# SET THEORY COURSE WINTER TERM 2020-21, EXERCISE SHEET NO. 7, DEC 10, 2020

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**Hand in by Dec 15, 2020.**

Recall that $TC(\{x\})$ is the transitive closure of $\{x\}$, i.e., the $\subset$-minimal transitive set $M$ with $x \in M$. For a cardinal $\kappa$, let

$$H_\kappa = \{x \colon \overline{\overline{TC(\{x\})}} < \kappa\}.$$

$H_\kappa$ is the collection of all sets which are *hereditarily smaller than* $\kappa$.

**Problem 1.** Show in ZFC that for every cardinal $\kappa$, $H_\kappa$ is a set.

For a set $A$ and a cardinal $\lambda$, let $[A]^\lambda = \{X \subset A \colon \overline{\overline{X}} = \lambda\}$.

**Problem 2.** Show that for all cardinals $\kappa$ and $\lambda$, $\kappa^\lambda = \overline{\overline{[\kappa]^\lambda}}$.

**Problem 3.** Show that for every ordinal $\alpha$ and for every regular cardinal $\lambda$ there is some cardinal $\kappa > \alpha$ with $\aleph_\kappa = \kappa$ and $\mathrm{cf}(\kappa) = \lambda$. Show that if $\kappa$ is the least cardinal with $\aleph_\kappa = \kappa$, then $\mathrm{cf}(\kappa) = \omega$.

**Problem 4.** Show that $\aleph_\omega^{\aleph_1} = \aleph_\omega^{\aleph_0} \cdot 2^{\aleph_1}$.

**Problem 5.** Let $\kappa \geq \omega_1$ be regular. Show that for every regular cardinal $\mu < \kappa$, $S_\kappa^\mu = \{\xi < \kappa \colon \mathrm{cf}(\xi) = \mu\}$ is stationary.

**Problem 6.** Let $S \subset \omega_1$ be stationary, and let $\alpha < \omega_1$. Show that there is some closed $c \subset S$ with $\mathrm{otp}(c) = \alpha + 1$.

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