SET THEORY COURSE WINTER TERM 2020-21, EXERCISE SHEET NO. 6, NOV 30, 2020

RALF SCHINDLER

Hand in by Dec 08, 2020.

Recall the definitions of $\alpha + \xi$, $\alpha \cdot \xi$, and α^ξ from Problem # 4 of the previous exercise sheet.

Problem 1. Show that for every ordinal $\alpha > 0$ there are unique positive natural numbers n and $k_1, \ldots, k_n$ and ordinals $\beta_1 < \ldots < \beta_n$ such that

$$\alpha = (\omega^{\beta_n} \cdot k_n) + \ldots + (\omega^{\beta_1} \cdot k_1).$$

This representation is called the *Cantor normal form* of α.

Let DC (dependent choice) be the following principle. Let $a \neq \emptyset$, and let $r \subset a \times a$ be such that for every $x \in a$ there is some $y \in a$ with $(y, x) \in r$; then there is some function $f : \omega \to a$ such that $(f(n+1), f(n)) \in r$ for all $n < \omega$.

Problem 2. Show that ZFC implies DC. Show in ZF plus DC that a relation $r \subset a \times a$ is well-founded iff there is no function $f : \omega \to a$ such that $(f(n+1), f(n)) \in r$ for all $n < \omega$.

Let $\mathcal{L}_\in$ denote the language of set theory. Let $\mathcal{L}_{\in,\dot{v}}$ be $\mathcal{L}_\in$, enriched by adding a further constant, $\dot{c}$. Ackermann's set theory is the system given by the following axioms: extensionality, foundation, separation (like in ZF) plus the following two statements.

(Str) $\forall x \in \dot{c} \forall y (y \in x \vee y \subset x \to y \in \dot{c})$

(Refl) For all formulae φ of $\mathcal{L}_\in$,

$$\varphi \leftrightarrow \varphi^{\dot{c}}.$$

In (Refl), $\varphi^{\dot{c}}$ arises from φ by replacing every occurence of $\forall v$ by $\forall v \in \dot{c}$ and every occurence of $\exists v$ by $\exists v \in \dot{c}$.

Problem 3. Show that every axiom of ZF is provable in Ackermann's set theory.

Problem 4. Use the compactness theorem of first order logic to show that if ZF has a model, then Ackermann's set theory has a model. In fact, show that Ackermann's set theory is *conservative over* ZF, i.e., every theorem of Ackermann's set theory which is formulated in $\mathcal{L}_\in$ is a theorem of ZF.

Institut für mathematische Logik und Grundlagenforschung, Universität Münster, Einsteinstr. 62, 48149 Münster, Germany

URL: http://wwwmath.uni-muenster.de/logik/Personen/rds