Some Pathological Sets in Special Model of Set Theory

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Abstract

We Produce a model of ZF+DC in which there are Bernstein sets, Luzin sets, and Sierpinski sets, but there is no Vitali sets and hence no Hamel basis.

Definition.

- $B \subset \omega^\omega$ is called Bernstein iff $B \cap P \neq \emptyset \neq P \setminus B$ for all perfect $P \subset \omega^\omega$.
- Letting $E_0 \subset (\omega^\omega)^2$ be the Vitali equivalence relation defined by $x \mathsf{E}_0 y$ iff $\exists n_0 \forall n \geq n_0, x(n) = y(n)$, $V \subset \omega^\omega$ is called Vitali if V picks exactly one element from each E_0 equvalence class, i.e. $\forall x \exists y \forall z ((z \mathsf{E}_0 x \wedge z \in V) \leftrightarrow z = y)$

Building upon [1], [3] proves that **ZF** + "there is a Bernstein set" does not yield a "Vitali set" V^{rec} in the redefined sense that V^{rec} picks exactly one element from each Turing degree. We here show that a slight variation of the argument of [1] amd [3] show that **ZF** + "there is a Bernstein set" does not yield a "Vitali set" in the original sense as defined above.

Theorem 1. **ZF** + "there is a Bernstein set" does not prove "there is a Vitali set".

Proof . Let G be a $C(\omega_1)$-generic over $\mathbf{L}$, where $\omega_1 = \omega_1^L$ and $C(\omega_1)$ is the finite support product of ω_1 Cohen forcing, cf. [2, p105].

For $\alpha < \omega_1$ let $\beta(\alpha)$ be the least $\beta > \omega$, $\beta > \alpha$ such that $L_\beta \models$ "$\overline{\overline{\alpha}} \leq \aleph_0$" and let $e_\alpha : \omega \leftrightarrow L_\alpha$ be the $L_{\beta(\alpha)}$-least bijection. Let $\mathsf{E}_\alpha \subset \omega \times \omega$ be such that $(\omega; \mathsf{E}_\alpha) \cong^{e_\alpha} (L_\alpha; \in)$ and let g_α be the set of all $n < \omega$ such that there are $\xi < \alpha$, $k, m < \omega$ with $e_\alpha(n) = (\xi, k, m)$ and $p(\xi)(k) = m$ for some $p \in G$. I.e., E_α is a canonical code for L_α and g_α codes $G\alpha$ relative to E_α. We may define $z_\alpha : \omega \to \omega$ by

$$z_\alpha(2l) = \begin{cases} 1, & \text{iff } (l)_0 \mathsf{E}_\alpha (l)_1; \\ 0, & \text{iff } (l)_0 \not\mathsf{E}_\alpha (l)_1. \end{cases}$$

and

$$z_\alpha(2l+1) = \begin{cases} 1, & \text{iff } l \in g_\alpha; \\ 0, & \text{iff } l \notin g_\alpha(l)_1. \end{cases}$$

.

Here, $(l)_0$ and $(l)_1$ is the first and second component, resp., of l where construed as a pair of natural number, i.e., fixing a canonical $e : \omega \leftrightarrow \omega \times \omega$, $e(l) = ((l)_0, (l)_1)$.

Claim 1 For every $x \in \omega^\omega \cap L[G]$ there is some $\alpha < \omega_1$ such that $x \in L[z_\alpha]$.

Proof: Given x, there is some α with $x \in L_\alpha[G\alpha]$. But $L_\alpha[G\alpha] \in L[z_\alpha]$

Claim 2 Let $s \in C(\omega_1)$ be any condition. Let G^S be he collection of all $p \in C(\omega_1)$ for which there is a $q \in G$ with $dom(p(\xi)) = dom(q(\zeta))$ for all $\xi < \omega_1$ and

$$p(\xi)(k) = \begin{cases} s(\xi)(k), & \text{if } k \in dom(s(\xi)); \\ q(\xi)(k), & o.w. \end{cases}$$

Then G^s is $C(\omega_1)$-generic over L, and $L[G^s] = L[G]$. Also, $s \in G^s$. *Proof*: cf. [2]

Claim 3. Let $s \in C(\omega_1)$ be any condition. Let $\alpha < \omega_1$, let

$$g^s_\alpha = \{n : \exists \xi < \alpha, \exists k, m < \omega [e_\alpha(n) = (\xi, k, m) \wedge p(\xi)(k) = m \ \ for \ \ some \ \ p \in G^s]\}$$

and let $z^s_\alpha : \omega \to \omega$ be defined by

$$z^s_\alpha(2l) = \begin{cases} 1, & \text{iff } (l)_0 \mathsf{E}_\alpha (l)_1; \\ 0, & \text{iff } (l)_0 \not\mathsf{E}_\alpha (l)_1. \end{cases}$$

$$z^s_\alpha(2l+1) = \begin{cases} 1, & \text{iff } l \in g^s_\alpha; \\ 0, & \text{iff } l \notin g^s_\alpha. \end{cases}$$

(I.e., z^s_α is defined as z_α above except for using g^s_α instead of g_α)
Then

$$z^s_\alpha \mathsf{E}_0 z_\alpha$$

.

Proof: Immediate, as there are only finitely many pairs (ξ, k) such that there are $p \in G^s$ and $q \in G$ with $p(\xi)(k) \neq q(\xi)(k)$.

Definition. Let us write $d_\alpha = \{z : z\mathsf{E}_0 z_\alpha\}$ for the E_0-equivalence class of z_α. By claim 3, $\{z^s_\alpha : s \in C(\omega_1)\} \subset d_\alpha$ and by Claim 1:

Claim1′. For every $x \in \omega^\omega \cap L[G]$ there is some $\alpha < \omega_1$ s.t. $x \in L[Z]$ for all $z \in d_\alpha$. Let us now consider the model

$$N = HOD^{L[G]}_{\omega^\omega \cap L[G] \cup \{(d_\alpha : \alpha < \omega_1\}}$$

i.e. the class of all $X \in L[G]$ which inside $L[G]$ are hereditarily ordinal definable from parameters in $(\omega^\omega \cap L[G]) \cup \{(d_\alpha : \alpha < \omega_1)\}$, cf. [Sch. p. 86].

$$N \models ZF$$

.

Claim 4. $N \models \neg AC$, i.e., the axiom of choice fails in N, in fact: There is no well-ordering of the reals in N.

Proof: Suppose $L[G]$ has a well-ordering of its reals which is definable from $\vec{\alpha} \in OR$, $\vec{y} \in \omega^\omega \cap L[G]$ and $(d_\alpha : \alpha < \omega_1)$. Let $\alpha < \omega_1$ be such that $\vec{y} \in L_\alpha[G\alpha]$. Let $\tilde{\alpha} > \alpha, \tilde{\alpha} < \omega_1$. Then $z_{\tilde{\alpha}}$ must be definable in $L[G]$ from $\vec{\alpha}, \gamma, \vec{y}$, and $(d_\alpha : \alpha < \omega_1)$ for some ordinal γ. Let us assume w.l.o.g. that $\vec{y} \in \omega^\omega \cap L$; the argument in the general case is just a simple variant of the argument that is to come. There is a formula ϕ such that for all $k, m < \omega$, $z_{\tilde{\alpha}}(k) = m)$ iff

$$L[G] \models \phi(k, m, \vec{\alpha}, \gamma, \vec{y}, (d_\alpha : \alpha < \omega_1)).$$

Let the formula ψ define $(d_\alpha : \alpha < \omega_1)$ from G over $L[G]$, i.e.,
$L[G] \models \forall \vec{d}(\vec{d} = (d_\alpha : \alpha < \omega_1) \leftrightarrow \psi(\vec{d}, \dot{G}))$. Hence $z_{\tilde{\alpha}}(k) = (m)$ iff

$$L[G] \models \text{"}(k, m, \vec{\alpha}, \gamma, \vec{y}, \vec{d}), \quad where \quad \psi(\vec{d}, \dot{G})\text{"}$$

iff

$$\exists p \in G_L^{C(\omega_1)}\text{"}\phi(\check{k}, \check{m}, \check{\vec{\alpha}}, \check{\gamma}, \check{\vec{y}}, \vec{d}), \quad where \quad \psi(\vec{d}, \dot{G})\text{"}$$

Suppose $s \in C(\omega_1)$ is any condition such that $s_L^{C(\omega_1)}\text{"}\neg\phi(\check{k}, \check{m}, \check{\vec{\alpha}}, \check{\gamma}, \check{\vec{y}}, \vec{d})$, where $\psi(\vec{d}, \dot{G})$. Using Claim 2, G^s is $C(\omega_1)$-generic over L, $s \in G^s$, and $L[G^s] = L[G]$. This gives that

$$L[G^s] = L[G] \models \text{"}\neg\phi(k, m, \vec{\alpha}, \gamma, \vec{y}, \vec{d}), \quad where \quad \psi(\vec{d}, G^s)\text{"}$$

. However, Claim 3 beys us that if $\psi(\vec{d}, G^s)$ holds true in $L[G]$, then in fact $\vec{d} = (d_\alpha : \alpha < \omega)$ and therefore $L[G] \models \neg\phi(k, m, \vec{\alpha}, \gamma, \vec{y}, (d_\alpha : \alpha < \omega_1))$. Contradiction!
We have shown that $z_{\tilde{\alpha}}(k) = m$ iff

$$\exists p \in Gp_L^{C(\omega_1)}\text{"}\phi(\check{k}, \check{m}, \check{\vec{\alpha}}, \check{\gamma}, \check{\vec{y}}, \vec{d})\text{"}, where \psi(\vec{d}, \dot{G})\text{"}$$

iff

$$1_{C(\omega_1)}^{C(\omega_1)}{}_L\text{"}\phi(\check{k}, \check{m}, \check{\vec{\alpha}}, \check{\gamma}, \check{\vec{y}}, \vec{d})\text{"}$$

where $\psi(\vec{d}, \dot{G})$. But then $z_{\tilde{\alpha}} \in L$, cf. [Sch., p. 118]. However, $L[z_{\tilde{\alpha}}]$ contains a Cohen real over L. Contradiction!
We have verified Claim 4.

Claim 5. $N \models$ *"There is no Vitali Set".*

Proof. Suppose there is some $V \in N$ such that $V \cap d_\alpha$ is singleton for each $\alpha < \omega_1$. There is then a sequence $(z^*_\alpha : \alpha < \omega_1)$ in N such that $z^*_\alpha \in d_\alpha$ for every $\alpha < \omega_1$. Let $<_\alpha$ be the canonical well-ordering of $L[z^*_\alpha]$ as being defined inside $L[z^*_\alpha]$. For $x \in \omega^\omega \cap L[G]$ let $\alpha(x)$ be the least $\alpha < \omega_1$ such that $x \in L[z^*_\alpha]$. By claim $\grave{1}$, $\alpha(x)$ is always well-defined. We may then define a well-ordere $<$ of $\omega^\omega \cap N$ inside N as follows $x < y$ iff $\alpha(x) < \alpha(y)$ or $\alpha(x) = \alpha(y.)$

Contradiction with Claim 4.

Claim 6. $N \models$ *"There is a Bernstein Set".*

Proof: Let $B = \{b \in \omega^\omega : \exists$ even $\alpha[b \in L[z]$ for all/some $z \in d_{\alpha+1} \wedge b \notin L[z]$ for all/some $z \in d_\alpha]\}$ and $' = \{b \in \omega^\omega : \exists odd\alpha[b \in L[z]$ for all/some $z \in d_{\alpha+1} \wedge b \notin L[Z] forall/some z \in d_\alpha\}$, as being defined in N. Obviously, $B \cap B' = \emptyset$. Let $P \subset \omega^\omega$ be a perfect set in N, say $P = [T]$ for some perfect tree T, $T \in L[z], z \in d_\alpha$, α even. We work in N. Pick $z^* \in d_{\alpha+1}$. We may easily find some $b \in \omega^\omega$ such that $L[T, b] = L[z^*]$. In particular, $b \in L[z^*]$. If $b \in L[z]$, then $L[z^*] = L[T, b] \subset L[z]$, which condracts $z^* \in d_{\alpha+1}$ and $z \in d_\alpha$. Hence $n \notin L[z']$ for any $z' \in d_\alpha$. We have shown that $B \cap P \neq \emptyset$. Virtually the same argument shows that $B' \cap P \neq \emptyset$. But then B is Bernstein.

We may verify that there are Luzin and Sierpinski sets in N.

Definition

- $L \subset \omega^\omega$ is called Luzini iff L is uncountable and $\overline{\overline{L \cap M}} \leq \aleph_0$ for every meager set.
- $S \subset \omega^\omega$ is called Sierpinski iff S is uncountable and $\overline{\overline{S \cap N}} \leq \aleph_0$ for all null set.

In what follows, we shall feel free using the above introduced notions. For each $\alpha < \omega_1$, let $\kappa(\alpha) < \omega_1$ be the least κ such that $L_\kappa[z] \models ZFC^-$ for all/some $z \in d_\alpha$.

Lemma 1. $N \models$ *"There is a Sierpinski set"*.

Proof: Let us define a normal function $f : \omega_1 \to \omega_1$ as follows, working entirely inside N. For $\alpha < \omega_1$, let H_α be the collection of all G_δ null sets which have a reale code in $L_{\kappa(\alpha)[z]}$ for some/all $z \in d_\alpha$. Notice that H is countable, so that $\bigcup H$ is a null set for all $\alpha < \omega_1$. Given $\alpha < \omega_1$, let $f(\alpha)$ be the least $\beta > \alpha$ such that there is some $x \in \omega^\omega$ such that for all/some $z \in d_\beta$ and for all/some $\bar{z} \in d_\alpha$:

$$x \in L_{\kappa(\beta)} \setminus (L_{\kappa(\alpha)} \cup \bigcup H)$$

For limit λ, let $f(\lambda) = \sup_{\alpha<\lambda} f(\alpha)$. We then let S be the collection of reals x such that for some $\alpha < \omega_1$ with β being $f(\alpha)$, i.e. $S = \{x \in \omega^\omega : \exists \alpha < \omega_1[x \in L_{\kappa(f(\alpha))[z]} \setminus (L_{\kappa(\alpha)[\bar{z} \cup \bigcup H)}$ for some/all $z \in d_{f(\alpha)}$ and $\bar{z} \in d_\alpha])\}$. It is easy to see that S is a Sierpinski set. Virtually the same proof shows:

Lemma 2. $N \models$ *" There is a Luzini Set"*.

Proof: As the proof of the previous lemma, replacing H with the collection of all meager sets which have a real code in $L_{\kappa(\alpha)[z]}$, for some/all $z \in d_\alpha$. For the record, let us also state:

Lemma 3. $N \models$ *"There is no Hamel basis"*.
This immediately follows from above mentioned results about Bernstein sets and Vitali sets together with the following.

Definition. Recall that a Hamel basis is a basis for R construed as a vector space over Q.

Lemma (*Folklore*). In ZFC, if there is a Hamel basis, then there is a Vitali Set.

Proof: Fix a Hamel basis B. For each x, there is a unique finite $b_x \subset B$ of leas size such that $[x]_{\mathsf{E}_0} \subset < b_x >$. Using a well-ordering of the finite sequences of rational, we may then for each $x \in \omega^\omega$ pick $y \in [x]_{E_0}$ such that if

$$y = \sum \vec{r}\, b_x, \qquad \vec{r} \in^{<\omega} Q,$$

then $\vec{r}$ is the least $\vec{r'}$ such that $\sum \vec{r'} \in [x]_{\mathsf{E}_0}$. this gives a Vitali set.

We showed there are Bernstein, Luzini and Sierpinski sets in N, but no Vitali sets and no Hamel basis.

References

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