# SET THEORY COURSE WINTER TERM 2020-21, EXERCISE SHEET NO. 2, NOV 05, 2020

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**Hand in by Nov 10, 2020.**

Let $A$, $B$ sets of natural numbers. We say that $A$ and $B$ are almost disjoint iff $A \cap B$ is finite. A collection $D \subset \mathcal{P}(\mathbb{N})$ is called almost disjoint iff $A$ is infinite for every $A \in D$ and $A$ and $B$ are almost disjoint for all $A$, $B \in D$, $A \neq B$. A collection $D \subset \mathcal{P}(\mathbb{N})$ is called maximal almost disjoint (m.a.d.) iff $D$ is almost disjoint and no $D' \supset D$, $D' \subset \mathcal{P}(\mathbb{N})$, is still almost disjoint.

**Problem 1.** Show that there is a m.a.d. collection $D \subset \mathcal{P}(\mathbb{N})$ with only *one* element, in fact for every positive $n \in \mathbb{N}$ there is a m.a.d. collection $D \subset \mathcal{P}(\mathbb{N})$ with exactly $n$ elements. Show that there is a countable collection $D \subset \mathcal{P}(\mathbb{N})$ such that $A$ is infinite for every $A \in D$ and $A$ and $B$ are disjoint (!) for all $A$, $B \in D$, $A \neq B$. Show that no disjoint countable collection $D \subset \mathcal{P}(\mathbb{N})$ is m.a.d.

Let $\{0,1\}^*$ denote the set of all finite 0-1-sequences, and let $\{0,1\}^\infty$ denote the set of all infinite 0-1-sequences.

**Problem 2.** Show that $\{0,1\}^* \sim \mathbb{N}$. Make use of the set

$$\{\{x \restriction \{0,\ldots,n-1\} \colon n \in \mathbb{N}\} \colon x \in \{0,1\}^\infty\}$$

to conclude that there is a m.a.d. collection $D \subset \mathcal{P}(\mathbb{N})$ with $D \sim \mathbb{R}$.

**Problem 3.** Show that for every $\epsilon > 0$ there is a dense open set $\mathcal{O} \subset \mathbb{R}$ such that $\mu(\mathcal{O}) \leq \epsilon$. Conclude that there are $A$, $B \subset \mathbb{R}$ such that $A$ is meager, $B$ is null, and $A \cup B = \mathbb{R}$.

For $a$, $b \in \mathbb{R}$ let

$$[a,b]^{\frac{2}{3}} = [a, \frac{2}{3}a + \frac{1}{3}b] \cup [\frac{1}{3}a + \frac{2}{3}b, b],$$

and for $a_0 < b_0 < a_1 < b_1 < \ldots < a_i < b_i$ write

$$([a_0,b_0] \cup \ldots \cup [a_i,b_i])^{\frac{2}{3}} = [a_0,b_0]^{\frac{2}{3}} \cup \ldots \cup [a_i,b_i]^{\frac{2}{3}}.$$

Finally, for $a < b$ write $[a,b]_0 = [a,b]$, $[a,b]_{n+1} = ([a,b]_n)^{\frac{2}{3}}$ for $n \in \mathbb{N}$, and

$$[a,b]_\infty = \bigcap_{n \in \mathbb{N}} [a,b]_n.$$

$[0,1]_\infty$ is called Cantor's discontinuum.

**Problem 4.** Show that $[0,1]_\infty$ is perfect, nowhere dense, and null.

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