

III.

Last time : We saw that the models \mathcal{M}_ξ from the K^c construction are countably iterable as long as they are all domestic.

It was also said that we may not use a reflection argument to see that they are in fact fully iterable ; actually, there are counterexamples.

Today, I first want to give an example of an application where full iterability would be needed. We'll then study the problem of the full iterability of K^c which will lead to the core model induction technique.

The pcf "conjecture" states that for a set a of regular cardinals,

$$\overline{\overline{\text{pcf}(a)}} = \overline{a}.$$

It has to be wrong if $2^{\aleph_0} < \aleph_\omega$, but

$$\aleph_\omega^{\aleph_0} > \aleph_\omega.$$

We get models with Woodin cardinals from this hypothesis, which is not known to be consistent.

However, we also get models with Woodin cardinals from a hypothesis which Gitik has shown to be consistent and which is related to the pcf "conjecture."

Theorem (Gitik, Sch, Shelah)

Let κ be a singular cardinal of uncountable cofinality. Suppose

$$\{ \alpha < \kappa : 2^\alpha = \alpha^+ \}$$

to be stationary and co-stationary.

Then for every $n < \omega$, there is an inner model with n Woodin cardinals.

I do not want to sketch the proof of this theorem, but I want to show you an aspect of the proof in order to convince you that full iterability of inner models is an issue.

The above hypothesis formulates a strong version of the failure of SCH.

The hypothesis of the theorem gives many increasing sequences

$$(\kappa_i : i < \omega)$$

of singular cardinals below κ s.t.

$$\text{cf}(\prod \kappa_i^+) > (\sup_i \kappa_i)^+ = \lambda^+$$

The plan is to show that for $W = K^c$

(or $W =$ a better model than K^c) s.t.

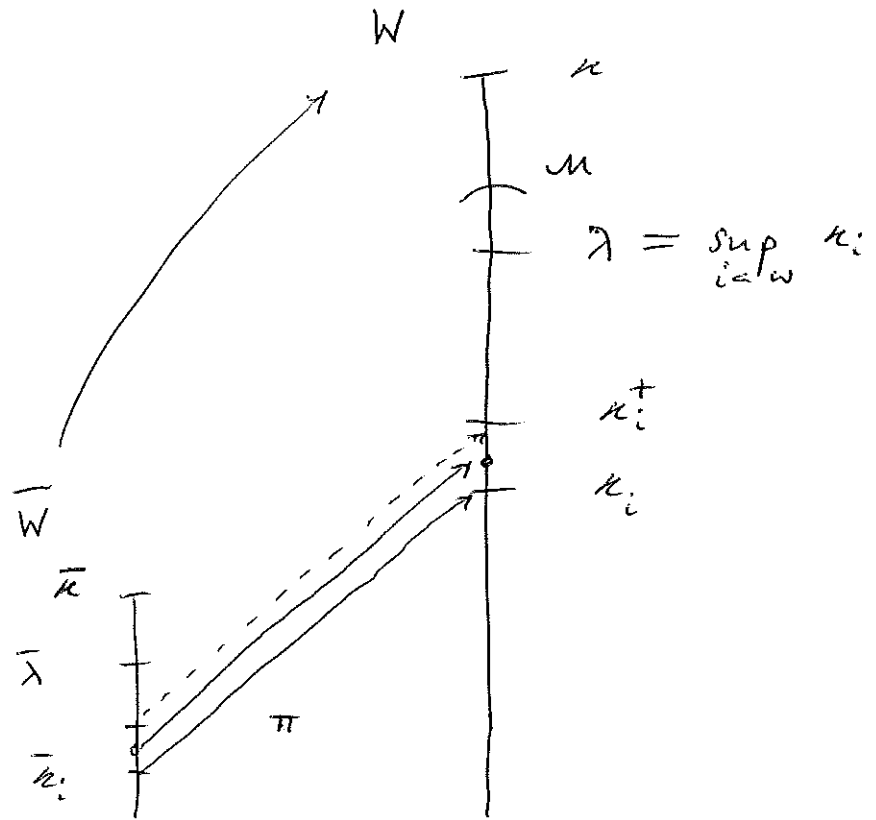
$W \models \text{GCH}$,

$$\{ f \upharpoonright \{\kappa_i : i < \omega\} : f \in W \}$$
$$f : \lambda \rightarrow \lambda$$

is cofinal in $\prod \kappa_i^+$.

This will certainly yield a contradiction.

The key idea is to use a covering argument.



Let $f \in \prod_{i < \omega} \kappa_i^+$; $f: \omega \rightarrow \lambda$, $f(i) < \kappa_i^+$

f.o. $i < \omega$.

Pick $\pi: \bar{W} \rightarrow W$ s.t. \bar{W} is transitive,

$\text{Card}(\bar{W}) = \aleph_1$, $f(i) \in \text{ran}(\pi)$ f.o. $i < \omega$.

The plan is to argue that there be some $\mu \triangleleft W$ s.t. for all but finitely many $i < \omega$,

$$\text{ran}(\pi) \cap \kappa_i^+ \subset \text{Hull}^\mu(\kappa_i \cup \{p\}),$$

some fixed $p \in \mu$.

We may then set

$$\tilde{f}(\xi) = \sup \left(\text{Hull}^M(\xi \cup \{p\}) \cap \xi^{+\kappa} \right),$$

where $\xi < \lambda$.

Then $\tilde{f} : \lambda \rightarrow \lambda$, $\tilde{f} \in W$, and because

$$\kappa_i^{+W} = \kappa_i^+$$

f.a. $i < \omega$ (as all the κ_i are singular) and

$$f(i) \in \text{ran}(\pi) \cap \kappa_i^+ \subset \text{Hull}^M(\kappa_i \cup \{p\}),$$

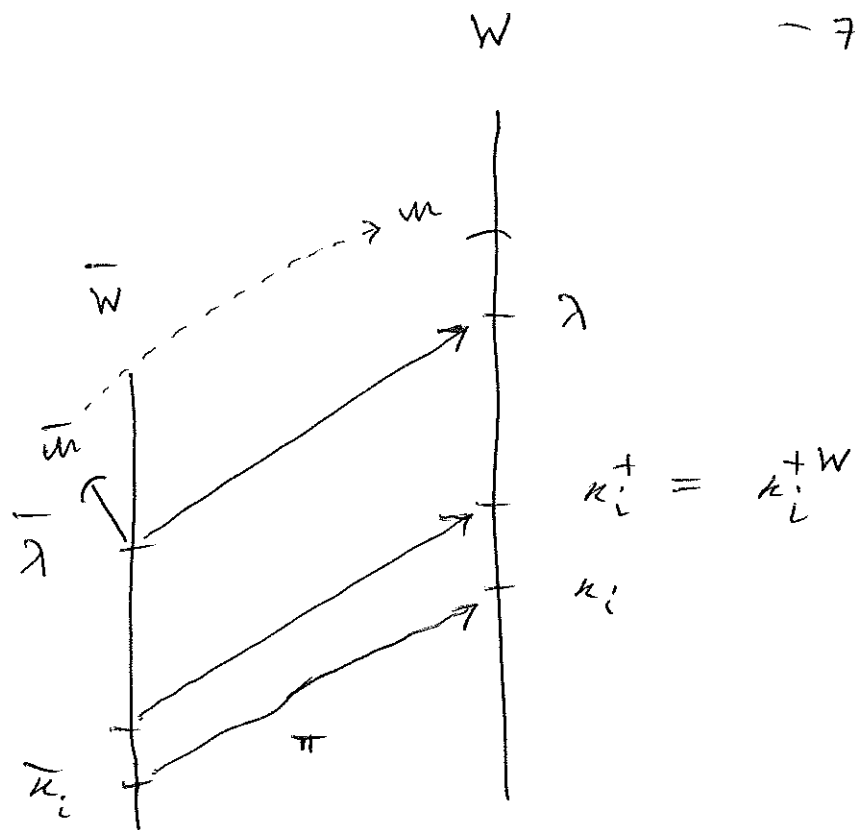
we get that

$$f(i) < \tilde{f}(\kappa_i).$$

i.e., $\tilde{f} \upharpoonright \{\kappa_i : i < \omega\}$ majorizes f , and

$\tilde{f} \in W$. \tilde{f} is thus as desired.

Where do we get such an M from?



The plan for this is :

- Iterate \bar{W}, W .
- Show that (π may have been chosen in such a way that) \bar{W} does not move in the iteration.
- The iteration produces an $\bar{\mu}$ s.t.

$$\pi^{-1}(k_i^+) \subset \text{Hull}^{\bar{\mu}}(\bar{k}_i \cup \{\bar{p}\}), \text{ some } \bar{p}.$$

- Then, setting $\mu = \text{Ult}(\bar{\mu}; \pi \upharpoonright \bar{\lambda})$,

$$\text{ran}(\pi) \cap k_i^+ \subset \text{Hull}^{\mu}(k_i \cup \{p\}),$$

where $p = \pi_{\bar{\mu}\mu}(\bar{p})$.

Point is: We obviously need more than countable iterability of W to show that this works.

We in fact need the full iterability of W !

On the other hand, the iterability proof for (the models from the) K^c (construction) really just produces cble. iterability.

We have to use a reflection argument to show

cble. iterability \Rightarrow full iterability.

In order for this reflection argument to work out, we need that V is closed under operators which certify branches thru iteration trees.

Let the premouse \mathcal{M} be ctblly. itrable.

How would you try proving that \mathcal{M} is fully itrable?

(1) We need a candidate for a full iteration strategy for \mathcal{M} . Call it Σ .

(2) We need to argue: if the iteration

$$\tilde{\mathcal{I}} = (\mathcal{M}_\alpha, \pi_{\beta\alpha} : \beta \leq_T \alpha < \gamma)$$

is according to Σ , then all the models from $\tilde{\mathcal{I}}$ are transitive, and if γ is a limit ordinal, then $\Sigma(\tilde{\mathcal{I}}) \downarrow$.

We need to reflect a potential failure of (2) down into H_{w_1} .



Pick $\sigma: H \rightarrow V$, H c.t.l.e. and transitive.

Let $\bar{M}, \bar{I} = \sigma^{-1}(M, I)$.

$$\bar{I} = (\bar{M}_\alpha, \bar{\pi}_{\beta\alpha} : \beta \leq_{\bar{I}} \alpha < \bar{j})$$

is a c.t.l.e. iteration of \bar{M} .

Suppose I is according to Σ , all the models are transitive, j (and hence \bar{j}) is a limit ordinal, and we search for a cofinal branch thru \bar{I} (which is according to Σ).

Let $\bar{\Sigma}$ be an iteration strategy for \bar{M} w.r.t. countable iterations of \bar{M} . So

$$\bar{\Sigma}(\bar{I}) \downarrow, \text{ say } = b.$$

Say there is an initial segment

$$Q \trianglelefteq M_{\bar{I}} = \text{the direct limit model according to } b$$

which can be identified in H , i.e., is an element of H and is definable in H .

Then by absoluteness, for the right θ ,

$$H^{\text{Col}(\omega, \theta)} \models \text{"there is a cofinal branch } b' \text{ thru } \bar{I} \text{ s.t. } Q \trianglelefteq M_{\bar{I}'} \text{"}$$

and if Q identifies b , then $b' = b \in H$ by homogeneity and b is definable in H via Q .

But then $\sigma(b)$ is a perfect candidate for $\Sigma(\bar{I})$.

Example: If there is no inner model with a Woodin cardinal and M has no definable Woodin cardinal, then this argument works with

$Q =$ the least initial segment of $L[M(\mathcal{I})]$ which kills the Woodinness of $\delta(\mathcal{I})$.

(Here, $\delta(\mathcal{I}) = \sup$ of the indices of the extenders used in \mathcal{I} ; $M(\mathcal{I}) =$ the "common part model" of \mathcal{I} ; $N \triangleleft M(\mathcal{I})$ iff $N \triangleleft M_\alpha$ for a tail end of α 's, where M_α is the α^{th} model from the iteration \mathcal{I} .)

On the other hand, under unfavorable circumstances, models with Woodin cardinals need not be fully iterable:

Theorem (Woodin). Let M be a fully
iterable premouse, $M \models \text{"}\delta \text{ is a Woodin cardinal.}"$

There is then a poset $\mathbb{P} \in H_{\delta^+}^M$ which
has the δ -c.c. in M s.t. for every set
 A of ordinals whatsoever there is some iterate

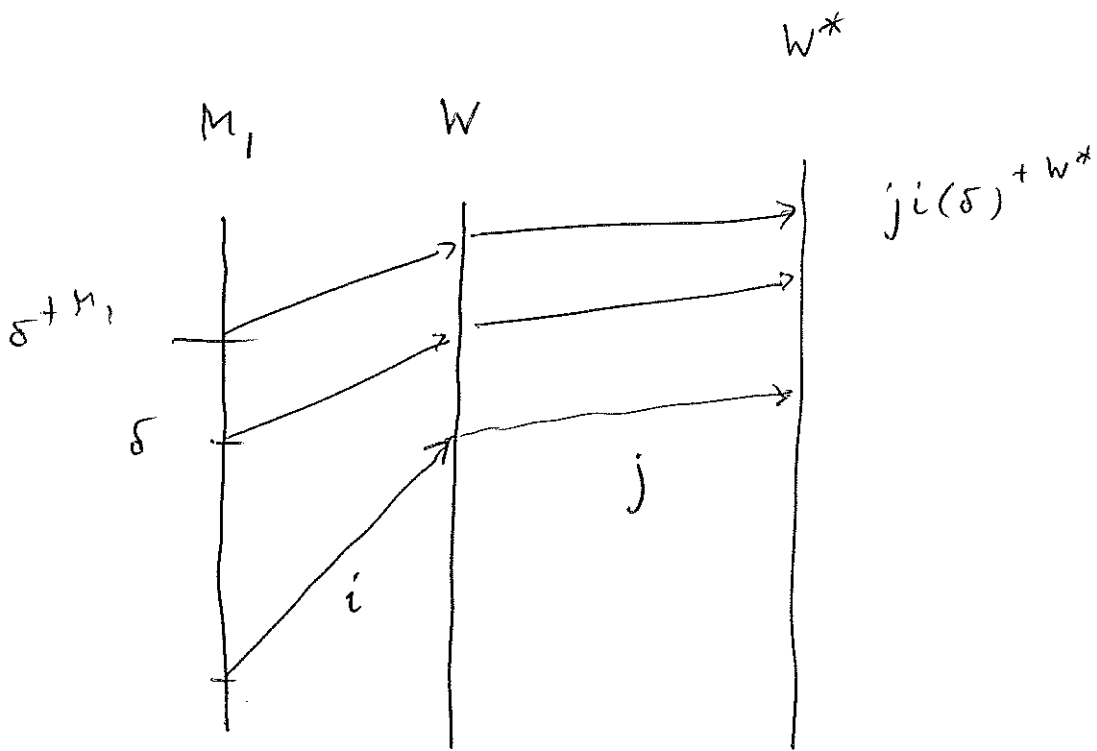
$$i : M \longrightarrow M^*$$

s.t. A is $i(\mathbb{P})$ -generic over M^* .

Now let $M_1 = L[E]$ be the least premouse
with a Woodin cardinal, δ . Basically, $E \subset \delta$.

Suppose that

$$M_1 \models \text{"I'm fully iterable."}$$



Let $W =$ the iterate of M_1 obtained by hitting the least measure of M_1 (and its images) δ^+ times, and let W^* be a further iterate s.t. E is generic over W^* . Then

$$W^*[E] = L[E] = M_1.$$

$j^i \in M_1$, so $j^i \uparrow \delta^+$ witnesses that
in M_1 ,

$$\begin{aligned} cf(j^i(\delta^+ M_1)) &= cf(j^i(\delta)^+ W^*) \\ &= cf(j^i(\delta)^+ M_1) = \delta^+. \end{aligned}$$

Contradiction!

There is hence nothing that might guarantee in general that K^c , albeit always being countably iterable, is fully iterable.

As in the example of M_1 , it might just be that V is not saturated by the relevant Q -structures which identify cofinal branches thru iterations of K^c .

The idea of the Core model induction, first developed by H. Woodin and later extended by J. Steel and others, is to inductively show V is closed under the relevant Q -structures and always work in local universes in which the K^c produced there is either fully iterable or provides the "next Q -structure."

Let us discuss this in the case of the above example in which κ is a singular cardinal, $\text{cf}(\kappa) > \omega$, and

$$\{ \alpha < \kappa : 2^\alpha = \alpha^+ \}$$

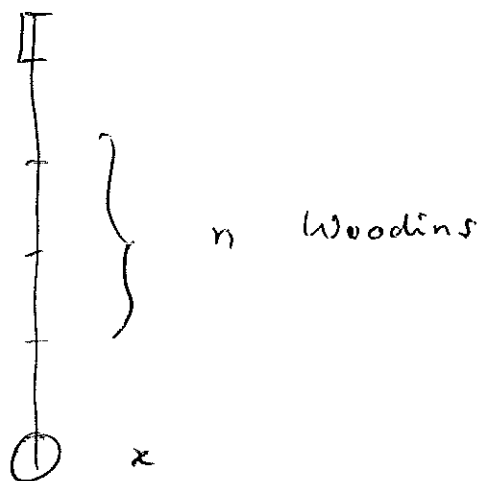
is stationary and costationary.

We may then first show that every set in H_κ has a $\#$.

Now suppose that for every set x in H_κ , $M_n^\#(x)$ exists, but $M_{n+1}^\#(x_0)$ does not exist, some $x_0 \in H_\kappa$.

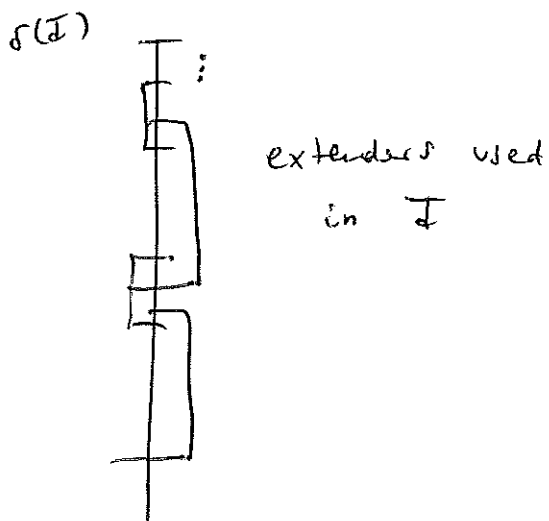
Say $x_0 = \emptyset$.

Here, $M_n^\#(x) =$ the least premouse over x which has a measure above n Woodin cardinals and which is ctly. iterable.



In this situation, let \mathcal{I} be an iteration of K^c , say, where \mathcal{I} has limit length $< \kappa$, and \mathcal{I} lives on K^c / λ , some $\lambda < \kappa$.

Let $\mu(\mathcal{I})$ be the common part model of \mathcal{I} , and let $\delta(\mathcal{I})$ be its height.



Then (an initial segment of) $M_n^\#(\mu(\mathcal{I}))$ will serve as the \mathcal{Q} -structure which identifies the correct branch thru \mathcal{I} .

Uses :

Theorem. (Martin, Steel) If $b \neq c$ are cofinal branches thru \mathcal{I} , then $\delta(\mathcal{I})$ is

Woodin in $wfp(\mu_b^{\mathcal{I}}) \cap wfp(\mu_c^{\mathcal{I}})$.

The reflection argument from above then thus shows that K^c/κ is κ -iterable.

We may then isolate a model W , namely the true core model K of height κ , for which the covering argument which we discussed above can be made work.

We'll have

$$K \cong X \prec K^c/\kappa$$

for an appropriate hull X . K will have the following property:

$$\text{If } \sigma : W \longrightarrow K,$$

then either W loses the coiteration against K (i.e., is strictly weaker than K), or else $W = K$. (Rigidity)

Other properties of K :

Forcing absoluteness : $K^{V^P} = K$

for all $P \in H_\kappa$.

Weak covering : $\cf(\lambda^{+\kappa}) \geq \bar{\lambda}$

whenever $\aleph_2 \leq \lambda < \kappa$.

This is a theorem of Mitchell, Schimmerling, Steel.

Local definability : $K \upharpoonright \lambda$ may be defined inside H_λ , where $\aleph_0 < \lambda < \kappa$.

K inherits the full iterability from K^c .

Thru results of Martin, Steel, and Woodin,
the above argument shows Projective
Determinacy, i.e., that all sets of reals
which are in $\mathcal{J}_2(\mathbb{R})$ are determined.

$$[\mathcal{J}_1(\mathbb{R}) = V_{\omega+1}, \text{ etc.}]$$

The core model induction now uses $L(\mathbb{R})$
as its guide in that:

We show inductively that (an initial
segment of) V is closed under
mice which correspond to the determinacy
of all sets of reals in $\mathcal{J}_\alpha(\mathbb{R})$, $\alpha \geq 2$.

Either the "next" mouse with a Woodin
cardinal exists, or else we may isolate
 K to derive a contradiction.

The mouse closure will serve as a basis for the models we are about to produce to have terms in them which capture a given set of reals of the next complexity class; we'll use:

Definition. Let M be a countable mouse with a Woodin cardinal, δ . Let $A \subset \mathbb{R}$, let $\tau \in M^{\text{CoI}(\omega, \delta)}$, and let Σ be the iteration strategy for M . We then say that τ, Σ capture A iff for all

$$i: M \rightarrow M^* \quad (M^* \text{ still cttk.})$$

according to Σ and for all $g \text{ CoI}(\omega, i(\delta))$ -generic over M^* , $g \in V$,

$$A \cap M^*[g] = \tau^g.$$

In the above situation,

$$A = \bigcup \left\{ \tau^g : g \in V \text{ generic over a } \right. \\ \left. \text{c.c.t.l.e. iterate } M^* \text{ of } M \right\}$$

Theorem. (Neeman) Let M, A, τ, Σ be as above. Then A is determined.

The core model induction has various cases.

Notice:

"there is a set of reals which is not determined" is Σ_1 ,

if we count $\forall x \in \mathbb{R}$ and $\exists x \in \mathbb{R}$ as bounded quantification.

Therefore, if α is least s.t. $J_\alpha \not\models AD$

(AD = the axiom of determinacy), then

α begins a Σ_1 -gap:

Definition. Let $\alpha \leq \beta$. Then $[\alpha, \beta]$ is a Σ_1 -gap

(in $L(\mathbb{R})$) iff

- $J_\alpha(\mathbb{R}) \prec_{\Sigma_1}^{\mathbb{R}} J_\beta(\mathbb{R})$
- $J_\alpha(\mathbb{R}) \not\prec_{\Sigma_1}^{\mathbb{R}} J_\alpha(\mathbb{R})$ for all $\bar{\alpha} < \alpha$
- $J_\beta(\mathbb{R}) \not\prec_{\Sigma_1}^{\mathbb{R}} J_\beta(\mathbb{R})$ for all $\bar{\beta} > \beta$.

The Σ_1 -gaps partition the class of all ordinals.

The core model induction works by induction on the gaps.

Main cases:

- (1) α is inadmissible and the previous gap, if there is one, is not strong
- (2) α ends a weak proper gap or it begins one, and there is a previous gap which is strong.

In the inadmissible gap case we can proceed as discussed above.

In the weak gap case we have to construct a new kind of premice, hybrids.

Say $[\beta, \alpha]$, $\beta < \alpha$, is the weak gap.

Let $m < \omega$ be least s.t. a new set of reals, A , is $\sum_m J_\alpha(\mathbb{R})$ -definable.

Then $A = \bigcup_{n < \omega} A_n$, where $A_n \in J_\alpha(\mathbb{R}) \forall n$.

The inductive hypothesis will give us a "suitable" premouse with an iteration strategy with condensation, i.e.

a ctte. mouse \mathcal{N} with an iteration strategy Σ , $\mathcal{N} \models \delta$ is Woodin", and terms τ_n , $n < \omega$, s.t. τ_n, Σ capture $A_n \forall n$,

The hybrids look like ordinary mice except for that where we closed under rud before we will now in addition feed in information about how to iterate \mathcal{W} according to Σ .

Hybrid premise: $\exists_j [\mathcal{W}, \vec{E}, \Sigma]$.

As Σ satisfies condensation, we may do a K^c, Σ construction in much the same way as we did a K^c construction before.

Once we found a hybrid mouse with a Woodin cardinal which has an iteration strategy Γ which moves Σ correctly, we may use Neeman's theorem to deduce

A is determined:

Let $M = \mathcal{J}_\gamma [W, \vec{E}, \Sigma]$ be a hybrid mouse with a Woodin cardinal, δ .

We may define a term $\tau \in M^{\text{CoI}(\omega, \delta)}$ in such a way that for $x \in \mathbb{R} \cap M^{\text{CoI}(\omega, \delta)}$,

$\Vdash x \in \tau$ iff

if x is made generic over an iterate of \mathcal{W} using Σ , then x is in the interpretation of the image of one τ_n , new.

τ will then capture A .

We need that M be iterable in a way that Σ is moved correctly, and

that Σ , as given to M , will extend to Σ , restricted to $M^{\text{CoI}(\omega, \delta)}$, in a definable way.

Applications of the core model induction :

Theorem (Woodin) If there is an ω_1 -dense ideal on ω_1 , then $AD^{L(\mathbb{R})}$ holds.

Theorem. (Steel) If PFA holds, then $AD^{L(\mathbb{R})}$ holds.

(The stacking technique today gives a stronger result, but it might be that an extension of the core model induction produces a stronger result than the stacking technique.)

Theorem (Bartsh, Schindler) If every uncountable cardinal is singular, then $AD^{L(\mathbb{R})}$ holds.

Extensions of the core model induction technique beyond $L(\mathbb{R})$:

Theorem (Ketchum) Suppose CH + there is an ω_1 -dense ideal on $\omega_1 + \varepsilon$. There is then a model of $AD + \theta_0 < \theta$ of the form $L(\mathbb{R}, A)$, some $A \subset \mathbb{R}$.

The set A in this theorem is actually an iteration strategy for a "full" mouse producing HOD / θ of the maximal model of $AD + \theta_0 = \theta$.

More generally:

Theorem, (Sargsyan) Suppose CH + there is an ω_1 -dense ideal on $\omega_1 + \varepsilon$. There is then a model of $AD_{\mathbb{R}} + \theta$ is regular.

By work of Woodin, this gives an equiconsistency.

The proof of the Ketchersid-Sargsyan result uses an extension of the core model induction technique beyond $L(\mathbb{R})$.

Given a model $L(\mathbb{R}, \Gamma) \models AD + \theta_\alpha = \theta$, one starts out by analyzing its HOD / θ and representing it as a direct limit of a countable hod-mouse \mathcal{N} . One then finds an iteration strategy Σ for \mathcal{N} which cannot be in $L(\mathbb{R}, \Gamma)$; using condensation for Σ , one runs a core model induction to show AD in $L(\mathbb{R}, \Sigma)$. But $L(\mathbb{R}, \Gamma)$ was taken to be maximal, and therefore $AD + \theta_\alpha < \theta$ holds true in $L(\mathbb{R}, \Sigma)$.

Questions.

(1) Suppose κ is a limit cardinal with $\omega < \cf(\kappa) < \kappa$, and

$$\{\alpha < \kappa : 2^\alpha = \alpha^+\}$$

is stationary and costationary in κ .

Does AD hold in $L(\mathbb{R})$?

Is there a model of $AD_{\mathbb{R}} + \Theta$ regular?

(2) Suppose that every uncountable cardinal is singular.

Is there a model of $AD_{\mathbb{R}} + \Theta$ regular?

How do you go beyond $AD_{\mathbb{R}} + \Theta$ regular from these hypotheses?

Further questions :

(3) Let κ be a singular strong limit cardinal, and suppose \square_κ fails.

Is there a model of $AD_{\mathbb{R}} + \theta$ regular?

(4) Suppose PFA holds.

Is there a model of $AD_{\mathbb{R}} + \theta$ regular?

Is there an inner model with a supercompact cardinal?

(5) Suppose κ is strongly compact.

Is there an inner model with a supercompact cardinal?

(4) + (5) are certainly holy grails of inner model theory.