Projective sets and large cardinals

Ralf Schindler, talk at Kiel, Dec. 02

We shall be interested in the complexity of Hamel bases in situations where certain projective sets of reals are Lebesgue measurable and have the Baire property. Large cardinals will become significant for these investigations.

Recall: A Hamel basis is a basis for the vector space $\mathbb{R}$ over $\mathbb{Q}$. A set $A \subset \mathbb{R}$ is nowhere dense iff f.a. open $\mathcal{O} \neq \emptyset$ there is some open $\bar{\mathcal{O}} \subset \mathcal{O}$, $\bar{\mathcal{O}} \neq \emptyset$, with $\bar{\mathcal{O}} \cap A = \emptyset$; a set A is meager iff $A = \bigcup_{n=0}^{\omega} A_n$, where each A_n is nowhere dense; a set A has the Baire property iff there is an open $\mathcal{O}$ s.t. $A \triangle \mathcal{O}$ is meager. Shelah has shown that if ZF is consistent then so is ZF + "every set of reals has the Baire property." There is no Hamel basis in a model witnessing this:

<u>Lemma</u> 1 (folklore ?) Suppose that every $A \subset \mathbb{R}$ has the Baire property. Then there is no Hamel basis.

<u>Proof</u>: Suppose B is a Hamel basis. W.l.o.g., $1 \in B$ (!). If $\vec{q} = (q_0, \dots, q_n)$ is a sequence of rationals then let

$$A_{\vec{q}} = \{x \in \mathbb{R} : \exists y_1 \dots \exists y_n \text{ in } B \ \ x = q_0 + \sum_{i=1}^{n} q_i y_i\}.$$

For some $\vec{q}$, $A_{\vec{q}}$ is not meager (o.w. $\mathbb{R}$ would be meager). Let $\mathcal{O}$ be open, $\mathcal{O} \neq \emptyset$, s.t. $A_{\vec{q}} \Delta \mathcal{O}$ is meager. Pick $a, b, p \in \mathbb{Q}$ s.t. $p \neq 0$, $a + p < b$, and $(a, b+p) \subset \mathcal{O}$. Then $(a,b) \setminus A_{\vec{q}}$ is meager; thus $(a+p, b+p) \setminus (A_{\vec{q}} + p)$ is meager, too, where $A_{\vec{q}} + p$ is the shift $\{x + p : x \in A_{\vec{q}}\}$. Hence

$$((a+p, b) \setminus A_{\vec{q}}) \cup ((a+p, b) \setminus (A_{\vec{q}} + p)) = (a+p, b) \setminus (A_{\vec{q}} \cap (A_{\vec{q}} + p))$$

is also meager. In particular, $A_{\vec{q}} \cap (A_{\vec{q}} + p) \neq \emptyset$. However, if $x \in A_{\vec{q}} \cap (A_{\vec{q}} + p)$ then $x \in A_{\vec{q}}$ and $x - p \in A_{\vec{q}}$ and their difference is rational (and $\neq 0$). Contradiction! $\dashv$

Rmk.: the same proof shows that the Vitali set doesn't have the Baire property.

Recall that $A \subset \mathbb{R}^k$ is analytic (or $\underset{\sim}{\Sigma}^1_1$) iff A is the projection of a Borel set (in $\mathbb{R}^{k+1}$); a set $A \subset \mathbb{R}^k$ is $\underset{\sim}{\Pi}^1_n$ iff $\mathbb{R}^k \setminus A$ is $\underset{\sim}{\Sigma}^1_n$; and $A \subset \mathbb{R}^k$ is $\underset{\sim}{\Sigma}^1_{n+1}$ iff A is the projection of a $\underset{\sim}{\Pi}^1_n$ set (in $\mathbb{R}^{k-1}$).

Corollary 1. Let $n \geq 1$, and suppose that every $\underset{\sim}{\Sigma}^1_n$ set has the Baire property. There is then no Hamel basis which is $\underset{\sim}{\Sigma}^1_n$.

Proof: If B is $\underset{\sim}{\Sigma}^1_n$ then the sets $A_{\vec{q}}$ from the proof of Lemma 1 are all $\underset{\sim}{\Sigma}^1_n$, too. $\dashv$

Corollary 2. There is no analytic Hamel basis.

Proof: Every analytic set has the Baire property (this is due to Luzin + Sierpiński and will also be implied by results below). $\dashv$

We shall now often think of $\mathbb{R}$ as ${}^\omega\omega$. Let $A \subset \mathbb{R}$ be $\underset{\sim}{\Sigma}^1_1$. There is then a tree T on $\omega \times \omega$ s.t. $x \in A \iff x \in p[T] = \{ \bar{x} : \exists y \, \forall n \, (\bar{x} \restriction n, y \restriction n) \in T \}$.

Let $T_s = \{ t : lh(t) \leq lh(s) \wedge (s \restriction lh(t), t) \in T \}$. Let us define a tree $\mathcal{U}$ by

$(s, f) \in \mathcal{U} \iff$

$f : (T_s, \supset) \xrightarrow{\sim} (\kappa, <)$ order preserving,

where $\kappa \geq \aleph_1$ is a fixed cardinal. We have $x \in p[\mathcal{U}] = \{ \bar{x} : \exists \tilde{f} \, \forall n \, (\bar{x} \restriction n, \tilde{f} \restriction T_{\bar{x} \restriction n}) \in \mathcal{U} \} \iff x \notin p[T]$. This holds true not only for reals x in V, but also for reals x in any $V^{\mathbb{P}}$ with $\mathbb{P} \in V$ being a poset of size $< \kappa$.

<u>Definition</u> 1. $A \subset \mathbb{R}$ is called <u>universally</u> <u>Baire</u> iff there are trees T_0, T_1 s.t.

(a) $A = p[T_0]$, and

(b) $V^{\mathbb{P}} \models p[T_1] = \mathbb{R} \setminus p[T_0]$ for all posets $\mathbb{P} \in V$.

<u>Lemma</u> 2. (Feng, Magidor, Woodin) If $A \subset \mathbb{R}$ is universally Baire then A has the Baire property.

<u>Proof</u>: Let T_0, T_1 witness A is universally Baire. Let $\pi : \bar{M} = L_\alpha[\bar{T}_0, \bar{T}_1] \longrightarrow L[T_0, T_1]$ be sufficiently elementary where $\bar{M}$ is countable. Then the set

$$\mathcal{C} = \{x \in \mathbb{R}^V : x \text{ is a Cohen real over } \bar{M}\}$$

is comeager.

Claim. For $x \in \mathcal{C}$, $x \in p[\bar{T}_0] \Leftrightarrow x \in p[T_0]$.

Prf.: "$\Rightarrow$" is trivial, as $p[\bar{T}_0] \subset p[T_0]$.

"$\Leftarrow$": Suppose $x \notin p[\bar{T}_0]$. Then $x \in p[\bar{T}_1]$, as $p[\bar{T}_0]^{\bar{M}}$ is universally Baire in $\bar{M}$. But then $x \in p[T_1]$, as $p[\bar{T}_1] \subset p[T_1]$. Thus $x \notin p[T_0]$. $\dashv$

Let $x \in \mathcal{C}$. Then x is obtained by forcing (over $\bar{M}$) with $\mathcal{B}_c$ = the Borel sets mod the meager ideal. Let G_x denote the generic. Let $\dot{x}$ be a canonical name for x; then

$$|| \dot{x} \in p[\check{\bar{T}}_0] ||_{\mathcal{B}_c}$$

is an element of $\mathcal{B}_c$, and if $y \in \mathbb{R} \cap \bar{M}$ is a Borel code for a representative of $|| \dot{x} \in p[\check{\bar{T}}_0] ||_{\mathcal{B}_c}$ then we shall write B for the Borel set in V coded by y.

We may now reason as follows:

For $x \in \mathcal{C}$, $x \in A \Leftrightarrow x \in p[T_0] \Leftrightarrow x \in p[\bar{T}_0]$

$\Leftrightarrow \bar{M}[x] = \bar{M}[G_x] \models x \in p[\bar{T}_0] \Leftrightarrow$

$$\| \dot{x} \in p[\check{\bar{T}}_0] \|_{\mathcal{B}_c} \in G_x \iff x \in B.$$

I.e., $A \triangle B$ is meager, and A has the Baire property. $\dashv$

Using T, U from p. 3 f., Lemma 2 shows every analytic set has the Baire property (cf. the remark in the proof of Corollary 2).

Recall: A set $A \subset \mathbb{R}$ is <u>Lebesgue</u> <u>measurable</u> iff for all $\varepsilon > 0$ there is a closed $F \subset A$ and there is an open $G \supset A$ s.t. $\mu^*(G \setminus F) < \varepsilon$ (here, μ^* denotes the outer measure). An argument as for Lemma 2, but using random reals instead, yields:

<u>Lemma</u> 3. (Feng, Magidor, Woodin) If $A \subset \mathbb{R}$ is universally Baire then A is Lebesgue measurable. $\dashv$

By Corollary 2 above, the simplest possible complexity of a Hamel basis is $\underset{\sim}{\Pi}^1_1$. More generally, by Corollary 1, if all $\underset{\sim}{\Sigma}^1_n$ sets have the Baire property then the simplest possible complexity of a Hamel basis is $\underset{\sim}{\Pi}^1_n$.

Lemma 4 (Miller) In L, there is a Π^1_1 Hamel basis.

Proof: Let us work in L. Let $(\beta_i : i<\omega_1)$ enumerate the ordinals $\beta<\omega_1$ with $\rho_1(L_\beta) = \omega$ (i.e., those β s.t. $L_\beta = h^{L_\beta}(\omega \cup \{p\})$ for some $p \in L_\beta$, where h denotes a Σ_1 Skolem funktion). We recursively define $(x_i^0, x_i^1 : i<\omega_1)$ as follows.

given $(x_j^0, x_j^1 : j<i)$, let (x_i^0, x_i^1) be the $<_L$-least pair such that $\{x_i^0, x_i^1\} \subset L_{\beta_i+1}$, $x_i^0 \notin \mathbb{Q}[\{x_j^0, x_j^1 : j<i\}]$, $x_i^1 \notin \mathbb{Q}[\{x_j^0, x_j^1 : j<i\} \cup \{x_i^0\}]$, and there is some $a_i = a \subset \omega$ recursive in x_i^0 as well as recursive in x_i^1 s.t. $(\omega; a) \cong (L_{\beta_i}; \in)$.
Let us verify that $(x_i^0, x_i^1 : i<\omega_1)$ is well-defined.

Inductively, $x_j^0, x_j^1 \in L_{\beta_j+1}$, so $\{x_j^0, x_j^1 : j<i\} \subset L_{\beta_i}$, and hence $\mathbb{Q}[\{x_j^0, x_j^1 : j<i\}] \subset L_{\beta_i}$. There is therefore some x with $x \notin \mathbb{Q}[\{x_j^0, x_j^1 : j<i\}]$, $x \in L_{\beta_i+1}$ (as $\rho_\omega(L_\beta) = \omega$). Let $a = a_i \subset \omega$ s.t. $(\omega; a) \cong (L_{\beta_i}; \in)$ (given by $\rho_1(L_\beta) = \omega$). Working in L_{β_i+1}, we may replace a by something more complicated to make sure a is not recursive in any finite join of elements of $\{x_j^0, x_j^1 : j<i\} \cup \{x\}$. Let us identify a with its characteristic function. Set

$$v = 0{,}\, a(0)\, a(0)\, a(1)\, a(1)\, a(2)\, a(2) \ldots \leq \tfrac{1}{9},$$

and let $r \in \mathbb{Q}$ be s.t. $\frac{1}{9} < r \cdot x < 1$; $0 < r \cdot x - v < 1$. Write $r \cdot x - v = 0{,}\, u(0)u(1)u(2)\ldots$, and define

$$x^0 = 0{,}\, a(0)\, u(1)\, a(1)\, u(3)\, a(2)\, u(5) \ldots, \quad \text{and}$$

$$x^1 = 0{,}\, u(0)\, a(0)\, u(2)\, a(1)\, u(4)\, a(2) \ldots .$$

Then $x^0 + x^1 = r \cdot x$, so $x \in \mathbb{Q}[\{x^0, x^1\}]$. Of course a is recursive in both x^0 and x^1. $x^0 \notin \mathbb{Q}[\{x^0_j, x^1_j : j<i\}]$, as otherwise x^0 would be recursive in a finite join of elements of $\{x^0_j, x^1_j : j<i\}$. Moreover, if $x^1 \in \mathbb{Q}[\{x^0_j, x^1_j : j<i\} \cup \{x^0\}]$ then $x \in \mathbb{Q}[\{x^0_j, x^1_j : j<i\} \cup \{x^0\}]$, thus $x^0 \in \mathbb{Q}[\{x^0_j, x^1_j : j<i\} \cup \{x\}]$ (as $x \notin \mathbb{Q}[\{x^0_j, x^1_j : j<i\}]$), and a would be recursive in a finite join of elements of $\{x^0_j, x^1_j : j<i\} \cup \{x\}$; hence $x^1 \notin \mathbb{Q}[\{x^0_j, x^1_j : j<i\} \cup \{x^0\}]$.

We have shown that, given $(x^0_j, x^1_j : j<i)$, x^0_i and x^1_i are well-defined.

Let $i<\omega_1$, and let $h = 0,1$. As a_i is recursive in x^h_i (recall that $(\omega; a_i) \cong (L_{\beta_i}; \in)$), there is a code for L_{β_i+1} which is $\Delta^1_1(x^h_i)$ (i.e., "$k \in a_i$" is $\Delta^1_1(x^h_i)$). We have that

(*) $\{ y \in \mathbb{R} :$ y codes a (countable) initial segment of $L \}$

is Π^1_1. This now gives that $B = \{x^0_i, x^1_i : i<\omega_1\}$ is Π^1_1, because now

$x \in B$ iff $\exists y \in \Delta^1_1(x)$ s.t. y codes some $L_{\beta+1}$ and $L_{\beta+1} \models$ "there is a sequence $(\bar{x}^0_i, \bar{x}^1_i : i \leq \theta)$ s.t. $(\bar{x}^0_i, \bar{x}^1_i)$ is $<_L$-least with $\bar{x}^0_i \notin \mathbb{Q}[\{\bar{x}^0_j, \bar{x}^1_j : j<i\}]$, $\bar{x}^1_i \notin \mathbb{Q}[\{\bar{x}^0_j, \bar{x}^1_j : j<i\} \cup \{\bar{x}^0_i\}]$, and there is some $a \subset \omega$ recursive in both $\bar{x}^0_i$ and $\bar{x}^1_i$ s.t. $(\omega; a) \cong (L_{\bar{\beta}_i}; \in)$ (where $\bar{\beta}_i$ is the i^{th} β with $\rho_1(L_\beta) = \omega$), for all $i \leq \theta$, and $x = \bar{x}^0_\theta$ or $x = \bar{x}^1_\theta$."

Of course, B is a Hamel basis. $\dashv$

We shall now be interested in lifting the situation with L ("all analytic sets are Lebesgue measurable and have the property of Baire + there is a Π^1_1 Hamel basis") to higher levels of the projective hierarchy. Our key tools will be Lemmas 2 + 3 above together with the following generalization of Lemma 4.

Lemma 5. Let $A \subset OR$ be a set or a class. Suppose that $L[A]$ is sound (in particular, if $\rho_1(L_\alpha[A]) = \omega$ then $L_\alpha[A] = h^{L_\alpha[A]}(\omega \cup \{p\})$ for some $p \in L_\alpha[A]$). Suppose further that

(**) $\{y \in \mathbb{R} :$ y codes a (cble.) initial segment $L_\alpha[A]$ of $L[A]$ with $\rho_1(L_\alpha[A]) = \omega\}$

is $\underset{\sim}{\Pi}^1_n$ (where $n \geq 1$). Then $L[A] \models$ "there is a $\underset{\sim}{\Pi}^1_n$ Hamel basis."

<u>Proof</u>: the proof of Lemma 4 goes thru. We may basically just replace the use of (*) by a use of (**).

$\dashv$

We'll need the following slight refinement of Def. 1:

<u>Definition</u> 2. $A \subset \mathbb{R}$ is called λ-<u>universally</u> <u>Baire</u> (where $\lambda \geq \aleph_0$ is a cardinal) iff there are trees T_0, T_1 s.t.

(a) $A = p[T_0]$, and

(b) $V^{\mathbb{P}} \models p[T_1] = \mathbb{R} \setminus p[T_0]$ for all posets $\mathbb{P} \in H_\lambda$.

In order to prove A is Lebesgue measurable and has the Baire property it suffices that A be $\aleph_1$-universally Baire.

Let $A \subset \mathbb{R}$ be $\underset{\sim}{\Sigma}^1_2$. Let $x \in A \Leftrightarrow \exists y\ (x,y) \in B$, where B is $\underset{\sim}{\Pi}^1_1$. Let T be a tree on $\omega \times \omega \times \omega$ s.t. $(x,y) \notin B \Leftrightarrow (x,y) \in p[T] = \{(\bar{x},\bar{y}) : \exists z \forall n (\bar{x} \restriction n, \bar{y} \restriction n, z \restriction n) \in T\}$.

Let $T_{(s,t)} = \{u : lh(u) \leq lh(s) = lh(t) \wedge (s \restriction lh(u), t \restriction lh(u), u) \in T\}$, where $s, t \in {}^{<\omega}\omega$; let $T_{(x,y)} = \{u : (x \restriction lh(u), y \restriction lh(u), u) \in T\}$, where $x, y \in {}^{\omega}\omega$. We have $x \in A \Leftrightarrow \exists y\ (x,y) \notin p[T]$

$\Leftrightarrow \exists y\ T_{(x,y)}$ is well-founded.

We let $(s,t,f) \in S \iff$

$$f : (T_{(s,t)}, \supset) \xrightarrow{\sim} (\kappa, <) \text{ order preserving},$$

where $\kappa \geq \aleph_1$ is a fixed cardinal. We have $x \in p[S] = \{\bar{x} : \exists y \exists \tilde{f} \, \forall n \, (\bar{x}\restriction n, y\restriction n, \tilde{f} \restriction T_{(\bar{x}\restriction n, y\restriction n)}) \in S\}$ $\iff x \in A$. The fact that $x \in p[S] \iff$ $\exists y \; T_{(x,y)}$ is well-founded holds true in all $V^{\mathbb{P}}$, where $\mathbb{P} \in V$ is a poset of size $< \kappa$. (Cf. p.4) S is called the "Shoenfield tree."

Now suppose that κ is a measurable cardinal. We aim to construct the "Martin-Solovay tree" for the complement of A. Our construction will be different from the standard construction, though (which exploits the idea of "shifting indiscernibles").

Let $x \notin A \iff \varphi(x,z)$, where φ is Π^1_2 and $z \in \mathbb{R}$ is a parameter. The tree T constructed on p.4 can easily be used to prove that for any x,

$$x \notin A \iff W \models \varphi(x,z)$$

for all inner models W with $x, z \in W$. (W is called an inner model iff W is a transitive proper class with $W \models ZFC$).

We invite the reader to verify that there is a tree $\tilde{T}$ searching for $(x, M, \pi, \mathbb{P}, G)$ s.t.

- $x \in \mathbb{R}$,
- M is a ctble model of ZFC^-,
- $\pi : M \longrightarrow V_{\kappa+2}$ is elementary,
- $\mathbb{P} \in V^M_{\pi^{-1}(\kappa)}$ is a poset in M,
- G is $\mathbb{P}$-generic / M,
- $x \in M[G]$, and
- $M[G] \models \varphi(x)$.

Set $p[\tilde{T}] = \{ x : \exists M, \pi, \mathbb{P}, G \ (x, M, \pi, \mathbb{P}, G) \in [\tilde{T}] \}$.

Claim 1. $x \in p[\tilde{T}] \iff x \notin A$ (in V).

Proof: "$\Leftarrow$" is easy, so let us prove "$\Rightarrow$". Let $(x, M, \pi, \mathbb{P}, G) \in [\tilde{T}]$. Let us w.l.o.g. assume that M is transitive. $M \models$ "$\exists$ normal measure on $\pi^{-1}(\kappa)$", let $\bar{U}$ be a witness, and let $U = \pi(\bar{U})$. As $\pi : M \to V_{\kappa+2}$, standard arguments from inner model theory show that M is "iterable" by $\bar{U}$ and images thereof; this gives an inner model $\tilde{M}$ with $V^{\tilde{M}}_{\pi^{-1}(\kappa)} = V^M_{\pi^{-1}(\kappa)}$. In particular, G is still $\mathbb{P}$-generic / $\tilde{M}$, $x \in \tilde{M}[G]$, and $\tilde{M}[G] \models \varphi(x)$.

But then $V \models \varphi(x)$, as desired, by Shoenfield absoluteness (which can be shown using the tree T from p. 4). $\dashv$ (Claim 1)

<u>Claim</u> 2. $x \in p[\tilde{T}] \Leftrightarrow x \notin p[S]$ in any $V^{\mathbb{P}}$, where $\mathbb{P} \in V_\kappa$ is a poset.

<u>Proof:</u> Fix $\mathbb{P}$. "$\Leftarrow$" is shown by taking a Skolem hull of $V^{\mathbb{P}}_{\kappa+2}$. Let $\pi : M \to V_{\kappa+2}$ thus obtained. Then $x \notin p[\pi^{-1}(S)]$ (as otherwise $x \in p[S]$), and hence $M^{\pi^{-1}(\mathbb{P})} \models \varphi(x)$. "$\Rightarrow$": This is basically as in the proof of Claim 1 (note that $\tilde{M}[G] \models \varphi(x) \Rightarrow x \notin p[\pi^{-1}(S)]$). $\dashv$ (Claim 2)

We have shown the following results.

<u>Lemma</u> 6 (Martin-Solovay) Let κ be a measurable cardinal. Then every $\underset{\sim}{\Sigma}^1_2$ set of reals is κ-universally Baire.

<u>Corollary</u> 7 Let κ be a measurable cardinal. Then every $\underset{\sim}{\Sigma}^1_2$ set of reals is Lebesgue measurable and has the Baire property.

Lemma 8. Let $L[\mu]$ denote the least inner model which contains a measurable cardinal. In $L[\mu]$, there is a Π^1_2 Hamel basis.

Proof: $\{y \in \mathbb{R} : y$ codes an initial segment of $L[\mu]$ which projects to $\omega\}$ is Π^1_2. Lemma 8 then follows from Lemma 5. ⊣

We could now use results of Martin and Steel and lift the situation with L / $L[\mu]$ to higher levels, using Woodin cardinals. However, our plan is to get better results by exploiting weaker large cardinals and work of Woodin.

Recall that a cardinal κ is called α-strong iff for all $X \in H_{\alpha^+}$ there is some $\pi : V \longrightarrow M$, where M is an inner model, $X \in M$, and κ is the critical point of π; κ is called strong iff κ is α-strong for all α.

Lemma 9. (Woodin) Let κ be λ-strong. Let $A \subset \mathbb{R}$ be κ-universally Baire, and let T and T' witness this, i.e., $A = p[T]$ and $V^{\mathbb{P}} \models$ "$p[T'] = \mathbb{R} \setminus p[T]$" for all posets $\mathbb{P} \in H_\kappa$. Let

H be $\mathrm{Col}(\omega, 2^{2^\kappa})$ - generic / V. Then in $V[H]$ the following holds true :

there are trees $\mathcal{U}$ and $\mathcal{U}'$ s.t.

- $p[\mathcal{U}] = \exists^{\mathbb{R}} p[T] = \{x : \exists y \ (x,y) \in p[T]\}$, and
- $\mathcal{U}$ and $\mathcal{U}'$ witness that $p[\mathcal{U}]$ is λ^+-universally Baire, i.e., $V[H]^{\mathbb{P}} \models$ "$p[\mathcal{U}'] = \mathbb{R} \setminus p[\mathcal{U}]$" for all posets $\mathbb{P} \in H_{\lambda^+}^{V[H]}$.

Proof of Woodin's lemma: Will be shown by amalgamating various $\mathcal{U}'$'s working for particular forcings. Let $\overline{\overline{\mathbb{P}}} < \lambda^+$ in $V[H]$, say $\mathbb{P} \in H_{\lambda^+}$ there. Let

$$\pi : V \longrightarrow M$$

at κ s.t. still $\mathbb{P} \in M[H]$.

We easily get $\mathcal{U}$ s.t. $p[\mathcal{U}] = \exists^{\mathbb{R}} p[T]$.

Moreover, by an argument as above, we'll have that $p[\mathcal{U}] = p[\pi(\mathcal{U})]$.

We now want to get, in $V[H]$, a tree $\mathcal{U}'$ s.t. for any K $\mathbb{P}$-gen. $/ M[H]$,

$$M[H][K] \models p[\pi(\mathcal{U})] = {}^{\omega}\omega \setminus p[\mathcal{U}'] .$$

[As any real in any size $< \lambda^+$ extension of $V[H]$ is in some such $M[H][K]$ we can then just take the "union" of all $\mathcal{U}'$'s.]

We have the long extender E given by π, i.e.

$$X \in E_a \iff a \in \pi(X)$$

for $a \in [\pi(\kappa)]^{<\omega}$, $X \in \mathcal{P}([\kappa]^{\bar{\bar{a}}})$. Set $\nu_a := \pi(E_a)$, being a measure on M.

Notice $(s, a) \in \pi(\mathcal{U}) \iff a \in \pi(\mathcal{U}_s) \iff$

$$\mathcal{U}_s \in E_a \iff \pi(\mathcal{U}_s) \in \nu_a .$$

In $V[H]$, enumerate the ν_a's as $(\sigma_i : i < \omega)$ s.t. every ν_a occurs inf. often. For any $i<\omega$ there is $\pi_i : M \rightarrow_{\sigma_i} \mathrm{Ult}(M; \sigma_i)$, and if $k, i < \omega$ are s.t. σ_k projects to σ_i then there is $\pi_{ik} : \mathrm{Ult}(M; \sigma_i) \rightarrow \mathrm{Ult}(M; \sigma_k)$.

We may then define, in $V[H]$:

$$(s, (\alpha_0, \dots, \alpha_{n-1})) \in \mathcal{U}' \quad \text{iff}$$

$$\forall i < k < n \quad (\pi(\mathcal{U}_{s\restriction \# i}) \in \sigma_i \ \wedge$$

$$\pi(\mathcal{U}_{s \restriction \# k}) \in \sigma_k \ \wedge$$

$$\sigma_k \text{ projects to } \sigma_i$$

$$\longrightarrow \quad \pi_{ik}(\alpha_i) > \alpha_k \ ),$$

where $\# i =$ the length of a s.t. $\sigma_i = \nu_a$.

<u>Claim.</u> If K is $\mathbb{P}$-gen. / $M[H]$ then

$$M[H][K] \models p[\pi(\mathcal{U})] = {}^{\omega}\omega \setminus p[\mathcal{U}'] .$$

Pf.: Assume first $x \in p[\pi(\mathcal{U})] \cap p[\mathcal{U}']$, say $(x, f) \in [\pi(\mathcal{U})]$ and $(x, \vec{\alpha}) \in [\mathcal{U}']$. For all n, $\pi(\mathcal{U}_{x \restriction n}) \in \nu_{f \restriction n}$, hence using $\vec{\alpha}$ we see that

$$\operatorname{dir\,lim}_n \ \mathrm{Ult}(M; \nu_{f\restriction n})$$

is ill-founded. However, this direct limit can be embedded into $\mathrm{Ult}(M; \pi(E))$ which is well-founded. Contradiction!

Now let $x \notin p[\pi(\mathcal{U})]$. Define, for $i < \omega$,

$$f_i(\vec{\gamma}) := |(x \restriction i, \vec{\gamma})|_{\pi(\mathcal{U})_x} .$$

For any i, π_i canonically extends to $\tilde{\pi}_i$ and σ_i can be fattened to $\tilde{\sigma}_i$ a measure on $M[H][K]$; for appropriate k, i π_{ik} extends to $\tilde{\pi}_{ik}$.

We may then set $\alpha_k := [f_{\# k}]_{\tilde{\sigma}_k}$.

It is straightforward to check that $(x, \vec{\alpha}) \in [\mathcal{U}']$: if $\pi(\mathcal{U}_{s \restriction \# i}) \in \sigma_i$, $\pi(\mathcal{U}_{s \restriction \# k}) \in \sigma_k$, σ_k projects to σ_i, then $\pi_{ik}(\alpha_i) = \tilde{\pi}_{ik}(\alpha_i) =$

$$= \tilde{\pi}_{ik}([\vec{\gamma} \mapsto |(x \restriction \# i, \vec{\gamma})|_{\pi(\mathcal{U})_x}]_{\tilde{\sigma}_i}) =$$

$$= [\vec{\varepsilon} \mapsto |(x \restriction \# i, \vec{\varepsilon} \restriction \# i)|_{\pi(\mathcal{U})_x}]_{\tilde{\sigma}_k} >$$

$$> [\vec{\varepsilon} \mapsto |(x \restriction \# k, \vec{\varepsilon})|_{\pi(\mathcal{U})_x}]_{\tilde{\sigma}_k} =$$

$$= \alpha_k .$$

This proves Woodin's lemma. $\dashv$

Corollary 10. Let $\kappa_1 < \kappa_2 < \ldots < \kappa_n$ be strong cardinals. then in $V^{Col(\omega, 2^{2^{\kappa_n}})}$, every $\underset{\sim}{\Sigma}^1_{n+2}$ set of reals is universally Baire.

Corollary 11. Let $\kappa_1 < \kappa_2 < \ldots < \kappa_n$ be strong cardinals. then in $V^{Col(\omega, 2^{2^{\kappa_n}})}$, every $\underset{\sim}{\Sigma}^1_{n+2}$ set of reals is Lebesgue measurable and has the Baire property.

We now want a version of Lemma 8.

Lemma 12. Let $L[E]$ denote the least inner model which contains n strong cardinals, where $n \geq 1$. Let $\kappa_1 < \ldots < \kappa_n$ be the strong cardinals of $L[E]$. Let g be $Col(\omega, 2^{2^{\kappa_n}})$ - generic / $L[E]$, and let $x \in \mathbb{R} \cap L[E][g]$ coding $L_{2^{2^{\kappa_n}}}[E]$. (In particular, $L[E][g] = L[E][x]$.) then in $L[E][g]$, there is a $\Pi^1_{n+2}(x)$ Hamel basis.

Proof. By Lemma 5, it suffices to prove that, setting $\lambda = (2^{2^{\kappa_n}})^{L[E]} = (\kappa_n^{++})^{L[E]}$,

$$\{ y \in \mathbb{R} : y \text{ codes an initial segment of } L[E] \text{ which is longer than } L_\lambda[E] \text{ and projects to } \lambda \}$$

is $\Pi^1_{n+2}(x)$. this was shown by Hauser.

$\dashv$

We therefore have, assuming the consistency of infinitely many strong cardinals, for each $n \in \omega \setminus \{0\}$ a model M_n of set theory in which every set of reals in $\utilde{\Sigma}^1_n \cup \utilde{\Pi}^1_n$ is Lebesgue measurable and has the Baire property and in which there is a $\utilde{\Pi}^1_n$ Hamel basis.

It is an open question if the large cardinals are needed. (Work of Shelah shows one needs one inaccessible.) It can be verified that the models M_n have a $\utilde{\Delta}^1_{n+1}$ well-ordering of the reals.

The attentive reader will have noticed that for $n \geq 3$ the Hamel basis of M_n is *not* lightface definable (the definition needs a parameter coding the ground model up to the double successor of the largest strong cardinal, cf. p. 19). In fact, by the homogeneity of the collapse, the Hamel basis *cannot* be lightface definable; we need an inhomogeneous forcing in order to produce a lightface definable Hamel basis. Such a forcing indeed exists, but there will be a gap between the complexity of the Hamel basis and the complexity of the pointclass consisting of the sets which admit the regularity properties we discuss.

theorem 13. (S. Friedman, Sch) Let $n \in \omega \setminus \{0\}$. Let $L[E]$ be the minimal inner model with n strong cardinals. There is then a real a, set generic over $L[E]$, s.t. in $L[E][a]$ the following hold true:

(a) every $\underset{\sim}{\Sigma}^1_{n+2}$ set of reals is universally Baire,

(b) there is a $\Delta^1_{n+3}(a)$-well ordering of $\mathbb{R}$, and

(c) a is a Π^1_{n+4} singleton (and hence there is a lightface Δ^1_{n+5}-well ordering of $\mathbb{R}$).

Proof (sketch): Let $\kappa_1 < \ldots < \kappa_n$ be the strong cardinals of $L[E]$.

Set $\lambda = \kappa_n^{++L[E]}$. Inside $L[E]$, we define a "nice" sequence of λ^+-Suslin trees, $(T_k : k<\omega)$. We first force with $\prod_k T_k$, adding cofinal branches B_k thru T_k (the forcing has the λ^+-c.c.). We then force with $Col(\omega, \lambda)$, adding G. We'll write $\omega_1 = \omega_1^{L[E][\vec{B}][G]} = \lambda^{+L[E]} = \kappa_n^{+++L[E]}$.

Any two distinct cofinal branches thru $({}^{<\omega}2, \subset) \in L$

give a pair of a.d. subsets of ω (reals). Let $(a_k : k<\omega) \in L$ be given by the first (along $<_L$) ω many branches. Write

$$x^{dec} = \{ k<\omega : x \cap a_k \text{ is finite} \}$$

for reals x.

Pick a real $g \subset \omega$, $g \in L[E][\vec{B}][G]$ coding $J_\lambda[E]$. We want to force over $L[E][\vec{B}][g]$ to obtain a real a s.t. $g = a^{dec}$, and a is a Π^1_{n+3} singleton inside $L[E][a]$. We shall also have (a) and (b) in $L[E][a]$ by arguments pretty much as before.

Let $(a_i : i<\omega_1) \in L[E][g]$ be the sequence of pairwise a.d. reals given by the first (along $<_{L[E][g]}$) ω_1 many branches thru $({}^{<\omega}2, \subset)$. Notice the 2 def.ns of $(a_k : k<\omega)$ given coincide.

the forcing $\mathbb{R}$ consists of $p = (\ell(p), r(p))$ where $\ell(p) : k \to 2$, some $k<\omega$, and $r(p) \subset \omega_1$ finite.

We set $q \le p$ iff $\ell(q) \supset \ell(p)$, $r(q) \supset r(p)$,

AND

$$k < \mathrm{dom}(\ell(p)) \wedge k \in g \implies$$

$$\{ m \in \mathrm{dom}(\ell(q)) \setminus \mathrm{dom}(\ell(p)) : \ell(q)(m) = 1 \} \cap a_k = \emptyset,$$

$$k < \mathrm{dom}(\ell(p)) \wedge \ell(p)(k) = 1 \wedge \alpha \in r(p) \cap B_{2k} \Rightarrow$$

$$\{ m \in \mathrm{dom}(\ell(q)) \setminus \mathrm{dom}(\ell(p)) : \ell(q)(m) = 1 \} \cap a_{\alpha+\omega+2k} = \emptyset,$$

and

$$k < \mathrm{dom}(\ell(p)) \wedge \ell(p)(k) = 0 \wedge \alpha \in r(p) \cap B_{2k+1} \implies$$

$$\{ m \in \mathrm{dom}(\ell(q)) \setminus \mathrm{dom}(\ell(p)) : \ell(q)(m) = 1 \} \cap a_{\alpha+\omega+2k+1} = \emptyset.$$

Let $a \subset \omega$ be given by a generic. By the 1st two lines above, $a^{dec} = g$.
Set $D_k = \{ \alpha : a \cap a_{\alpha+\omega+k}$ is finite $\}$. We also have (by the last 4 lines):

$$k \in a \Rightarrow B_{2k} = D_{2k} \wedge D_{2k+1} = \emptyset, \quad \text{and}$$

$$k \notin a \Rightarrow B_{2k+1} = D_{2k+1} \wedge D_{2k} = \emptyset.$$

It is crucial that moreover we'll have that

$k \in a \Rightarrow T_{2k+1}$ is Aronszajn in $L[E][a]$, and

$k \notin a \Rightarrow T_{2k}$ is Aronszajn in $L[E][a]$.

these properties of a make it a Π^1_{n+4} singleton ("David's trick"). We let $\phi(x) :\equiv$

x^{dec} codes $J_\lambda[E]$, and

for all $\mathcal{N}$, $J_\lambda[E] \triangleleft \mathcal{N} \triangleleft J_{\omega_1}[E]$, with

ⓐ λ is the 2nd largest cardinal of $\mathcal{N}$,

ⓑ $\mathcal{N}[x] \models ZF^-$,

we have that, if $(T_n^{\mathcal{N}} : n<\omega)$ and $(a_i^{\mathcal{N},x} : i < \omega_1^{\mathcal{N}[x]})$ are defined inside $\mathcal{N}$, $\mathcal{N}[x]$, as (T_n), (a_i) was defined above in $L[E]$, $L[E][g]$, and if we set $B_k^{\mathcal{N},x} = \{\alpha : x \cap a^{\mathcal{N},x}_{\alpha+\omega+k}$ is finite$\}$, then

ⓐ′ $k \in x \Rightarrow B^{\mathcal{N},x}_{2k}$ is a cof. branch thru $T^{\mathcal{N}}_{2k}$, and

ⓑ′ $k \notin x \Rightarrow B^{\mathcal{N},x}_{2k+1}$ is a cof. branch thru $T^{\mathcal{N}}_{2k+1}$.

$\phi(x)$ can be checked to be Π^1_{n+4}. We're left with having to verify $\phi(x) \Longleftrightarrow x = a$ in order to establish (c).

Let $x \neq a$, let w.l.o.g. $\ell \in x \setminus a$. In particular, $T_{2\ell}$ is an Aronszajn tree in $L[E][a]$. We may pick $\sigma : \mathcal{N}[x] \to J_{\omega_2}[E][x]$ with $\mathcal{N}$ ctble., c.p.$(\sigma) > \lambda$. There is a condensation lemma telling us that $\mathcal{N} \lhd L[E]$. But then $\mathcal{N}[x] \models$ "$T^{\mathcal{N}}_{2k}$ is not Aronszajn," by ⓐ′, so $J_{\omega_2}[E][x] \models$ "T_{2k} is not Aronszajn." Contradiction!

To prove that $\phi(a)$ holds one first observes that $(T_k^{\mathcal{N}} : k < \omega) = (T_k \cap \mathcal{N} : k < \omega)$ and $(a_i^{\mathcal{N}, a} : i < \omega_1^{\mathcal{N}[a]}) = (a_i : i < \omega_1^{\mathcal{N}[a]})$ for any $\mathcal{N}$ as in $\phi(x)$. In particular, $B_k^{\mathcal{N}, a} = B_k \cap \mathcal{N}$ for $k < \omega$. But then ⓐ′, ⓑ′ will be obvious.

Now (a) follows from Woodin's lemma. (b) holds, as the order of constructibility of $L[E][a]$ is $\Delta^1_{n+3}(a)$, when restricted to reals:

$x <_{L[E][a]} y$ iff

there is an initial segment $J_\alpha[E]$ of $L[E]$ s.t. $L_\lambda[E] \trianglelefteq J_\alpha[E] \triangleleft L_{\lambda+L[E]}[E]$, $\rho_\omega(J_\alpha[E]) \leq \lambda$, and $J_\alpha[E] \models$ "$x < y$."
By the proof of lemma 12, the set of codes of the relevant $J_\alpha[E]$ is $\Pi^1_{n+2}(a)$.

$\dashv$

Lemma 14. (Sch) Let n, $L[E]$, and a be as in theorem 13, then in $L[E][a]$ there is a Π^1_{n+4} Hamel basis.

Prf.: Work in $L[E][a]$. Let $(\beta_i : i<\omega_1)$ enumerate the ordinals $\beta<\omega_1$ with $\rho_1(J_\beta[E]) = \lambda$ (where $\lambda = \kappa_n^{++}$, $\kappa_1 < \ldots < \kappa_n$ being the strong cardinals of $L[E]$). We recursively define $(x^0_i, x^1_i : i<\omega_1)$ as follows.
Given $(x^0_j, x^1_j : j<i)$, let (x^0_i, x^1_i) be the $<_{L[E][a]}$-least pair such that $\{x^0_i, x^1_i\} \subset J_{\beta_i+1}[E]$, $x^0_i \notin \mathbb{Q}[\{x^0_j, x^1_j : j<i\}]$, $x^1_i \notin \mathbb{Q}[\{x^0_j, x^1_j : j<i\} \cup \{x^0_i\}]$, and there is some $b_i = b \subset \omega$ s.t. $(\omega; b) \cong (J_{\beta_i}[E]; \in)$ and $a \oplus b$ is recursive in x^0_i as well as in x^1_i.
An argument as in the proof of Lemma 4 shows that $(x^0_i, x^1_i : i<\omega_1)$ is well-defined. Of course, $\mathcal{B} = \{x^0_i, x^1_i : i<\omega_1\}$ is a Hamel basis.

Let $i<\omega_1$, and let $h=0,1$. As b_i is recursive in x_i^h (recall that $(\omega; b_i)\cong(J_{\beta_i}[E];\in)$), there is a code for $J_{\beta_i+1}[E]$ which is $\Delta^1_1(x_i^h)$. Moreover, a is recursive in x_i^h. We also have that there is some Π^1_{n+4} formula $\varphi(v)$ with $a'=a \Leftrightarrow \varphi(a')$, and there is some Π^1_{n+2} formula $\psi(v,w)$ with

y codes some $J_\beta[E]$, where $\lambda\leq\beta<\lambda^+$ and $\rho_\omega(J_\beta(E))=\lambda$

$\Leftrightarrow \psi(y,a)$

(cf. p. 19). We therefore get that B is Π^1_{n+4}, because $x\in B$ iff

$$\exists a' \text{ recursive in } x\ \exists y\in\Delta^1_1(x)\ [\varphi(a')\wedge\psi(y,a')\wedge \text{if } y \text{ codes } J_{\beta+1}[E'] \text{ then } J_{\beta+1}[E']\models \text{"}x\in B\text{"}].$$

[Here, "$x\in B$" has to be written out according to p. 26; cf. p. 9.]

$\dashv$

Summary:

Model	L	L[μ]	L[E]	
Sets measurable and having Baire property	$\underset{\sim}{\Sigma}^1_1$	$\underset{\sim}{\Sigma}^1_2$	$\underset{\sim}{\Sigma}^1_{n+2}$	
Complexity of Hamel basis	Π^1_1	Π^1_2	$\underset{\sim}{\Pi}^1_{n+2}$	Π^1_{n+4}
Complexity of w.o. of $\mathbb{R}$	Δ^1_2	Δ^1_3	$\underset{\sim}{\Delta}^1_{n+3}$	Δ^1_{n+5}

Many questions remain open. In particular, we don't know the consistency strength of "all $\underset{\sim}{\Sigma}^1_n$ sets are Lebesgue measurable and have the property of Baire + there is a $\underset{\sim}{\Pi}^1_n$ Hamel basis". (Cf. p. 20.) We do have equiconsistencies in the presence of Projective Uniformization, though. Recall that Projective Uniformization holds iff for all projective $R \subset \mathbb{R} \times \mathbb{R}$ there is a projective function $F \subset R$ with the same domain as R (i.e., $\forall x\, (\exists y\, (x,y) \in R \rightarrow \exists y\; F(x) = y)$).

Theorem 15 (Steel, Sch) The following are equiconsistent.

(a) ZFC + all projective sets are Lebesgue measurable and have the Baire property + Projective Uniformization holds, and

(b) ZFC + there are $\kappa_1 < \kappa_2 < \dots$ with supremum λ such that each κ_n is λ-strong.

This indicates a relation between strong cardinals and regularity properties for projective sets of reals.

One could now study definable Hamel bases in models of $\neg$CH (cf. work of Harrington), or one could study such bases in models of "all Δ^1_n sets are measurable, but not all of them ~~are~~ have the Baire property (cf. work of Judah + Spinas).