A MODEL WITH EVERYTHING EXCEPT FOR A WELL-ORDERING OF THE REALS

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Abstract. We construct a model of $\mathsf{ZF} + \mathsf{DC}$ containing a Luzin set, a Sierpiński set, as well as a Burstin basis but in which there is no well ordering of the continuum.

1. Introduction

In this paper we study subsets of the real line $\mathbb{R}$ with specific properties whose classic constructions were performed by assuming various forms of the Axiom of Choice (AC). The first *pathological* set was constructed by F. Bernstein in 1908 (cf. [5]); he constructed a set $B \subset \mathbb{R}$ of cardinality the continuum such that neither B nor $\mathbb{R} \setminus B$ contains a perfect subset of reals. Such a set can be obtained by assuming the existence of a well-ordering of $\mathbb{R}$. Later in 1914, Luzin constructed an uncountable set $\Lambda \subset \mathbb{R}$ having countable intersection with every meager set (cf. [19]). His construction required the continuum hypothesis (CH, in the strong form according to which $\mathbb{R}$ may be well-ordered in order type ω_1). In 1924, Sierpiński developed a similar construction to the one given by Luzin; under the assumption of the same form of CH, he constructed an uncountable set $S \subset \mathbb{R}$ having countable intersection with every measure zero set (cf. [27]).

However CH is not a necessary assumption for the existence of Luzin and Sierpiński sets (see [22]). Moreover a Luzin set may exist in a model in which the set of reals is not well-ordered. In fact, D. Pincus and K. Prikry [23] proved that in the Cohen-Halpern-Lévy model H, a model in which the reals cannot be well-ordered (in fact, in H there is an uncountable set of reals with no countable subset), there is a Luzin set as well as a Vitali set. Additionally, Pincus and Prikry asked whether a Hamel basis, i.e., a basis for $\mathbb{R}$ construed as a vector space over the field of rational numbers $\mathbb{Q}$, exists in H or, in general, if the existence of a Hamel basis is compatible with the non-existence of a well-ordering of the reals. Recently, M. Beriashvili, R. Schindler, L. Wu and L. Yu (cf. [4]) answered this question in the affirmative, by showing that in H there is a Hamel basis and, furthermore, in H there is also a Bernstein set (see [4, Theorems 1.7 and 2.1]). Thus the model H has many pathological sets of reals, but in H the continuum cannot be well ordered. There is no Sierpiński set in H, though (see [4, Lemma 1.6]).

Let us informally refer to a model M as a "Solovay model" iff M is obtained via a symmetric collapse over a model in which what is to become ω_1^M is either inaccessible or a limit of large cardinals (e.g., Woodin cardinals). The paper [24] shows that if U is a selective ultrafilter on ω which was added by forcing over a Solovay model M, then $M[U]$ satisfies the Open Coloring Axiom (see [24, p. 247]), hence $M[U]$ inherits from M the property that every uncountable set of reals has a perfect subset and in particular $M[U]$ does not contain a well–ordering of the reals, see [24, Theorem 5.1].

The paper [17] further explores this topic and studies which consequences of having a well–ordering of $\mathbb{R}$ remain false when adding certain ultrafilters on ω over a Solovay model or when adding a Vitali set. Also, [17] produces a model of ZF plus DC plus "there is a Hamel basis" plus "there is no well–ordering of the reals." The verification in [17] that the extension of the Solovay model via forcing with countable linearly independent sets of reals (called $\mathbb{Q}_H$ in the current paper,

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see Definition 4.9 below) doesn't have a well–ordering of its reals uses large cardinals, specifically Woodin's stationary tower forcing. The forcing $\mathbb{Q}_H$ used by [17] does not work in the absence of large cardinals, though, see Corollary 4.11 below.

The current paper improves the result obtained in [4] by showing that there is a model W of $\mathsf{ZF} + \mathsf{DC}$ such that in W the reals cannot be well-ordered and W contains Luzin as well as Sierpiński sets and also a Burstin basis, i.e., a set which is simultaneously a Hamel basis and a Bernstein set. Notice that from the existence of a Hamel basis one can derive that in W there is also a Vitali set (see [4, Lemma 1.1]).

2. Basic definitions and results

2.1. Pathological sets within ZFC.

Definition 2.1. Let $A \subseteq \mathbb{R}$ uncountable. We say that A is

- (i) a *Vitali set* if A is the range of a selector for the equivalence relation $\sim_{\mathbb{Q}}$ defined over $\mathbb{R} \times \mathbb{R}$ by $x \sim_{\mathbb{Q}} y \iff x - y \in \mathbb{Q}$;
- (ii) a *Sierpiński set* if for every $N \in \mathcal{N}$ -the ideal of null-sets with respect to Lebesgue measure over $\mathbb{R}$- we have $|A \cap N| \leq \omega$;
- (iii) a *Luzin set* if for every $M \in \mathcal{M}$ -the ideal of the Borel meager sets- we have $|A \cap M| \leq \omega$;
- (iv) a *Bernstein set* if for every perfect set $P \subseteq \mathbb{R}$ we have $A \cap P \neq \varnothing \neq (\mathbb{R} \smallsetminus A) \cap P$;
- (v) a *Hamel basis* if A is a maximal linearly independent subset of $\mathbb{R}$ when we consider it as a vector space over $\mathbb{Q}$.
- (vi) a *Burstin basis* if A is a Hamel basis which has nonempty intersection with every perfect set.

The existence of a Hamel basis in a model of $\mathsf{ZF} + \mathsf{DC}$ implies the existence of nonmeasurable sets and the existence of sets without the Baire property. In particular, we have the next result connecting Hamel bases and Vitali sets. For a proof, see [4, Lemma 1.1].

Lemma 2.2. (Folklore) *Suppose $V \models \mathsf{ZF}$ and suppose that a Hamel basis H exists. Then there is a Vitali set.*

Lemma 2.3. (Luzin, 1914, and Sierpiński,1924) *Assume V is a model of $\mathsf{ZFC} + \mathsf{CH}$. Then, there are Λ and S in V such that Λ is a Luzin set and S is a Sierpiński set.*

Proof. Let $\{N_i : i < \omega_1\}$ be an enumeration of all G_δ null sets. Recursively define $\langle x_i : i < \omega_1 \rangle$ such that $x_i \notin \bigcup \{N_j : j < i\} \cup \{x_j : j < i\}$. Then, $S = \{x_i : i < \omega_1\}$ is a Sierpiński set.

The same procedure gives us a Luzin set, starting out with an enumeration $\{M_i : i < \omega_1\}$ of all F_σ-meager sets. □

Remark 2.4. As we may write $\mathbb{R} = N \cup M$ where N is null and M is meager, no set can be both a Sierpiński set as well as a Luzin set.

The construction of a Bernstein set under AC is based on the enumeration of all perfect subsets of $\mathbb{R}$. We omit the proof and instead present below the construction of a Burstin basis in V under AC (see Theorem 2.6).

Proposition 2.5. (Folklore) *Every Burstin basis is a Bernstein set.*

Proof. Suppose $B \subseteq \mathbb{R}$ is a Burstin basis such that $P \subseteq B$ for some perfect $P \subseteq \mathbb{R}$. As B is linearly independent, the set $2P = \{2p : p \in P\}$ has empty intersection with B. On the other hand, $2P$ is a perfect set, so $2P \cap B \neq \varnothing$, which gives a contradiction. It follows that B is totally imperfect, so $(\mathbb{R} \smallsetminus B) \cap P \neq \varnothing$ as well, i.e., B is a Bernstein set. □

It is easy to construct a Hamel basis H such that $H \cap P = \emptyset$ for some perfect set P; no such H can then be a Burstin basis. It is also not hard to construct a Hamel basis H which contains a perfect set (see e.g. [15, Example 1, p. 477f.]); no such H can be a Burstin basis either.

Theorem 2.6. (Burstin, 1916) *Assume $V \models \mathsf{ZFC}$. Then there is a Burstin basis B.*

Proof. Suppose $\{P_i : i \leq 2^{\aleph_0}\}$ is an enumeration of all perfect subsets of $\mathbb{R}$. By transfinite recursion we are going to define a set $\{b_\alpha : \alpha < 2^{\aleph_0}\} \subseteq \mathbb{R}$ such that

- (i) $b_\alpha \in P_\alpha$ for every $\alpha < 2^{\aleph_0}$

(ii) for every $\beta < 2^{\aleph_0}$, the set $\{b_\alpha : \alpha < \beta\}$ is linearly independent

Suppose that $\beta < 2^{\aleph_0}$ and we already have defined the collection $\{b_\alpha : \alpha < \beta\}$ satisfying (i) and (ii) above.

Consider the set $\operatorname{span}\{b_\alpha : \alpha < \beta\}$. Note that

$$|\operatorname{span}\{b_\alpha : \alpha < \beta\}| \leq |\beta| + \omega < 2^{\aleph_0}$$

Thus, $P_\beta \setminus \operatorname{span}\{b_\alpha : \alpha < \beta\} \neq \varnothing$ and we may pick an element b_β from this set.

According to this procedure, we have constructed a linearly independent family $\{b_\alpha : \alpha < 2^{\aleph_0}\}$ satisfying (i). We can extend this family to a maximal one, call it B, and in this way, B will be a Hamel basis over $\mathbb{R}$.

By construction, B intersects every perfect subset of $\mathbb{R}$, so B is in fact a Burstin basis. □

2.2. **The Marczewski ideal and new generic reals.** Before the appearance of the forcing technique, in 1935 E. Marczewski introduced the σ-ideal s^0. This ideal is related to Sacks forcing in much the same way that Cohen forcing is related with the ideal of meager subsets of $\mathbb{R}$ and Random forcing is related with the ideal of Lebesgue null subsets of $\mathbb{R}$.

Definition 2.7. (Marczewski, 1935) A set $X \subseteq {}^\omega 2$ is in s^0 if and only if for every perfect tree $T \subseteq {}^{<\omega}2$, there is a perfect subtree $S \subseteq T$ with $[S] \cap X = \varnothing$.

It is easy to see that s^0 is an ideal which does not contain any perfect set. Furthermore, any subset X of the reals with $|X| < 2^{\aleph_0}$ is in the Marczewski ideal, as well as every universal measure zero set and every perfectly meager set[1]. However, s^0 contains sets of size continuum (cf. [22, Theorem 5.10]). Moreover, by a "fusion" argument we can see that s^0 is a σ-ideal, i.e. closed under countable unions.

Remark 2.8. We say that $X \subseteq {}^\omega 2$ is s-measurable if for each $T \in \mathbb{S}$ there is $S \leq T$ such that either $[S] \cap X = \varnothing$ or $[S] \subseteq X$. Note that the algebra of the s-measurable sets modulo the ideal s^0 corresponds, in fact, to Sacks forcing.

Definition 2.9. Suppose that $M \subseteq N$ are models of ZFC. We say that the pair (M, N) satisfies *countable covering for reals* if for every $A \subseteq {}^\omega 2^M$, $A \in N$, such that A is countable in N, there is a set $B \subseteq {}^\omega 2^M$, $B \in M$, such that $A \subseteq B$ and B is countable in M.

In the 1960's, K. Prikry asked whether the existence of a non constructible real implies the existence of a perfect set of non constructible reals (cf. [20]). In order to find a solution to Prikry's problem, Marcia J. Groszek and Theodore A. Slaman have shown the following result in [14, Theorem 2.4][2]:

Theorem 2.10. (Groszek-Slaman) *Suppose that $M \subseteq N$ are models of* ZFC *such that ${}^\omega 2^N \setminus {}^\omega 2^M \neq \varnothing$ and $M \models$* CH*. Then every perfect set $P \subseteq {}^\omega 2^N$ in N has an element which is not in M.*

In [14, §1], the authors state without proof that the conclusion in 2.10 can be strengthened to: for every perfect set $P \subseteq {}^\omega 2^N$ in N there is a perfect set $P' \subseteq P$ in N such that $P' \cap M = \varnothing$, which is equivalent to saying that ${}^\omega 2^M \in s_0^N$ (s_0^N being s_0 of N). In what follows we present a proof of this strengthened version of [14, Theorem 2.4].

Theorem 2.11. (Groszek-Slaman) *Let $W \subseteq V$ be an inner model such that $W \models$* CH*. If ${}^\omega 2^V \setminus {}^\omega 2^W \neq \varnothing$ holds, we have*

$$V \models {}^\omega 2^W \in s^0$$

Proof. We may assume that $\omega_1^W = \omega_1^V$, as otherwise W has only countably many reals and the result is trivial.

Claim 1. *The pair (W, V) satisfies countable covering for reals.*

[1] A set $N^* \subseteq {}^\omega 2$ has universal measure zero if for every measure μ defined on the Borel sets of ${}^\omega 2$, there is B a μ-null Borel set such that $N^* \subseteq B$. Analogously, we say that $M^* \subseteq {}^\omega 2$ is perfectly meager if for every perfect tree $T \subseteq {}^{<\omega}2$, the set $M^* \cap [T]$ is meager relative to the topology of [T].

[2] See also [28, Theorem 3]

Proof. Suppose that $A \in V$ is a countable set such that $A \subseteq {}^{\omega}2^W$. Since $\omega_1^W = \omega_1^V$ and $W \models \mathsf{CH}$ we can take a well-ordering of ${}^{\omega}2^W$ in W of length ω_1. Then, there is some $\alpha < \omega_1^W$ such that $A \subseteq \{a_i : i < \alpha\}$ where $\{a_i : i < \omega_1^W\}$ is an enumeration of ${}^{\omega}2^W$ according to the fixed well-ordering. Therefore, $B = \{a_i : i < \alpha\} \in W$ is countable in W and covers A. □

Let us fix a perfect set $P \subseteq {}^{\omega}2$ in V. We aim to find a perfect subset $\bar{P} \subseteq P$ such that $\bar{P} \cap {}^{\omega}2^W = \varnothing$, or, equivalently $\bar{P} \subseteq V \setminus W$. Let $T \subseteq {}^{<\omega}2$ be a perfect tree such that $P = [T]$. We call $x \in [T]$ *eventually trivial* if and only if there is some finite $s \subsetneq x$ such that x is the leftmost or the right most branch of T_s. We consider two cases:

Case 1. Suppose that there is some $s \in T$ such that if $x \in [T_s]$ is not eventually trivial then $x \in V \setminus W$. In this situation we have that $[T_s] \cap W$ is a subset of all eventually trivial elements of $[T_s]$; since the latter set is countable there is some perfect set $\bar{P} \subseteq [T_s]$ consisting only of elements of $V \setminus W$. But then $\bar{P} \subseteq [T_s] \subseteq P$.

Case 2. Now suppose that for all $s \in T$, there is some $x \in [T_s] \cap W$ which is not eventually trivial. For each $s \in T$, pick $x_s \in [T_s] \cap W$ not eventually trivial. Let $\vec{g} = \langle g_n \mid n < \omega \rangle \in W$ be a sequence of elements of ${}^{\omega}2 \cap W$ such that for all $s \in T$, there is some $n < \omega$ such that $x_s = g_n$. $\vec{g}$ exists by Claim 1. We shall also assume that $g_0 = x_{s_0}$ for some $s_0 \in T$.

First, we prove $P \cap (V \setminus W) \neq \varnothing$. Fix $r \in ({}^{\omega}2 \cap V) \setminus W$ and construct $x, y \in {}^{\omega}2$ and subsequences $\vec{g}^{\,x}, \vec{g}^{\,y}$ of $\vec{g}$ such that $x, y \in [T]$ and

(1*) $r \leq_T x,\ \vec{g}^{\,x}$, and
(2*) $\vec{g}^{\,x},\ \vec{g}^{\,y} \leq_T x,\ y,\ \vec{g}$

Thus, we have that $r \leq_T x, y, \vec{g}$. But then, $x \in V \setminus W$ or $y \in V \setminus W$ and hence P will have a member in $V \setminus W$. In a second round we shall actually produce a perfect $\bar{P} \subseteq P$, $\bar{P} \subseteq V \setminus W$.
We shall produce recursively strict initial segments of x given by $\vec{g}^{\,x} = \langle g_n^x \mid n < \omega \rangle$, y and $\vec{g}^{\,y} = \langle g_n^y \mid n < \omega \rangle$ as follows.

We start with $g_0^x = g_0 = g_0^y$. We shall maintain inductively that $m = m(n), k = k(n)$ are such that $k \geq m \geq n$. Suppose we are given $x \restriction m(n)$, g_n^x, $y \restriction k(n), g_n^y$ such that

(a) $x \restriction m(n) \subsetneq g_n^x$,
(b) $g_n^x = x_s$ for some $s \in T$,
(c) $y \restriction k(n) \subsetneq g_n^y$, and
(d) $g_n^y = x_{s'}$ for some $s' \in T$.

For $n = 0$, we may just let $m = 0 = k$ and then (a) through (d) will be satisfied.
Now say $g_n^y = g_j$. Pick $m' > m(n), k(n)$ such that $g_l \restriction m' \neq g_j \restriction m'$ for all $l < j$. By item (b), we may also assume that $g_n^x \restriction m'$ is a splitting node in T and $g_n^x(m') \neq r(n)$.
Then set

$$x \restriction m' + 1 = g_n^x \restriction m'^\frown r(n)$$

and pick g_{n+1}^x such that for $s'' := x \restriction m' + 1 \in T$ we have $g_{n+1}^x = x_{s''}$ and $x \restriction m' + 1 \subsetneq x_{s''}$.
Say $g_{n+1}^x = g_i$. Pick $k' > m' + 1$ such that $g_l \restriction k' \neq g_i^x \restriction k'$ for all $l < i$. By (d), we may also assume that $g_n^y \restriction k'$ is a splitting node.
Then, set

$$y \restriction k' + 1 = g_n^y \restriction k'^\frown (1 - g_n^y(k'))$$

and pick g_{n+1}^y such that for $s''' := y \restriction k' + 1 \in T$ we have $g_{n+1}^y = x_{s'''}$ and $y \restriction k' + 1 \subsetneq x_{s'''}$.
Then, we are back to (a) through (d) with $x \restriction m' + 1$, g_{n+1}^x, s'', $y \restriction k' + 1$, g_{n+1}^y, s''', $m(n+1) = m' + 1$, and $k(n+1) = k' + 1$ replacing $x \restriction m$, g_n^x, s, $y \restriction k$, g_n^y, s', $m(n) = m$, and $k(n) = k$, respectively.
This finishes the construction of $x, \vec{g}^{\,x}, y, \vec{g}^{\,y}$. For every $n < \omega$, $r(n) = 1 - g_n^x(m')$, where m' is maximal such that $x \restriction m' = g_n^x \restriction m'$. This shows (1*) on p. 4.
To show (2*) on p. 4, notice that $g_n^y = g_j$ for the least j such that $y \restriction m' = g_j \restriction m'$, where m' is maximal with $x \restriction m' = g_n^x \restriction m'$; also, $g_{n+1}^x = g_i$ for the least i such that $x \restriction k' = g_i \restriction k'$, where k' is maximal with $y \restriction k' = g_n^y \restriction k'$.
We have shown that $P \cap (V \setminus W) \neq \varnothing$.
Let us now prove the full theorem, varying the argument above. By recursion on the length of $s \in {}^{<\omega}2$ we construct $x^s, y^s \in T$ and subsequences $\vec{g}^{\,x^s}, \vec{g}^{\,y^s}$ of $\vec{g}$ such that

(1) $x^{s\frown 0}, x^{s\frown 1}$ and $y^{s\frown 0}, y^{s\frown 1}$ are incompatible;
(2) $x^s \subsetneq x^{s'}$, $y^s \subsetneq y^{s'}$ for $s \subsetneq s'$;
(3) $\vec{g}^{\,x^s} = \langle g_n^{x^s} : n < \mathrm{lh}(s)+1\rangle$, $\vec{g}^{\,y^s} = \langle g_n^{y^s} : n < \mathrm{lh}(s)+1\rangle$ are sequences of elements from $\vec{g}$, in fact from $\{x_s : s \in T\}$, of length $\mathrm{lh}(s)+1$;
(4) $\vec{g}^{\,x^s} \subsetneq \vec{g}^{\,x^{s'}}$, $\vec{g}^{\,y^s} \subsetneq \vec{g}^{\,y^{s'}}$ for $s \subsetneq s'$;
(5) if for $z \in {}^{\omega}2$ we write $v^z = \bigcup\{v^s : s \subseteq z\}$, where $v \in \{x, y\}$, we have also
(6) for all $z, z' \in {}^{\omega}2$: $\vec{g}^{\,x^z} = \bigcup\{\vec{g}^{\,x^s} : s \subseteq z\}$, $\vec{g}^{\,y^z} = \bigcup\{\vec{g}^{\,s} : s \subseteq z\}$
(6-a) $r \leq_T x^z, \vec{g}^{\,y^z}$, and
(6-b) $\vec{g}^{\,x^z}, \vec{g}^{\,y^{z'}} \leq_T x^z, y^{z'}, \vec{g}$.

In particular, $r \leq_T x^z, y^{z'}, \vec{g}$ for all $z, z' \in {}^{\omega}2$. But then $\{x^z : z \in {}^{\omega}2\} \subseteq V \smallsetminus W$ or $\{y^z : z \in {}^{\omega}2\} \subseteq V \smallsetminus W$, because if $x^z, y^{z'} \in W$ we would have $r \in W$. By (1), both $\{x^z : z \in {}^{\omega}2\}$ and $\{y^z : z \in {}^{\omega}2\}$ are perfect, so one of them is a perfect set $\bar{P} \subseteq P$ consisting entirely of reals in $V \smallsetminus W$, as desired.
The construction of $x^s, \vec{g}^{\,x^s}, y^s, \vec{g}^{\,y^s}$ is basically as above, just building in (1). Again, we start out with $x^\varnothing = \varnothing = y^\varnothing, \vec{g}^{\,x^\varnothing} = \langle \vec{g}_0\rangle = \vec{g}^{\,y^\varnothing}$. Suppose we already have defined $x^s, \vec{g}^{\,x^s}, y^s, \vec{g}^{\,y^s}$ for all $s \in {}^{<\omega}2$ of length $\leq n$.
Fix s of length n, and let us define $x^{s\frown 0}$, $g_{n+1}^{x^{s\frown 0}}$, $x^{s\frown 1}$, $g_{n+1}^{x^{s\frown 1}}$. Let $j = \max\{\bar{\imath} : g_n^{y^t} = g_{\bar{\imath}}, \mathrm{lh}(t) = n\}$, and pick $m' > \max\{\mathrm{lh}(x^t), \mathrm{lh}(y^t) : \mathrm{lh}(t) = n\}$ such that $g_l \restriction m' \neq g_{l'} \restriction m'$ for all $l, l' \leq j$, $l \neq l'$ and $m_1 > m_0 \geq m'$ are both such that $g_n^{x^s} \restriction m_0$, $g_n^{x^s} \restriction m_1$ are splitting nodes in T and $g_n^{x^s}(m_0) \neq r(n) \neq g_n^{x^s}(m_1)$.
Then set

$$x^{\,s\frown 0} = g_n^{x^s} \restriction m_0 \frown r(n)$$
$$x^{\,s\frown 1} = g_n^{x^s} \restriction m_1 \frown r(n)$$

and pick $g_{n+1}^{x^{s\frown 0}}, g_{n+1}^{x^{s\frown 1}}$ such that there are $s'', \bar{s}'' \in T$ with $x^{s\frown 0} \subsetneq x_{s''} = g_{n+1}^{x^{s\frown 0}}$, $x^{s\frown 1} \subsetneq x_{\bar{s}''} = g_{n+1}^{x^{s\frown 1}}$.
This defines all x^t, $g_{n+1}^{x^t}$, $\mathrm{lh}(t) = n+1$. Again, fix s of length n, and let us define $y^{s\frown 0}$, $g_{n+1}^{y^{s\frown 0}}$, $y^{s\frown 1}$, $g_{n+1}^{y^{s\frown 1}}$.

Let $i = \max\{\bar{\imath} : g_{n+1}^{x^t} = g_{\bar{\imath}}, \mathrm{lh}(t) = n+1\}$ and pick $k' > \max\{\mathrm{lh}(y^{\bar{t}}), \mathrm{lh}(x^t) : \mathrm{lh}(\bar{t}) = n, \mathrm{lh}(t) = n+1\}$, such that $g_l \restriction k' \neq g_{l'} \restriction k'$ for $l, l' \leq i$, $l \neq l'$, and $k_1 > k_0 \geq k'$ are both such that $g_n^{y^s} \restriction m_0$, $g_n^{y^s} \restriction m_1$ are splitting nodes in T.
Then set

$$y^{\,s\frown 0} = g_n^{y^s} \restriction k_0 \frown (1 - g_n^{y^s}(k(_0))$$
$$y^{\,s\frown 1} = g_n^{y^s} \restriction k_1 \frown (1 - g_n^{y^s}(k_1))$$

and pick $g_{n+1}^{y^{s\frown 0}}$, $g_{n+1}^{y^{s\frown 1}}$ such that there are $s''', \bar{s}''' \in T$ with $y^{s\frown 0} \subsetneq x_{s'''} = g^{\,y^{s\frown 0}}, y^{s\frown 1} \subsetneq x_{\bar{s}'''} = g^{\,y^{s\frown 1}}$.
This defines all $y^t, \vec{g}_{n+1}^{\,y^t}$ where $\mathrm{lh}(t) = n+1$. This finishes the construction.
The proofs of items (6-a) and (6-b) on p. 5 are like the proofs of (1*) and (2*) on p. 4: for each n, $r(n) = x^z(m)$, where m is largest such that $x^z \restriction m = g_n^{x^z} \restriction m$. This shows (6-a). Moreover, $g_n^{y^z} = g_j$ for the least j such that $y^z \restriction m' = g_j \restriction m'$ where m' is maximal with $x^{z'} \restriction m' = g_n^{x^{z'}} \restriction m'$. Also, $g_{n+1}^{x^z} = g_i$ for the least i such that $x^z \restriction k' = g_i \restriction k'$ where k' is maximal with $y^{z'} \restriction k' = g_n^{y^{z'}} \restriction k'$. This shows item (6-b). □

2.3. **Side-by-side product of Sacks forcing and its properties.** This section recapitulates known facts about Sacks forcing. See [2], [12]. As we are going to use side-by-side products of Sacks forcing which are less common than iterations (for instance, side-by-side products of Sacks forcing are not discussed in [1]), we include the proofs of these facts to make our paper more self-contained.

Definition 2.12. Sacks forcing $\mathbb{S}$ is defined in the following way.

$$\mathbb{S} = \{T : T \text{ is a perfect tree on } 2\}$$

For $S, T \in \mathbb{S}$ we stipulate $S \leq T$ if and only if $S \subseteq T$. If $S \in \mathbb{S}$ and $p \in S$, we define the subtree $S_p = \{t \in S : t \subset p \text{ or } p \subset t\}$

A node $p \in T$ is called a splitting node if $p^\frown 0, p^\frown 1 \in T$. The set of splitting points of T is denoted by $\operatorname{split}(T)$. We define $\operatorname{stem}(T)$ as the unique element in $\operatorname{split}(T)$ comparable with any other node of T. A node $p \in T$ is in $\operatorname{split}_n(T)$ if $p \in \operatorname{split}(T)$ and p has exactly n predecessors in $\operatorname{split}(T)$. In particular, $\operatorname{split}_0(T) = \{\operatorname{stem}(T)\}$. Notice that for $T \in \mathbb{S}$, $|\operatorname{split}_n(T)| = 2^n$.

For every $n \in \omega$ and $S \in \mathbb{S}$ we write $\operatorname{Lev}_n(S) = \{t \in S : \exists s \in \operatorname{split}_n(S)\, t \subset s\}$, and for $S, T \in \mathbb{S}$ we stipulate $S \leq_n T$ if and only if $S \leq T$ and $\operatorname{Lev}_n(S) = \operatorname{Lev}_n(T)$.

Definition 2.13. If κ is an ordinal and $X \subset \kappa$ (e.g., $X = \kappa$), let $\mathbb{S}_X$ be the κ-side-by-side countable support product of Sacks forcing, i.e., $\mathbb{S}_X$ is the set of all functions $p : X \to \mathbb{S}$ such that $\operatorname{supp}(p) := \{\alpha \in X : p(\alpha) \neq 1_{\mathbb{S}}\}$ is at most countable. If $p, q \in \mathbb{S}_X$, we stipulate

$$p \leq q \iff \forall \alpha < \kappa (p(\alpha) \leq_{\mathbb{S}} q(\alpha))$$

This implies in particular that $\operatorname{supp}(q) \subseteq \operatorname{supp}(p)$.

For now we are only interested in the case that $X = \kappa$ is a cardinal, the more general case will only show up in the proof of Lemma 5.1. If g is $\mathbb{S}_\kappa$-generic over V, and $\alpha < \kappa$, then

$$s_\alpha = \bigcup_{p \in g} \operatorname{stem} p(\alpha)$$

is a real which is $\mathbb{S}$-generic over V. Therefore forcing with $\mathbb{S}_\kappa$ adds κ-many Sacks reals which are independent over the ground model, i.e. for any $A \subset \kappa$ in V,

$$^{\omega}2^{V[\langle x_\alpha : \alpha \in A \rangle]} \cap {}^{\omega}2^{V[\langle x_\alpha : \alpha \in \kappa \setminus A \rangle]} = {}^{\omega}2^{V}$$

The product forcing $\mathbb{S}_\kappa$ has properties very similar to those of $\mathbb{S}$. By defining a suitable notion of levels and fusion, it can be shown that $\mathbb{S}_\kappa$ satisfies the Baumgartner Axiom A[3] and therefore it is proper and does not collapse ω_1. For our purposes, the most remarkable property of $\mathbb{S}_\kappa$ is that it inherits from $\mathbb{S}$ also the so called *Sacks property.* See [13, Definition 6.34] and [1, Definition 6.3.37].

Definition 2.14. Let $g : \omega \to \omega$ be an increasing function. We say $F : \omega \to [\omega]^{<\omega}$ is a *g-slalom* if $|F(n)| \leq g(n)$ for all $n \in \omega$.

Definition 2.15. Let $\mathbb{P}$ be a forcing notion and suppose $g \in {}^{\omega}\omega \cap V$ is an increasing function. We say that $\mathbb{P}$ has the *Sacks property* if whenever G is $\mathbb{P}$-generic over V, for every $f \in {}^{\omega}\omega \cap V[G]$ there exists a g-slalom $F \in V$, such that $V[G] \models \forall n (f(n) \in F(n))$.[4]

Lemma 2.16. *Let κ be a cardinal. Suppose that $p \in \mathbb{S}_\kappa$ and for $\theta \gg \kappa$ let $X \prec V_\theta$ be a countable elementary substructure with $p, \mathbb{S}_\kappa \in X$. Let $\langle \tau_n \mid n < \omega \rangle \in V$ be a sequence of terms for ordinals, $\{\tau_n : n < \omega\} \subseteq X$ (possibly but not necessarily $\langle \tau_n \mid n < \omega \rangle \in X$). Then, there is some $q \leq p$ and some $F : \omega \to [X \cap \mathsf{OR}]^{<\omega}$, $F \in V$, such that for all $n < \omega$:*

1. $q \Vdash \tau_n \in (F(n))^\vee$,
2. $|F(n)| \leq 2^{2n}$, *and*
3. $F(n) \subset X$.

Proof. Suppose that $\alpha = X \cap \omega_1$. Since $\operatorname{supp}(p)$ is an element of X, $\operatorname{supp}(p)$ also is a subset of X. Let $e : \omega \longleftrightarrow \alpha$ be a fixed bijection. We aim to produce a sequence $\langle p_n \mid n < \omega \rangle$ such that $p_0 = p$ and $p_{n+1} \leq p_n$, $p_n \in X$ for all $n \in \omega$. In this way, we also will have $\operatorname{supp}(p_n) \subseteq \alpha$ for every $n < \omega$. Suppose p_n is already defined. Working in X, we shall produce $p_{n+1} \leq p_n$ such that for all $k < n$,

(i) $p_{n+1}(e(k)) \leq_n p_n(e(k))$, and
(ii) there is some $a_n \in [X \cap \mathsf{OR}]^{\leq 2^{2n}}$ such that $p_{n+1} \Vdash \check{\tau}_n \in \check{a}_n$.

The condition q defined as $q(e(k)) = \bigcap_{n<\omega} p_n(e(k))$ for each $k < \omega$ and the function F given by $F(n) = a_n$ satisfy the conclusion of our lemma.
We may produce p_{n+1} by means of some sequence $\langle q_m \mid m \leq 2^{2n} \rangle$ defined as follows inside X. Let $q_0 = p_n$. Fix some enumeration $\langle \vec{s}_m \mid m < 2^{2n} \rangle$ of all tuples $\vec{s} = (s_{e(0)}, \ldots s_{e(n-1)})$ such that $s_{e(k)} \in \operatorname{split}_n p_n(e(k))$ for all $k < n$.

[3]For the details, see [12, §6]

[4]For equivalent definitions of Sacks property, the reader can see [13, Fact 6.35].

Suppose $m < 2^{2n}$ and q_m has been chosen. We aim to define q_{m+1}. Write $\vec{s}_m = (s_{e(0)}, \dots s_{e(n-1)})$. For each $k < n$, let $\bar{m}_k \leq m$ be maximal such that $s_{e(k)} \in q_{\bar{m}_k}$, and define $\bar{q}$ in such a way that $\text{supp}(\bar{q}) = \text{supp}(q_m)$ and

$$\bar{q}(\xi) = \begin{cases} (q_{\bar{m}_k}(e(k)))_{s_{e(k)}} & \text{if } \xi = e(k) \\ q_m(\xi) & \text{if } \xi \neq e(k) \text{ for all } k < n \end{cases}$$

Let $q_{m+1} \leq \bar{q}$ be a condition deciding $\check{\tau}_n$, and put the $\xi \in X \cap \mathsf{OR}$ with $q_{m+1} \Vdash \check{\tau}_n = \check{\xi}$ into a_n. This defines $\langle q_m \mid m \leq 2^{2n} \rangle$. Let us define p_{n+1} as follows. For each $k < n$ and $s \in \text{split}_n(p_n(e(k)))$, let $\bar{m}_{k,s} \leq m$ be maximal such that $s \in q_{\bar{m}_{k,s}}(e(k))$. Then $(q_{\bar{m}_{k,s}}(e(k)))_s = q_{\bar{m}_{k,s}}(e(k))$.

Let p_{n+1} have the same support as $q_{2^{2n}}$ and

$$p_{n+1}(\xi) = \begin{cases} \bigcup\{q_{\bar{m}_{k,s}}(e(k)) : s \in \text{split}_n\, p_n(e(k))\} & \text{if } \xi = e(k) \\ q_{2^{2n}}(\xi) & \text{if } \xi \neq e(k) \text{ for all } k < n \end{cases}$$

It is easy to see that this sequence is as desired. □

The following two corollaries are implicit in the statement of [2, Theorem 1.11]. See also [12, Lemma 6.2].

Corollary 2.17. *For every cardinal κ the countable support product $\mathbb{S}_\kappa$ satisfies the Sacks property.*

Proof. Let $f \in {}^\omega\omega \cap V^{\mathbb{S}_\kappa}$ and let $p \in \mathbb{S}_\kappa$ such that $p \Vdash \tau \in {}^\omega\omega$ where τ is a $\mathbb{S}_\kappa$-name for f. Let $\theta > 2^{2^\kappa}$ and let $X \prec \mathsf{V}_\theta$ be a countable elementary substructure such that $p, \tau, \mathbb{S}_\kappa \in X$. Suppose that $\alpha = X \cap \omega_1$. By Lemma 2.16, there is a 2^{2n}-slalom $F : \omega \to [\omega]^{<\omega}$ in V and a condition $q \leq p$ with $\text{supp}(q) \subseteq \alpha$ such that

$$q \Vdash \forall n\, \tau(n) \in F(n)^\vee.$$

Given any increasing function $g : \omega \to \omega$, a simple variant of the argument for Lemma 2.16 with an appropriate bookkeeping produces a g-slalom F and a condition $q \leq p$ with the same properties. Therefore $\mathbb{S}_\kappa$ has the Sacks property. (See also [13, 6.35].) □

Corollary 2.18. *For every cardinal κ, the countable support product $\mathbb{S}_\kappa$ is a proper forcing. If g is $\mathbb{S}_\kappa$-generic over V and if $x \in {}^\omega 2 \cap V[g]$, then there is some $\tau \in V^{\mathbb{S}_\kappa}$ which is countable in V such that $x = \tau^g$.*

Proof. First part: Let $p \in \mathbb{S}_\kappa$. Suppose that $\theta \gg \kappa$ and let $N \prec H_\theta$ be a countable substructure with $\mathbb{S}_\kappa \in N$, $p \in N$.

Let $\{\tau_n : n \in \omega\} \in V$ be an enumeration of all $\mathbb{S}_\kappa$-names for ordinals in N. By lemma 2.16, there exists some $q \leq p$ and some $F : \omega \to [N \cap \mathsf{OR}]^{<\omega}$ in V such that for all $n \in \omega$,

$$q \Vdash \tau_n \in F(n)^\vee \subset \check{N}.$$

I.e., $q \Vdash \dot{\alpha} \in \check{N} \cap \mathsf{OR}$ for every $\mathbb{S}_\kappa$-name $\dot{\alpha} \in N$ for an ordinal. This implies that $\mathbb{S}_\kappa$ is proper.

Second part: Let $x = \sigma^g$, where $\sigma = \bigcup\{\{(n,h)^\vee\} \times A_{n,h} \colon (n,h) \in \omega \times 2\} \in V^{\mathbb{S}_\kappa}$ and for each $(n,h) \in \omega \times 2$, A_n is a maximal antichain of $p \in \mathbb{S}_\kappa$ such that $p \Vdash \sigma(\check{n}) = \check{h}$. In $V[g]$, for each $n < \omega$ there is some unique $h = h_n \in 2$ and $p = p_n \in \mathbb{S}_\kappa$ such that $p \in A_{n,h} \cap g$. Let $X \supset \{p_n \colon n < \omega\}$, where $X \in V$ is countable in V. (This choice of X is possible as $\mathbb{S}_\kappa$ is proper.) Then $\tau = \bigcup\{\{(n,h)^\vee\} \times (A_{n,h} \cap X) \colon (n,h) \in \omega \times 2\}$ is as desired. □

[16] gives more information on how reals in $V^{\mathbb{S}_\kappa}$ may be represented.

3. Luzin and Sierpiński sets in the Sacks model

Let $\mathbb{S}_{\omega_1}$ be the countable support product of ω_1-many copies of Sacks forcing. From the fact that $\mathbb{S}_{\omega_1}$ has the Sacks property we shall show that in the generic extension obtained after forcing with $\mathbb{S}_{\omega_1}$ the Luzin and Sierpiński sets in the ground model are also Luzin and Sierpiński sets in the generic extension.

We use the following result. See [1, Lemma 2.3.10].

Lemma 3.1. *Let $N \subseteq {}^\omega\omega$ be null and let $\{\varepsilon_n : n \in \omega\}$ be a sequence of positive reals. Then there is a sequence $\langle C_n \subseteq {}^\omega\omega : n \in \omega\rangle$ of finite unions of basic open sets such that*

(i) *for all* $n < \omega$, $\mu(C_n) < \varepsilon_n$ *and*
(ii) $N \subseteq \bigcup_{n\in\omega} C_n$

Proof. Since N is null, there is a collection of basic open sets $\{O_n : n \in \omega\}$ such that $N \subset \bigcup\{O_n : n \in \omega\}$ and $\mu(\bigcup_{n\in\omega} O_n) < \varepsilon_0$.

Then let $k(n) = \min\{m : \mu(\bigcup_{i\geq m} O_i) < \varepsilon_n\}$. Without loss of generality, we can assume that the sequence $\langle\varepsilon_n : n \in \omega\rangle$ is decreasing, so k is monotone. We have $k(0) = 0$. Then for each n set

$$C_n = \bigcup\{O_i : k(n) \leq i < k(n+1)\}.$$

It is straightforward to see that the collection $\{C_n : n \in \omega\}$ satisfies (i) and (ii). □

The followig is implicit in [1, Theorem 2.3.12], see also [12, Lemma 3.1].

Lemma 3.2. *Let* $\mathbb{P}$ *be a forcing notion satisfying the Sacks property and let* G *be a* $\mathbb{P}$*-generic filter over* V*. Then:*

(1) *For every null set* $N \subseteq {}^\omega\omega$ *in* $V[G]$ *there is a* G_δ*-null set* $\bar{N} \subseteq {}^\omega\omega$ *coded in* V *such that* $N \subseteq \bar{N}$*.*
(2) *Similarly, for every meager set* $M \subseteq {}^\omega\omega$ *in* $V[G]$*, there is a meager set* $\bar{M} \subseteq {}^\omega\omega$ *coded in* V *such that* $M \subseteq \bar{M}$*.*

Proof. We prove the statement (1). Let us fix in V an enumeration $\{C_n : n < \omega\}$ of all finite unions of basic open sets in ${}^\omega\omega$. Let us write $\varepsilon_m = \frac{1}{m+1}$ for $m < \omega$.
Let $N \subseteq {}^\omega\omega$ be a null set in $V[G]$. By 3.1 there is a function $f : \omega\times\omega \to \omega$ in $V[G]$ such that for every $m < \omega$,

$$N \subseteq \bigcup_{n\in\omega} C_{f(n,m)} \quad \text{and} \quad \mu(C_{f(n,m)}) \leq \frac{\varepsilon_m}{2^{2n+1}\cdot 2^m},\ n \in \omega$$

Since $\mathbb{P}$ has the Sacks property, there is some $F : \omega\times\omega \to [\omega]^{<\omega}$ in V such that for every $(n, m) \in \omega\times\omega$, $f(n, m) \in F(n, m)$ and $|F(n, m)| \leq 2^{n+m}$, see (the proof of) Lemma 2.16.[5] For $m < \omega$ set

$$\bar{N}_m = \bigcup_{n\in\omega}\bigcup\{C_k : k \in F(n,m) \text{ and } \mu(C_k) \leq \frac{\varepsilon_m}{2^{2n+1}\cdot 2^m}\}.$$

Since only ground model parameters are used in the definition of $\bar{N}_m$ and this definition is uniform, $\langle\bar{N}_m : m < \omega\rangle$ is a sequence of open sets which is coded in the ground model, and thus $\bigcap_{m<\omega}\bar{N}_m$ is a G_δ set which is coded in the ground model.
We have that $N \subseteq \bar{N}_m$ for each $m < \omega$, i.e., $N \subseteq \bigcap_{m<\omega}\bar{N}_m$. But since $|F(n, m)| \leq 2^{n+m}$ for each $(n, m) \in \omega\times\omega$, it follows that

$$\mu(\bigcup\{C_k : k \in F(n,m) \text{ and } \mu(C_k) \leq \frac{\varepsilon_m}{2^{2n+1}\cdot 2^m}\}) \leq 2^{n+m}\cdot\frac{\varepsilon_m}{2^{2n+1}\cdot 2^m} = \frac{\varepsilon_m}{2^{n+1}}$$

for each $m < \omega$. Therefore $\mu(\bar{N}_m) \leq \sum_{n\in\omega}\frac{\varepsilon_m}{2^{n+1}} = \varepsilon_m$ for each $m < \omega$. It follows that $\bigcap_{m<\omega}\bar{N}_m$ is a G_δ null set which is coded in V and covers N. □

Remark 3.3. Let $\mathcal{N}$ and $\mathcal{M}$ stand for the null and meager ideals over ${}^\omega\omega$ respectively. Since $\mathrm{add}(\mathcal{N}) \leq \mathrm{add}(\mathcal{M})$ and $\mathrm{cof}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$, if a forcing notion $\mathbb{P}$ satisfies item (1) above, then $\mathbb{P}$ satisfies (2) as well. See [1, Theorem 2.3.1].

Corollary 3.4. *If* $\mathbb{P}$ *has the Sacks property, then* $\mathbb{P}$ *preserves Luzin and Sierpiński sets.*

Proof. Suppose that there is a Luzin set Λ in V and let G be $\mathbb{P}$-generic over V. First, observe that, since ω_1 is not collapsed by $\mathbb{P}$, Λ remains uncountable in $V[G]$. Now, let M be a (Borel code for a) meager set in $V[G]$. In view of Lemma 3.2, there is a (Borel code) for a G_δ-null set $\bar{M}$ in V such that $V[G] \models M \subset \bar{M}$. Thus, since $V \models |\Lambda\cap\bar{M}| \leq \omega$, it follows that $V[G] \models |\Lambda\cap M| \leq \omega$. Hence,

$$V[G] \models \Lambda \text{ is a Luzin set.}$$

The proof of the preservation of Sierpiński sets is completely analogous. □

[5]The particular size of $F(n, m)$ is of course not really relevant. f may be coded by a function from ω to ω; applying Lemma 2.16 to the latter yields e.g. a $2^{2\cdot\lfloor\sqrt{n}\rfloor}$-slalom witnessing an instance of the Sacks property, which when translated back gives an F as described.

4. Adding generically a Burstin basis

We now define a partial order $\mathbb{P}_B$ generically adding a Burstin basis.

Definition 4.1. We say $p \in \mathbb{P}_B$ if and only if there exists $x \in \mathbb{R}$ such that

(1) $p \in L[x]$, and
(2) $L[x] \models$ "p is a Burstin basis."

We stipulate $p \leq_{\mathbb{P}_B} q$ iff $p \supseteq q$.

Notice that by Theorem 2.6 we have $\mathbb{P}_B \neq \varnothing$.

If $\mathbb{R} \cap V \subset L[x]$ for some real x, then $\mathbb{P}_B$ has a dense set of atoms. We are interested in situations where the set of all reals is not constructible from a single real. Variants of $\mathbb{P}_B$ will be discussed at the end of this chapter.

The following is an immediate consequence of Theorem 2.11.

Lemma 4.2. *Let x, y be reals such that $y \notin L[x]$, and let $\{z_0, z_1, \dots\} \in L[x,y] \cap [\mathbb{R}]^\omega$. Then*
$$\operatorname{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \dots\}) \in (s^0)^{L[x,y]},$$
i.e., for every perfect set P in $L[x,y]$ there is a perfect set $\bar{P} \subset P$, $\bar{P} \in L[x,y]$ such that
$$\bar{P} \cap \operatorname{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \dots\}) = \varnothing$$

Proof. We may assume that $\{z_0, z_1, \dots\} = \operatorname{span}(\{z_0, z_1, \dots\})$, so that if $z \in \operatorname{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \dots\})$, then $z \in (\mathbb{R} \cap L[x]) + z_n$, for some $n < \omega$. Given $P \in L[x,y]$ a perfect set, we shall construct recursively a sequence $T_0 \supseteq T_1 \supseteq \cdots T_n \supseteq T_{n+1} \supseteq \cdots$ of perfect trees, such that

(1) $P = [T_0]$,
(2) $\operatorname{Lev}_n(T_{n+1}) = \operatorname{Lev}_n(T_n)$ and,
(3) $[T_{n+1}] \cap ((\mathbb{R} \cap L[x]) + z_n) = \varnothing$.

Let T_0 be the perfect tree such that $P = [T_0]$. By Theorem 2.11 we have that $L[x,y] \models$ "${}^\omega 2 \cap L[x] \in s^0$". Since $P - z_0 = \{x - z_0 : x \in P\}$ is also perfect in $L[x,y]$, there is some $\tilde{P} \subset P - z_0$ perfect, $\tilde{P} \in L[x,y]$, such that $\tilde{P} \subseteq L[x,y] \smallsetminus L[x]$. Therefore $P' := \tilde{P} + z_0 \subseteq P$ is perfect and if $u \in \tilde{P}$ (equivalently, $u + z_0 \in \tilde{P} + z_0 = P'$), then $u \notin L[x]$, so $u + z_0 \notin (\mathbb{R} \cap L[x]) + z_0$. Thus, $P' \cap (\mathbb{R} \cap L[x] + z_0) = \varnothing$. Take then T_1 as the perfect tree such that $P' = [T_1]$.

Now suppose that we have constructed $T_0, T_1, \dots, T_n$ satisfying (1)-(3) above. For any $s \in \operatorname{Lev}_n(T_n)$ let us consider the subtree $(T_n)_s$ of T_n. By the argument from the previous paragraph, there is some perfect set $P_{n,s} \subset [(T_n)_s]$ such that $P_{n,s} \cap (\mathbb{R} \cap L[x] + z_n) = \varnothing$. Let
$$P_{n+1} = \bigcup \{P_{n,s} : s \in \operatorname{Lev}_n(T_n)\}.$$
Notice that $P_{n+1} \cap (\mathbb{R} \cap L[x] + z_n) = \varnothing$, hence by taking T_{n+1} as the perfect tree such that $P_{n+1} = [T_{n+1}]$ condition (3) holds. Also, by construction, $\operatorname{Lev}_n(T_{n+1}) = \operatorname{Lev}_n(T_n)$.

Now, set $T = \bigcap \{T_n : n \in \omega\}$. By condition (2), we have that T is a perfect tree. Thus $\bar{P} := [T]$ is a perfect set such that $\bar{P} \cap \operatorname{span}(\mathbb{R} \cap L[x] \cup \{z_0, z_1, \dots\}) = \varnothing$, as required. □

Lemma 4.3. *Let $b \in L[x]$ be linearly independent, $x \in \mathbb{R}$. Let $y \in \mathbb{R} \setminus L[x]$. There is then some $p \supset b$, $p \in L[x,y]$ such that $L[x,y] \models$ "p is a Burstin basis".*

Proof. Let $\langle P_i \mid i < \omega_1 \rangle$ be an enumeration of all perfect sets of $L[x,y]$. Working in $L[x,y]$ we define recursively $\langle b_i \mid i < \omega_1 \rangle$ as follows. Let $\{y_i : i < \omega_1\} \in L[x,y]$ enumerate the reals of $L[x,y]$. Given $\{b_j : j < i\}$, we will have that $\bar{b} = \bigcup \{b_j : j < i\}$ is at most countable. By Lemma 4.2 there is some $\bar{P} \subset P_i$ perfect such that $\bar{P} \cap \operatorname{span}((\mathbb{R} \cap L[x]) \cup \bar{b}) = \varnothing$. Pick $\bar{x} \in \bar{P}$ and set
$$b_i = \begin{cases} \bar{b} \cup \{\bar{x}\} & \text{if } y_i \in \operatorname{span}((\mathbb{R} \cap L[x]) \cup \bar{b} \cup \{\bar{x}\}) \\ \bar{b} \cup \{\bar{x}, y_i\} & \text{otherwise} \end{cases}$$
Finally, if $c \in L[x]$ is such that $c \supseteq b$ and $L[x] \models$ "c is a Hamel basis", take
$$p := c \cup \bigcup \{b_i : i < \omega_1\}$$
By construction p is a Hamel basis for $L[x,y]$. Moreover for each $i < \omega_1$, $b_i \subset p$ hence $P_i \cap p \neq \varnothing$. This shows that p is a Burstin basis in $L[x,y]$. □

Lemma 4.3 has the following immediate corollary, extendability for $\mathbb{P}_B$:

Lemma 4.4. *If $p \in \mathbb{P}_B$, say $L[x] \models$ "p is a Burstin basis," and if y is a real not in $L[x]$, then there is some $q \leq_{\mathbb{P}_B} p$ such that q is a Burstin basis in $\mathbb{R} \cap L[x, y]$.*

Also, lemma 4.3 shows that $\mathbb{P}_B$ is countably closed under favourable circumstances. What is more than enough for our purposes is the following. Hypothesis (1) of Lemma 4.5 is satisfied e.g. if V is a forcing extension of L via some proper forcing. Hypotheses (1) and (2) are certainly satisfied in $V = L[g]$, where g is $\mathbb{S}_{\omega_1}$-generic over L, cf. Corollary 2.18.

Lemma 4.5. *Assume that*

(1) for every countable set X of ordinals there is a set $Y \supset X$, $Y \in L$, such that Y is countable in L, and
(2) there is no real x such that $\mathbb{R} \subset L[x]$.

Then $\mathbb{P}_B$ is ω-closed. In particular, forcing with $\mathbb{P}_B$ does not add any new reals.

Proof. Consider a sequence $(p_n : n < \omega)$ of conditions in $\mathbb{P}_B$ such that $p_{n+1} \leq_{\mathbb{P}_B} p_n$ for all $n < \omega$. For each $n < \omega$, let $x_n \in \mathbb{R}$ be such that $p_n \in L[x_n]$ is a Burstin basis for $\mathbb{R} \cap L[x_n]$. Pick $z \in \mathbb{R}$ such that $x_n \in L[z]$ for all $n < \omega$.

Claim. There is some $x \in \mathbb{R}$ such that $\{p_n : n < \omega\} \in L[x]$.

To prove the claim, notice that $\{p_n : n < \omega\} \subset L[z]$. Let $F : \mathrm{OR} \to L[z]$ be bijective and definable over $L[z]$, and let $X = \{\xi : \exists n < \omega\, F(\xi) = p_n\}$. By hypothesis (1) there is some $Y \supset X$, $Y \in L$, and Y is countable in L. Let $f : \omega \to Y$ be bijective, $f \in L$, and write $x^* = f^{-1}"X$. Then $x^* \subset \omega$ and $X = f"x^* \in L[x^*]$. But then $\{p_n : n < \omega\} \in L[z, x^*]$, and if $x \in \mathbb{R}$ is such that $L[z, x^*] \subset L[x]$, then x verifies the Claim.

Now let $b = \bigcup\{p_n : n < \omega\}$, let x be as in the Claim, and let us make use of hypothesis (2) to pick some $y \in \mathbb{R} \setminus L[x]$. We have that $b \in L[x]$, so that by Lemma 4.3 we can extend the linearly independent set b to a Burstin basis p over $L[y]$. Then, for every $n < \omega$ we have that $p \leq_{\mathbb{P}_B} p_n$, so $\mathbb{P}_B$ is ω-closed. □

Notation. For $\vec{x}$, $\vec{y}$ two real vectors of the same length, let $\vec{x} \cdot \vec{y} := \sum_{i<\mathrm{lh}(x)} x_i y_i$.

Remark 4.6. We have that

$$\begin{aligned} p \in \mathbb{P}_B &\iff \exists x (L[x] \models \text{"}p \text{ is a Burstin basis"}) \\ &\iff \exists \vec{x} \in [p]^{<\omega} \exists \vec{q} \in [\mathbb{Q}]^{<\omega} (\forall y \in \mathbb{R}^{L[\vec{q} \cdot \vec{x}]} \exists \vec{p}_y \in [p]^{<\omega} \exists \vec{q}_y \in [\mathbb{Q}]^{<\omega} \\ &\qquad y = \vec{q}_y \cdot \vec{p}_y \wedge \forall \vec{z} \in [p]^{<\omega} \forall \vec{q} \in [\mathbb{Q}]^{<\omega} (\vec{q} \cdot \vec{z} = 0 \to \vec{q} = \vec{0}) \wedge \\ &\qquad L[\vec{q} \cdot \vec{x}] \models \text{"}P \cap p \neq \varnothing \text{ for every perfect set } P\text{"}) \end{aligned}$$

Since the matrix of this formula is Π^1_2 we have that

$$p \in \mathbb{P}_B \iff \exists \vec{x} \in [p]^{<\omega} \exists \vec{q} \in [\mathbb{Q}]^{<\omega} \psi(\vec{x}, \vec{q}, p) \tag{1}$$

where ψ is Π^1_2.

Remark 4.7. In what follows, we will call

$$\dot{b} := \{(\check{x}, p) : x \in p \in \mathbb{P}_B\}$$

the *canonical name* for the generic Burstin basis b. By the previous remark,

$$\begin{aligned} (\check{x}, p) \in \dot{b} &\iff x \in p \wedge \exists \vec{x} \in [p]^{<\omega} \exists \vec{q} \in [\mathbb{Q}]^{<\omega} \psi(\vec{x}, \vec{q}, p) \\ &\iff \theta(x, p), \end{aligned}$$

where θ is Σ^1_3. It is easy to verify that "$(\check{x}, p) \in \dot{b}$" is absolute between transitive class sized models of set theory.

Let us discuss some variants of $\mathbb{P}_B$.

Definition 4.8. We say $p \in \mathbb{P}_H$ if and only if there exists $x \in \mathbb{R}$ such that

(1) $p \in L[x]$, and
(2) $L[x] \models$ "p is a Hamel basis."

We stipulate $p \leq_{\mathbb{P}_H} q$ iff $p \supseteq q$.

If $\mathbb{R} \cap V \subset L[x]$ for some real x, then like $\mathbb{P}_B$, $\mathbb{P}_H$ has a dense set of atoms. If there is no real x with $\mathbb{R} \cap V \subset L[x]$, then the content of Lemma 4.3 is exactly that $\mathbb{P}_B$ is dense in $\mathbb{P}_H$, which implies that $\mathbb{P}_H$ and $\mathbb{P}_B$ will be forcing equivalent and forcing with $\mathbb{P}_H$ will not just add a Hamel basis but in fact a Burstin basis.

Hence if we aim to generically add a Hamel basis which in the extension contains a perfect set, then forcing with $\mathbb{P}_H$ won't work. E.g., let $P \in L$ be a perfect set in L which is also linearly independent, see [15, Example 1, p. 477f.]. If $M \supset L$ is any inner model, then let us write P_M for M's version of P. Then P_M is perfect in M, $P_M \cap L = P$, and by Π^1_1 absoluteness, P_M is linearly independent in M. We may then let $p \in \mathbb{P}^P_H$ if and only if there exists $x \in \mathbb{R}$ such that $p \in L[x]$, $p \supset P_{L[x]}$, and $L[x] \models$ "p is a Hamel basis"; $p \leq_{\mathbb{P}^P_H} q$ iff $p \supseteq q$. If $p \in \mathbb{P}^P_H \cap L[x] \subset L[y]$, x, $y \in \mathbb{R}$, then $p \cup P_{L[y]}$ is linearly independent by Π^1_1 absoluteness, so that $\mathbb{P}^P_H$ will generically add a Hamel basis which contains the version of P of the model over which we force. The proof of Lemma 5.1 will go through for $\mathbb{P}^P_H$ instead of $\mathbb{P}_B$.

The following forcing, $\mathbb{Q}_H$, is the obvious candidate for adding a Hamel basis.

Definition 4.9. We say $p \in \mathbb{Q}_H$ if and only if p is a countable linearly independent set of reals. We stipulate $p \leq_{\mathbb{Q}_H} q$ iff $p \supseteq q$.

It is clear that if ω_1 is inaccessible to the reals (i.e., $\mathbb{R} \cap L[x]$ is countable for all reals x), then $\mathbb{Q}_H$ is dense in $\mathbb{P}_H$ (and hence also in $\mathbb{P}_B$), so that under this hypothesis all the three forcings are forcing equivalent with each other. On the other hand, in the absence of large cardinals, in contrast to $\mathbb{P}_B$ and $\mathbb{P}_H$ (see Lemma 5.1 below) forcing with $\mathbb{Q}_H$ over $L(\mathbb{R})$ will add a well-ordering of $\mathbb{R}$, see Corollary 4.11 below, so that $\mathbb{Q}_H$ definitely is the wrong candidate for forcing a Hamel basis for our purposes. (The forcing $\mathbb{Q}_H$ would be called P_ψ in [17], where ψ expresses linear independence, see [17, Introduction].)

Lemma 4.10. *Let $\vec{x} = (x_\alpha : \alpha < \omega_1)$ be a sequence of pairwise distinct reals such that $\{x_\alpha : \alpha < \omega_1\}$ is linearly independent. Let g be $\mathbb{Q}_H$-generic over V, and let $h = \bigcup g$. Then inside $L(\mathbb{R}, \vec{x}, h)$, there is a well-order of $\mathbb{R}$ of order type ω_1. In particular, $L(\mathbb{R}, \vec{x}, h)$ is a model of* ZFC.

Proof. Of course $\mathbb{Q}_H$ is ω-closed, so that V and $V[g]$ have the same reals. Hence h is a Hamel basis inside $L(\mathbb{R}, h)$.

Let $p \in \mathbb{Q}_H$, and let $x \subset \omega$. There is a countable limit ordinal λ such that $p \cup \{x_{\lambda+n} : n < \omega\}$ is linearly independent. Let

$$q = p \cup \{x_{\lambda+n} : n \in x\} \cup \{2 \cdot x_{\lambda+n} : n \in \omega \setminus x\}.$$

Then $q \in \mathbb{Q}_H$, $q \leq_{\mathbb{Q}_H} p$, and $x = \{n < \omega : x_{\lambda+n} \in q\}$.

In $L(\mathbb{R}, \vec{x}, h)$ let us define $f : \mathscr{P}(\omega) \to \omega_1$ by $f(x) =$ the least countable limit ordinal λ such that $x = \{n < \omega : x_{\lambda+n} \in h\}$. Trivially, f is injective, and by the density argument from the previous section f is a well-defined total function. This shows that in $L(\mathbb{R}, \vec{x}, h)$, there is a well-order of $\mathbb{R}$ of order type ω_1.

As there is a surjection $F : \mathbb{R} \times \mathrm{OR} \to L(\mathbb{R}, \vec{x}, h)$ which is Σ_1-definable over $L(\mathbb{R}, \vec{x}, h)$ from the parameters $\mathbb{R}$, $\vec{x}$, and h, the existence of a well-order of $\mathbb{R}$ inside $L(\mathbb{R}, \vec{x}, h)$ yields that $L(\mathbb{R}, \vec{x}, h)$ is a model of ZFC. □

Corollary 4.11. *Assume that ω_1 is not inaccessible to the reals, let g be $\mathbb{Q}_H$-generic over V, and let $h = \bigcup g$. Then in $L(\mathbb{R}, h)$, there is a well-order of $\mathbb{R}$ of order type ω_1 and $L(\mathbb{R}, h)$ is a model of* ZFC.

Proof. By our hypothesis, there is a real x such that we may pick $\vec{x} \in L[x]$ and $\vec{x}$ is as in the hypothesis of Lemma 4.10. □

5. The main theorem

The following Lemma is dual to Corollary 4.11.

Lemma 5.1. *Let g be $\mathbb{S}_{\omega_1}$-generic over L, let h be $\mathbb{P}_B$-generic over $L[g]$ and let $b = \bigcup h$ be the Burstin basis added by h. Let*

$$W = L(\mathbb{R}, b)^{L[g,h]}$$

Then $W \models$ "There is no well-ordering of $\mathbb{R}$ ".

Proof. That b is indeed a Burstin basis in $L[g,h]$ as well as in W follows from Lemmas 4.4 and 4.5.

Let us assume for contradiction that

$$L[g,h] \models \text{“}\varphi(\cdot\,,\cdot\,,\vec{x},\vec{\alpha},b) \text{ defines a well-ordering of } {}^{\omega}2\text{”}$$

where $\vec{x} \in \mathbb{R} \cap L[g,h] = \mathbb{R} \cap L[g]$ and $\vec{\alpha} \in \mathsf{OR}$.

Then, there is some $p \in h \subset \mathbb{P}_B$ such that

$$p \Vdash^{\mathbb{P}_B}_{L[g]} \text{“}\varphi(\cdot\,,\cdot\,,\check{\vec{x}},\check{\vec{\alpha}},\dot{b}) \text{ defines a well-ordering of } {}^{\omega}2\text{”}$$

where $\dot{b}$ is the canonical $\mathbb{P}_B$-name for the generic Burstin basis b as defined in Remark 4.7; but then we may rewrite this as

$$p \Vdash^{\mathbb{P}_B}_{L[g]} \text{“}\varphi(\cdot\,,\cdot\,,\check{\vec{x}},\check{\vec{\alpha}},\{(\check{y},q) : \theta(y,q)\}) \text{ defines a well-ordering of } {}^{\omega}2,\text{”}$$

with θ being the Σ^1_3 formula from Remark 4.7. We may pick $\xi < \omega_1$ with $p, \vec{x} \in L[g \restriction \xi]$, see Corollary 2.18. Now since $\mathbb{S}_\xi \times \mathbb{S}_{\omega_1 \setminus \xi}$ is isomorphic to $\mathbb{S}_{\omega_1}$ via the isomorphism $(p_0,p_1) \mapsto p_0 \cup p_1$, standard arguments show that $g \restriction [\xi,\omega_1)$ is $(\mathbb{S}_{\omega_1\setminus\xi})^L$-generic over $L[g\restriction\xi]$ and so we can write

$$p \Vdash^{\mathbb{P}_B}_{L[g\restriction\xi][g\restriction[\xi,\omega_1)]} \text{“}\varphi(\cdot\,,\cdot\,,\check{\vec{x}},\check{\vec{\alpha}},\{(\check{y},q) : \theta(y,q)\}) \text{ defines a well-ordering of } {}^{\omega}2\text{”} \tag{2}$$

The following only uses that $\mathbb{S}_{\omega_1}$ is a countable support product of uncountably many copies of the same forcing.

Claim 2. *$\mathbb{S}_{\omega_1}$ is weakly homogeneous, i.e., given $p, p' \in \mathbb{S}_{\omega_1}$ there is an isomorphism $\pi : \mathbb{S}_{\omega_1} \to \mathbb{S}_{\omega_1}$ such that $p || \pi(p')$.*

Proof. Let $p, p' \in \mathbb{S}_{\omega_1}$. Since $\mathrm{supp}(p)$ is countable there is some $\gamma < \omega_1$ such that $\mathrm{supp}(p) \subset \gamma$. Set $\pi : \mathbb{S}_{\omega_1} \to \mathbb{S}_{\omega_1}$ defined as follows:

$$\pi(r)(\beta) = \begin{cases} 1_{\mathbb{S}} & \text{if } \beta < \gamma \\ r(\alpha) & \text{if } \beta = \gamma + \alpha \end{cases}$$

Note that $\mathrm{supp}(p) \cap \mathrm{supp}(\pi(p')) = \varnothing$, hence $p || \pi(p')$. □

Since $\mathbb{S}_{\omega_1}$ is weakly homogeneous and $\mathbb{S}_{\omega_1\setminus\xi} \cong \mathbb{S}_{\omega_1}$, (2) gives us

$$\mathbb{1} \Vdash^{\mathbb{S}_{\omega_1}}_{L[g\restriction\xi]} \check{p} \Vdash^{\mathbb{P}_B}_{L[\check{g\restriction\xi}][\dot{g}]} \text{“}\varphi(\cdot\,,\cdot\,,\check{\check{\vec{x}}},\check{\check{\vec{\alpha}}},\{(\check{y},q) : \theta(y,q)\}) \text{ defines a well-ordering of } {}^{\omega}2.\text{”}$$

Let g^* be $(\mathbb{S}_{\omega_1})^L$-generic over $L[g]$ so that $g \restriction [\xi,\omega_1)$ and g^* are (or may be construed as) mutually $(\mathbb{S}_{\omega_1})^L$-generics over $L[g \restriction [\xi,\omega_1)]$, and let h^* be $\mathbb{P}_B$-generic over $L[g\restriction\xi, g^*]$ with $p \in h^*$. We have that

$$L[g\restriction\xi, g^*][h^*] \models \text{“}\varphi(\cdot\,,\cdot\,,\vec{x},\vec{\alpha},b^*) \text{ defines a well-ordering of } {}^{\omega}2,\text{”}$$

where $b^* := \bigcup h^*$ is the Burstin basis added by h^*. Since

$$\mathbb{R} \cap L[g\restriction\xi, g^*][h^*] = \mathbb{R} \cap L[g\restriction\xi, g^*] \neq \mathbb{R} \cap L[g] = \mathbb{R} \cap L[g][h]$$

we can find some β, some $n < \omega$, and $i \in \{0,1\}$ such that

(i) $L[g,h] \models$ “the n^{th} digit of the β^{th} element of ${}^{\omega}2$ given by $\varphi(\cdot\,,\cdot\,,\vec{x},\vec{\alpha},b)$ is i”
(ii) $L[g\restriction\xi, g^*][h^*] \models$ “the n^{th} digit of the β^{th} element of ${}^{\omega}2$ given by $\varphi(\cdot\,,\cdot\,,\vec{x},\vec{\alpha},b^*)$ is $1-i$”

Thus there exist two conditions $p_0 \in h$ and $p_1 \in h^*$ below p such that

(i)* $p_0 \Vdash^{\mathbb{P}_B}_{L[g]}$ “the $\check{n}^{th}$ digit of the $\check{\beta}^{th}$ element of ${}^{\omega}2$ given by $\varphi(\cdot\,,\cdot\,,\check{\vec{x}},\check{\vec{\alpha}},\{(\check{y},q) : \theta(y,q)\})$ is $\check{i}$”
(ii)* $p_1 \Vdash^{\mathbb{P}_B}_{L[g\restriction\xi,\, g^*]}$ “the $\check{n}^{th}$ digit of the $\check{\beta}^{th}$ element of ${}^{\omega}2$ given by $\varphi(\cdot\,,\cdot\,,\check{\vec{x}},\check{\vec{\alpha}},\{(\check{y},q) : \theta(y,q)\})$ is $1 \check{-} i$”

Pick $\zeta \geq \xi$, $\zeta < \omega_1$ such that $p_0 \in L[g\restriction\zeta]$ and $p_1 \in L[g\restriction\xi, g^*\restriction\zeta]$, say $\xi + \zeta = \zeta$. Then (i)* and (ii)* above give us

$$(*)\begin{cases} \mathbb{1} \Vdash^{(\mathbb{S}_{\omega_1})^L}_{L[g\restriction\zeta]} \check{p}_0 \Vdash^{\mathbb{P}_B}_{L[g\restriction\zeta][\dot{g}]} \text{“the } \check{\check{n}}^{th} \text{ digit of the } \check{\check{\beta}}^{th} \text{ element of } {}^\omega 2 \text{ given by} \\ \qquad\qquad \varphi(\cdot\,,\cdot\,, \check{\check{\vec{x}}}, \check{\check{\vec{\alpha}}}, \{(\check{y}, q) : \theta(y,q)\}) \text{ is } \check{\check{i}}\text{”} \\ \mathbb{1} \Vdash^{(\mathbb{S}_{\omega_1})^L}_{L[g\restriction\xi, g^*\restriction\zeta]} \check{p}_1 \Vdash^{\mathbb{P}_B}_{L[g\restriction\xi, g^*\restriction\zeta][\dot{g}]} \text{“the } \check{\check{n}}^{th} \text{ digit of the } \check{\check{\beta}}^{th} \text{ element of } {}^\omega 2 \text{ given by} \\ \qquad\qquad \varphi(\cdot\,,\cdot\,, \check{\check{\vec{x}}}, \check{\check{\vec{\alpha}}}, \{(\check{y}, q) : \theta(y,q)\}) \text{ is } 1 \check{\check{-}} i\text{”} \end{cases}$$

Now we want to make sure that the conditions p_0 and $p_1 \in L[g, g^*]$ are compatible.

Claim 3. *$p_0 \cup p_1$ is linearly independent.*

Proof. We may assume without loss of generality that

$$L[g\restriction\xi] \models \text{“}p \text{ is a Burstin basis.”}$$

In particular, it is true in $L[g\restriction\xi]$ that p is a Hamel basis. Suppose that there are $\vec{y} \in p$, $\vec{y_0} \in p_0 \smallsetminus p$, $\vec{y_1} \in p_1 \smallsetminus p$ and some vectors of rational numbers $\vec{q}, \vec{q_0}, \vec{q_1}$ such that

$$\vec{q} \cdot \vec{y} + \vec{q_0} \cdot \vec{y_0} + \vec{q_1} \cdot \vec{y_1} = 0 \tag{3}$$

By mutual genericity we have

$$\vec{q} \cdot \vec{y} + \vec{q_0} \cdot \vec{y_0} = -\vec{q_1} \cdot \vec{y_1} \in L[g\restriction\zeta] \cap L[g\restriction\xi, g^*\restriction\zeta] = L[g\restriction\xi]$$

Since p is a Hamel basis for the reals of $L[g\restriction\xi]$, there exists some $\vec{z_1} \in [p]^{<\omega}$, $\vec{r_1} \in [\mathbb{Q}]^{<\omega}$ such that

$$\vec{r_1} \cdot \vec{z_1} = -\vec{q_1} \cdot \vec{y_1}$$

Since $p_1 \supset p$ is linearly independent it follows that $\vec{r_1} = 0 = \vec{q_1}$. Coming back to the equation (3), we now have that

$$\vec{q} \cdot \vec{y} + \vec{q_0} \cdot \vec{y_0} = 0$$

Since $q_0 \supset p$ is also linearly independent, we conclude that $\vec{q} = 0 = \vec{q_0}$. Hence $p_0 \cup p_1$ is linearly independent. □

We may construe $g\restriction[\zeta, \omega_1)^\frown g^*$ as $(\mathbb{S}_{\omega_1})^L$-generic over $L[g\restriction\xi, g^*\restriction\zeta]$ as well as over $L[g\restriction\zeta]$. Therefore by $(*)$ it follows that

$$(**)\begin{cases} p_0 \Vdash^{\mathbb{P}_B}_{L[g][g^*]} \text{“the } \check{n}^{th} \text{ digit of the } \check{\beta}^{th} \text{ element of } {}^\omega 2 \text{ given by} \\ \qquad\qquad \varphi(\cdot\,,\cdot\,, \check{\vec{x}}, \check{\vec{\alpha}}, \{(\check{y}, q) : \theta(y,q)\}) \text{ is } \check{i}\text{”} \\ p_1 \Vdash^{\mathbb{P}_B}_{L[g][g^*]} \text{“the } \check{n}^{th} \text{ digit of the } \check{\beta}^{th} \text{ element of } {}^\omega 2 \text{ given by} \\ \qquad\qquad \varphi(\cdot\,,\cdot\,, \check{\vec{x}}, \check{\vec{\alpha}}, \{(\check{y}, q) : \theta(y,q)\}) \text{ is } 1 \check{-} i\text{”} \end{cases}$$

By claim 3 and lemma 4.3, there is some $q \leq p_0, p_1$, $q \in {\mathbb{P}_B}^{L[g, g^*]}$. But then, q forces the contradictory statements from the matrices of $(**)$. This concludes the proof. □

The previous proof in fact shows the following.

Lemma 5.2. *Let g be $\mathbb{S}_{\omega_1}$-generic over L, let h be $\mathbb{P}_B$-generic over $L[g]$ and let $b = \bigcup h$ be the Burstin basis added by h. Inside $L[g, h]$, there are Turing-cofinally many $x \in \mathbb{R}$ such that if $X \subset L[x]$, $X \in \mathrm{OD}_{x,b}$, then $X \in L[x]$.*

By standard arguments, Lemma 5.2 then implies.

Lemma 5.3. *Let g be $\mathbb{S}_{\omega_1}$-generic over L, let h be $\mathbb{P}_B$-generic over $L[g]$ and let $b = \bigcup h$ be the Burstin basis added by h. Let $W = L(\mathbb{R}, b)^{L[g,h]}$. Then*

$${}^\omega W \cap L[g, h] \subset W.$$

In particular, W is a model of DC, the principle of dependent choice.

Theorem 5.4. *Let g be $\mathbb{S}_{\omega_1}$-generic over L, and let b be $\mathbb{P}_B$ generic over $L[g]$. Let*

$$W = L(\mathbb{R}, b)^{L[g, b]}.$$

Then, $W \models \mathsf{ZF} + \mathsf{DC}$ and in W there are Luzin, Sierpiński, Vitali sets and a Burstin basis but in W there is no a well-ordering of $\mathbb{R}$.

Proof. Clearly Lemma 5.3 gives $W \models \mathsf{ZF} + \mathsf{DC}$. Now, as $\mathbb{P}_B$ is ω-closed, $\mathbb{R} \cap W = \mathbb{R} \cap L[g]$, so that $W \models$ "b is a Burstin basis". This means that in W, we have a Bernstein set and a Hamel basis. Hence, in view of 2.2, there is a Vitali set in W induced by b. By Corollary 3.4, W has a Luzin as well as a Sierpiński set. Finally, by 5.1, in W there is no well-ordering of the reals, as required. □

6. Further remarks: ultrafilters on ω, mad families, Mazurkiewicz sets, etc.

Let g be $\mathbb{S}_{\omega_1}$-generic over L.

By [18, Theorem 6], in $L[g]$ there is an ultrafilter on ω which is generated by an ultrafilter in L. In fact, if $U \in L$ is a selective ultrafilter on ω, then U generates an ultrafilter in $L[g]$ (see [29]). This implies that the model $W = L(\mathbb{R}, b)^{L[g,b]}$ from Theorem 5.4 has ultrafilters on ω.

The same remark applies to maximal almost disjoint (mad) families as well as to maximal independent families. See [6, Section 11.5] on mad families in the iterated Sacks forcing extension; an argument which works for maximal independent families in the iterated Sacks forcing extension as well as in $L[g]$ will appear in [8], the argument for mad families is simpler than the one for maximal independent families.

A set $M \subseteq \mathbb{R}^2$ is a *Mazurkiewicz* set if M intersects every straight line in exactly two points. Mazurkiewicz showed in ZFC that Mazurkiewicz sets exist, see [21]. We may force with a poset $\mathbb{P}_M$ consisting of "local" Mazurkiewicz sets over $L[g]$ in much the same way as Definition 4.1 gave a forcing whose conditions are "local" Burstin bases. If m is the set added by $\mathbb{P}_M$, then m will be a Mazurkiewicz set in $L(\mathbb{R}, m)^{L[g,m]}$ and this model will not have a well-ordering of the reals. This result is proved in [3].

We may in fact force with the product $\mathbb{P}_B \times \mathbb{P}_M$ over $L[g]$ and get a model with a Burstin base and a Mazurkiewicz set with no well-order of the reals.

In the same fashion, one may add further "maximal independent" sets generically over $L[g]$, e.g. selectors for Σ^1_2 definable equivalence relations, without adding a well-ordering of $\mathbb{R}$. (Cf. [9].)

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