R \arrow[ur] \arrow[rr] & & M_{\alpha+1} \arrow[ul, "{(R_{\alpha+1},\Lambda_{\alpha+1})}"'] \trianglerighteq M^*_{\alpha+1} \arrow[ul, bend right, "{(R^{M^*_{\alpha+1}},\Psi^{M^*_{\alpha+1}})}"'] &
\end{tikzcd}$$

$R^{M^*_{\alpha+1}}$ it via $\Psi^{M^*_{\alpha+1}}$ to $\pi_{\alpha+1}(M^*_{\alpha+1})$ and
$(\Lambda_{\alpha+1})_{\pi_{\alpha+1}(M^*_{\alpha+1})} = (\Psi^{M^*_{\alpha+1}})_{\pi_{\alpha+1}(M^*_{\alpha+1})} = \Sigma_{\pi_{\alpha+1}(M^*_{\alpha+1})}$

Since $\pi_{\alpha+1}\restriction M^*_{\alpha+1}$ is the iteration embedding according to $\Sigma_{M^*_{\alpha+1}}$ we get $\Psi_{\alpha+1} = (\Sigma_{\pi_{\alpha+1}(M^*_{\alpha+1})})^{\pi_{\alpha+1}} = \Sigma_{M^*_{\alpha+1}}$

$\boxtimes$

<u>α limit</u> Easy (Exercise)
this shows that $\vec{T}$ is via Σ because $\vec{T}_\eta$ is according to $\Psi_\eta = \Sigma_{M^*_\eta}$. $\boxtimes$

To finish, need to show that given $\vec{T}$ according to Λ s.t. $\vec{T} = \langle M_\alpha, M^*_\alpha, T_\alpha, i_{\alpha\beta} \mid \alpha<\beta\le\eta\rangle$ with M^*_η, T_η undefined then we can [find M^*_η] continue $\vec{T}$. Let $M^*_\eta \trianglelefteq M_\eta$ be the least init segment for which we haven't defined Λ. Let $\{\langle R_\alpha,\Lambda_\alpha\rangle \mid \alpha<\eta\}$, $\langle \Psi_\alpha \mid \alpha<\eta\rangle$ witness that $\vec{T}$ is according to Λ. Define a strategy for M^*_η as follows. Given $\vec{U}$ on M^*_η let $E\in\mathcal{E}$ be an extender with $lh(E) > rank(\vec{U})$ and $\exists$ strong cardinal $\lambda \le lh(E)$, $\lambda > rank(R^{M^*_\eta})$. Let b be the branch of $\vec{U}$ iff $\exists\sigma: M^*_\eta \to \pi_\eta(M^*_\eta)$ s.t. $\pi_\eta\restriction M^*_\eta = \sigma\circ i^{\vec{U}}_b$. There is always such a b and is according to $\Sigma_{M^*_\eta}$. Why:

$$\begin{tikzcd}
 & j_E(R) & & \\
 & & \pi_\eta(M^*_\eta) & \\
 & & & (R^{M^*_\eta}, \Psi^{M^*_\eta}) \arrow[ul] \\
R \arrow[uur] \arrow[rr] & & M_\eta \arrow[uu, bend right] & \\
 & & \unrhd & \\
 & & M^*_\eta \arrow[uuur, dashed] \arrow[r, "\vec{u}"'] & M_b^{\vec{u}}
\end{tikzcd}$$

$\pi_\eta \restriction M^*_\eta$ is acc to $\Sigma_{M^*_\eta}$.

Let

$$b = (\Psi^{M^*_\eta})^{\pi_\eta(\vec{u})}_{\pi^E_\eta(M^*_\eta)}$$

by hull condensation

$b = \Sigma_{M^*_\eta}(\vec{u})$ and $\pi_\eta(M^*_\eta)$ is an iteration of $M_b^{\vec{u}}$.
So let $\sigma : M_b^{\vec{u}} \longrightarrow \pi^E_\eta(M^*_\eta)$ be the iteration map
then $\pi_\eta \restriction M^*_\eta = \sigma \circ i_b^{\vec{u}}$ (By commutativity.) $\boxtimes$

Derived models of Mice

Direction 1

~~*~~ M a mouse with λ a limit of Woodins. Study $D(M,\lambda)$

Direction 2.

Given a model V of AD^+ find (Prikry force) a premouse M s.t. $V = D(M,\lambda)$.

Connect the theory of M with that of $D(M,\lambda)$.

Mouse operators

Examples

(1) a is countable transitive, most often self-wellordered (1)

$M^{\#}_{adr}(a)$ = the minimal active mouse M over a s.t. $M \models AD_{\mathbb{R}}$ hypo

$AD_{\mathbb{R}}$-hypo = $\exists \lambda$ λ limit of Woodins and $<\lambda$ strongs.

Remark $M \models AD_{\mathbb{R}}$-hypo $\Rightarrow$ $D(M,\lambda) \models AD_{\mathbb{R}}$

We already proved this without requiring M a mouse

We will show: If $AD_{\mathbb{R}}$ holds then there is a "Prikry generic" G s.t. in $V[G]$ there is a class premouse $M \models AD_{\mathbb{R}}$-hypo

$a \mapsto M^{\#}_{adr}(a)$ is an operator of interest.

(1) Means a has a well-ordering that is ~~rudimen~~ rud$(a \cup \{a\})$

(2) $M^{\#}_{dc}(a)$ = minimal mouse satisfying
$\exists \lambda$ λ limit of Woodins and
$\mathrm{otp}(\{\xi < \lambda \mid \xi \text{ is } <\lambda\text{-strong}\}) = \lambda$
$D(M_{dc}(a) \models AD_{\mathbb{R}} + DC \implies cf(\Theta) > \omega$

(3) $M^{\#}_{inlim(a)}$: $\exists \lambda$ λ inaccessible limit of Woodins
and $<\lambda$-strongs
$D(M, \lambda) \models \Theta = \Theta_{\omega_1}$ still

(4) $M^{\#}_{wlim}(a)$: minimal mouse with a Woodin limit
of Woodins
Will show: $D(M_{wlim}(a), \lambda) \models AD_{\mathbb{R}} + \Theta = \Theta_\Theta$
So $\Theta_{\omega_1} < \Theta$.

<u>Open question</u> : Does $D(M_{wlim}(a), \lambda) \models \Theta$ regular?
Known some $L(\Gamma, \mathbb{R})$ satisfies $AD_{\mathbb{R}} + \Theta$ regular
where $L(\Gamma, \mathbb{R}) \subseteq D(M, \lambda)$. (Sargsyan.)

<u>Reference</u> Steel: Derived models associated to mice.

Operators have the form $a \mapsto (M(a), \lambda(a))$
~~Most~~ Important property: are "tractable".
If $M = M(a)$ g is $\mathrm{col}(\omega, \nu)$-generic ($\nu < \lambda$)
and $b \in HC^{M[g]}$ then $M[g]$ can rebuild $M(b)$ using
its sequence of certificates. So "$M(a)$ reconstructs
itself below λ".

Today: Start with one of these operators $a \mapsto (M(a), \lambda(a))$ Investigate $\mathcal{D}(M(a), \lambda(a))$. We are assuming that each $M(a)$ has a Hom_∞ iteration strategy①. Since $M(a)$ projects to $\underline{a}$, such a strategy is unique. Assume there are arbitrary large Woodin cardinals in V.

For such $M = M(a)$ and I an $\mathbb{R}$-genericity iteration of M: $M = M_0^I \xrightarrow{\mathcal{T}_0} M_1^I \xrightarrow{\mathcal{T}_1} M_2^I \cdots \rightarrow M_\infty^I$ where

- $\mathcal{T}_0 {}^\frown \mathcal{T}_1 {}^\frown \cdots \mathcal{T}_n {}^\frown \cdots$ is normal by $\Sigma = \Sigma_M$ (unique)
- $\mathbb{R}^V = \mathbb{R}^*_G$ for some G on $\mathrm{col}(\omega, < \lambda_\infty^I)$
- I itself is V-generic over the natural poset of $\langle \mathcal{T}_0 \ldots \mathcal{T}_k \rangle$'s.

Then we have, letting $\mathrm{Hom}^*_I = \mathrm{Hom}^*_G$ for any and all G s.t. $\mathbb{R}^*_G = \mathbb{R}^V$.

(a) $\mathrm{Hom}^*_I \subseteq V$ (as $\Sigma_M \in V$)

(b) $\mathrm{Hom}^*_I = \mathrm{Hom}^*_J$ even if J is for some $M(b)$, $b \neq a$.

~~(a)~~ See the paper.

<u>Lemma</u> For any of our $a \mapsto M(a)$: $\mathcal{D}(M(a), \lambda(a)) \models MC$, i.e. $x \in OD(y)$ iff x is an element of a y-mouse.

$M(a)$ satisfies: $M(a) \restriction \omega_1^{M(a)}$ is ω_1-iterable.

$M(a) \models$ UBH for the trees based on its extender sequence; this uses the ~~f~~ iterability of $M(a)$ in V.

$M(a)$ re-builds $M(a) | \gamma$ ($\gamma < \omega_1^{M(a)}$) with background extenders arbitrarily high (this generalizes for $M(a)[g]$ for small g.)

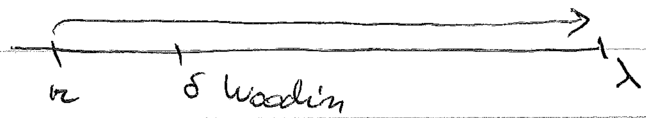
① Denote it by $\Sigma_{M(a)}$

So $M(a)$ can iterate $M(a)|\gamma$ using UBH and CBH for trees of size $<\kappa$.
Gives Σ for $M(a)|\gamma$ which is $\mathrm{Hom}_{<\kappa}$.

<u>Corollary</u> If $a \mapsto (M(a), \lambda(a))$ is a tractable operator s.t. $M(a) \models \lambda(a)$ is a limit of cutpoints then $D(M(a), \lambda(a)) \models \Theta_0 = \Theta$.

<u>Proof</u> If not: have $f \in D(M^I_\infty, \lambda^I_\infty) = L(\mathrm{Hom}^*_G, \mathbb{R}^*_G)$ s.t. $f : \mathbb{R} \to \mathbb{R}$, $f(x) \notin OD(x)^{D(M,\lambda)}$ all x and $f \in \mathrm{Hom}^*$.
Then have: η cutpoint of M, $(T, U) \in M[g]$ with $p[T] = \{(x, m, n) \mid f(x)(m) = n\}$. Let $z \in \mathbb{R}$ be a code of $\langle M|\eta, g\rangle$. Then $f(z) \in M(z)$:
$f(z) \in M[g]$ because of T.
$M[g]|\omega_1^{M[g]} = M(z)|\omega_1^{M[z]}$ so $f(z)$ is OD in $D(M(a), \lambda(a))$.
$M(a)|\omega_1^{M(a)}$ can be computed in D for any ~~tractable~~ countable transitive a. ☒

<u>Next goal</u>: Identify the sets with Wadge rank Θ_α for $\Theta_\alpha < \Theta$ in $D(M^I_\infty, \lambda^I_\infty)$.
For one of our M-operators:
Set "at" Θ_0: its operator $a \mapsto M^\#(a)$ in D, otherwise $M^\#(a) \in M(a)$. Why it is in D? Use a κ that is strong up to λ.



$M^\#(M|\kappa)$ make element of $M[g]$ g on $Col(\omega, \eta)$
η next Woodin $> \kappa$. Now stretch $M^\#(M_\infty|\kappa)$ using extenders with critical pts κ. (Do rebuilding of the stretched model inside M.) This gives iteration strategy below the bottom Woodin.

3.8.2010 5:00 DISCUSSION JOHN STEEL

Given Σ an IS for P and a transitive self-well-ordered $\underline{a} \in P$
Σ satisfies hull condensation and condenses well =
= support closed subtrees of $\mathcal{T}$ by Σ are also by Σ.

$F^\Sigma(M)$ is defined for M a model over $\underline{a}$ and is itself
a model over $\underline{a}$ $(M, \in, B, E, S) = \mathcal{M}$
where $S^{\mathcal{M}}_\xi$ = the ξ-th level of $\mathcal{M}$.

F^Σ is a <u>model operator</u>

Given a model M over $\underline{a}$ we get $F^\Sigma(M)$ as follows:
Let $\mathcal{T}$ be the least tree on P in the canonical well-order
of M ($\underline{a}$ is self-well-ordered) such that:
→ for λ limit $< lh(\mathcal{T})$ $\exists \xi$ $[0,\lambda]_T = \dot{B}^{S^M_\xi}$
and no $\dot{B}^{S^M_\xi}$ is itself a cofinal branch of $\mathcal{T}$.
→ Let $\lambda = lh(\mathcal{T})$

<u>Case 1</u> Let γ be least s.t. $\rho_\omega(\mathcal{J}_\gamma(M)) < \rho_\omega(M)$ ~~or~~
$$F^\Sigma(M) = (\mathcal{J}^M_\gamma, \in, \phi, \phi, S^M{}^\frown\langle M\rangle)$$
<u>Case 2</u> $\gamma = \lambda$
$$F^\Sigma(M) = (\mathcal{J}^M_\gamma, \in, \{o(M)+\xi \mid \xi \in \Sigma(\mathcal{T})\}, \phi, S^M{}^\frown\langle M\rangle)$$
this structure is amenable. Check conditions on F^Σ

Here we assume $M \models cof(\lambda)$ not measurable.
With this, one can do $K^{c,F}$ constructions and K^F-dichotomy.

① Things that take more: generic interpretability: Take an F^Σ-mouse M. Does M have a term for $\Sigma \cap M[g]$.

② <u>Another related topic</u>: Translations of Σ-mice M over some a to Σ-mice over $M[\langle a, g\rangle]$ where g is $col(\omega, a)$-generic.

③ a not self-well-ordered, e.g. Σ-mice over $\mathbb{R}$.
One approach: "tell the mouse it is Σ_q-mouse of HOD".
Or if (P, Σ) is a hod mouse
<u>Sample problem</u> Let (P, Σ) be a hod pair ~~such that~~ with $M_\infty(P, \Sigma) = HOD \upharpoonright \Theta_\alpha$. Show $K^\Sigma(\mathbb{R}) = V \upharpoonright \Theta_{\alpha+1}$.
Special case we know this: Assuming MSC:
$K(\mathbb{R}) = V \upharpoonright \Theta_0$.
Get fine structure of scales this way.

① FROM D36
<u>Remark</u> To see that (β, σ, δ) codes at most one real.
Suppose $(N^r_\xi)_\xi$, $(N^s_\xi)_\xi$ ~~code~~ witness coding r, s.
If ξ is a limit pt of the club on which these sequences agree: $m(N^r_\xi, N^s_\xi) \longrightarrow \omega$ as $\eta \to \xi$. This is the key.

~~[illegible]~~
<u>Corollary</u> If M is an inner model, Both M, V models of BPFA, $\omega_2^M = \omega_2^V$ then $\mathcal{P}(\omega_1) \subseteq M$.

<u>Definition</u> (Moore). Mapping reflection principle, MRP.
Let θ be a regular cardinal, X an uncountable set ~~[illegible]~~, $M \prec H_\theta$ countable and $[X]^\omega \in M$. A set $\Sigma(M) \subseteq$ ~~[illegible]~~ $[X]^\omega$ is M-stationary iff
$\forall E \subseteq [X]^\omega$ club if $E \in M$ then $\Sigma(M) \cap E \cap M \neq \emptyset$.
($\Sigma(M)$ is not required to be in M). ▲

SUMMARY

1. We defined HOD pairs
2. We proved comparison (2 arguments: $\diamond$ and AD^+ using W_2^*
3. We computed HOD under $AD^+ + SMC$ up to $\Theta_{\alpha+1}$ under the assumption that $\Theta_{\alpha+2}$ exists. This under the minimality assumption $\neg(AD_{\mathbb{R}} + \Theta$ regular)
4. We proved: $AD^+ + \neg(AD_{\mathbb{R}} + \Theta$ regular$) \Rightarrow SMC$
5. We proved: "Wodin limit of Woodins" $\Rightarrow Con(AD_{\mathbb{R}} + \Theta$ regular)
   Actually from divergent models

Exercise (Not easy) the proof of 4 can be used to get (5) more directly

DID NOT DO

1. Branch condensation in limit cases (Getting a pair (P, Σ) of λ^P limit, Σ with BC)
2. the internal theory of HOD mice - how to interpret strategies
3. Given Γ s.t. $L(\Gamma, \mathbb{R}) \models AD^+ + SMC$ and more Suslin cardinals $> \Gamma$ then there is a pair (P, Σ) s.t. $\Gamma(P, \Sigma) = \Gamma$. Here we want: $\Gamma = \mathcal{P}(\mathbb{R})^{L(\Gamma, \mathcal{P}(\mathbb{R}))}$

<u>Reference</u> The Bounded proper forcing axiom and well-orderings of reals : Caicedo Veličković, Math Res Lett 13 (3) 2006 393 to 40 p

<u>Theorem</u> BPFA $\Rightarrow \exists \Delta_1(A)$ w.o. of $\mathbb{R}$ for some $A \subseteq \omega_1$.

<u>Conjecture 1</u> (Woodin?) MM $\Rightarrow$ there is a definable w.o. of $\mathbb{R}$
<u>Conjecture 2</u> Assume MM. Let M be an inner model with the same cardinals. ~~as~~ then ${}^{\omega_1}M \subseteq M$.

<u>Proof of Thm</u>: First, a coding of reals. Let $\vec{C}$ be a C-sequence: $\vec{C} = \langle C_\xi \mid \xi < \omega_1 \cap \lim \rangle$. $C_\xi \subseteq \xi$, $otp(C_\xi) = \omega$, $\sup C_\xi = \xi$.

<u>Def</u> (Oscillation map) (Todorcević) $osc: (\mathcal{P}(\omega))^3 \to 2^{\leq\omega}$.

Let $x, y, z \subseteq \omega$. Let $\sim_x$ on $\omega - x$ be the equivalence relation given by $n \sim_x m$ iff $[\min(n,m), \max(n,m)] \cap x = \emptyset$.
Let $(I_k)_{k<t}$ be the increasing enumeration of the corresponding intervals – equivalence classes that meet both y, z. Define $osc(x,y,z): t \to 2$ by
$osc(x,y,z)(k) = 0 \iff \min(I_k \cap y) \leq \min(I_k \cap z)$.

<u>Def</u> Let $\omega_1 < \beta < \gamma < \delta$ be limit ordinals. Suppose

(1) $N \subseteq M \subseteq \delta$ are countable sets of ordinals

(2) $\sup(\rho \cap N) < \sup(\rho \cap M)$ for $\rho = \omega_1, \beta, \gamma, \delta$

(3) $\sup(\rho \cap M)$ is a limit ordinal, ρ as in (2)

Under these conditions the pair (N, M) <u>codes</u> x or <u>a finite string (of ordinals)</u> as follows.

Let $\pi: M \to \delta_M$ be the transitive collapse

$\alpha_M = \pi(\omega_1)$, $\beta_M = \pi(\beta)$, $\gamma_M = \pi(\gamma)$

$\alpha_N = \sup \pi[\omega_1 \cap N]$

The height of α_N in α_M is $m = m(N,M) = |\alpha_N \cap C_{\alpha_M}|$

Set $x = \{ |\pi(\xi) \cap C_{\beta_M}| : \xi \in \beta \cap N \}$

$y = \quad$ — " — $\quad C_{\gamma_M} \quad$ — " —

$z = \quad$ — " — $\quad C_{\delta_M} \quad$ — " —

Note: x, y, z are finite.

Define $S_{\beta,\gamma,\delta}(N,M) = \mathrm{osc}(\cancel{x,y},z) \restriction n$

$= \mathrm{osc}(x-n, y-n, z-n) \restriction n.$

if $\mathrm{dom}(\mathrm{osc}(x-n, y-n, z-n)) \geq m$

otherwise set $S_{\beta,\gamma,\delta}(N,M) = *$

Also let $S_{\beta,\gamma,\delta}(N,M) = *$ if $\ell > n$.

The triple (β,γ,δ) codes a real $r \in 2^\omega$ iff $\exists (N_\xi \mid \xi < \omega_1)$ continuous sequence of countable sets with union δ s.t. $\forall \xi$ limit $\exists \nu < \xi$ s.t. $r = \bigcup_{\nu < \eta < \xi} S_{\beta\gamma\delta}(N_\eta, N_\xi)$

Note (β,γ,δ) codes at most one real.

Theorem (C-V) (BPFA)

(1) If $\omega_1 < \beta < \gamma < \delta < \omega_2$ are of cofinality ω_1 then there is an increasing continuous sequence $(N_\xi \mid \xi < \omega_1)$ consisting of countable sets with $\bigcup N_\xi = \delta$ s.t. $\forall \xi$ limit $\forall n\ \exists \nu < \xi\ \exists S^n_\xi \in \{0,1\}^n \cup \{*\}^{\xi < \omega_1}$ s.t.

$$S_{\beta,\gamma,\delta}(N_\xi, N_\xi) \restriction m = S^n_\xi \quad \text{for all } \eta \in (\nu, \xi)$$

(2) For each real r there are $\omega_1 < \beta < \gamma < \delta < \omega_2$ of cofinality ω_1 s.t. (β,γ,δ) codes r.

~~Goto D31~~

<u>Corollary</u> (BPFA) There is a Δ_1 w.o. of $\mathcal{P}(\omega_1)$ with parameter a C-sequence. The length of w.o. is ω_2.

<u>Proof</u> Let $\vec{C}$ be a C-sequence. If N is transitive model of enough ZFC and $\vec{C} \in N$ then $\omega_1^N = \omega_1$ and there is ω_1-sequence of a.d. reals so that N correctly identifies.

Let $T_{\vec{C}}$ be the theory $ZFC^- + MA_{\omega_1} + (1)_{\vec{C}} + (2)_{\vec{C}} + (\forall X)(|X| \leq \omega_1)$ (1, 2 are as in Thm with codings relative to $\vec{C}$).

Key: If $\vec{C} \in M_1, M_2 \models T_{\vec{C}}$ and $\mathrm{On} \cap M_1 = \mathrm{On} \cap M_2$ then $M_1 = M_2$.

Suppose $M \models T_{\vec{C}}$. Let $\beta, \gamma, \delta \in \mathrm{On}^M$ and suppose (β, γ, δ) code $r \in V$. Then $r \in M$:

Let $(N_\xi \mid \xi < \omega_1)$ witness (β, γ, δ) code r.

Since $M \models (1)_{\vec{C}}$ there is in N a sequence $(P_\xi \mid \xi < \omega_1)$ witnessing this for (β, γ, δ). Then $N_\xi = P_\xi$ on a club of ξ.

Now by $M \models (2)_{\vec{C}}$ it follows:

$\mathbb{R}^M = \{r \mid \exists (\beta, \gamma, \delta) \in M \ ((\beta, \gamma, \delta) \text{ codes } r)\}$

Using a.d. coding: $\mathcal{P}(\omega_1)^M$ is characterized this way.

Similarly: if $M_1 \models T_{\vec{C}}$ and $M_2 \models T_{\vec{C}}$ and $o(M_1) < o(M_2)$ then $M_1 \subseteq M_2$. We can define a w.o. of $\mathbb{R}$ by $r \lhd s$ iff letting θ_r be the least s.t. $\exists M \models T_{\vec{C}}$, $o(M) = \theta_r$, $r \in M = M_r$

we have $\theta_r < \theta_s$ or

($\theta_r = \theta_s$ and $M_r \models$ "the least anti-lex triple coding r < that coding s")

This is $\Delta_1(\vec{C})$: $r \lhd s$ iff some/all model of of set they thinks $r \lhd s$.

GO TO D32

(A) The Ellentuck topology on $[X]^\omega$: $A \subseteq [X]^\omega$ is open iff A is a union of sets of the form
$[x, B] = \{ Y \mid x \subseteq Y \subseteq B \mid Y \text{ countable} \}$
for $x \subseteq B$ ~~for~~ x finite, $B \in [X]^\omega$

Σ is an open stationary mapping iff $\exists$ ~~Θ~~ $\Theta = \Theta_\Sigma$, $X = X_\Sigma$ s.t. $[X]^\omega \in H_\Theta$, $\mathrm{dom}(\Sigma)$ is a club in $[H_\Theta]^\omega$ consisting of countable elementary substructures and
$\forall M \in \mathrm{dom}(\Sigma)$: $\Sigma(M) \subseteq [X]^\omega$ is M-stationary and open in the Ellentuck topo.

Example Let $M \prec H_\Theta$ be countable, $\alpha = \omega_1 \cap M$. Let $\alpha = K_0 \cup K_1$ be a partition. Then (for $X = \omega_1$) either $[\alpha]^\omega - K_0$ or $[\alpha]^\omega - K_1$ is M-stationary. Because $\omega_1 - \omega \subseteq [\omega_1]^\omega$ is club. Exercise.

Now note that if $S_{\beta\gamma\delta}(N, M) = s \in 2^n$ then there is $t \subseteq N$ finite s.t. $\forall Y \in [t, N]$ $S_{\beta\gamma\delta}(Y, M) = s$

Definition (MRP) If Σ is an open stationary map then there is a reflecting sequence for Σ, i.e. a sequence $\langle N_\xi \mid \xi < \omega_1 \rangle$ continuous, increasing, $|N_\xi| = \omega$ ~~of~~ ~~of~~ each $N_\xi \in \mathrm{dom}(\Sigma)$ s.t.
$\forall \xi$ limit $\exists \nu < \xi$ s.t. $N_\eta \cap X_\Sigma \in \Sigma(N_\xi)$ all $\eta \in (\nu, \xi)$

Theorem (Moore) PFA $\to$ MRP
Proof-idea: Let $\mathbb{P}_\Sigma$ = the set of $p : \alpha + 1 \to \mathrm{dom}\,\Sigma$ s.t.

$\alpha < \omega_1$, p continuous, $\in$-increasing

$\forall \beta \leq \alpha$ limit $\exists \gamma < \beta\ \forall \delta \in (\gamma, \beta)\ p(\delta) \cap X_\Sigma \in \Sigma(p(\beta))$

The poset $\mathbb{P}_\Sigma$ is proper. This gives: if G is sufficiently generic then $\bigcup G$ has domain ω_1.

Proof of the main theorem In fact: MRP $\Rightarrow$ (1), (2). So there are proper forcings giving us these statements and they are $\Sigma_1(\vec{C})$, so BPFA suffices.

Proof of (1) Recall: $\omega_1 < \beta < \gamma < \delta < \omega_2$, $\beta, \gamma, \delta \in S^{\omega_2}_{\omega_1}$

WTF $(N_\xi \mid \xi < \omega_1)$ countable N_ξ, increasing + continuous, $\bigcup_{\xi < \omega_1} N_\xi = \delta$ s.t. $\forall \xi\ \forall n\ \exists s^n_\xi\ \exists \nu < \xi\ \forall \eta \in (\nu, \xi)\ s_{\beta\gamma\delta} \restriction (N_\eta, N_\xi) \restriction n = s^n_\xi$.

Given (β, γ, δ): fix $(N_\xi \mid \xi < \omega_1)$ inc + cont, N_ξ ctble, $|N_\xi| = \omega$, $\bigcup_\xi N_\xi = \delta$ s.t. $\sup(N_\xi \cap \rho)$ is limit for $\rho = \omega_1, \beta, \gamma, \delta$ (Easy). Also want the sups are increasing.

Fix n: For α limit let

$K^s_\alpha = \{\eta < \alpha \mid s_{\beta\gamma\delta}(N_\eta, N_\alpha) \restriction n = s\}$ for $s \in 2^n \cup \{*\}$.

Set $\Sigma(M) = \{N \in [\delta]^\omega \mid s_{\beta\gamma\delta}(N, M \cap \delta) = s_M\}$

where s_M is s.t. for $\alpha = M \cap \omega_1$ with $K^{s_M}_\alpha$ is M-stat.

MRP gives a sequence for Σ: $(N^n_\xi \mid \xi < \omega_1)$

Let $C \subseteq \omega_1$ be a club ~~where~~ on which N_ξ, N^n_ξ coincide. Then $(N_\xi \mid \xi \in C)$ works.

Proof of (2) ~~[illegible]~~ θ regular large, $M \prec H_\theta$ countable, $r \in 2^\omega$.

$\Sigma^r(M) = \{N \in [M \cap \omega_4]^\omega \mid s_{\omega_2\omega_3\omega_4}(N, M \cap \omega_4) \sqsubseteq r\}$

If Σ^r is open stationary and $\langle N_\xi \mid \xi < \omega_1 \rangle$ is sufficiently generic for $\mathbb{P}_{\Sigma^r}$ then letting $(M_\xi \mid \xi < \omega_1)$ be the sequence of transitive collapses, this gives (β, γ, δ) coding r as witnessed by the M_ξ's. So BPFA gives this.

To see that $\Sigma^r(M)$ is stationary use the game:

I	β_0	γ_0	δ_0	. . .
II	κ_0, ξ_0	λ_0, τ_0	μ_0, ν_0	. . .

Let $F : [\omega_4]^{<\aleph_0} \longrightarrow \omega_4$, $\cancel{F \in M}$, $\alpha \in \omega_1$

$\mathcal{G}^F_\alpha$: $\beta_i, \gamma_i, \delta_i$ are increasing below $\omega_2, \omega_3, \omega_4$

$\beta_i \le \kappa_i \le \xi_i < \omega_2 \qquad \xi_i < \beta_{i+1}$

$\gamma_i \le \lambda_i \le \tau_i < \omega_3 \qquad \tau_i < \gamma_{i+1}$

$\delta_i \le \mu_i \le \nu_i < \omega_4 \qquad \nu_i < \delta_{i+1}$

$X = Cl_F(\{\kappa_i, \lambda_i, \mu_i \mid i \in \omega\} \cup \alpha)$

II wins iff $X \cap \omega_1 = \alpha$

$X \cap [\beta_i, \beta_{i+1}] \subseteq [\beta_i, \xi_i)$

The game is open for I.

Lemma For club many α II has a winning strategy in $\mathcal{G}^F_\alpha$.

Granting the lemma: Given M we want $\Sigma^r(M)$ be M-stationary. For this let $F \in M$, w.m.a the club of the lemma be in Γ. Let $\sigma_\alpha : \alpha \mapsto$ winning strategy.

Let $(\kappa^\alpha_0, \xi^\alpha_0) = \sigma_\alpha(\phi)$ $\quad \xi_0 = \sup_\alpha \xi^\alpha_0 \in M \cap \omega_2$.

Let $h = ht_{\omega_2}(\xi_0) \overset{\text{def}}{=} h_{\pi(\omega_2)}(\pi(\xi_0))$ where $\pi : M \longrightarrow \bar{M}$ transitive collapse.

I was unable to record the game

We showed $D(M^I_\infty, \lambda^I_\infty)$ via Σ_M for our mouse operators independent of I, a.

Question: How general is this? Is this true for any sound mouse projecting to ω with $M \models \lambda$ limit of Woodins?
Also: $\Sigma_M \notin D$. When is Σ_M the Wadge-least $\notin D$?

Showed $D(M,\lambda) \models MSC$. In fact TFAE any $a, b \subseteq a$ (a ctbl transitive)

(1) b is $OD(a)^{D(M^I_\infty, \lambda^I_\infty)}$

(2) $b \in M(a)$

(3) b is in an ω-iterable mouse over a.

(2)→(3) last time (3)→(1) trivial, (1)→(2) by symmetricity of the collapse.

Definition M - one of our operators with hypothesis φ.
$M^0(a) = M(a)$
$M^1(a)$ = sharp of the minimal M-closed model of φ
$M^{\alpha+1}(a)$ = Sharp of the minimal M^α-closed model of φ
$M^\alpha(a) = \bigcup_{\beta<\alpha} M^\beta(a)$ α limit, $\alpha \in a$. (Projects to a!)
here $\alpha < \omega_1^a$

Then:

(1) For $M = M^\#_{adr}$, $\forall n \in \omega$
$a \mapsto M^n(a) \in D(M^I_\infty, \lambda^I_\infty)$
(a ctbl transitive)
moreover: F^n has Wadge rank approximately Θ_n in $D(M^I_\infty, \lambda^I_\infty)$

Moreover: $A \in \mathcal{P}(\mathbb{R})^{D(M^I_\infty, \lambda^I_\infty)}$ iff $\exists n\ A \leq_W F^n$.

(2) For M one of the remaining three: For $\alpha < \omega_1$:
$F^\alpha : a \mapsto M^\alpha(a)$
is in $D(M^I_\infty, \lambda^I_\infty)$ of Wadge rank approximately Θ_α.
(a) In cases 2, 3 $M^\#_{de}$, $M^\#_{wlim}$ these are $\leq_W$ cofinal
(So $D(M^I_\infty, \lambda^I_\infty) \models \Theta = \Theta_{\omega_1}$ in these cases)
(b) Let $M^{\omega_1}(a) = M^{\omega_1^a}(a)$ ω_1^a = 1^{st} admissible in $\underline{a}$ ordinal.

Let $F : a \mapsto M^{\omega_1}(a)$ then for $M = M^\#_{wlim}$: $F \in D$
and $|F|_W \approx \Theta_{\omega_1}$. In particular: $\Theta_{\omega_1} < \Theta$

For instance: M = one of the 4 operators: Consider $a \mapsto M^0(a)$.
(1) Do the $\mathbb{R}$-genericity iteration I s.t. of
κ_0 = the least $<\lambda$ - strong
δ_1 = the least Woodin $> \kappa_0$ critical pts $> \kappa_0$.
Let g be generic over the iterate N of M s.t. $M \in N[g]$
In $N[g]$ compute M^0-operator $\upharpoonright V_\lambda^{N[g]}$
($a \in V_\nu^{N[g]}$, $\nu < \lambda$. Take $P = \mathrm{Ult}(M, E)$
where E on the N-sequence with $\mathrm{cr}(E) = \kappa$, $\mathrm{lh}(E) > \nu$.
$P(\kappa)^M = P(\kappa)^N$! So $a \in P[g]$. So use $P[g]$-extenders
to rebuild $M(a) \in V_\lambda^{N[g]}$.)
__Then:__ (Exercise) $N[g]$ has T, U that are $<\lambda$-a.c.
s.t. in M^I_∞: $p[j(T)]$ = the operator $a \mapsto M^0(a)$
(by condensation.)
Note: $a \mapsto M^0(a)$ cannot be $OD(a_0)$ in D as otherwise
$M^0(a) \in M^0(a)$.

On the other hand: If

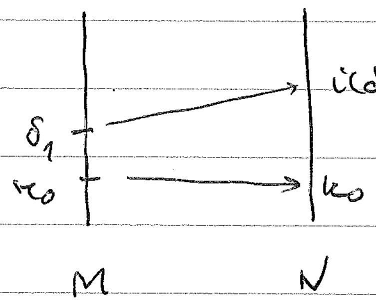
$T^n(a)$ = the theory of first n indiscernibles of

Then for any real x: $M^0(x) \equiv_T \bigoplus_n T^n(x)$. Each $T^n(x) \in M^0(x)$

Use this to see that T^n is $OD(z)^P$.

Why is $a \mapsto M^1(a)$ at Θ_1 in $D(M^I_a, \Sigma^I_a)$

<u>Exercise</u> If g is M-generic / $Col(\omega, \kappa)$ then $M[g]$ does not have a.c. (T, U) with $p[T] = M$-operator.



$M \in N[g]$. Notice: $N[g] = M^1(\underbrace{\langle N \mid i(\delta_1), g\rangle}_{\sim \text{real } z})$

Hint from R to L: From z we get g and $N \mid i(\delta_1)$. ETS: can reconstruct extenders with cr.pt. κ_0; these can be identified as follows: reconstruct how subsets of κ_0 are moved. For this use M which is ω-sound, so κ-sound and its canonical function $f: \kappa_0 \xrightarrow{\text{onto}} \mathcal{P}(\kappa_0)^M = \mathcal{P}(\kappa_0)^N$. $M^1(z)$ knows how f should be shifted, ~~as it knows~~ by extenders, as it knows $a \mapsto M(a)$. Use this to repeat our previous argument but with $M^1(z)$ replacing $M^0(\emptyset)$.

Now start with $V \models AD^+ + \Theta_0 < \Theta$. We get M s.t. $M \models$ $AD_{\mathbb{R}}$-hypo. (We use MSC here.)

First: Prikry force hod premouse P with Woodin cardinals $\delta_0, \delta_1, \ldots$ such that $(L_P^{+W}(P \mid \delta_0), \Sigma_0)$ is a hod pair. This can be done, as we assume $\Theta_0 < \Theta$.

Think of Σ_0 as ~~a~~ an M^0-operator. $(W(\Sigma_0) = \Theta_0)$.

Let κ_1 = the least cardinal in P strong to δ_1. ~~Build~~

Wait: Build $L[E]$ in P in which κ_1 is strong up to $\lambda = \sup_i \delta_i$.

Change in notation: Write N^* for P.
Let $N = L[E]^{N^*|\delta_1}$ built over $N^*|\delta_0$. Extend N to a $L[\vec{E}]$ model of height = ORD s.t.

$N \models \lambda$ is a limit of Woodins + κ_1 is $<\lambda$-strong

We will show: at sufficiently good $\eta < \delta$ there is a "translation process" whereby for any $\gamma < \delta_1$ s.t. $\eta < \gamma$ and $N|\gamma \models$ "η is Woodin" so $N^*|\eta$ is κ-generic over $N|\gamma$ for η-generator extender algebra.

$(N|\xi)[N^*|\eta] \approx$ a Σ_0-mouse over $N^*|\eta$. for all $\xi < \gamma$
"almost" ("proved by induction on ξ.")

This goes in both directions. ← "above" denote by Tr_η.

The translation is tight to the fixed η, so Tr_η (has a definition from η) Now apply the definition of Tr_η to levels of N^* past δ, and $\exists \kappa < \lambda$ with Tr_{δ_1} = the result of replacing ? κ is λ-strong η by δ_1 in the definition ?.

Then $o(\kappa_1) = \lambda$ in the resulting model by a reflection argument.

The $*$-transform: Given M ω-sound, $\rho^\omega_M = \omega$, (ω_1+1)-iterable (typically $M = M(a)$ for one of our operators.) Let $\mathcal{T}$ be a tree on M according to the strategy with the last model N.

Let δ be a cardinal in N not measurable in N. Let

$$\Phi(\mathcal{T}) = \langle (P_\kappa, \kappa) \mid \kappa = cr(E) \text{ some } E \text{ on } N \text{ sequence with } lh(E) \geq \delta(\mathcal{T}) \rangle$$

δ

$\kappa + 2$

N

$P_\kappa = M^{\mathcal{T}}_\xi$ where ξ least s.t. $\kappa < \nu(E^{\mathcal{T}}_\xi)$

Let g be $col(\omega, \delta)$-generic $/N$ and $\Phi(\mathcal{T}) \in N[g]$.

Then we define for all $\gamma \leq On^N$:

$$(N|\gamma)[g]^{*,\delta} \simeq \text{a mouse over } \langle N|\delta, g \rangle$$

s.t. $N|\gamma$ is "fine structurally equivalent" to $(N|\gamma)[g]^{*,\delta}$ modulo $\langle N|\delta, g\rangle$.

Given Q s.t. $N|\delta \trianglelefteq Q$ and can be reached by extenders overlapping δ from N, define $Q[g]^{*,\delta}$,

(a) if $o(Q) = \omega\alpha + \omega$ $\quad Q[g]^* = \text{rud } Q^-[g]^*$,

(b) $o(Q)$ limit then $Q[g]^* = \bigcup_{\alpha < o(Q)} (Q|\alpha)[g]^*$

(c) if Q is active with last extender E, say $Q = (R, E)$:

 (i) if $cr(E) > \delta \Rightarrow Q[g]^* = (R^*, E^*)$

 (ii) if $cr(E) < \delta$, $cr(E) = \kappa$ then

 $Q[g]^* = Ult(P_\kappa, E)[g]^*$

Details: Thesis of Eric Closson. Shows that the fine structure correspondence.

5.8.2010 9:30 JOHN STEEL

One application: For M any of the first three operators:

(1) $D(M, \lambda) \models \theta = \Theta_\omega$

(2)(3) $D \models \theta = \Theta_{\omega_1}$

Proof sketch: If $A \in D(M^I_\infty, \lambda^I_\infty)$, $A \subseteq \mathbb{R}$.

$M_0 \to \cdots \to M_n \to \cdots$ genericity iteration

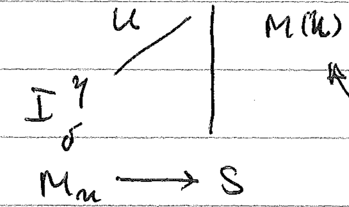
λ^{M_n} $(M_n, (T, U), \Sigma_{M_n})$ Suslin captures A.

(T,U) Generators of $T_0, \ldots, T_{n-1}$ are below δ.

δ, g WMA: $\Phi(\langle T_0, \ldots, T_{n-1}\rangle)$ (i.e. the relevant

M_n part of the phalanx) has been made generic earlier, so it is in $M_n[g]$. Let η = the next Woodin $> \delta$.

We can read off $\Sigma_{M_n} \restriction$ trees in the window (δ, η) using the $*$-transform. nondropping



Find $Q(U)$. (= the Q-structure)

the common part model.

Look at $(S[g])^{\#,\delta(U)}$. This is a Q-structure in the other hierarchy, and is $\trianglelefteq M^\beta(M(U))$ for $\beta \leq \alpha$ where α = the number of cardinals strong past η in M_η.

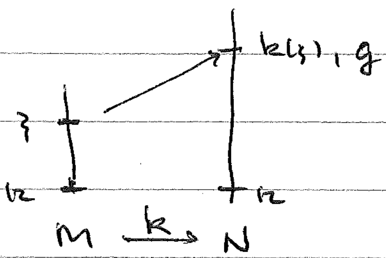
Gives: A is projective in the M^α operator for this α.

$\Box$ Case (1)

<u>2nd application</u> For $M = M^{\#}_{\mathrm{wlim}}$: $D(M,\lambda) \models \Theta_{\omega_1} < \Theta$.

For this want to show: $a \mapsto M^{\omega_1}(a)$ is in $D(M^I_\infty, \lambda^I_\infty)$ some I. Let $M \models \kappa$ is A-reflecting in λ = top Woodin for $A = Th^{M|\lambda}(\lambda)$

ζ = least Woodin $> \kappa$. Choose k, g s.t. $M \in N[g]$.



<u>Claim</u> For any cardinal η of $N[g]$ and any $\alpha \leq \omega_1^{N[g]|\zeta}$:

$$M^\alpha(N[g]|\eta) \in N[g]$$

Given this: can then use the proof to see $M^{\omega_1}|V_\lambda^{N[g]} \in N[g]$, then get $M^{\omega_1} \in D$ using condensation.

<u>Proof of Claim</u> Show by induction on $\alpha < \lambda$: For any such η: $M^\alpha(N[g]|\eta) \in N[g]$, in fact: $M^\alpha(N[g]|\eta) \trianglelefteq Q[g]^{\#,\eta}$ for some proper init segment Q of $Ult(N, F_\eta)$ where F_η = 1st extender overlapping η from N with $cr(F_\eta) > \kappa$ or $\emptyset$ if DNE.

<u>Exercise</u> α limit: Show the induction step at α. (Stacking the things)

Now do the successor step $\alpha \mapsto \alpha+1$.

Let $A = Th^{P_\eta}(k(\lambda))$. Let E on P_η sequence witness k is A-reflecting past η^+. Let $Q = P_\eta \| lh(E)$ (1). Note $Q[g]^{*,\eta} = Ult(M, E)[g]^{*,\eta}$. Now $Ult(M, E)$ reaches "woodin lim of Woodins" hypo. So it is enough to see the transform (= RHS) is M^α-closed. ETS:

$Ult(N, E)[g]^{*,\eta}$ is M^α-closed. But $N[g]^{*,\eta}$ is so closed and we have enough reflection via E:

$P = P_\eta$ $P_\eta = Ult(N, F_\eta)$

$P[g] \models \forall \beta \leq \alpha\ \forall \nu$ s.t. $k(\xi) < \nu < k(\lambda)$, ν successor card with $\beta \leq \omega_1^{(P|\nu)[g]}$: $P_\nu[g]^{*,\nu}$ has a proper i.s. satisfying "I am $M^\beta(P_\nu|\nu)[g]$"

Call this sentence $\varphi(k(\xi), M, g, k(\lambda), \alpha)$. Let p be a condition on $Col(\omega, k(\xi))$ s.t.

$$p \Vdash^{P_\xi}_{Col(\omega, k(\xi))} \varphi(k(\xi), \overset{\tau^g = M}{\tau}, \dot{g}, k(\lambda), \alpha)$$

so

$$p \Vdash^{Ult(P_\xi, E)}_{Col(\omega, k(\xi))} \varphi(k(\xi), \tau, \dot{g}, \underset{k(\lambda)}{\underset{\shortparallel}{i_E(k(\lambda))}}, i_E(\alpha))$$

Here we can replace $Ult(P_\xi, E)$ by $Ult(M, E)$, since we have $M \to P_\xi$. Also $\sigma : Ult(M, E) \to Ult(P_\xi, E)$ with $cr(\sigma) \geq \eta$. Also note $i^M_E(\alpha) \geq \alpha$.

<u>Rem</u> We actually needed less about M.

<u>Def</u> Let λ be a limit of Woodins and $k < \lambda$. Say k is strong of rank α in λ (for $\alpha < \lambda$)

(1) With the top extender E

iff $\forall\beta<\alpha$ κ is $<\lambda$-strong as witnessed by extenders E s.t. $Ult(V,E) \models \kappa$ is strong of rank β in $i_E(\lambda)=\lambda$.

The proof above can be refined to show: if M is ~~the~~ an iterable mouse projecting to ω + sound (i.e. M is a sharp-mouse) with

(φ) $M \models \lambda$ is a limit of Woodins + $\exists\kappa<\lambda$ strong of rank λ in λ

then $D(M,\lambda) \models \Theta_{\omega_1} < \Theta$.

For the minimal such M we get

$$D(M,\lambda) \models \Theta_{\omega_1+1} = \Theta.$$

<u>Open</u> Do we need a mouse here? Could we ~~do~~ get the result for V instead?

<u>Probably</u>: $Con(ZFC+\varphi) \Rightarrow Con(AD^+ + \Theta_{\omega_1+1}=\Theta)$

<u>Probably$^+$</u>: the converse

THE REVERSE DIRECTION

<u>Conjecture? Theorem</u>: Assume $AD^+ + SMC +$ ~~...~~ $\Theta=\Theta_\alpha$ some $\alpha\leq\omega_1$.

Then in some $V[G]$ there is a premouse M with $M\models$ "λ is WLW" such that $V = L(A_H, \mathbb{R}^*_H)^{M[H]}$ for some H on $Col(\omega,<\lambda)$ generic ~~for~~ over M.

<u>Proof in the case $\Theta=\Theta_0$</u>: We Prikry force over V. Let a be countable transitive, $x\in\mathbb{R}$ and a coded by a real recursive in x. Let

$\mathcal{F}_a^x$ = the set of all P_z, s.t. $z \leq_T x$ and P_z is
0-suitable (i.e. 1 Woodin + Σ_1^2-full) premouse
over a and P_z is short tree iterable

If T is a tree for universal Σ_1^2 set, we can simultaneously compare all P_z in $\mathcal{F}_a^x$ inside $L[T,x]$. At the same time, making all reals $y \leq_T x$ generic for the extender algebra at our common Woodin. $L[T,x]$ can find correct branches for short trees $\mathcal{U}$. It has all mice over $M(\mathcal{U})$ projecting to $\delta(\mathcal{U})$ and their IS restricted to itself. We get a 0-suitable premouse Q_a^x with $\delta^{Q_a^x} = \omega_1^{L[T,x]}$. (This is like the $L[x]$ argument given earlier.) Also: $Q_a^x = Q_a^d$ for the $d = [x]_T$.
Q_a^x = the stack of all iterable mice over $M(\mathcal{U})$
projecting to $\delta(\mathcal{U})$; $\mathcal{U}$ any of those
$lh = \omega_1^{L[T,x]}$ trees.

For a.e. T-degrees $d_0 < d_1 < \cdots < d_n$ we have

$Q_0^{\vec{d}} = Q_{\emptyset}^{d_0}$ $\qquad$ $F(\vec{d})$ = this stack

$Q_1^{\vec{d}} = Q_{Q_0^{\vec{d}}}^{d_1}$ $\qquad$ Define a measure ν_a and ν_n:

$\vdots$ $\qquad$ $A \in \nu_n$ iff for a.e. d_0 for a.e. $d_1 \ldots$

$Q_n^{\vec{d}} = Q_{Q_{n-1}^{\vec{d}}}^{d_n}$ $\qquad$ $F(\vec{d}) \in A$

For any a:

$X \in \nu_a$ iff for a.e. d : $Q_a^d \in X$

Measure one tree is a tree with a stem s.t.

$\forall (Q_0 \ldots Q_n) \in T : \{Q_{n+1} \mid \langle Q_0 \ldots Q_{n+1}\rangle \in T\} \in \nu_{\langle Q_0 \ldots Q_n\rangle}$

such that:

$\langle Q_0, .. Q_{n+1} \rangle \in T \Rightarrow$ each Q_i is suitable over Q_{i-1}
and $Q_{i-1} \triangleleft Q_i$
and $o(Q_{i-1})$ is a cardinal in Q_i

Let $T \leq S$ iff T is a subtree of S
Let $\mathbb{P}_0$ = this forcing in $L(\mathcal{U}, \mathbb{R})$ where $\mathcal{U}$ is the universal Σ^2_1 set
$\mathbb{P}$ = this forcing in V

__Easy__ Every dense set in $\mathbb{P}_0$ is predense in $\mathbb{P}$. ~~so~~ ~~so $\mathbb{P}$ generic~~

Let G be $\mathbb{P}$-generic $/ V$ (hence ~~over~~ $\mathbb{P}_0$-generic $/ L(\mathcal{U}, \mathbb{R})$.)
Let $Q_\infty = \langle Q_i \,(i \in \omega) \rangle$ = union of stems in $T \in G$. Show:
$L(\mathcal{U}, \mathbb{R}^V)$ is a derived model of $L[Q_\infty]$. Let
λ = sup of the Woodins of Q_i's $= o(Q_\infty)$
Note: As before, no bounded subsets are added to $L[Q_\infty]$
i.e. $\mathcal{P}(\delta_i^{Q_\infty}) \cap L[Q_\infty] \subseteq Q_\infty$

(1) There is $h \subseteq Col(\omega, <\lambda)$ s.t. $\mathbb{R}^*_H = \mathbb{R}^V$.

__Remark__ ~~$\forall x$~~ $\mathcal{F}^x_a \neq \emptyset$ $\forall a$ and a cone of x.
This is done by a reflection argument to Σ^2_1:
If not, $V \models \varphi$ some φ saying "no". Get
$L_\alpha(P_\beta(\mathbb{R})) \models ZF^- + \Theta = \Theta_0$ exists $+ \varphi$
with $L_\alpha(P_\beta(\mathbb{R})) \subsetneq \underset{\sim}{\Delta}^2_1$; $M_{\underset{\sim}{\Delta}^2_1} \prec_{\Sigma_1} V$

Using N_x^* capturing good, for our purpose we can take inductive-like pointclass Γ where scale(Γ) and $L_\alpha(P_\beta, \mathbb{R}) \subsetneq \underset{\sim}{\Delta}$ we can construct by full backgrounds $L[\vec{E}]^{N_x^* | \delta_x}(a)$ and show it must reach a Θ-suitable short tree iterable Q_a (there "short" means "short in the sense of $L_\alpha(P_\beta(\mathbb{R}))$"! Note: there are club many Γ Woodins here. (Exercise.) Then $L_\alpha(P_\beta(\mathbb{R})) \models$ "Q_a is iterable + Θ-suitable"

Q_∞ h s.t. $\mathbb{R}_h^* = \mathbb{R}^V$, h generic $/ V[G]$ for p.o. whose conditions are $h_0 \ldots h_n$ for ~~h_n~~'s $h_n : \omega \xrightarrow{\text{onto}} o(Q_n)$ Q_n-generic / col$(\omega, \delta_n^{Q_n})$ over $Q_n[h_0 \ldots h_{n-1}]$.

<u>Claim</u> $\text{Hom}_h^* = (\underset{\sim}{\Delta}_1^2)^V$

<u>Pf</u> Easy: $\subseteq$ Every Hom^* set is Suslin co-Suslin in the derived model. Or we get a $\#$ for V by forcing.

$\supseteq$ For any $A \in (\underset{\sim}{\Delta}_1^2)^V$ let $\vec{\varphi}, \vec{\psi}$ be scales for A, $\neg A$ which are $\Delta_1^2(x)$. ~~By~~ ~~because~~ let $x \in Q_\infty[h_i]$ at each δ_j, $j > i$ we have capturing terms for $\vec{\varphi}, \vec{\psi}$ in $Q_\infty[h_i]$, by Σ_1^2-fullness. These give a.c. trees $(T, U) \in L[Q_\infty][h_i]$ projecting to A, $\neg A$ on $\mathbb{R}^V$. So we are done in $L(V, \mathbb{R})$