THE SOLIDITY AND NONSOLIDITY OF INITIAL SEGMENTS OF THE CORE MODEL

GUNTER FUCHS AND RALF SCHINDLER

Abstract. It is shown that $K|\omega_1$ need not be solid in the sense of [FS]: It is consistent that there is no inner model with a Woodin cardinal yet there is an inner model W and a Cohen real x over W such that $K|\omega_1 \in W[x] \setminus W$. However, if $0^\P$ does not exist and $\kappa \geq \omega_2$ is a cardinal, then $K|\kappa$ is solid. We draw the conclusion that solidity is not forcing absolute in general, and that under the assumption of $\neg 0^\P$, the core model is contained in the solid core, introduced in [FS].

It is also shown, assuming $0^\P$ does not exist, that if there is a forcing that preserves ω_1, forces that every real has a sharp, and increases $\boldsymbol{\delta}^1_2$, then ω_1 is measurable in K.

1. Introduction

In [FS, Def. 4.1, 4.7], we introduced the concepts of solidity and generic solidity as follows.

Definition 1.1. A set a is *solid* if it cannot be added by set-forcing to an inner model, i.e., if for every b, $\mathbb{P}$ and g such that $\mathbb{P} \in L[b]$, g is $\mathbb{P}$-generic over $L[b]$ and $a \in L[b][g]$, it follows that $a \in L[b]$. A set is *generically solid* if it is solid in every set-forcing extension of the universe.

The motivation for these definitions was that solid sets should be canonical in some sense, so that it would be worthwhile to analyze the class $\mathfrak{C}$, the *solid core*, also defined in [FS, Def. 4.12], as

$$\mathfrak{C} = \bigcup_{a \text{ solid}, a \subseteq \mathrm{On}} L[a]$$

Our main result on the solid core was [FS, Thm. 4.21], saying that if there is an inner model with a Woodin cardinal, then there is a "minimal" fine structural one, such that if one iterates the least normal measure of this model out of the universe, the resulting model is the solid core. In particular, under the assumption of an inner model with a Woodin cardinal, the solid core is a fine structural extender model.

The obvious question is what can be said about the solid core in the absence of an inner model with a Woodin cardinal. We showed in [FS, Thm. 4.22] that it is consistent that $K \neq \mathfrak{C}$. We shall give some more information here, namely that

2010 *Mathematics Subject Classification.* 03E40, 03E45, 03E55, 03E57.

Key words and phrases. Core model theory, forcing, solidity.

The first author's research for this paper was supported in part by PSC CUNY research grant 69656-00 47.

under the assumption that $0^\P$ does not exist, it follows that $K \subseteq \mathfrak{C}$. This is Lemma 3.2.

Generic solidity was introduced in order to arrive at a concept that is forcing absolute, but it was originally unclear whether solidity itself is forcing absolute. We show in Lemma 3.3, again under the assumption that $0^\P$ does not exist, that there may be solid sets that are not generically solid.

All of these conclusions about the relationship between the core model and the solid core, as well as solidity versus generic solidity come from an analysis of the solidity/nonsolidity of initial segments of K. In section 2, we show that under certain assumptions, $K||\alpha$ may not be solid, if $\alpha \leq \omega_1$. The case $\alpha = \omega_1$ is more complicated than the countable case, and uses a forcing due to Jensen, and in order to provide a self-contained account, we give a detailed description of this forcing in the appendix, Section 5. Section 3 shows that the assumptions we made in order to produce a model where $K|\omega_1$ is not solid were optimal, and then, using a similar argument, it shows under $\neg 0^\P$, that $K|\kappa$ is solid if κ is a cardinal greater than or equal to ω_2.

It turned out that our methods show that, assuming $0^\P$ does not exist, if there is a forcing that preserves ω_1, forces that every real has a sharp, and increases $\boldsymbol{\delta}^1_2$, then ω_1 is measurable in K. This is proven in Section 4.

The authors would like to thank Gabriel Fernandes for drawing the beautiful diagrams. We also thank the referee for her/his valuable list of comments.

2. The non–solidity of initial segments of K

We aim to prove Lemma 2.2 according to which $K||\alpha$ is not generically solid under very general circumstances. Lemma 2.2 will be a consequence of the following somewhat technical statement.

Lemma 2.1. *Let M and M^+ be countable models of* ZFC$^-$*, such that $M = (H_\theta)^{M^+}$, where θ is a regular cardinal in M^+. Let $\vec{U}$ be a sequence of normal ultrafilters in M (and hence in M^+) such that M^+ is normally iterable with respect to $\vec{U}$. Let $a \in \mathrm{V}^M_\kappa$, where U_0 is on κ. Let $\varphi(x,y)$ be a Σ_0-formula, and suppose there is a normal iteration $\mathcal{I}$ of M, using ultrafilters from $\vec{U}$ and their images, with a last model, of length less than θ, such that $\varphi(\mathcal{I}, a)$ holds. Then there is a countable model $\bar{M}$ and a normal iteration $\bar{\mathcal{I}}$ of $\bar{M}$, with last model M, such that $\varphi(\bar{\mathcal{I}}, a)$ holds.*

Proof. Let $\mathcal{I}$ be as stated, $\mathcal{I} = \langle\langle M_i \mid i \leq \rho\rangle, \langle \pi_{i,j} \mid i \leq j \leq \rho\rangle\rangle$, where $\rho < \theta$. Let $\mathcal{I}^+ = \langle\langle M_i^+ \mid i \leq \rho\rangle, \langle \pi^+_{i,j} \mid i \leq j \leq \rho\rangle\rangle$ be the iteration of M^+ which results from lifting $\mathcal{I}$ to $M^+ = M_0^+$. It follows that $M_i = \pi^+_{0,i}(M)$, for all $i \leq \rho$. Let's view $\mathcal{I}$ as a subset of M_ρ, say, by identifying it with $\{\langle x, y, i\rangle \mid i < \rho, x \in M_i, y \in M_\rho \text{ and } y = \pi_{i,\rho}(x)\}$.

Let g be $\mathrm{Col}(\omega, M_\rho)$-generic over V. Then in $H^{\mathrm{V}[g]}_{\omega_1}$, the statement $(*)$ that there is a $\mathcal{I}' \subseteq M_\rho$ such that the following Σ_0-statement $\psi(\mathcal{I}', a, M_\rho)$ is true:

$$\varphi(\mathcal{I}', a) \text{ and the last model of } \mathcal{I}' \text{ is } M_\rho$$

holds.

Let's say that a real x codes an element u of H_{ω_1} if, letting $E_x = \{\langle m, n\rangle \mid x(m,n) = 0\}$, E_x is well-founded, and, letting the function $\pi_x : \omega \longrightarrow \mathrm{V}$ be defined by $\pi_x(n) = \{\pi_x(m) \mid mE_xn\}$, we have that $\pi_x(0) = u$. Clearly, every member of H_{ω_1} has a real coding it, and we have that

(+) for every Σ_0-formula $\theta(\vec{y})$, there is a Σ^1_1-formula θ^c that expresses θ "in the codes", meaning that whenever $\vec{u} \in H_{\omega_1}$ and $u^c_0, \ldots, u^c_{n-1}$ are real codes for $u_0, \ldots, u_{n-1}$, respectively, we have that

$$\langle H_{\omega_1}, \in\rangle \models \theta(\vec{u}) \quad \text{iff} \quad \theta^c(\vec{u}^c).$$

In general, expressing that a real codes an element of H_{ω_1} is a Π^1_1 statement, because the binary relation coded by the real has to be well-founded. But the existential statement we are dealing with concerns the existence of a subset (the iteration $\mathcal{I}'$) of a set (M_ρ) for which we already may fix a real code.

To be precise, let $\psi^c(x, y, z)$ be the Σ^1_1 statement expressing $\psi(x, y, z)$ "in the codes". We can then express the statement $(*)$ above in the codes by saying that there is a real x and a $w \subseteq \omega$ such that for every $n \in w$, $\langle n, 0\rangle \in E_z$ and $E_x = \{\langle m+1, n+1\rangle \mid \langle m, n\rangle \in E_z\} \cup \{\langle m+1, 0\rangle \mid m \in w\}$ and such that $\psi^c(x, y, z)$ holds. This insures that if z is a code for u, then x is a code for a subset of u. To give some detail here, note that in this situation, we have that 0 is not an E_x-predecessor of any $k \in \omega$, and that E_x is well-founded. It follows by E_x-induction that for any $m \in \omega$, $\pi_x(m+1) = \pi_z(m)$. Using this, one can see that x is a code for a subset of the set coded by z, because $\pi_x(0) = \{\pi_x(l) \mid lE_x0\} = \{\pi_x(m+1) \mid m \in w\} = \{\pi_z(m) \mid m \in w\} \subseteq \{\pi_z(m) \mid mE_z0\} = \pi_z(0)$. Moreover, assuming for simplicity that E_z is extensional (there is always such a z), every subset v of $\pi_z(0)$ can be realized by such an x, by letting $w = \{m < \omega \mid \pi_z(m) \in v\}$.

Let's call the resulting statement $\psi'(y, z)$. It's a Σ^1_1-statement. (This shows that the part of (+) above holds not only for θ that are Σ_0, but of the form $\exists v \subseteq u_0 \quad \bar{\theta}(v, \vec{u})$, where $\bar{\theta}$ is Σ_0.)

Now, there are reals M^c_ρ and a^c in $M^+_\rho[g]$, coding M_ρ and a. The Σ^1_1-statement $\psi'(a^c, M^c_\rho)$ then holds in $\mathrm{V}[g]$, and hence also in $M^+_\rho[g]$, by Σ^1_1-absoluteness. But this statement can now be "decoded" in $M^+_\rho[g]$, with the result that in $M^+_\rho[g]$, the statement $(*)$ holds, i.e., that there is an $\mathcal{I}' \subseteq M_\rho$ such that $\varphi(\mathcal{I}', a)$ holds and the last model of $\mathcal{I}'$ is M_ρ. This is a statement about members of M^+_ρ, and is thus forced by the trivial condition of the collapse $\mathrm{Col}(\omega, M_\rho)$, in M^+_ρ. Recalling that $M_\rho = \pi^+_{0,\rho}(M)$, we can apply $(\pi^+_{0,\rho})^{-1}$ to the parameters in this statement, so that it is forced by $\mathrm{Col}(\omega, M)$ over M^+ that there is a countable iteration $\bar{\mathcal{I}}$ that satisfies $\varphi(\bar{\mathcal{I}}, a)$ and has last model $\check{M}$. Now, there is in V an $h \subseteq \mathrm{Col}(\omega, M)$ which is M^+-generic, and in $M^+[h]$, there is an iteration of some countable model, with last model M, and exhibiting the desired property. □

Lemma 2.2. *Suppose there is no inner model with a Woodin cardinal, and that K has infinitely many measurable cardinals. Let α be greater than or equal to the supremum of first ω many measurable cardinals, and assume that there is a partial extender on the extender sequence of K with critical point greater than α. Then $K||\alpha$ is not generically solid.*

Proof. Let ν be least such that E^K_ν has critical point greater than α. Let $\bar{\delta} = \nu^{++}$, $\delta = \bar{\delta}^+$. Let g be $\mathrm{Col}(\omega, \delta)$-generic. Let $x \subseteq \omega$ be a Cohen real over $K[g]$. In $K[x]$, let $M^+ = (K||\delta)[x]$ and $M = (K||\bar{\delta})[x]$. For $n < \omega$, let κ_n be n-th measurable cardinal of K, and let U_n be the normal measure on κ_n. Let U'_n be the canonical extension of U_n to M^+. Let us say that a length $\omega+1$ iteration $\mathcal{I} = \langle\langle N_i \mid i \leq$

$\omega\rangle, \langle\pi_{i,j} \mid i \leq j \leq \omega\rangle\rangle$ of a transitive model N of ZFC$^-$ *encodes* x if for every $i < \omega$, letting $\pi_{i,i+1} : N_i \longrightarrow_{W_i} N_{i+1}$, W_i is the normal ultrafilter on the $f(i)$-th measurable cardinal of M_i, where $f : \omega \longrightarrow x$ is the monotone enumeration of x. Clearly, in $K[g][x]$, there is an an iteration of $M[x]$ that encodes x. Since $M[x]$ and $M^+[x]$ are countable in $K[g][x]$, Lemma 2.1 applies, telling us that there is a countable $\bar{M}$ and an iteration $\bar{\mathcal{I}} = \langle\langle\bar{M}_i \mid i \leq j \leq \omega\rangle, \langle\bar{\pi}_{i,j} \mid i \leq j \leq \omega\rangle\rangle$ of $\bar{M}$ that encodes x, such that $\bar{M}_\omega = M[x]$. By elementarity, $\bar{M} = \bar{M}'[x]$, for a ground model $\bar{M}'$ of $\bar{M}$, and by restricting the ultrafilters used in $\bar{\mathcal{I}}$, we get an iteration $\bar{\mathcal{I}}' = \langle\langle\bar{M}'_i \mid i \leq j \leq \omega\rangle, \langle\bar{\pi}'_{i,j} \mid i \leq j \leq \omega\rangle\rangle$ of $\bar{M}'$, with last model M. Let $\bar{\pi}'_{0,\omega}(\bar{\nu}) = \nu$. Let W be the inner model which results from iterating $E^{\bar{M}'}_{\bar{\nu}}$ out of the universe (it is iterable, because it embeds into M). Clearly, x is Cohen-generic over W, and $M \in W[x]$, because x tells W how to iterate an initial segment of itself to reach M. But $M \notin W$, because otherwise, W could compare an initial segment of itself to M in order to recover the iteration $\bar{\mathcal{I}}'$, thus recovering x, which is not in W, since x is a Cohen real over W. Actually, this shows that $M|\alpha$ is generic over W - it is certainly added by adding x, and $M|\alpha$ cannot be in W because comparing $M|\alpha$ with $\bar{M}'|\bar{\delta}$ adds x. But $M|\alpha = K|\alpha$. So $K|\alpha$ is not solid in V$[g]$. □

Remark 2.3. The previous lemma shows that for α as in its assumption, there is a forcing extension in which α is countable and $K||\alpha$ is not solid, because set-forcing does not change K.

We want to find out how large an α such that $K||\alpha$ is not solid can be. The methods up to now only yield countable α with that property. We will use the following theorem, due to Jensen, in order to produce models where $K||\omega_1$ is not solid.

Theorem 2.4 ([Jen90, Theorem 1, p. 4]). *Let U be a normal ultrafilter on the measurable cardinal κ, and let $\theta \geq \kappa^+$ be such that $2^{<\theta} = \theta$. There is then a poset $\mathbb{P} = \mathbb{P}(U, \theta)$ such that if g is $\mathbb{P}$–generic over V, then in* V$[g]$*, there is a countable, transitive structure $\mathcal{M}$ together with a linear iteration*

$$(M_i, \pi_{ij} \colon i \leq j \leq \kappa) \in V[g]$$

of $M_0 = \mathcal{M}$ such that
- (a) $\kappa = \omega_1^{V[g]}$,
- (b) $M_\kappa = (H^V_\theta; \in, U)$, *and*
- (c) $M_{i+1} = \mathrm{ult}(M_i; \pi^{-1}_{i\kappa}(U))$ *for every $i < \kappa$.*

In the Appendix to this paper, Section 5, we will present a proof of Theorem 2.4.

Theorem 2.5. *Let $L[E]$ be a 1–small extender model with no Woodin cardinal such that in $L[E]$, $(\kappa_i \colon i < \omega{+}2)$ is a strictly increasing sequence of measurable cardinals. There is then a forcing extension $V = L[E][g]$ of $L[E]$ such that $\omega_1^V = \kappa_\omega$ and V has a definable (from a set parameter) inner model W such that there is $x \in \mathbb{R} \cap V$, a Cohen real over W, with $L[E]|\kappa_\omega \in W[x] \setminus W$.*

Proof. Inside $L[E]$, let $\theta = (\kappa_{\omega+1})^{+++}$. Let $U_i \in L[E]$ be a normal measure on κ_i, $i < \omega + 2$. By Jensen's Theorem 2.4, applied to U_ω and θ, there is a poset $\mathbb{P}$ such that if g is $\mathbb{P}$–generic over $L[E]$, then in $L[E][g]$, there is a countable mouse M^+ together with a linear iteration

$$(M_i, \pi_{ij} \colon i \leq j \leq \kappa_\omega) \in L[E][g] \tag{1}$$

of $M_0 = M^+$ such that

(a) $\kappa_\omega = \omega_1^{L[E][g]}$,
(b) $M_{\kappa_\omega} = (H_\theta)^{L[E]} = J_\theta[E]$, and
(c) $M_{i+1} = \mathrm{ult}(M_i; \pi^{-1}_{i\kappa_\omega}(U_\omega))$ for every $i < \kappa_\omega$,

in particular, $(H_\theta)^{L[E]}$ is the $\kappa_\omega^{\mathrm{th}}$ iterate of M via a linear iteration using the measure $\pi^{-1}_{0\kappa_\omega}(U_\omega)$ and its images.

Now let x be a Cohen real over $L[E][g]$. Inside $L[E][g][x]$, the iteration of $M^+[x]$ encoding x can be formed (see the proof of Lemma 2.2). Note that $M^+[x]$ has a largest cardinal, which it thinks is $\bar\theta := \bar\kappa^{++}_{\omega+1}$, where $\langle \bar\kappa_i \mid i \leq \omega{+}1\rangle$ enumerates the measurable cardinals of $M^+[x]$. Let $M[x] = (H_{\bar\theta})^{M^+[x]}$. By Lemma 2.1, there is in $L[E][g]$ a countable model $\bar M[x]$, with an iteration encoding x and with last model $M[x]$. By restricting this iteration to $\bar M$, we see that there is an iteration encoding x, with first model $\bar M$ and last model M. Now, $\bar M$ has an $(\omega+1)$-st measurable cardinal. Since $\bar M$ embeds into M, which embeds into a segment of $J_\theta[E]$, we can let W be the model obtained from iterating a measure on that cardinal out of the universe. It follows as in the proof of Lemma 2.2 that $L[E]||\omega_1$ is generic over W, since x is generic over W, and the first $\omega+1$ steps of an iteration from an initial segment of W to $L[E]||\omega_1$ can be read off of x. Moreover, $L[E]||\omega_1$ is not in W, since otherwise, W could compare it with an initial segment of itself, thereby constructing x, which is Cohen-generic over W. □

3. The solidity of initial segments of K

In this section, we show under suitable anti-large cardinal assumptions that longer initial segments of K are solid. First, the hypothesis of Theorem 2.5 is essentially optimal, as our next result shows. For more on $0^\P$, the reader is referred to [Jena, §4.1], or [Zem02, pp. 272].

Lemma 3.1. *Assume that $0^\P$ does not exist, and let K denote the core model. Suppose that $K|\omega_1^{\mathrm{V}}$ is not solid. Then ω_1^{V} is a measurable cardinal in K, and*

$$K|\omega_1^{\mathrm{V}} \models \text{“there are infinitely many measurable cardinals.”}$$

Proof. Since $K|\omega_1^{\mathrm{V}}$ is assumed not to be solid, let W be an inner model, $\mathbb{P} \in W$ a poset, and let g be $\mathbb{P}$–generic over W such that

$$K|\omega_1^{\mathrm{V}} \in W[g] \setminus W. \tag{2}$$

By forcing absoluteness, $K^{W[g]} = K^W$. Let us look at the comparison of K with K^W. Let $\mathcal{T}$ and $\mathcal{U}$ denote the iterations of K and K^W, respectively, arising in the comparison of K with K^W.

Claim 1. $\mathcal{T}$ does not use any extenders of length less than ω_1^{V}.

Proof. Suppose not, and let F be the first extender used in $\mathcal{T}$. Let α_0 be the least α such that $\mathcal{M}^{\mathcal{U}}_\alpha|\mathrm{lh}(F) = K|\mathrm{lh}(F)$, where $\mathrm{lh}(F)$ is the length of F, which is the same as its index. So by assumption, $\mathrm{lh}(F) < \omega_1^{\mathrm{V}}$. Let M be the longest initial segment of $\mathcal{M}^{\mathcal{U}}_{\alpha_0}$ such that $\mathcal{P}(\mathrm{crit}(F)) \cap M \subset \mathcal{M}^{\mathcal{U}}_{\alpha_0}|\mathrm{lh}(F) = K|\mathrm{lh}(F)$. Let $n < \omega$ be such that $\rho_{n+1}(M) \leq \mathrm{crit}(F) < \rho_n(M)$ if there is such an n, and let $n = 0$ otherwise.

We claim that

$$\mathrm{ult}_n(M; F) \text{ is iterable.} \tag{3}$$

Notice that $F \in K|\omega_1^V \in W[g]$, so that by $K^{W[g]} = K^W$ and $\neg 0^\P$, we may argue inside $W[g]$ to see that (3) gives that F is on the sequence of M, cf. [Jena]. But then F cannot be used in $\mathcal{T}$.

It thus remains to verify (3). Let N be the longest initial segment of K such that $\mathcal{P}(\mathrm{crit}(F)) \cap N \subset K|\mathrm{lh}(F)$. Let $m < \omega$ be such that $\rho_{m+1}(N) \leq \mathrm{crit}(F) < \rho_m(N)$ if $N \lhd K$, and let $m = 0$ otherwise. Let $\mathcal{T}'$ and $\mathcal{U}'$ denote the iterations of M and N, respectively, arising from the comparison of M with N. By the universality of K, $\mathcal{M}^{\mathcal{T}'}_\infty \trianglelefteq \mathcal{M}^{\mathcal{U}'}_\infty$ and there is no drop in $\mathcal{T}'$. By $\neg 0^\P$, both $\mathcal{T}'$ and $\mathcal{U}'$ are above $\mathrm{crit}(F)$. For future reference, let us write

$$i \colon M \to R$$

for $\pi^{\mathcal{T}'}_{0,\infty} \colon M \to \mathcal{M}^{\mathcal{T}'}_\infty$, so that $i \restriction \mathrm{crit}(F) = \mathrm{id}$ and $R \trianglelefteq \mathcal{M}^{\mathcal{U}'}_\infty$.

Let us write k for the ultrapower map

$$\pi^N_F \colon N \to \mathrm{ult}_m(N; F),$$

and let us use k to copy $\mathcal{U}'$ onto $\mathrm{ult}(N;F)$, producing an iteration $k\mathcal{U}'$ of $\mathrm{ult}(N;F)$, together with canonical copy maps $k_\alpha \colon \mathcal{M}^{\mathcal{U}'}_\alpha \to \mathcal{M}^{k\mathcal{U}'}_\alpha$ for $\alpha < \mathrm{lh}(\mathcal{U}')$.

$$\begin{tikzcd}
 & Ult(M;F) \arrow[rr, "i'"] & & Ult(R;F) \arrow[dr, "j"] & \\
M \arrow[ur, "\pi^M_F"] \arrow[rr, "i"] & & R \arrow[ur, "\pi^R_F"] \arrow[rr, "\widetilde{k}"] & & R^* \\
 & & K^W \arrow[u, dashed, "\mathcal{U}'"] & & Ult(K^W;F) \arrow[u, dashed, "k\mathcal{U}'"']
\end{tikzcd}$$

FIGURE 1. The situation of the proof of claim 1.

The maps k_α are recursively defined as follows. $k_0 = k$. If $\alpha + 1 < \mathrm{lh}(\mathcal{U}')$, then

$$k_{\alpha+1}([a,f]^{\mathcal{M}^{\mathcal{U}'}_\alpha || \theta_\alpha}_{E^{\mathcal{U}'}_\alpha}) = [k_\alpha(a), k_\alpha(f)]^{\mathcal{M}^{k\mathcal{U}'}_\alpha || k_\alpha(\theta_\alpha)}_{k_\alpha(E^{\mathcal{U}'}_\alpha)}$$

for appropriate a and f, where θ_α indexes the initial segment of $\mathcal{M}^{\mathcal{U}'}_\alpha$ to which $E^{\mathcal{U}'}_\alpha$ gets applied. Note that since we are working below $0^\P$, the iterations are linear. If $\lambda < \mathrm{lh}(\mathcal{U}')$ is a limit, then

$$k_\lambda(x) = \pi^{k\mathcal{U}'}_{\alpha,\lambda} \circ k_\alpha \circ (\pi^{\mathcal{U}'}_{\alpha,\lambda})^{-1}(x)$$

whenever $x \in \mathrm{ran}(\pi^{\mathcal{U}'}_{\alpha,\lambda})$.

It is straightforward to verify inductively that for $\alpha < \mathrm{lh}(\mathcal{U}')$,

(4) $$k_\alpha \restriction \mathcal{P}(\mathrm{crit}(F)) = k \restriction \mathcal{P}(\mathrm{crit}(F)),$$

so that for each $\alpha < \mathrm{lh}(\mathcal{U}')$, k_α factors as

$$k_\alpha = j_\alpha \circ \pi^{\mathcal{M}^{\mathcal{U}'}_\alpha}_F,$$

where

$$\pi_F^{\mathcal{M}_\alpha^{\mathcal{U}'}} : \mathcal{M}_\alpha^{\mathcal{U}'} \to \mathrm{ult}(\mathcal{M}_\alpha^{\mathcal{U}'}; F)$$

is the ultrapower map given by F and j_α is the factor map defined by

$$j_\alpha(\pi_F^{\mathcal{U}'_\alpha}(f)(a)) = k_\alpha(f)(a).$$

Equation (4) is in fact trivial to verify, except for the case $\alpha = 1$. Note that $E_0^{\mathcal{U}'} = F$. So we can argue as follows.

Let $X \in \mathcal{P}(\mathrm{crit}(F)) \cap N$. Then $X = [\{\mathrm{crit}(F)\}, \{\xi\} \mapsto X \cap \xi]^{N||\theta_0}_{E_0^{\mathcal{U}'}}$, so that $k_1(X) = [\{k_0(\mathrm{crit}(F))\}, \{\xi\} \mapsto k_0(X) \cap \xi]^{\mathrm{ult}_m(N;F)||k_0(\theta_0)}_{k_0(E_0^{\mathcal{U}'})} = k_0(X)$.

Let us write

$$\tilde{k} = k_{\mathrm{lh}(\mathcal{U}')-1} \restriction R,$$

so that again $\tilde{k}$ factors as

$$j \circ \pi_F^R,$$

where

$$\pi_F^R \colon R \to \mathrm{ult}_n(R; F)$$

is the ultrapower map given by F and j is some factor map.

Write $R^* = k_{\mathrm{lh}(\mathcal{U}')-1}(R)$. Then R^* is iterable, as it is an iterate of $\mathrm{ult}(N; F)$ (via $k\mathcal{U}'$) and hence of N. But now

$$i' : [a, f]_F^M \mapsto \tilde{k} \circ i(f)(a),$$

where $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f \colon [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \to M$ is one of the functions used to define $\mathrm{ult}_n(M; F)$, embeds $\mathrm{ult}_n(M; F)$ into R^*, see figure 1. This embedding shows that $\mathrm{ult}_n(M; F)$ is iterable. □(Claim 1)

Claim 2. There are infinitely many measurable cardinals in K below ω_1^V.

Proof. Let α_0 be the least α such that $K|\omega_1^V \trianglelefteq \mathcal{M}_\alpha^{\mathcal{U}}$. α_0 is well–defined by Claim 1.

Suppose now that Claim 2 is false. Then $\mathcal{U} \restriction (\alpha_0 + 1)$ can only use measures of order 0 and in fact for all $\alpha < \alpha_0$, $\mathcal{M}_\alpha^{\mathcal{U}}|\mathrm{crit}(E_\alpha^{\mathcal{U}})$ can have only finitely many measurable cardinals. This is easily seen to give that $\mathcal{U} \restriction (\alpha_0 + 1)$ is computable from a finite amount of information, namely, an extender index, the number of times it and its images are iterated, then the next index, etc., finitely many times. This means that $\mathcal{U} \restriction (\alpha_0 + 1) \in K^W \subset W$. But then $K|\omega_1^V \in W$. Contradiction! □(Claim 2)

Claim 3. There is a countable mouse M which wins the comparison against $K|\omega_1^V$, i.e., $K|\omega_1^V <^* M$.[1]

Proof. Of course, $\mathcal{U}$ must use an extender of length less than ω_1^V, as otherwise $K|\omega_1^V \triangleleft K^W$ by Claim 1, and then $K|\omega_1^V \in W$. Let F be the first extender used in $\mathcal{U}$, so that $\mathrm{lh}(F) < \omega_1^V$. Let $N \trianglelefteq K$ be the longest initial segment of K such that $\mathcal{P}(\mathrm{crit}(F)) \cap N \subset K|\mathrm{lh}(F) = K^W|\mathrm{lh}(F)$, and let $n < \omega$ be such that $\rho_{n+1}(N) \leq \mathrm{crit}(F) < \rho_n(N)$ if $N \triangleleft K$, and $n = 0$ otherwise.

Then $\mathrm{ult}_n(N; F)$ makes sense, but it can't be iterable as otherwise we would have that $F = E^K_{\mathrm{lh}(F)}$. This is true because if $\mathrm{ult}_n(N; F)$ were iterable, then by $\neg 0^\P$, K and $\mathrm{ult}_n(N; F)$ would compare to the same mouse Q and the first extender

[1] Here and in what follows, $<^*$ denotes the mouse order, cf. e.g. [SW98].

used on the K–side of that comparison would have to be identical with F, cf. [Jena] (see figure 2).

$$\begin{tikzcd}
K & \trianglerighteq & N \arrow[r] & Ult_n(N;F) \arrow[dr, dashed] & \\
 & & & & Q \\
K \arrow[urrrr, dashed] & & & &
\end{tikzcd}$$

FIGURE 2. If $\mathrm{Ult}_n(N, F)$ were iterable, it would have a common iterate with K.

By taking a countable hull we may thus find a countable mouse M such that $M \trianglerighteq K|\mathrm{lh}(F)$, $\rho_n(M) > \mathrm{crit}(F)$, and

$$\mathrm{ult}_n(M;F) \text{ is not iterable.} \tag{5}$$

Let us now assume Claim 3 were false, which gives that $M \leq^* K|\omega_1^V$ and hence (as M is countable) $M <^* K|\omega_1^V$. By Claim 1 and $\neg 0^{\P}$, $K|\omega_1^V \leq^* K^W|\omega_1^V$, and hence

$$M <^* K^W|\omega_1^V. \tag{6}$$

We now argue similarly as in the proof of Claim 1, so we give fewer details. Let $\mathcal{T}'$ and $\mathcal{U}'$ denote the iterations of M and $K^W|\omega_1^V$, respectively, arising from the comparison of M with $K^W|\omega_1^V$. By (6), $\mathcal{M}_\infty^{\mathcal{T}'} \trianglelefteq \mathcal{M}_\infty^{\mathcal{U}'}$ and there is no drop in $\mathcal{T}'$. By $\neg 0^{\P}$, both $\mathcal{T}'$ and $\mathcal{U}'$ are above $\mathrm{crit}(F)$. Let us write

$$i\colon M \to R$$

for $\pi_{0,\infty}^{\mathcal{T}'}\colon M \to \mathcal{M}_\infty^{\mathcal{T}'}$, so that $i \restriction \mathrm{crit}(F) = \mathrm{id}$ and $R \trianglelefteq \mathcal{M}_\infty^{\mathcal{U}'}$.

By the universality of K and $\neg 0^{\P}$, $\mathcal{U}$ cannot have any drops, so that F must be total on K^W. Let us write k for the ultrapower map

$$\pi_F^{K^W}\colon K^W \to \mathrm{ult}(K^W;F),$$

and let us use k to copy $\mathcal{U}'$ onto $\mathrm{ult}(K^W;F)$, producing an iteration $k\mathcal{U}'$ of $\mathrm{ult}(K^W;F)$, together with canonical copy maps $k_\alpha\colon \mathcal{M}_\alpha^{\mathcal{U}'} \to \mathcal{M}_\alpha^{k\mathcal{U}'}$ for $\alpha < \mathrm{lh}(\mathcal{U}')$. In particular, $k_0 = k$.

For each $\alpha < \mathrm{lh}(\mathcal{U}')$, k_α factors as

$$k_\alpha = j_\alpha \circ \pi_F^{\mathcal{M}_\alpha^{\mathcal{U}'}},$$

as before, where

$$\pi_F^{\mathcal{M}_\alpha^{\mathcal{U}'}}\colon \mathcal{M}_\alpha^{\mathcal{U}'} \to \mathrm{ult}(\mathcal{M}_\alpha^{\mathcal{U}'};F)$$

is the ultrapower map given by F and j_α is the factor map described above. Let us write

$$\tilde{k} = k_{\mathrm{lh}(\mathcal{U}')-1} \restriction R,$$

so that again $\tilde{k}$ factors as

$$j \circ \pi_F^R,$$

where

$$\pi_F^R\colon R \to \mathrm{ult}(R;F)$$

$$\begin{tikzcd}
 & Ult(M;F) \arrow[rrr] & & & Ult(R;F) \arrow[dr, "j"] & \\
M \arrow[ur, "\pi^M_F"] \arrow[rrr, "i"] & & & R \arrow[ur, "\pi^R_F"] \arrow[rr, "\widetilde{k}"] & & R^* \\
 & & K \arrow[r, phantom, "\trianglerighteq"] & N \arrow[u, dashed, "\mathcal{U}'"] & & Ult(N;F) \arrow[u, dashed, "k\mathcal{U}'"']
\end{tikzcd}$$

is the ultrapower map given by F and j is some factor map.

Write $R^* = k_{\mathrm{lh}(\mathcal{U}')-1}(R)$. Then R^* is iterable, as it is an iterate of $\mathrm{ult}(K^W;F)$ (via $k\mathcal{U}'$), and hence of K^W. But again, the map

$$i' : [a,f]^M_F \mapsto \tilde{k} \circ i(f)(a),$$

where $a \in [\mathrm{lh}(F)]^{<\omega}$ and $f\colon [\mathrm{crit}(F)]^{\mathrm{Card}(a)} \to M$ is one of the functions which is used to define $\mathrm{ult}_n(M;F)$, embeds $\mathrm{ult}_n(M;F)$ into R^*, showing that $\mathrm{ult}_n(M;F)$ is iterable. This contradicts (5). $\square$(Claim 3)

Claim 4. ω_1^V is a measurable cardinal in K.

Proof. Let M witness Claim 3, and let $\mathcal{T}$ and $\mathcal{U}$ denote the iterations of M and $K|\omega_1^V$, respectively, arising from the comparison of M and $K|\omega_1^V$. There can be no drop on the $\mathcal{U}$–side, and by replacing M by an appropriate iterate of itself if necessary, we may also assume that there is no drop on the $\mathcal{T}$–side. We must have that the comparison lasts exactly $\omega_1^V + 1$ steps, $\mathcal{M}^{\mathcal{U}}_\infty \triangleleft \mathcal{M}^{\mathcal{T}}_\infty$, and $\mathcal{M}^{\mathcal{U}}_\infty \cap \mathrm{OR} = \omega_1^V$.

We may let $\mathcal{U}$ act on all of K, and we shall write $\mathcal{U}'$ for the resulting iteration of K. In particular, $\mathcal{M}^{\mathcal{U}'}_\infty|\omega_1^V \triangleleft \mathcal{M}^{\mathcal{T}}_\infty$. We have that ω_1^V must be inaccessible in K.

In fact, there is a club $C \subset \omega_1^V$ such that if $\alpha \in C$, then $\alpha = \mathrm{crit}(\pi^{\mathcal{T}}_{\alpha,\omega_1^V})$ and $\pi^{\mathcal{T}}_{\alpha,\omega_1^V}(\alpha) = \omega_1^V$. For any $\alpha \in C$, α is then measurable in $\mathcal{M}^{\mathcal{T}}_\alpha$, so that ω_1^V is measurable in $\mathcal{M}^{\mathcal{T}}_\infty$. Moreover, by standard arguments,

$$\text{if } X \in \mathcal{P}(\omega_1^V) \cap \mathcal{M}^{\mathcal{T}}_\infty, \text{ then for some } \eta < \omega_1,\ C \setminus \eta \subset X \text{ or } (C \setminus \eta) \cap X = \emptyset. \tag{7}$$

We must have that

$$\mathcal{P}(\omega_1^V) \cap \mathcal{M}^{\mathcal{U}'}_\infty \subset \mathcal{M}^{\mathcal{T}}_\infty. \tag{8}$$

To show (8), notice that $\mathcal{T}$, $\mathcal{U}'$ give the first $\omega_1^V + 1$ steps of the comparison of M with K. We have that ω_1^V is measurable in $\mathcal{M}^{\mathcal{T}}_\infty$, so that by $\neg 0^\P$, $\mathcal{M}^{\mathcal{T}}_\infty|\omega_1^V = \mathcal{M}^{\mathcal{U}'}_\infty|\omega_1^V = \mathcal{M}^{U}_\infty$ does not have a strong cardinal (as witnessed by extenders on its sequence), which implies that the rest of the comparison of M and K would only use extenders with critical points $\geq \omega_1^V$ and hence if (8) were false, then the comparison of M with K would continue with a drop on the M–side. But then again by $\neg 0^\P$, also the final iterate on the M–side would be a dropping iterate of

$\mathcal{M}^{\mathcal{T}}_\infty$ and hence a dropping iterate of M, and it would end–extend the final iterate on the K–side. This gives a contradiction to the universality of K.

By (7) and (8), if $\mathcal{C}$ denotes the club filter on ω_1^V, then $\bar{\mathcal{C}} = \mathcal{C} \cap \mathcal{M}^{\mathcal{U}'}_{\omega_1^V}$ is a $\mathcal{M}^{\mathcal{U}'}_{\omega_1^V}$–ultrafilter. As $\bar{\mathcal{C}}$ is countably closed, $\mathrm{ult}(\mathcal{M}^{\mathcal{U}'}_{\omega_1^V};\bar{\mathcal{C}})$ is iterable. By $\neg 0^\P$, the comparison of $\mathcal{M}^{\mathcal{U}'}_{\omega_1^V}$ with $\mathrm{ult}(\mathcal{M}^{\mathcal{U}'}_{\omega_1^V};\bar{\mathcal{C}})$ must be above ω_1^V on both sides, and standard arguments then show that the map

$$\pi_{\bar{\mathcal{C}}} \colon \mathcal{M}^{\mathcal{U}'}_{\omega_1^V} \to \mathrm{ult}(\mathcal{M}^{\mathcal{U}'}_{\omega_1^V};\bar{\mathcal{C}})$$

actually arises via an iteration of $\mathcal{M}^{\mathcal{U}'}_{\omega_1^V}$, cf. [Jena]. In particular, ω_1^V is a measurable cardinal in $\mathcal{M}^{\mathcal{U}'}_{\omega_1^V}$.

As ω_1^V is inaccessible in K, $\pi^{\mathcal{U}'}_{0,\omega_1^V}(\omega_1^V) = \omega_1^V$, so that we finally get from elementarity that ω_1^V is a measurable cardinal in K. $\square$(Claim 4)

$\square$(Lemma 3.1)

Lemma 3.2. *Asume that $0^\P$ does not exist. Let $\kappa \geq \omega_2^V$ be a cardinal. Then $K|\kappa$ is solid. As a result, $K \subset \mathfrak{C}$.*

Proof. This follows from the proofs of Claims 1 and 3 in the proof of Lemma 3.1 which go through as before with ω_1^V being replaced by κ and "countable" in the statement of Claim 3 being replaced by "of size less than κ." But the new version of Claim 3, for $\kappa \geq \omega_2^V$, contradicts the universality of $K|\kappa$, cf. [SW98]. $\square$(Lemma 3.1)

We don't know, but we conjecture that Lemmas 3.1 and 3.2 remain true when their hypothesis on the non–existence of $0^\P$ is weakened to "there is no inner model with a Woodin cardinal." The key problem is that the proof of (8) in the proof of Lemma 3.1 used $\neg 0^\P$ in a substantial way.

It is easy to see that the statement "x is solid" is downward absolute to inner models ([FS, Lemma 4.2]), but Question 4.5 of [FS] asked whether it is forcing absolute, i.e., whether a solid set will remain solid after set forcing. We are now ready to answer this question in the negative.

Lemma 3.3. *Suppose $\neg 0^\P$. Let $\kappa \geq \omega_2$ be a cardinal. Suppose that there are infinitely many $\gamma < \kappa$ such that γ is measurable in K, and that there is a partial extender on the K-sequence with critical point greater than κ. Then $K||\kappa$ is solid, but in a forcing extension of* V*, $K||\kappa$ is not solid.*

Proof. $K||\kappa$ is solid by Lemma 3.2, but it is not generically solid by Lemma 2.2. $\square$

4. Increasing $\boldsymbol{\delta}^1_2$

The proof of Lemma 3.1 has another interesting consequence which we would like to point out. The paper [CS09] produces a stationary set preserving forcing which increases the size of $\boldsymbol{\delta}^1_2$, starting from the hypothesis that NS_{ω_1} is precipitous and $\mathcal{P}(\omega_1)^\#$ exists. The following Lemma basically says that, at least if $0^\P$ does not exist, the hypothesis on the precipitousness of NS_{ω_1} is necessary in the sense that if there is a stationary set preserving forcing which increases the size of $\boldsymbol{\delta}^1_2$ and $0^\P$ does not exist, then in V or in some forcing extension of V there is some inner model whose ω_1 is the ω_1 of V and in which NS_{ω_1} is precipitous. More precisely:

Lemma 4.1. *Assume that* $0^\P$ *does not exist. Suppose that* $\mathbb{P} \in V$ *is a poset such that if* g *is* $\mathbb{P}$*–generic over* V*, then* $\omega_1^{V[g]} = \omega_1^V$*, in* $V[g]$*, every real has a sharp, and*

$$(\boldsymbol{\delta}_2^1)^{V[g]} > (\boldsymbol{\delta}_2^1)^V.$$

Then ω_1^V *is measurable in* K*.*

If we drop the hypothesis that in $V[g]$, every real has a sharp, then Lemma 4.1 becomes false, see [Har77].

Proof. By the proof of Claim 4 in the proof of Lemma 3.1, it suffices to again prove Claim 3 from the proof of Lemma 3.1, this time from the hypothesis of Lemma 4.1:

Claim 5. There is a countable mouse M which wins the comparison against $K|\omega_1^V$, i.e., $K|\omega_1^V <^* M$.

Proof. Let $x \in \mathbb{R} \cap V[g]$ be such that

$$(\omega_1^V)^{+L[x]} \geq (\boldsymbol{\delta}_2^1)^V. \tag{9}$$

By Covering, we have that

$$\kappa^{+L[y]} = \kappa^{+K^{L[y]}}$$

for every real y and for every y–indiscernible κ. This easily gives that $K^{L[x]} <^* K^{L[x^\#]}$ and in fact there is a club proper class $C \subset \mathrm{OR}$ such that if $\mathcal{T}$ and $\mathcal{U}$ are the iterations of $K^{L[x]}$ and $K^{L[x^\#]}$, respectively, arising from the comparison of $K^{L[x]}$ and $K^{L[x^\#]}$, then $\pi^{\mathcal{U}}_{\alpha,\beta}(\alpha) = \beta$ for all $\alpha \leq \beta$, α, $\beta \in C$. $K^{L[x^\#]}$ cannot have a strong cardinal, as otherwise the measure of $x^{\#\#}$ could be used to produce $0^\P$ (cf. [FNS10]). But then it is easy to see, using $\neg 0^\P$ again, that if $\alpha \in C$ and $\gamma > \alpha$ is any cardinal of $\mathcal{M}^{\mathcal{U}}_\alpha$ such that $\mathcal{M}^{\mathcal{U}}_\alpha|\gamma \models$ "α is not a strong cardinal," then $\mathcal{M}^{\mathcal{U}}_\alpha|\gamma$ wins the comparison against $K^{L[x]}$. There is thus some δ such that

$$K^{L[x]} <^* K^{L[x^\#]}|\delta.$$

By indiscernibility, there is then some $\delta < \omega_1^V$ such that

$$K^{L[x]}|\omega_1^V <^* K^{L[x^\#]}|\delta$$

and by indiscernibility again, for the same $\delta < \omega_1^V$,

$$K^{L[x]} <^* K^{L[x^\#]}|\delta. \tag{10}$$

We must have that

$$(\omega_1^V)^{+\mathcal{M}^{\mathcal{U}}_{\omega_1^V}} \geq (\omega_1^V)^{+\mathcal{M}^{\mathcal{T}}_{\omega_1^V}} \geq (\omega_1^V)^{+K^{L[x]}} \geq (\boldsymbol{\delta}_2^1)^V, \tag{11}$$

as otherwise $(\omega_1^V)^{+\mathcal{M}^{\mathcal{U}}_{\omega_1^V}} < (\omega_1^V)^{+\mathcal{M}^{\mathcal{T}}_{\omega_1^V}}$ and the comparison $\mathcal{T}$, $\mathcal{U}$ would continue with a drop on the $\mathcal{T}$–side after stage ω_1^V in contradiction with (10).

We thus obtained a countable mouse $M \in V[g]$, namely $K^{L[x^\#]}|\delta$, such that there is an iteration $\mathcal{S}$ of M of length $\omega_1^V + 1$, namely $\mathcal{U} \restriction \omega_1^V + 1$, such that

$$\mathcal{M}^{\mathcal{S}}_{\omega_1^V} \cap \mathrm{OR} \geq (\boldsymbol{\delta}_2^1)^V. \tag{12}$$

Let us fix M and $\mathcal{S}$ with this property. We claim that M witnesses that Claim 5 is true.

To go for a contradiction, suppose that $M <^* K|\omega_1^V$, so that there is also some countable ordinal β such that $M <^* K|\beta$. Let $\mathcal{T}'$ and $\mathcal{U}'$ be the iterations of M

and $K|\beta$, respectively, arising from the comparison of M and $K|\beta$. There is no drop on the $\mathcal{T}'$–side, and we may write k for the embedding

$$\pi^{\mathcal{T}'}_{0,\infty}\colon M \to \mathcal{M}^{\mathcal{U}'}_{\infty} \trianglelefteq \mathcal{M}^{\mathcal{T}'}_{\infty}.$$

We may use k to copy the iteration $\mathcal{S}$ onto $\mathcal{M}^{\mathcal{U}'}_{\infty}$, which produces an iteration $k\mathcal{S}$ of $\mathcal{M}^{\mathcal{U}'}_{\infty}$ together with a last copying map

$$k_{\infty}\colon \mathcal{M}^{\mathcal{S}}_{\omega_1^V} \to \mathcal{M}^{k\mathcal{S}}_{\omega_1^V}.$$

$$\begin{tikzcd}
\mathcal{M}^{\mathcal{S}}_{\omega_1^V} \arrow[r, "k_\infty"] & \mathcal{M}^{k\mathcal{S}}_{\omega_1^V} & & \\
M \arrow[u, dashed] \arrow[r, "k"] & \mathcal{M}^{\mathcal{U}'}_{\infty} \arrow[u, dashed] & \trianglelefteq & \mathcal{M}^{\mathcal{T}'}_{\infty} \\
& & & K|\beta \arrow[u, dashed]
\end{tikzcd}$$

It is trivial to see that then (12) gives that

$$\mathcal{M}^{k\mathcal{S}}_{\omega_1^V} \cap \mathrm{OR} \geq (\boldsymbol{\delta}^1_2)^V. \tag{13}$$

But $\mathcal{M}^{k\mathcal{S}}_{\omega_1^V}$ is an iterate of $K|\beta$ via $\mathcal{U}'{}^\frown k\mathcal{S}$. As $K|\beta \in V$, we may let $z \in V$ be a real which codes $K|\beta$. A boundedness argument, cf. [Woo99, p. 56f.], then gives that

$$\mathcal{M}^{k\mathcal{S}}_{\omega_1^V} \cap \mathrm{OR} < (\omega_1^V)^{+L[z]} < (\boldsymbol{\delta}^1_2)^V.$$

This contradicts (13). □(Claim 5)

□(Lemma 4.1)

5. Appendix

In order to make this paper more self-contained, we include here a proof of Theorem 2.4, which we restate below. The theorem was originally proved in [Jen90] with an application to the core model in mind, which is why it assumed that V $= K$. The notes [Jenb] contain a more general framework of forcings called $\mathcal{L}$-forcings, and we are following the exposition in [Jenb, §2], albeit without using infinitary languages (we replace the consistency of such a language with the existence of a model in a collapse extension of V), and in a less general form that is streamlined to prove the result we need. The approach we choose here is also similar to the presentation of the forcing used in [CS09].

Theorem (R. Jensen, [Jen90])**.** *Let U be a normal ultrafilter on the measurable cardinal κ, and let $\theta \geq \kappa^+$ be a cardinal with $2^{<\theta} = \theta$. There is then a poset $\mathbb{P} = \mathbb{P}(U, \theta)$ such that if g is $\mathbb{P}$–generic over* V*, then in* V$[g]$*, there is a countable transitive structure $\mathcal{M}$ together with a linear iteration*

$$(M_i, \pi_{ij} \colon i \leq j \leq \kappa) \in V[g]$$

of $M_0 = \mathcal{M}$ such that

(a) $\kappa = \omega_1^{V[g]}$,
(b) $M_\kappa = (H_\theta^V; \in, U)$, *and*
(c) $M_{i+1} = \mathrm{ult}(M_i; \pi_{i\kappa}^{-1}(U))$ *for every* $i < \kappa$.

The proof of this theorem will take up the remainder of this section. Fixing U and θ, let us pick a regular cardinal ρ such that $2^{2^{<\theta}} < \rho$. Therefore, $H_\theta \in H_\rho$, and in fact $\mathcal{P}(H_\theta)$ is in H_ρ as well. It will follow that the forcing $\mathbb{P}(U, \theta)$ we are about to define will also be an element of H_ρ. For notational convenience, let us assume that $2^{<\rho} = \rho$, so that $\mathrm{Card}(H_\rho) = \rho$. We can always force $2^{<\rho} = \rho$ with $<\rho$-closed forcing in a first step, if necessary.

Let us fix a well-order, denoted by $<$, of H_ρ of order type ρ such that $<\restriction H_\theta$ is an initial segment of $<$ of order type θ. (In what follows, we shall also write $<$ for $<\restriction H_\theta$.) We shall write

$$\mathcal{H} = \langle H_\rho; \in, H_\theta, U, < \rangle \text{ and } \mathcal{M} = \langle H_\theta; \in, U, < \rangle.$$

The models we deal with will always be models of the languange of set theory, and we shall tacitly assume that if $\mathfrak{A}$ is a model, then the well-founded part of $\mathfrak{A}$, $\mathrm{wfp}(\mathfrak{A})$, is transitive.

Definition 5.1. Conditions p in $\mathbb{P}(U, \theta)$ are triples

$$p = \langle \langle \kappa_i^p; i \in \mathrm{dom}(p) \rangle, \langle \pi_i^p; i \in \mathrm{dom}(p) \rangle, \langle \tau_i^p; i \in \mathrm{dom}_-(p) \rangle \rangle$$

such that the following hold true.

(i) Both $\mathrm{dom}(p)$ and $\mathrm{dom}_-(p)$ are finite, and $\mathrm{dom}_-(p) \subseteq \mathrm{dom}(p) \subseteq \kappa$.
(ii) $\langle \kappa_i^p; i \in \mathrm{dom}(p) \rangle$ is a sequence of ordinals.
(iii) $\langle \pi_i^p; i \in \mathrm{dom}(p) \rangle$ is a sequence of finite partial maps from κ to θ.
(iv) $\langle \tau_i^p; i \in \mathrm{dom}_-(p) \rangle$ is a sequence of complete $\mathcal{H}$-types over H_θ, i.e., for each $i \in \mathrm{dom}_-(p)$ there is some $x \in H_\rho$ such that, having φ range over $\mathcal{H}$-formulae with free variables $u, \vec{v}$,
$$\tau_i^p = \{ \langle \ulcorner \varphi \urcorner, \vec{z} \rangle \ ; \ \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi[x, \vec{z}] \}.$$
(v) If i, $j \in \mathrm{dom}_-(p)$ with $i < j$, then there are $n < \omega$, $\vec{u} \in \mathrm{ran}(\pi_j^p)$ with
$$\tau_i^p = \{ (m, \vec{z}) \ ; \ (n, \vec{u}^\frown m^\frown \vec{z}) \in \tau_j^p \}.$$
(vi) In $V^{\mathrm{Col}(\omega, 2^\theta)}$, there is a model $\mathfrak{A}$ that that *certifies p with respect to $\mathcal{M}$*, meaning that $H_{\theta^+} \subset \mathrm{wfp}(\mathfrak{A})$, $H_{\theta^+}^{\mathrm{V}} \in \mathfrak{A}$, $\mathfrak{A} \models \mathsf{ZFC}^-$ (that is, ZFC − Power Set, with Collection instead of Replacement), κ is a regular cardinal in $\mathfrak{A}$, and in $\mathfrak{A}$, there is an iteration $\langle M_i^{\mathfrak{A}}, \pi_{i,j}^{\mathfrak{A}}, U_i^{\mathfrak{A}}, \kappa_i^{\mathfrak{A}}; i \leqslant j \leqslant \kappa \rangle$ such that
  (a) if $i < \kappa$, then $M_i^{\mathfrak{A}}$ is countable and $M_{i+1}^{\mathfrak{A}} = \mathrm{ult}(M_i^{\mathfrak{A}}; U_i^{\mathfrak{A}})$,
  (b) if $i \leq \kappa$, then $\kappa_i^{\mathfrak{A}} = \mathrm{crit}(U_i^{\mathfrak{A}})$ and $U_i^{\mathfrak{A}} = \pi_{0,i}^{\mathfrak{A}}(U_0^{\mathfrak{A}})$,
  (c) if $i < \kappa$, then $\mathrm{Hull}^{\mathcal{M}}(\mathrm{ran}(\pi_{i,\kappa}^{\mathfrak{A}})) \cap \theta \subseteq \mathrm{ran}(\pi_{i,\kappa}^{\mathfrak{A}})$,
  (d) $M_\kappa^{\mathfrak{A}} = \langle H_\theta; \in, U \rangle$,

(e) if $i \in \mathrm{dom}(p)$, then $\kappa_i^p = \kappa_i^{\mathfrak{A}}$ and $\pi_i^p \subseteq \pi_{i,\omega_1}^{\mathfrak{A}}$,

(f) if $i \in \mathrm{dom}_-(p)$, then for all $n < \omega$ and for all $\vec{z} \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})$,

$$\exists y \in H_\theta\ (n, y^\frown \vec{z}) \in \tau_i^p \Longrightarrow \exists y \in \mathrm{ran}(\pi_{i,\omega_1}^{\mathfrak{A}})\ (n, y^\frown \vec{z}) \in \tau_i^p.$$

If p, $q \in \mathbb{P}$, then we write $p \leqslant q$ iff $\mathrm{dom}(q) \subseteq \mathrm{dom}(p)$, $\mathrm{dom}_-(q) \subseteq \mathrm{dom}_-(p)$, for all $i \in \mathrm{dom}(q)$, $\kappa_i^p = \kappa_i^q$ and $\pi_i^q \subseteq \pi_i^p$, and for all $i \in \mathrm{dom}_-(q)$, $\tau_i^q = \tau_i^p$.

Conditions p should be seen as finite approximations to the desired iteration leading to $\langle H_\theta; \in, U\rangle$. Due to the presence of $<$, it is enough to know the action of the iteration maps on the ordinals. The third components τ_i^p will guarantee that the iteration maps extend to elementary maps into $\mathcal{H}$ with some $x \in H_\rho$ of interest in their range (cf. Lemma 5.5 below), which will be relevant in the verification that $\mathbb{P}(U, \theta)$ preserves κ as a cardinal.

Note that if $\mathfrak{A}$ certifies any condition p with respect to $\mathcal{M}$, then, as κ is a regular cardinal in $\mathfrak{A}$ and H_θ^{V} has size $\omega_1^{\mathfrak{A}}$ in $\mathfrak{A}$, it follows that $\omega_1^{\mathfrak{A}} = \kappa$.

Lemma 5.2. $\mathbb{P} \neq \emptyset$.

Proof. We need to verify that in $V^{\mathrm{Col}(\omega, 2^\theta)}$ there is a model which certifies the trivial condition $\langle\langle\rangle, \langle\rangle, \langle\rangle\rangle$ with respect to $\mathcal{M}$.

Let g be $\mathrm{Col}(\omega, < \rho)$-generic over V. Inside $V[g]$, $\langle V; \in, U\rangle$ is iterable via U and its images. Let us work inside $V[g]$ until further notice, and let $\langle M_i, \pi_{i,j}, U_i, \kappa_i; i \leqslant j \leqslant \rho\rangle$ be an iteration of $M_0 = \langle V; \in, U\rangle$ via U and its images of length $\rho + 1$.

The map $\pi_{0,\rho}\colon H_\theta \to M_\rho$ admits a canonical extension $\pi\colon V \to N$, where N is transitive and $\pi(H_\theta) = M_\rho$. Let us now leave $V[g]$ and pick some h which is $\mathrm{Col}(\omega, \pi(2^\theta))$-generic over $V[g]$. Of course, h is also $\mathrm{Col}(\omega, \pi(2^\theta))$-generic over N. Let $x \in \mathbb{R} \cap N[h]$ code $\pi((H_{\theta^+})^V)$ in a natural way. The existence of a model which certifies $\langle\langle\rangle, \langle\rangle, \langle\rangle\rangle$ with respect to $\pi(\mathcal{M})$ is then easily seen to be a $\Sigma_1^1(x)$ statement which holds true in $V[g, h]$, as witnessed by $V[g]$. By absoluteness, this statement is then also true in $N[h]$. That is, inside $N^{\mathrm{Col}(\omega, \pi(2^\theta))}$ there is a model which certifies $\langle\langle\rangle, \langle\rangle, \langle\rangle\rangle$ with respect to $\pi(\mathcal{M})$. By elementarity, in $V^{\mathrm{Col}(\omega, 2^\theta)}$ there is therefore a model which certifies $\langle\langle\rangle, \langle\rangle, \langle\rangle\rangle$ with respect to $\mathcal{M}$. $\square$

We will use the following lemma to show that the generic filter indeed produces a generic iteration leading to $\langle H_\theta; \in, U\rangle$. If $p \in \mathbb{P}$, then we shall just say that $\mathfrak{A}$ *certifies* p to express that $\mathfrak{A}$ certifies p with respect to $\mathcal{M}$.

Lemma 5.3. *Let $p \in \mathbb{P}$, as certified by $\mathfrak{A} \in \mathrm{V}^{\mathrm{Col}(\omega, 2^\theta)}$. Then the following hold. In i. to viii., the condition p' claimed to exist is again certified by $\mathfrak{A}$.*

(i) *Let u be finite with* $\mathrm{dom}(p) \subseteq u \subseteq \kappa$. *There is $p' \leqslant p$ with $u \subseteq \mathrm{dom}(p')$.*

(ii) *For $i \in \mathrm{dom}(p)$, $\xi < \theta$, there is $p' \leqslant p$, $\alpha \in \mathrm{dom}(\pi_i^{p'})$ with $\xi < \pi_i^{p'}(\alpha)$.*

(iii) *Let $i \in \mathrm{dom}(p)$, $\xi < \zeta \in \mathrm{dom}(\pi_i^p)$. There is a $p' \leq p$ with $\xi \in \mathrm{dom}(\pi_i^{p'})$.*

(iv) *Let $\xi \in H_\theta$. There is $p' \leqslant p$, $i \in \mathrm{dom}(p')$ with $\xi \in \mathrm{ran}(\pi_i^{p'})$.*

(v) *Let $i, j \in \mathrm{dom}(p)$, $i < j$ and $\xi \in \mathrm{ran}(\pi_i^p)$. There is a $p' \leq p$ such that $\xi \in \mathrm{ran}(\pi_j^{p'})$.*

(vi) *Let $i, i+1 \in \mathrm{dom}(p)$. Let $\xi \in \mathrm{ran}(\pi_{i+1}^p)$. There is a $p' \leqslant p$ such that ξ is definable over $\mathcal{M}$ from parameters in $\mathrm{ran}(\pi_i^{p'}) \cup \{\kappa_i^p\}$.*

(vii) *Let $\lambda \in \mathrm{dom}(p)$ be a limit ordinal, and let $\xi \in \mathrm{ran}(\pi_\lambda^p)$. Then there is a $p' \leq p$ and an $i < \lambda$ with $i \in \mathrm{dom}(p')$ such that $\xi \in \mathrm{ran}(\pi_i^{p'})$.*

(viii) *Let $i \in \mathrm{dom}(p)$, and let ξ be definable over $\mathcal{M}$ from parameters in* $\mathrm{ran}(\pi_i^p)$. *There is a $p' \leqslant p$ such that $\xi \in \mathrm{ran}(\pi_i^{p'})$.*

(ix) *Let $i \in \mathrm{dom}(p)$, and suppose that $X \in \mathcal{P}(\kappa) \cap V$ is definable over $\mathcal{M}$ from parameters in* $\mathrm{ran}(\pi_i^p)$. *Then $\kappa_i^p \in X$ iff $X \in U$.*

Proof. For i., define p' by setting $\mathrm{dom}(p') = u$, $\mathrm{dom}_-(p') = \mathrm{dom}_-(p)$, $\kappa_i^{p'} = \kappa_i^{\mathfrak{A}}$ for $i \in u$, $\pi_i^{p'} = \pi_i^p$ for $i \in \mathrm{dom}(p)$, $\pi_i^{p'} = \emptyset$ for $i \in \mathrm{dom}(p') \setminus \mathrm{dom}(p)$, and $\tau_i^{p'} = \tau_i^p$ for $i \in \mathrm{dom}_-(p')$.

For ii., let α be such that $\pi_{i,\kappa}^{\mathfrak{A}}(\alpha) > \xi$. Such an α exists, as the iteration map $\pi_{i,\kappa}^{\mathfrak{A}}$ is cofinal. We may now define p' to be like p, except that we set $\pi_i^{p'} = \pi_i^p \cup \{\langle \alpha, \pi_{i,\kappa}^{\mathfrak{A}}(\alpha)\rangle\}$.

For iii., define p' to be like p, except that $\pi_i^{p'} = \pi_i^p \cup \{\langle \xi, \pi_{i,\kappa}^{\mathfrak{A}}(\xi)\rangle\}$.

For iv., let $i < \kappa$, $i \notin \mathrm{dom}(p)$, and $\bar{\xi}$ be such that $\pi_{i,\kappa}^{\mathfrak{A}}(\bar{\xi}) = \xi$. Define p' to extend p by adding i into the domain, setting $\pi_i^{p'} = \{\langle \bar{\xi}, \xi\rangle\}$, and leaving the remaining parts of p unchanged.

For v., let $\bar{\xi}$ be such that $\pi_{j,\kappa}^{\mathfrak{A}}(\bar{\xi}) = \xi$, and define p' to be like p, except that we set $\pi_j^{p'} = \pi_j^p \cup \{\langle \bar{\xi}, \xi\rangle\}$.

For vi., since $M_{i+1}^{\mathfrak{A}} = \mathrm{Ult}(M_i^{\mathfrak{A}}, U_i^{\mathfrak{A}})$ there is an $f\colon \kappa_i^p = \pi_{0,i}^{\mathfrak{A}}(\kappa) \to M_i^{\mathfrak{A}}$, $f \in M_i^{\mathfrak{A}}$, such that $(\pi_{i+1}^p)^{-1}(\xi) = \pi_{i,i+1}^{\mathfrak{A}}(f)(\kappa_i^p)$, i.e., $\xi = \pi_{i,\kappa}^{\mathfrak{A}}(f)(\kappa_i^p)$. Due to the presence of $<$ in $\mathcal{M}$, the function $\pi_{i,\kappa}^{\mathfrak{A}}(f)$ is definable over $\mathcal{M}$ in some ordinal parameter $\lambda < \theta$. Let $\bar{\lambda}$ be such that $\lambda = \pi_{i,\kappa}^{\mathfrak{A}}(\bar{\lambda})$. We may define p' to be like p, except that $\pi_i^{p'} = \pi_i^p \cup \{\langle \bar{\lambda}, \lambda\rangle\}$.

For vii., because $\mathrm{ran}(\pi_{\lambda,\kappa}^{\mathfrak{A}}) = \bigcup_{i<\lambda} \mathrm{ran}(\pi_{i,\kappa}^{\mathfrak{A}})$, there is some $i < \lambda$ such that $\xi \in \mathrm{ran}(\pi_{i,\kappa}^{\mathfrak{A}})$. We may assume that $i \in \mathrm{dom}(p)$, by i. Let $\bar{\xi}$ be such that $\pi_{i,\kappa}^{\mathfrak{A}}(\bar{\xi}) = \xi$, and define p' to be like p, except that $\pi_i^{p'} = \pi_i^p \cup \{\langle(\bar{\xi}, \xi\rangle\}$.

For viii., it follows that $\xi \in \mathrm{ran}(\pi_{i,\kappa}^{\mathfrak{A}})$, by Definition 5.1, condition vi(c). Let $\pi_{i,\kappa}^{\mathfrak{A}}(\bar{\xi}) = \xi$, and define p' to be like p, except that $\pi_i^{p'} = \pi_i^p \cup \big\{\langle \bar{\xi}, \xi\rangle\big\}$.

For ix., let $X = \pi_{i,\kappa}^{\mathfrak{A}}(\bar{X})$. Then $X \in U$ iff $\bar{X} \in U_i^{\mathfrak{A}}$ iff $\kappa_i^p = \kappa_i^{\mathfrak{A}} \in \pi_{i,i+1}^{\mathfrak{A}}(\bar{X})$ iff $\kappa_i^p \in X$. $\square$

Now let G be $\mathbb{P}$-generic over V. Set, for $i < \kappa$,

$$\begin{aligned}
\kappa_i &= \kappa_i^p \text{ for some/all } p \in G \text{ with } i \in \mathrm{dom}(p),\\
\pi_i &= \bigcup\{\pi_i^p ; p \in G \wedge i \in \mathrm{dom}(p)\}, \text{ and}\\
\beta_i &= \mathrm{dom}(\pi_i).
\end{aligned}$$

By Lemma 5.3,i.,ii. and iii., $\pi_i\colon \beta_i \to \theta$ is cofinal and order preserving. By Lemma 5.3, iv., $\theta = \bigcup\{\mathrm{ran}(\pi_i); i < \omega_1\}$. For $i < \omega_1$, let X_i be the smallest $X \prec \mathcal{M}$ such that $\mathrm{ran}(\pi_i) \subseteq X$. By Lemma 5.3, viii., $\mathrm{ran}(\pi_i) = X_i \cap \theta$. Let $\tilde{\pi}_i\colon M_i \cong X_i \prec \mathcal{M}$ be the uncollapsing map, so that $\tilde{\pi}_i \supset \pi_i$. For $i \le j \le \omega_1$, let $\tilde{\pi}_{i,j} = \tilde{\pi}_j^{-1} \circ \tilde{\pi}_i$. Then $\tilde{\pi}_{i,j}\colon M_i \to M_j$ is well-defined by Lemma 5.3, v. For $i \le \omega_1$, let $U_i = \tilde{\pi}_i^{-1}(U)$ and $\kappa_i = \tilde{\pi}_i^{-1}(\kappa)$. Using Lemma 5.3, vi., vii., and ix., we then have the following.

Lemma 5.4. *$\langle M_i : \ i \le j \le \kappa\rangle$ is an iteration of M_0 such that if $i < \kappa$, then M_i is countable, and $M_\kappa = \langle H_\theta; \in, U\rangle$.*

It remains to be shown that κ stays regular (so that $\kappa = \omega_1^{\mathrm{V}[G]}$). To this end, let's explore the meaning of the third component of a condition in $\mathbb{P}$.

Lemma 5.5. *Let $p \in \mathbb{P}$ be a condition, and let $\mathfrak{A}$ be a model that satisfies everything in part vi. of Definition 5.1 except possibly condition vi.(f). Let $i \in \mathrm{dom}_-(p)$, and let $x \in H_\theta$ be such that τ_i^p is the complete $\mathcal{M}$-type of x over H_θ. Then the following are equivalent:*

(i) *$\mathfrak{A}$ satisfies Condition vi.(f) at i, that is, for every $n < \omega$ and all $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$, if there is a $y \in H_\theta$ such that $(n, y^\frown\vec{z}) \in \tau_i^p$, then there is such a y in $\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$.*

(ii) $\mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \cup \{x\}) \cap H_\theta = \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$.

(iii) *The map $\pi^{\mathfrak{A}}_{i,\kappa} \colon M_i \to \mathcal{M}$ extends to an elementary map $\tilde{\pi} \colon H \to \mathcal{H}$, where H is transitive, $M_i \in H$, $\tilde{\pi}(M_i) = \langle H_\theta; \in, U\rangle$, $\tilde{\pi} \restriction M_i = \pi^{\mathfrak{A}}_{i,\kappa}$ and $x_i \in \mathrm{ran}(\tilde{\pi})$.*

(iv) $\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \prec \langle H_\theta; \in, U, <, \tau_i^p\rangle$.

Proof. i. $\Rightarrow$ ii.: Let $y \in \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \cup \{x\}) \cap H_\theta$. Then y is definable over $\mathcal{H}$ from parameters $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$ and x. So, for some $n < \omega$, we have that y is unique with $(n, y^\frown\vec{z}) \in \tau_i^p$ (since H_θ is a constant of $\mathcal{M}$). Now, since $y \in H_\theta$ and $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$, it follows i that there is a $y' \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$ with $(n, y'^\frown\vec{z}) \in \tau_i^p$. So by the uniqueness of y, it follows that $y = y' \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$.

ii. $\Rightarrow$ iii.: Let $\tilde{\pi} \colon H \longrightarrow \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \cup \{x\}) \prec \mathcal{H}$ be the inverse of the Mostowski collapse, so that H is transitive. It is obvious that this map works.

iii. $\Rightarrow$ ii.: As $x \in \mathrm{ran}(\tilde{\pi})$ and $\tilde{\pi} \supset \pi^{\mathfrak{A}}_{i,\kappa}$, $\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \subset \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \cup \{x\}) \cap H_\theta \subset \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\tilde{\pi})) \cap H_\theta = \mathrm{ran}(\tilde{\pi}) \cap H_\theta = \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$, since $\tilde{\pi} \restriction M_i = \pi^{\mathfrak{A}}_{i,\kappa}$.

ii. $\Rightarrow$ iv.: We need to show that if $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$ and φ is a formula of the language associated with $\langle H_\theta; \in, U, <, \tau_i^p\rangle$ such that

$$\langle H_\theta; \in, U, <, \tau_i^p\rangle \models \exists v \varphi(v, \vec{z}), \tag{14}$$

then there is a $u \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$ with

$$\langle H_\theta; \in, U, <, \tau_i^p\rangle \models \varphi(u, \vec{z}).$$

There is a recursive map $\ulcorner\psi\urcorner \mapsto \ulcorner\psi^*\urcorner$ (assigning to each formula in the language of $\langle H_\theta; \in, U, <, \tau_i^p\rangle$ a formula of the language of $\langle H_\rho; \in, H_\theta, U, <, x\rangle$) such that for all $\vec{w} \in H_\theta$,

$$\langle H_\theta; \in, U, <, \tau_i^p\rangle \models \psi(\vec{w}) \quad \text{iff} \quad \langle H_\rho; \in, H_\theta, U, <, x\rangle \models \psi^*(\vec{w}).$$

Hence if (14) holds, then there is some $u \in H_\theta$ such that

$$\langle H_\rho; \in, H_\theta, U, <, x\rangle \models \varphi^*(u, \vec{z}).$$

There is then such a u in $H_\theta \cap \mathrm{Hull}^{\mathcal{H}}(\mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \cup \{x\})$, so that by ii., $u \in \mathrm{ran}(\pi_{i,\kappa})^{\mathfrak{A}}$. But then

$$\langle H_\theta; \in, U, <, \tau_i^p\rangle \models \varphi(u, \vec{z}).$$

iv. $\Rightarrow$ i.: Let $n < \omega$ and $\vec{z} \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\kappa})$, and suppose there is a $y \in H_\theta$ such that $(n, y^\frown\vec{z}) \in \tau_i^p$. Then

$$\langle H_\theta; \in, U, <, \tau_i^p\rangle \models \exists y (n, y^\frown\vec{z}) \in \tau_i^p,$$

so that by iv., there is a $y \in \mathrm{ran}(\pi^{\mathfrak{A}}_{i,\omega_1})$ with $(n, y^\frown\vec{z}) \in \tau_i^p$. $\square$

Lemma 5.6. *κ is a regular cardinal in $V^{\mathbb{P}}$.*

Proof. Let $p \in \mathbb{P}$ and $\dot{f} \in H_\rho$ be a $\mathbb{P}$-name such that $p \Vdash \dot{f} \colon \omega \to \check{\kappa}$. We may assume that $\dot{f} \in H_\rho$, and we need to see that there is a $p' \leqslant p$ and an $\alpha < \kappa$ such that $p' \Vdash \operatorname{ran}(\dot{f}) \subset \check{\alpha}$.

Let

$$R = \{(r, n, \delta); r \in \mathbb{P}, \delta < \kappa, \text{ and } r \Vdash_{\mathbb{P}} \dot{f}(\check{n}) = \check{\delta}\}.$$

Notice that $p, R, \leq_{\mathbb{P}} \in H_\rho$. Let τ be the complete $\mathcal{H}$-type of $\langle p, R, \leq_{\mathbb{P}}\rangle$ over H_θ. Let $\mathfrak{A} \in V^{\mathrm{Col}(\omega, 2^\theta)}$ certify p with respect to $\mathcal{M}$. Recall that $H_{\theta+} \in \mathfrak{A}$ and $\omega_1^{\mathfrak{A}} = \kappa$. Thus, $\tau \in \mathfrak{A}$. We have that $\langle \operatorname{ran}(\pi^{\mathfrak{A}}_{i,\kappa}); i < \kappa\rangle$ is a continuous tower of countable substructures of H_θ with $\bigcup\{\operatorname{ran}(\pi^{\mathfrak{A}}_{i,\kappa}); i < \kappa\} = H_\theta$. Since κ is regular in $\mathfrak{A}$, we can pick an $\alpha < \kappa$ such that $\kappa_\alpha = \alpha$, $\operatorname{dom}(p) \subseteq \alpha$ and

$$\operatorname{ran}(\pi^{\mathfrak{A}}_{i,\kappa}) \prec \langle H_\theta; \in, U, <, \tau\rangle \tag{15}$$

We now define p' by setting $\operatorname{dom}(p') = \operatorname{dom}(p) \cup \{\alpha\}$, $\operatorname{dom}_-(p') = \operatorname{dom}(p)_- \cup \{\alpha\}$, $\kappa_i^{p'} = \kappa_i^p$ for all $i \in \operatorname{dom}(p)$, $\kappa_\alpha^{p'} = \alpha$, $\pi_i^{p'} = \pi_i^p$ for all $i \in \operatorname{dom}(p)$, $\pi_\alpha^{p'} = \emptyset$, $\tau_i^{p'} = \tau_i^p$ for all $i \in \operatorname{dom}_-(p)$, and $\tau_\alpha^{p'} = \tau$.

To see that $p' \in \mathbb{P}$, let's first check that condition v. of Definition 5.1 is satisfied. So let $i \in \operatorname{dom}_-(p')$, $i < \alpha$. Then τ_i^p is (trivially) definable over $\mathcal{H}$ from the parameter p, so that because τ is the complete $\mathcal{H}$-type of $\langle p, R, \leq_{\mathbb{P}}\rangle$ over H_θ, we get that there is an $n < \omega$ such that

$$\tau_i^p = \{(m, \vec{z})\ ;\ (n, m^\frown \vec{z}) \in \tau\}.$$

Since by (15), α was explicitly chosen so that $\mathfrak{A}$ satisfies condition iv of Lemma 5.5 at α, it follows that $\mathfrak{A}$ still certifies p', and it's then clear that $p' \in \mathbb{P}$, and that $p' \leqslant p$.

We claim that $p' \Vdash \operatorname{ran}(\dot{f}) \subset \check{\alpha}$. Suppose not. Let $q \leqslant p'$ and $n < \omega$ be such that $q \Vdash \dot{f}(\check{n}) \geq \check{\alpha}$, and let $\mathfrak{B}$ certify q. Set

$$q' = \langle \langle \kappa_i^q; i \in \operatorname{dom}(q) \restriction \alpha\rangle, \langle \pi_i^q; i \in \operatorname{dom}(q) \restriction \alpha\rangle, \langle \tau_i^q; i \in \operatorname{dom}_-(q) \restriction \alpha\rangle\rangle.$$

Of course, $q \leqslant q' \leqslant p$. If $i \in \operatorname{dom}_-(q') = \operatorname{dom}_-(q) \restriction \alpha$, then there are $k = \ulcorner\varphi\urcorner < \omega$, $\vec{u} \in \operatorname{ran}(\pi_\alpha^q)$ such that

$$\begin{aligned} \tau_i^{q'} &= \{(m, \vec{z}) \mid (k, \vec{u}^\frown m^\frown \vec{z}) \in \tau_\alpha^q = \tau\} \\ &= \{(m, \vec{z}) \mid \vec{z} \in H_\theta \wedge \mathcal{H} \models \varphi(\langle p, R, \leq_{\mathbb{P}}\rangle, \vec{u}^\frown m^\frown \vec{z})\} \end{aligned}$$

so that $\tau_i^{q'} \in X := \operatorname{Hull}^{\mathcal{H}}(\operatorname{ran}(\pi^{\mathfrak{B}}_{\alpha,\kappa}) \cup \{\langle p, R, \leq_{\mathbb{P}}\rangle\})$. This implies that in fact

$$q' \in X \prec \mathcal{H}. \tag{16}$$

Now, the statement that there is a $q'' \leq_{\mathbb{P}} q'$ and a $\gamma < \kappa$ such that $q'' \Vdash_{\mathbb{P}} \dot{f}(\check{n}) = \check{\gamma}$ (or, equivalently, $(q'', n, \gamma) \in R$) is true in $\mathcal{H}$, and therefore, by (16), we may let q'' and γ be such objects which are in X.

By part ii. of Lemma 5.5, $X \cap H_\theta = \operatorname{ran}(\pi^{\mathfrak{B}}_{\alpha,\kappa})$, so since $\gamma < \kappa < \theta$, it follows that $\gamma \in \kappa \cap X = \kappa_\alpha^{q'} = \alpha$, so that $q'' \Vdash \dot{f}(\check{n}) < \check{\alpha}$. The same reasoning shows that $\operatorname{dom}(q'') \subseteq \alpha$. So since q and q'' force contradictory statements about $\dot{f}(\check{n})$, they must be incompatible. We derive a contradiction by constructing a common extension $q^* \leq q'', q$. Let

$$\tilde{\pi} : H \longrightarrow X \prec \mathcal{H}$$

be the uncollapse, where H is transitive. Since $X \cap H_\theta = \mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\kappa})$, $M^{\mathfrak{B}}_\alpha = \tilde{\pi}^{-1}(\langle H_\theta; \in, U\rangle) \in H$ and $\tilde{\pi} \restriction M^{\mathfrak{B}}_\alpha = \pi^{\mathfrak{B}}_{\alpha,\kappa}$. Since there is a model in $\mathrm{V}^{\mathrm{Col}(\omega,2^\theta)}$ that certifies q'', it follows that in $\mathcal{H}^{\mathrm{Col}(\omega,2^\theta)}$, there is an iteration

$$\langle M_i, \pi_{i,j}, U_i, \kappa_i; i \leqslant j \leqslant \kappa\rangle$$

such that $M_\kappa = \langle H_\theta; \in, U\rangle$ and for all $i \in \mathrm{dom}(q'')$, $\kappa^{q''}_i = \kappa_i$ and $\pi^{q''}_i \subseteq \pi_{i,\kappa}$. By the elementarity of $\tilde{\pi}$, there is hence in $H^{\mathrm{Col}(\omega,\tilde{\pi}^{-1}(2^\theta))} \subseteq V^{\mathrm{Col}(\omega,2^\theta)}$ an iteration $\langle M_i, \pi_{i,j}, U_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle$ with the properties that $M_\alpha = M^{\mathfrak{B}}_\alpha$ (because $\tilde{\pi}^{-1}(\langle H_\theta; \in, U\rangle) = M^{\mathfrak{B}}_\alpha$), that for all $i \in \mathrm{dom}(q'') \subseteq \alpha$, $\kappa^{q''}_i = \kappa_i$, and that for all such i, $\pi^{q''}_i \subseteq \pi^{\mathfrak{B}}_{\alpha,\kappa} \circ \pi_{i,\alpha}$ (since $\tilde{\pi}^{-1}(\pi^{q''}_i) \subseteq \pi_{i,\alpha}$, so $\pi^{q''}_i \subseteq \tilde{\pi} \circ \pi_{i,\alpha} = \pi^{\mathfrak{B}}_{\alpha,\kappa} \circ \pi_{i,\alpha}$). Because $M^{\mathfrak{B}}_\alpha$ is countable in $\mathfrak{B}$, $\theta + 1 \subset \mathrm{wfp}(\mathfrak{B})$, and $\mathfrak{B} \in V^{\mathrm{Col}(\omega,2^\theta)}$, there is therefore by absoluteness an iteration $\langle M_i, \pi_{i,j}, U_i, \kappa_i; i \leqslant j \leqslant \alpha\rangle$ with these properties in $\mathfrak{B}$.

Let $\langle M^*_i, \pi^*_{i,j}, U^*_i, \kappa^*_i; i \leqslant j \leqslant \kappa\rangle \in \mathfrak{B}$ be defined as follows. If $i \leq j \leq \alpha$, then we set $M^*_i = M_i$, $\pi^*_{i,j} = \pi_{i,j}$, $U^*_i = U_i$, and $\kappa^*_i = \kappa_i$. If $\alpha \leq i \leq j \leq \kappa$, then we set $M^*_i = M^{\mathfrak{B}}_i$ (there is no conflict for $i = \alpha$, as $M^{\mathfrak{B}}_\alpha = M_\alpha$), $\pi^*_{i,j} = \pi^{\mathfrak{B}}_{i,j}$, $U^*_i = U^{\mathfrak{B}}_i$, and $\kappa^*_i = \kappa_i$. Finally, if $i \leq \alpha \leq j$, then we set $\pi^*_{i,j} = \pi^{\mathfrak{B}}_{\alpha,j} \circ \pi_{i,\alpha}$. The existence of this iteration in $\mathfrak{B}$ clearly shows that $\mathfrak{B}$ certifies q''. However, as $\mathrm{dom}(q'') \supseteq \mathrm{dom}(q) \cap \alpha$, it also shows that $\mathfrak{B}$ certifies q.

Let us now define $q^* \in \mathbb{P}$ as follows. Let $\mathrm{dom}(q^*) = \mathrm{dom}(q) \cup \mathrm{dom}(q'')$ and $\mathrm{dom}_-(q^*) = \mathrm{dom}_-(q) \cup \mathrm{dom}_-(q'')$. For $i \in \mathrm{dom}(q^*)$ set $\kappa^{q^*}_i = \kappa^*_i$. For $i \in \mathrm{dom}_-(q'')$ set $\tau^{q^*}_i = \tau^{q''}_i$, and for $i \in \mathrm{dom}_-(q)$, set $\tau^{q^*}_i = \tau^q_i$. Also, for $i \in \mathrm{dom}(q'')$ set $\pi^{q^*}_i = \pi^{q''}_i$. Finally, when defining $\pi^{q^*}_j$ for $j \in \mathrm{dom}(q) \setminus \alpha$, we make a small adjustment in order to satisfy point v. of Definition 5.1. Since $q'' \in X$, there is a finite tuple $\vec{u} \in \mathrm{ran}(\pi^{\mathfrak{B}}_{\alpha,\kappa})$ so that q' is definable in $\mathcal{H}$ from $\vec{u}$ and $\langle p, R, \leq_{\mathbb{P}}\rangle$. Also, for every $i \in \mathrm{dom}_-(q'')$ there is a $k_i < \omega$ such that

$$\tau^{q''}_i = \tau^{q^*}_i = \{(m, \vec{z})\ ;\ (k_i, \vec{u}^\frown m^\frown \vec{z}) \in \tau^{q^*}_\alpha = \tau^{p'}_\alpha = \tau\}.$$

We may assume that $\mathrm{ran}(\pi^{q^*}_i) \subseteq \vec{u}$ for $i \in \mathrm{dom}(q'') \subseteq \alpha$. For $j \in \mathrm{dom}(q^*)$, $j \geqslant \alpha$, we then set

$$\pi^{q^*}_j = \pi^*_{j,\kappa} \restriction ((\pi^*_{j,\kappa})^{-1}(\vec{u}) \cup \mathrm{dom}(\pi^q_j)).$$

It is now straightforward to see that $q^* \in \mathbb{P}$. Notice that if $i \in \mathrm{dom}_-(q^*) \cap \alpha = \mathrm{dom}_-(q'')$ and $j \in \mathrm{dom}_-(q^*) \setminus \alpha = \mathrm{dom}_-(q) \setminus \alpha$, and if

$$\tau^{q^*}_\alpha = \tau^q_\alpha = \{(m, \vec{z}); (l, \vec{v}^\frown m^\frown \vec{z}) \in \tau^{q^*}_j = \tau^q_j\},$$

where $\vec{v} \in \mathrm{ran}(\pi^q_j) \subseteq \mathrm{ran}(\pi^{q^*}_j)$, then

$$\tau^{q^*}_i = \tau^{q''}_i = \{(m, \vec{z}); (k_i, \vec{u}^\frown m^\frown \vec{z}) \in \tau^{q^*}_\alpha\} = \{(m, \vec{z}); (l, \vec{v}^\frown k_i^\frown \vec{u}^\frown m^\frown \vec{z}) \in \tau^{q^*}_j\}$$

and $\vec{v}$, $\vec{u} \subseteq \mathrm{ran}(\pi^{q^*}_j)$.

Thus $q^* \in \mathbb{P}$, and $q^* \leqslant q, q''$, a contradiction. □

References

[CS09] Benjamin Claverie and Ralf Schindler. Increasing u_2 by a stationary set preserving forcing. *Journal of Symbolic Logic*, (74):187–200, 2009.

[FNS10] Gunter Fuchs, Itay Neeman, and Ralf Schindler. A criterion for coarse iterability. *Archive for Mathematical Logic*, 49(4):447–467, 2010.

[FS] Gunter Fuchs and Ralf Schindler. Inner model theoretic geology. *Accepted for publication in the Journal of Symbolic Logic.*

[Har77] Leo Harrington. Long projective wellorderings. *Annals of Mathematical Logic*, 12(1):1–24, 1977.

[Jena] Ronald B. Jensen. The core model for nonoverlapping extenders. *Handwritten notes.*

[Jenb] Ronald B. Jensen. $\mathcal{L}$-forcing. Handwritten notes.

[Jen90] Ronald B. Jensen. On some problems of Mitchell, Welch, and Vickers. *Handwritten notes*, 1990?

[SW98] J. R. Steel and P. D. Welch. Σ^1_3 absoluteness and the second uniform indiscernible. *Israel J. Math.*, 104:157–190, 1998.

[Woo99] W. H. Woodin. *The axiom of determinacy, forcing axioms, and the nonstationary ideal.* DeGruyter, 1999.

[Zem02] Martin Zeman. *Inner Models and Large Cardinals.* De Gruyter, 2002.

(G. Fuchs) Mathematics, The Graduate Center of The City University of New York, 365 Fifth Avenue, New York, NY 10016 & Mathematics, College of Staten Island of CUNY, Staten Island, NY 10314

E-mail address: `Gunter.Fuchs@csi.cuny.edu`

URL: `http://www.math.csi.cuny/edu/~fuchs`

(R. Schindler) Institut für mathematische Logik und Grundlagenforschung, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

E-mail address: `rds@math.uni-muenster.de`

URL: `https://ivv5hpp.uni-muenster.de/u/rds/`