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The number of Woodin cardinals in K

We report on a result that was proven in Brooklyn in March, 2016.

H. Woodin was the first to have produced situations in which there is a core model with a Woodin cardinal, starting from determinacy hypotheses. Similar constructions were given by Steel and Sargsyan. Later, Sargsyan/Zeman and Sargsyan/Schindler described situations with a K with Woodin cardinals, starting from purely "combinatorial" hypotheses.

For the purpose of this note, by a <u>core model</u> we mean a pure extender model (i.e., a class sized premouse) W such that

(a) W is fully iterable, and

(b) there are stationarily many λ with $\lambda^{+W} = \lambda^{+V}$.

(Usually, W would need to satisfy more properties to be called a core model, e.g., to embed into any universal weasel, to be forcing absolute, etc., properties which also make W unique.)

<u>Theorem 1.</u> Suppose that K is a core model. Then one of the following holds.

(a) K has no Woodin cardinal.

(b) K has exactly one Woodin cardinal and a strong cardinal above.

(c) K has a strong cardinal, and if κ is the least strong cardinal of K, then κ is a limit of Woodin cardinals in K.

There is no anti large cardinal hypothesis here. Also, the statement of the theorem is not supposed to rule ~~of~~ out the possibility that K has Woodin cardinals above the least strong of K.

None of the above cases is void.

(a) : Take $V = L$.

(b) : $V = M_{sw}$, cf. Sargsyan / Schindler, "Varsovian models I"

(c) : Let V be the "least" inner model with a strong cardinal which is a limit of Woodin cardinals. This example will be written up by Stefan Miedzianowski in his Ph. D. thesis.

The proof of Theorem 1 uses the following.

<u>Theorem 2.</u> (J. Steel) Suppose that K is a core model, and K has a Woodin cardinal. Then K has a strong cardinal.

Proof. Let δ be the least Woodin cardinal of K. Assume that K has no strong cardinal, and let $\eta > \delta$ be a V-cardinal such that there is no $E_\nu^K \neq \emptyset$ with $\mathrm{crit}(E_\nu^K) < \eta$ and $\nu > \eta$, and $\eta^{+K} = \eta^{+V}$.

Let $\mathcal{T}$ be a tree on K which starts out by iterating the least total measure of K η times and then makes an initial segment of $K|\eta^+$ generic over the image of $K|\delta$, i.e., $\mathcal{T}$ doesn't involve any drops, $lh(\mathcal{T}) < \eta^+$, writing $i = \pi^{\mathcal{T}}_{0, lh(\mathcal{T})-1}$, $K|i(\delta)$ is generic over $\mathcal{M}^{\mathcal{T}}_{lh(\mathcal{T})-1}$ for the extender algebra at $i(\delta)$, where $\eta < i(\delta) < \eta^+$, and $\mathcal{T}$ is "canonical" with these properties. It will be true that $\rho^K(\mathcal{M}(\mathcal{T}\restriction\lambda))$ will provide Q-structures along the way, i.e., for $\lambda < lh(\mathcal{T})$, as otherwise $\delta(\mathcal{T}\restriction\lambda)$ is a Woodin cardinal of $\rho^K(\mathcal{M}(\mathcal{T}\restriction\lambda))$, $\rho^K(\mathcal{M}(\mathcal{T}\restriction\lambda))[K|\delta(\mathcal{T}\restriction\lambda)] = K$, where $K|\delta(\mathcal{T}\restriction\lambda)$ is generic for the extender algebra, and $\delta(\mathcal{T}\restriction\lambda)$ is a cardinal in $\rho^K(\mathcal{M}(\mathcal{T}\restriction\lambda))[K|\delta(\mathcal{T}\restriction\lambda)]$ because the extender algebra has the $\delta(\mathcal{T}\restriction\lambda)$-c.c., but $\eta < \delta(\mathcal{T}\restriction\lambda) < \eta^+ = \eta^{+K}$. Therefore, $\mathcal{M}(\mathcal{T})$ will be definable over $K|i(\delta)$, and

$\rho^k(\mathcal{M}(\mathcal{I}))[K|i(\delta)] = K$, when $K|i(\delta)$ is generic for the extender algebra, $i(\delta)$ is a cardinal of $\rho^k(\mathcal{M}(\mathcal{I}))[K|i(\delta)]$, but $\eta < i(\delta) < \eta^+ = \eta^{+K}$: if $\rho^k(\mathcal{M}(\mathcal{I}))$ were set-sized and reached a Q-structure Q for $\mathcal{M}(\mathcal{I}) = i(K|\delta)$, then $Q \triangleleft \mathcal{M}^{\mathcal{I}}_{lh(\mathcal{I})-1}$ by universality, $\lightning$.

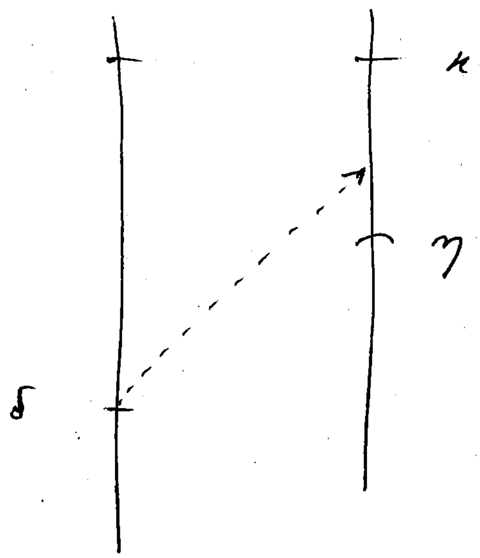
We derived a contradiction. $\dashv$ (Theorem 2)

Proof of Theorem 1. Let us assume that K has a Woodin cardinal, and let δ be the least Woodin cardinal of K. By theorem 2, K has a strong cardinal, so let κ be the least strong of K.

We need to see ~~if~~ that if K has a 2^{nd} Woodin cardinal, then κ is a limit of Woodin cardinals in K.

So let δ_1 be a Woodin cardinal of K,

$\delta < \delta_1 < \kappa$. Let us assume $\eta < \kappa$ to be such that ($\delta_1 < \eta$ and) K doesn't have any Woodin cardinal $\geq \eta$. We may also assume that η is a K-cardinal and there is no $E_\nu^K \neq \emptyset$ with $\mathrm{crit}(E_\nu^K) \leq \eta$ and $\nu > \eta$.



Let $\mathcal{T}$ be a tree on K which starts out by iterating the least total measure of K (and its images) η times and then makes an initial segment of K generic over the image of $K|\delta$, i.e.

- $\mathcal{T}$ doesn't involve any drops, $\mathcal{T}$ is normal,

- $\mathcal{I} \restriction \eta+1$ is given by iterating the least total K–measure (and its images) η times,

- and if $i \geq \eta$, $i+1 < \mathrm{lh}(\mathcal{I})$, then $E_i^{\mathcal{I}}$ is the least total $\mathcal{M}_i^{\mathcal{I}}$–extender which violates an axiom of the (δ–version of the) extender algebra of $\mathcal{M}_i^{\mathcal{I}}$ at $\pi_{0i}^{\mathcal{I}}(\delta)$.

This process must terminate, and we get $\theta = \mathrm{lh}(\mathcal{I})-1$ s.t. $K | \pi_{0\theta}^{\mathcal{I}}(\delta)$ is generic over $\mathcal{M}_\theta^{\mathcal{I}}$ for the extender algebra at $\pi_{0\theta}^{\mathcal{I}}(\delta)$, where $\theta < \eta^{+V}$.

For $\eta < \lambda < \theta$, let us write Q^λ for the least $Q \trianglelefteq \mathcal{M}_\lambda^{\mathcal{I}}$ s.t. $\delta(\mathcal{I} \restriction \lambda)$ is not definably Woodin over Q. Q^λ is always well-defined. Also, $\delta(\mathcal{I} \restriction \lambda)$ is a cardinal of $\mathcal{M}_\lambda^{\mathcal{I}}$, so that by the fact that $\delta(\mathcal{I} \restriction \lambda) < \pi_{0\lambda}^{\mathcal{I}}(\delta)$ and δ is the least Woodin cardinal of K, $\delta(\mathcal{I} \restriction \lambda)$ is not overlapped in Q^λ (i.e., Q^λ is a "tame" Q–structure), which means that there is no $E_\nu^{Q^\lambda} \neq \emptyset$ with $\mathrm{crit}(E_\nu^{Q^\lambda}) \leq \delta(\mathcal{I} \restriction \lambda)$ and $\nu > \delta(\mathcal{I} \restriction \lambda)$.

<u>Claim.</u> For $\eta < \lambda < \theta$, $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ reaches Q^λ.

Proof. Here, $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ denotes the P-construction over $\mathcal{M}(\mathcal{I}\restriction\lambda)$ performed inside K. If $\delta(\mathcal{I}\restriction\lambda)$ is overlapped in K, then by definition $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda)) = P^{ult(K\|\alpha;\,F)}(\mathcal{M}(\mathcal{I}\restriction\lambda))$, where $F = E^K_\nu$ for the least ν s.t. $crit(E^K_\nu) \leq \delta(\mathcal{I}\restriction\lambda)$ and $\nu > \delta(\mathcal{I}\restriction\lambda)$ and α is largest s.t. F measures $P(crit(F)) \cap K\|\alpha$.

The proof of the above claim is by induction on λ. The inductive hypothesis yields that $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ actually makes sense.

<u>Case 1.</u> $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ is class sized.

Then $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ is fully iterable above $\delta(\mathcal{I}\restriction\lambda)$ and computes successors correctly on a stationary class, so that $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ is universal. Hence $P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$ must absorb Q^λ, i.e., $Q^\lambda \trianglelefteq P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$. But then $Q^\lambda = P^K(\mathcal{M}(\mathcal{I}\restriction\lambda))$

by the definition of $\rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$. Contradiction!

<u>Case 2.</u> $\rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$ is set sized.

If $\delta(\mathcal{I}\restriction\lambda)$ is not overlapped in K, then we get $Q^\lambda = \rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$ as in Case 1.

So let $F = E_\nu^K$ be as on p. 8 and $\rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda)) = \rho^{ult(K\|\alpha;F)}(\mathcal{M}(\mathcal{I}\restriction\lambda))$, α also as on p. 8.

If $\alpha = \infty$ (i.e., F is total on K), then again we get $Q^\lambda = \rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$ as in Case 1.

We may thus assume that $\alpha < \infty$, i.e., F is not total on K, and α is the least $\alpha' \geq \nu$ s.t. $\rho_\omega(K\|\alpha') \leq crit(F) \leq \delta(\mathcal{I}\restriction\lambda)$.

By Schindler / Steel, "The self-iterability of $L[E]$," Lemma 1.5, 2nd part (cf. the argument on p. 11 of the .pdf file), if $\rho_\omega(K\|\alpha) < \delta(\mathcal{I}\restriction\lambda)$, then $\delta(\mathcal{I}\restriction\lambda)$ is not definably Woodin in $\rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$ in which case we also get $Q^\lambda = \rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$.

We may thus further assume that

$\rho_\omega(K\|\alpha) = \mathrm{crit}(F) = \delta(\mathcal{I}\restriction\lambda)$. We then get $Q^\lambda = \rho^k(\mathcal{M}(\mathcal{I}\restriction\lambda))$ by the argument for Lemma 1.6 (b) of Schindler / Steel, "The self-iterability of $L[E]$" (notice that Q^λ is "tame", so that the argument on p. 12 f. of the ~~.pdf~~ .pdf file applies).

We have verified the Claim. $\dashv$ (Claim)

By the Claim, we now have that $\mathcal{M}(\mathcal{I})$ is definable over $K | \pi_{0\theta}^{\mathcal{I}}(\delta)$ (where $\pi_{0\theta}^{\mathcal{I}}(\delta) = \delta(\mathcal{I})$).

[this might be literally wrong for the tree $\mathcal{I}$ described as on pp. 6–7; but we could easily build in "delays" so as to make it true. E.g., after a non-trivial limit stage λ, iterate linearily for a while so as to make sure that $\delta(\mathcal{I}\restriction\lambda')$ for the next nontrivial λ' is bigger than the least β s.t. $\rho^{K\|\beta}(\mathcal{M}(\mathcal{I}\restriction\lambda))$ reaches Q^λ.]

Let us write $\rho = \rho^k(\mathcal{M}(\mathcal{I}))$.

$\mathcal{P}$ is fully iterable above $\pi^I_{0\theta}(\delta)$, as is $\mathcal{M}^I_\theta$, so that by the universality of $\mathcal{M}^I_\theta$ and the fact that δ is the least Woodin cardinal of K, $\pi^I_{0\theta}(\delta)$ must be definably Woodin in $\mathcal{P}$.

<u>Case 1.</u> $\mathcal{P}$ is class sized.

Then $\lambda^{+\mathcal{P}} = \lambda^{+V}$ for a stationary class of λ, so that $\mathcal{P}$, $\mathcal{M}^I_\theta$ coiterate to a common $\mathcal{R}$:

$$\begin{tikzcd} \mathcal{M}^I_\theta \arrow[dr, "i"] & \\ & \mathcal{R} \\ \mathcal{P} \arrow[ur, "j"'] & \end{tikzcd}$$

$i \restriction \pi^I_{0\theta}(\delta)+1 = j \restriction \pi^I_{0\theta}(\delta)+1 = \mathrm{id}$, and by elementarity, $\mathcal{R}$, and hence also $\mathcal{P}$, has a Woodin cardinal $> \pi^I_{0\theta}(\delta)+1 > \eta$.

<u>Case 1.A.</u> $\pi^I_{0\theta}(\delta)$ is not overlapped in K.

Then $K|\pi^I_{0\theta}(\delta)$ is generic over $\mathcal{P}$ for the extender algebra at $\pi^{\mathcal{I}}_{0\theta}(\delta)$ and in fact $K = \mathcal{P}[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$. This implies that K has a Woodin cardinal $> \eta$. Contradiction!

<u>Case 1.B.</u> $\pi^{\mathcal{I}}_{0\theta}(\delta)$ is overlapped in K.

Let ι least s.t. $E^K_\iota \neq \emptyset$, $\mathrm{crit}(E^K_\iota) \leq \pi^{\mathcal{I}}_{0\theta}(\delta)$ and $\iota \geq \pi^{\mathcal{I}}_{0\theta}(\delta)$. By case hypothesis, $F = E^K_\iota$ is total on K and $\mathcal{P} = \mathcal{P}^{\mathrm{ult}(K;F)}(\mathcal{M}(\mathcal{I}))$.

By the choice of η (cf. p. 6), $\mathrm{crit}(F) > \eta$. Hence by elementarity, $\mathrm{ult}(K;F)$ does not have any Woodin cardinals $\geq \eta$, as K does not have any Woodin cardinal $\geq \eta$. But $\mathcal{P}[K|\pi^{\mathcal{I}}_{0\theta}(\delta)] = \mathrm{ult}(K;F)$, and $\mathcal{P}$ and hence also $\mathrm{ult}(K;F)$ has a Woodin cardinal $> \eta$. Contradiction!

<u>Case 2.</u> $\mathcal{P}$ is set sized.

As $\pi^{\mathcal{I}}_{0\theta}(\delta)$ is definitely Woodin in $\mathcal{P}$, we must then have that there is $E^K_\iota \neq \emptyset$

with $\operatorname{crit}(E_\nu^K) \le \pi^J_{0\theta}(\delta)$ and $\nu \ge \pi^J_{0\theta}(\delta)$,

and if ν is least such, $F = E_\nu^K$ is <u>not</u> total on K. Let $\alpha < \infty$ be largest such that F measures $\mathcal{P}(\operatorname{crit}(F)) \cap K|\alpha$.

But now we get contradictions as in the proof of the above claim in case 2.

If $\rho_\omega(K||\alpha) < \pi^J_{0\theta}(\delta)$, then $\pi^J_{0\theta}(\delta)$ would not be definably Woodin in $\mathcal{P}$.

Hence $\rho_\omega(K||\alpha) = \operatorname{crit}(F) = \pi^J_{0\theta}(\delta)$.

$K|\pi^J_{0\theta}(\delta)$ is generic over $\mathcal{M}^J_\theta$ at $\pi^J_{0\theta}(\delta)$ which is not overlapped in $\mathcal{M}^J_\theta$. Then $\mathcal{M}^J_\theta[K|\pi^J_{0\theta}(\delta)]$ may be construed as a premouse over $K|\pi^J_{0\theta}(\delta)$ which is fully iterable above $\pi^J_{0\theta}(\delta)$. Let us look at the comparison of $\mathcal{M}^J_\theta[K|\pi^J_{0\theta}(\delta)]$ with K:

$$\begin{array}{ccc} \mathcal{M}^J_\theta[K|\pi^J_{0\theta}(\delta)] & \longrightarrow & \\ & & \mathcal{R} \\ K & \rightsquigarrow & \end{array}$$

As $\rho(K\|\alpha) = \operatorname{crit}(E_\omega^K) \overset{F}{=} \pi^{\mathcal{I}}_{0\theta}(\delta)$, the comparison starts out with a drop on the K-side, and in fact every proper iterate on the K-side will be non-sound. There can then be no drop on the main branch of the $\mathcal{M}^{\mathcal{I}}_\theta[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$ – side, and on this side only extenders with critical points $> \pi^{\mathcal{I}}_{0\theta}(\delta)$ can be used. Hence K wins against $\mathcal{M}^{\mathcal{I}}_\theta[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$.

We may construe the comparison of K, $\mathcal{M}^{\mathcal{I}}_\theta[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$ as a comparison of $K\|\alpha$, $\mathcal{M}^{\mathcal{I}}_\theta[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$ and in fact as a comparison of $\operatorname{ult}(K\|\alpha; F)$, $\mathcal{M}^{\mathcal{I}}_\theta[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$. As $\pi^{\mathcal{I}}_{0\theta}(\delta)$ is definably Woodin in $\mathcal{P}$, $\mathcal{P} \cap OR = \operatorname{ult}(K\|\alpha; F) \cap OR$ and the comparison of $\operatorname{ult}(K\|\alpha; F)$ with $\mathcal{M}^{\mathcal{I}}_\theta[K|\pi^{\mathcal{I}}_{0\theta}(\delta)]$ may be construed as a comparison of $\mathcal{P} = \mathcal{P}^{\operatorname{ult}(K\|\alpha; F)}(\mathcal{M}(\mathcal{I}))$ with $\mathcal{M}^{\mathcal{I}}_\theta$ in which $\mathcal{P}$ wins even though it is set sized. But this is a contradiction as $\mathcal{M}^{\mathcal{I}}_\theta$ is an iterate of K and hence universal.

$\dashv$ (theorem 1)