Talks Tehran Oct, 2015

PFA$^-$

We're now going to discuss the obvious solution to the equation

$$\frac{x}{\text{PFA}} = \frac{\text{remarkable}}{\text{supercompact}}$$

Recall that PFA may be restated as follows. Let $\mathfrak{M} = (M; (R_i : i<\omega_1))$ be a model, and let φ be Σ_1. Suppose that

$$V^{\mathbb{P}} \models \varphi(\mathfrak{M})$$

for some proper poset $\mathbb{P}$. There is then some $\bar{\mathfrak{M}} = (\bar{M}; (\bar{R}_i : i<\omega_1))$ s.t. $\bar{M}$ is of size $\aleph_1$, $V \models \varphi(\bar{\mathfrak{M}})$, and there is some el. $j : \bar{\mathfrak{M}} \longrightarrow \mathfrak{M}$.

Let's "remarkabilize" this assertion.

Definition. PFA^- is the following statement.

Let $\mathfrak{M} = (M; (R_i : i < \omega_1))$ be a model, and let φ be Σ_1. Suppose that

$$V^{\mathbb{P}} \models \varphi(\mathfrak{M})$$

for some proper poset $\mathbb{P}$. There is then some $\bar{\mathfrak{M}} = (\bar{M}, (\bar{R}_i : i < \omega_1))$ s.t. $\bar{M}$ is of size $\aleph_1$, $V \models \varphi(\bar{\mathfrak{M}})$, and II wins $G(\bar{\mathfrak{M}}, \mathfrak{M})$.

~~Proof~~ Theorem 1. PFA^- is consistent relative to a remarkable cardinal.

Proof. We simply imitate the consistency proof of PFA, replacing the supercompact cardinal by a remarkable cardinal.

Let κ be remarkable. Let $f : \kappa \longrightarrow V_\kappa$ be a Laver function in the following sense. For all x and all α s.t. $x \in V_\alpha$ there is some $\beta < \kappa$ and some $\bar\kappa < \beta$ s.t. for some $\bar x \in V_\beta$,

- $f(\bar\kappa) = \bar x$, and ~~its~~
- in $V^{\mathrm{Col}(\omega, <\kappa)}$ there is some $j : V_\beta \longrightarrow V_\alpha$ with $\mathrm{crit}(j) = \bar\kappa$, $j(\bar\kappa) = \kappa$, and $j(\bar x) = x$.

(Cf. [Cheng – Gitman].) Let $\mathbb{P}$ be the usual proper iteration to force (or, try to force) PFA, using f.

We claim that PFA^- holds true in $V^{\mathbb{P}}$.

Let g be $\mathbb{P}$-gen. $/ V$. Let $\mathcal{M} = (M; (R_i : i<\omega_1)) \in V[g]$ be a model, say $\mathcal{M} \in V_\alpha^{V[g]}$.

Let $\mathbb{Q} \in V[g]$ be proper, let φ be Σ_1, and assume that $\Vdash^{\mathbb{Q}}_{V[g]} \varphi(\mathcal{M})$. Say also $\mathbb{Q} \in V_\alpha^{V[g]}$. We may further assume

$\dot{Q}$ to be s.t. $\dot{Q}^g = Q$. Let $\tau^g = \mathcal{M}$, say $\tau \in V_\alpha$.

Let us pick $\bar{\kappa} < \beta < \kappa$ s.t. for some pair $(\bar{\tau}, \bar{Q})$, in $V^{Col(\omega, <\kappa)}$ there is some $j : V_\beta \longrightarrow V_\alpha$ with $crit(j) = \bar{\kappa}$, $j(\bar{\kappa}) = \kappa$, and $j(\bar{\tau}, \dot{\bar{Q}}) = (\tau, \dot{Q})$ and $f(\bar{\kappa}) = \mathring{\bar{Q}}$.

We may lift, in $V^{Col(\omega, <\kappa)}$, $j : V_\beta \to V_\alpha$ to $\hat{j} : V_\beta[g \restriction \bar{\kappa}] \longrightarrow V_\alpha[g]$ by setting $\hat{j}(\sigma^{g \restriction \bar{\kappa}}) = j(\sigma)^g$.

Let $\bar{\tau}^{g \restriction \bar{\kappa}} = \bar{\mathcal{M}} = (\bar{M}, (\bar{R}_i : i < \omega_1))$, and let $\bar{Q} = \dot{\bar{Q}}^{g \restriction \bar{\kappa}}$. So $\hat{j}(\bar{\mathcal{M}}) = \mathcal{M}$ and $\hat{j}(\bar{Q}) = Q$.

By $f(\bar{\kappa}) = \dot{\bar{Q}}$, $\bar{Q}$ is the next forcing in the iteration $\mathbb{P}$.

By $\Vdash^{Q}_{V[g]} \varphi(\mathcal{M})$, $\Vdash^{\bar{Q}}_{V_\beta[g \restriction \bar{\kappa}]} \varphi(\bar{\mathcal{M}})$, so that $V[g] \models \varphi(\bar{\mathcal{M}})$.

As $\hat{j} \upharpoonright \bar{\mathfrak{M}} : \bar{\mathfrak{M}} \longrightarrow \mathfrak{M}$ is elementary,

$\underline{\mathrm{II}}$ wins $G(\bar{\mathfrak{M}}, \mathfrak{M})$.

We have indeed verified PFA^-. $\dashv$

In contrast to BPFA, PFA^- is an <u>un</u>-bounded forcing axiom. But:

<u>Lemma 1.</u> $\mathrm{PFA}^- \Longrightarrow \mathrm{BPFA}$.

Proof: Easy. $\dashv$

PFA^- is stronger than BPFA in the following sense.

<u>Theorem</u> 2. Suppose PFA^- is true. Then ω_2^V is remarkable in L.

Proof: It suffices to prove that if $\alpha > \omega_2^V$ is an L-cardinal, then there is an L-cardinal $\beta < \alpha$ s.t. in $V^{\mathrm{Col}(\omega, <\alpha)}$

there is some $j : L_\beta \to L_\alpha$ with $j(\mathrm{crit}(j)) = \kappa = \omega_2^V$.

Fix $\alpha > \kappa$. There is a proper forcing sealing the tree from the canonical $\square_\alpha$ – sequence of L. *) By PFA^-, there is then some $\beta < \kappa$ s.t. II wins $G(L_\beta, L_{\alpha^{+L}})$ and if $\bar\beta$ is the largest cardinal of L_β, then the canonical $\square_{\bar\beta}$ – sequence of L_β is sealed, which implies that β and hence $\bar\beta$ is an L – cardinal. By replacing $L_{\alpha^{+L}}$ by a slightly larger model which knows that ω_1^V, ω_2^V are ω_1 and ω_2, we may make sure that $j(\mathrm{crit}(j)) = \kappa = \omega_2^V$, when $j : L_\beta \to L_{\alpha^{+L}}$ comes from II's w.s., $j \in V^{Col(\omega, <\kappa)}$. ⊣

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*) We may assume $\neg 0^\#$, so that $\mathrm{cf}(\alpha^{+L}) > \omega$.

Appendix. Let us prove that in L, every remarkable cardinal κ carries a Laver function. Work in L.

Let us ~~rec.~~ rec. define $f : \kappa \to V_\kappa$ by: $f(\bar\kappa)$ = the least $\bar x \in V_\kappa$ s.t. if α is least with $\bar x \in V_\alpha$ then in $V^{Col(\omega, <\bar\kappa)}$ there is no $j : V_\beta \to V_\alpha$ with $j(crit(j)) = \bar\kappa$ and $j(\bar x') = \bar x$, some $\bar x' \in V_\beta$.

Assume f is not a Laver fcn., and let (x, α) be the least witness.

Pick $j : V_{\beta^+} \to V_{\alpha^+}$, $j(crit(j)) = \kappa$, $j(\bar x) = x$, some $\bar x \in V_\beta$, $j \in V^{Col(\omega, <\kappa)}$. By elementarity, $(\bar x, \beta)$ is least s.t. in $V^{Col(\omega, <\bar\kappa)}$ there is no $k : V_{\bar\beta} \to V_\beta$, $k(crit(k)) = crit(j)$, and $k(\bar x') = \bar x$, some $\bar x' \in V_{\bar\beta}$. But this means that

$f(crit(j)) = \bar x$. Contradiction!