Varsovian models II

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Abstract

Assume the existence of sufficent large cardinals. Let $M_{\mathrm{sw}n}$ be the minimal iterable proper class $L[E]$ model satisfying "there are $\delta_0 < \kappa_0 < \ldots < \delta_{n-1} < \kappa_{n-1}$ such that the δ_i are Woodin cardinals and the κ_i are strong cardinals". Let $M = M_{\mathrm{sw}2}$. We identify an inner model $\mathscr{V}_2^M$ of M, which is a proper class model satisfying "there are 2 Woodin cardinals", and is iterable both in V and in M, and closed under its own iteration strategy. The construction also yields significant information about the extent to which M knows its own iteration strategy. We characterize the universe of $\mathscr{V}_2^M$ as the mantle and the least ground of M, and as $\mathrm{HOD}^{M[G]}$ for $G \subseteq \mathrm{Coll}(\omega, \lambda)$ being M-generic with λ sufficiently large. These results correspond to facts already known for $M_{\mathrm{sw}1}$, and the proofs are an elaboration of those, but there are substantial new issues and new methods used to handle them. [1] [2]

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1 Introduction

The first generation of canonical inner models for large cardinals are those of the form $M = L[E]$ (or $L_\alpha[E]$) where E is a sequence of (partial) measures or extenders with various nice properties. The second generation are those of the form $M = L[E, \Sigma]$ (or $L_\alpha[E, \Sigma]$), with E as before, but Σ is a (partial) iteration strategy for M. We refer to the former as *mice* or *extender models*, and the latter as *strategy mice* or *strategic extender models*. Strategy mice arise naturally as HODs of determinacy models, and this phenomenon has been extensively studied. (The universe of) a strategy mouse $\mathscr{V}^{M_{\mathrm{sw}}}$ was also found in [8] to be the mantle of and a certain HOD associated to the mouse $M_{\mathrm{sw}} = M_{\mathrm{sw1}}$ (the "minimal" proper class mouse with a strong cardinal above a Woodin cardinal). While mice with Woodin cardinals (and which model ZFC, for example) can only compute restricted fragments of their own iteration strategies, strategy mice can be fully self-iterable.

One can contemplate the relationship between the two hierarchies; a key issue is the consistency strength of large cardinals when exhibited in the respective models: how do large cardinal hypotheses in (fully iterable) mice compare in consistency strength to those in (fully iterable) strategy mice, particularly for strategy mice which are closed under their own strategy? Continuing the line of investigation of [8], the present paper derives[3] the existence of a fully iterable proper class strategy mouse $\mathscr{V} = L[E, \Sigma]$, closed under its strategy, and containing two Woodin cardinals, from the existence and full iterability of the mouse $M^\#_{\mathrm{swsw}}$. This is the least active mouse N such that letting $\kappa = \mathrm{crit}(F)$ where F is the active extender of N, then $N|\kappa \vDash$ "There are ordinals $\delta_0 < \kappa_0 < \delta_1 < \kappa_1$ such that each δ_i is a Woodin cardinal and each κ_i is a strong cardinal". Letting M_{swsw} be the proper class model left behind after iterating F out of the universe, the strategy mouse $\mathscr{V}$ will be an inner model of M_{swsw}. (We also obtain Silver indiscernibles for $\mathscr{V}$.) The analysis also shows that M_{swsw} computes substantial fragments of its own iteration strategy, thereby contributing to the investigation of self-iterability in mice as in [12], but here beyond the tame level.

Now recall that if W is a model of ZFC, then $P \subseteq W$ is a *ground* of W iff P is also a model of ZFC and there is some poset $\mathbb{P} \in P$ and some g which is $(P, \mathbb{P})$-generic with $W = P[g]$. (Note this implies that P is transitive in the sense of W and contains all of the ordinals of W; by the Woodin/Laver ground definability result [2], [6], P is also definable from parameters over W.) The intersection of all grounds of W is called the *mantle* $\mathbb{M}^W$ of W. Recall W is called a *bedrock* iff W has no non-trivial grounds, or equivalently, $W = \mathbb{M}^W$. See [2] and [25] for more general background on these topics not specific to inner model theory.

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[3]Disclaimer: The "proofs" (and some definitions) presented here are not quite complete, because their full exposition depends on an integration, omitted here, of the method of $*$-translation (see [1]) with the techniques we develop. The integration itself is a straightforward matter of combining the two things. But because $*$-translation itself is already quite detailed, its inclusion would have added significantly to the length of the paper. It will be covered instead in [16].

The reason that mice modelling ZFC + Woodin cardinals do not compute their own iteration strategies is connected with the fact that they have proper (set-)grounds. The standard examples of such grounds arise from Woodin's genericity iterations. This phenomenon has led to *inner model theoretic geology*, which has proven to be an exciting and fruitful area of set theory. Its program is to analyze the collection of grounds and the mantle of given canonical inner models. See [3] and [8], which address exactly this kind of problem, and are precursors to the current work. See also [15], parts of which were motivated by the current work. The theme uncovered in these works is roughly that the mantle of a (sufficiently canonical) mouse tends to itself be a mouse or a strategy mouse, and hence can be analyzed in high detail.

The paper [3] proves that if $M = L[E]$ is a tame proper class mouse with a Woodin cardinal but no strong cardinal, and some further technical assumptions hold, then the mantle of M is itself a mouse, but is not a ground of $L[E]$; see [3, §3.4] and specifically [3, Theorem 3.33]. As an example, the mantle of M_1 (the minimal proper class mouse with one Woodin cardinal) is the model left behind after iterating the unique measure on the least measurable of M_1 out of the universe, and note this model has no measurable cardinals. The situation is entirely different if $L[E]$ has a strong cardinal.

Let $M_{\mathrm{sw}}{}^{\#}$ denote the minimal active mouse N such that letting $\kappa = \mathrm{crit}(F)$ where F is the active extender of N, then $N|\kappa \vDash$ "there is a strong cardinal above a Woodin cardinal", and suppose this mouse is fully iterable (for all set-sized trees). Let M_{swsw} be the proper class mouse left by iterating F out of the universe. It is shown in [8] that there are only set many grounds of M_{sw} and that the mantle of M_{sw} is itself a ground of M_{sw} and hence a bedrock. There is therefore some analogy here between M_{sw} and V in the presence of an extendible cardinal; see [26, Theorem 1.3]. The mantle of M_{sw}, however, also has an interesting structural analysis, as it is the universe of the strategy mouse $\mathscr{V}^{M_{\mathrm{sw}}}$ mentioned earlier. It is, moreover, a canonical "least" inner model which has a Woodin cardinal and knows how to fully iterate itself; see [8, Lemma 2.20].

In personal communication with the second author [27], W. Hugh Woodin expressed suspicion that the mantle of any proper class mouse $L[E]$ with a strong cardinal above a Woodin cardinal might perhaps contain non-trivial strategy information at its least Woodin cardinal and not at any larger Woodin.

A reasonable candidate for testing this suspicion and for extending the analysis of [8] is the big brother of M_{sw}, namely $M = M_{\mathrm{swsw}}$, introduced above, and studied in this paper. We will show that the strategy mouse $\mathscr{V}^{M}$, also introduced above, has universe the mantle of M, and so in fact, this mantle contains two Woodins together with non-trivial (and is closed under) strategy information for both of them. There is therefore a stronger analogy between hod mice (see [7]) and mantles of extender models $L[E]$ than was previously expected. This universe is also a ground of M, and hence is a bedrock. We will also show that $\mathscr{V}^{M}$ has universe the eventual generic HOD of M; that is, its universe is $\mathrm{HOD}^{M[G]}$ whenever λ is a sufficiently large ordinal and $G \subseteq \mathrm{Coll}(\omega, \lambda)$ is M-generic.

In some more detail, we will first isolate the *first Varsovian model* $\mathscr{V}_1 = \mathscr{V}_1^{M}$

of M and show that $\mathscr{V}_1$ is a ground of M, contains exactly two Woodin cardinals and a strong above them, and knows how to iterate itself fully for trees based on its least Woodin. This model is at first constructed in the form of "$L[\mathcal{M}_\infty, *]$", very much like in the construction of [8], which also mirrors Woodin's analysis of $\mathrm{HOD}^{L[x,G]}$. We then show that this model admits a stratification as a fine structural strategy premouse. The indexing used for the stratification is new, and this indexing is important in the overall analysis we give. It is moreover determined in a very strong sense by the hierarchy of M – the extender sequence of $\mathscr{V}_1$ is in fact given by simply restricting the extenders on the sequence of M above a certain point, some of which correspond to strategy. We then go on to isolate the *second Varsovian model* $\mathscr{V}_2 = \mathscr{V}_2^M$ of M, which will be constructed inside $\mathscr{V}_1$ (so $\mathscr{V}_2 \subseteq \mathscr{V}_1 \subseteq M$), using an elaboration of the construction of $\mathscr{V}_1$ in M. We then analyze the model and compute an iteration strategy for it, and establish the remaining facts mentioned above: the universe of $\mathscr{V}_2$ is the mantle and eventual generic HOD of M, $\mathscr{V}_2$ contains exactly two Woodin cardinals and knows fully how to iterate itself. We also show that the universe of $\mathscr{V}_1$ is the κ_0-mantle of M, where κ_0 is the least strong of M. The overall picture and process is expected to generalize to $n < \omega$ iterations (working in the appropriate starting mouse) and beyond.

The reader who is familiar with [7], for example, will encounter a lot of parallels between our analysis and the theory of hod mice; a key difference, though, is that our treatment is purely combinatorial and "inner model theoretic", using no descriptive set theory. Familiarity with [8] certainly helps, since the current paper is in large part an extension of that one, and some arguments covered in [8] are omitted here. But the reader who is reasonably familiar with inner model theory in general should be able to refer to [8] as needed.

The paper is organized as follows. There are some preliminaries and notation listed at the end of this section. In §2, we present the general method of assigning the (first) Varsovian model $\mathscr{V}^{L[E]}$ to an extender model $L[E]$, and prove key facts about it, under certain hypotheses. In §3, we describe some key properties of M_{swsw} and its iteration strategy, which will be essential throughout. In §4, the first Varsovian model $\mathscr{V}_1$ of M, and its iteration strategy $\Sigma_{\mathscr{V}_1}$, are defined and analyzed. This analysis is centered around the stratification of $\mathscr{V}_1$ as a strategy premouse. We also give natural characterizations of the universe of $\mathscr{V}_1$. In §5, we identify $\mathscr{V}_2$, also stratifying it as a strategy premouse. We show that $\mathscr{V}_2$ has two Woodin cardinals, is fully iterable, and is closed under its iteration strategy. We expect that the hypothesis used to construct such a model is in some sense optimal. We finally show in §5.8 that the universe of $\mathscr{V}_2$ is the mantle of M and is $\mathrm{HOD}^{M[G]}$ for sufficiently large collapse generics G.

The work presented here was started by the first two authors, extending their [8]. In the early stages, significant progress was made, but without a full development of the level-by-level fine structural correspondence presented in this paper between the models $M = M_{\mathrm{swsw}}$, $\mathscr{V}_1$ and $\mathscr{V}_2$; such a correspondence was considered to some extent, but then put aside in favour of other methods. During this time, the first author developed an approach to computing the man-

tle of M which does not use the level-by-level correspondence, but this has not been published. Later, the second author returned to the level-by-level correspondence, and developed some of the main ideas in its connection. Following this, in September 2017, the second and third authors then began discussing this approach. Over the next few months, building on what had already been established, they (mostly) completed the analysis via this approach, leading to the current presentation (some details being added over time somewhat later). Some of the evolution of ideas was documented by the second author's talks at the 4th Münster conference on inner model theory, July 17–Aug 01, 2017, and at the 1st Girona conference on inner model theory, July 16–27, 2018, and in the handwritten notes [10].

The early development, worked out by the first two authors, directly yielded parts of the present paper, as well as precursors to some other parts. Some version of probably the most central concept in the paper, the strategy mouse hierarchy used in Definition 4.39 (which is also a precursor of Definition 5.39), is due to the first two authors, as is §4.7; the setup for the first direct limit system in §§4.1,4.3,4.4 is much as in [8] and is basically due to them, although the approach used in §4.2 for computing short tree strategy, and some other uses of normalization, are due to the 3rd author. The 2nd author is responsible for the majority of §2, including Definition 2.11, for the computation of $\mathrm{HOD}_{\mathscr{E}}^{M[G]}$ in Theorem 4.28 via extending Lemma 4.24 (and the idea to consider $\mathrm{HOD}_{\mathscr{E}}$), and the modified P-construction (Definition 5.4). The 2nd and 3rd authors jointly established Lemma 5.8, the construction of the second direct limit system in §5.2, and the strategy mouse hierarchy used in Definition 5.39 (adapting 4.39). The (self-)iterability of $\mathscr{V}_1$ and $\mathscr{V}_2$ is also mostly due to the 2nd and 3rd authors, integrating some of the earlier work of the first two. The 3rd author is responsible for Lemma 4.24, that $\mathscr{V}_1 \subseteq \mathcal{M}_\infty[*]$, Lemmas 2.10, 4.41, 4.81, 4.84, 5.68, 5.72, Definitions 4.79, 5.67, §§4.10, 4.11, 5.1.2, 5.1.3, 5.8, and the original version of §4.8.

1.1 Notation and Background

General: Given structures P, Q, $\lfloor P \rfloor$ denotes the universe of P, and $P \mathrel{\widehat{=}} Q$ means $\lfloor P \rfloor = \lfloor Q \rfloor$.

Premice: All premice in the paper are Jensen-indexed (λ-indexed). Given premouse $N = (U, \mathbb{E}, F)$ with universe U, internal extender sequence $\mathbb{E}$ and active extender F, we write $\lfloor N \rfloor = U$, $\mathbb{E}^N = \mathbb{E}$, $F^N = F$, and $\mathbb{E}_+^N = \mathbb{E} \mathbin{^\frown} F$. Write $N^{\mathrm{pv}} = (U, \mathbb{E}, \emptyset)$ for its passivization, $N || \alpha$ for the initial segment P of N with $\mathrm{OR}^P = \alpha$ (inclusive of active extender) and $N | \alpha = (N || \alpha)^{\mathrm{pv}}$. Write $\mathrm{lh}(F) = \mathrm{OR}^N$. Given premice M, N, we write $M \trianglelefteq N$ iff $M = N || \alpha$ for some $\alpha \le \mathrm{OR}^N$, and $M \triangleleft N$ iff $M \trianglelefteq N$ but $N \ntrianglelefteq M$. Given also $m, n \le \omega$ such that M is m-sound and N is n-sound, we write $(M, m) \trianglelefteq (N, n)$ iff $M \trianglelefteq N$ and if $M = N$ then $m \le n$, and write $(M, m) \triangleleft (N, n)$ iff $(M, m) \trianglelefteq (N, n)$ but $(N, n) \ntrianglelefteq (M, m)$. For $\eta \le \mathrm{OR}^N$, we say η is a *cutpoint* of N iff for all $E \in \mathbb{E}_+^N$, if $\mathrm{crit}(E) < \eta$ then $\mathrm{lh}(E) \le \eta$, and a *strong cutpoint* iff for all $E \in \mathbb{E}_+^N$, if

$\text{crit}(E) \leq \eta$ then $\text{lh}(E) \leq \eta$. For $\xi < \delta \in \text{OR}^N$, $\mathbb{B}^N_{\delta,\xi}$ denotes the δ-generator extender algebra at δ, with axioms induced by extenders $E \in \mathbb{E}$ with $\nu(E)$ inaccessible in N and $\xi \leq \text{crit}(E)$. And $\mathbb{B}^N_\delta = \mathbb{B}^N_{\delta,0}$.

Hulls: In general, $\text{Hull}^M_t(X)$ denotes the structure whose universe is the collection of elements of M, definable over M from parameters in X, with definitions of "kind t", and whose predicates are just the restrictions of those of M. Here "kind t" depends on context, but the main example is that if M is a premouse, then the universe of $\text{Hull}^M_{n+1}(X)$ is the collection of all $y \in M$ such that for some $r\Sigma_{n+1}$ formula φ and $\vec{x} \in X^{<\omega}$, y is the unique $z \in M$ such that $M \vDash \varphi(z, \vec{x})$. When it makes sense, $\text{cHull}^M_t(X)$ denotes the transitive collapse of $\text{Hull}^M_{n+1}(X)$ (including the collapses of predicates).

Ultrapowers: Let E be an extender over N. Write $i^N_E : N \to \text{Ult}(N, E)$ for the ultrapower map, and $i^{N,n}_E \to \text{Ult}_n(N, E)$ for the degree-n ultrapower and associated map. Write $\kappa_E = \text{crit}(E)$ for the critical point of E, $\lambda_E = \lambda(E) = i_E(\kappa_E)$, and $\delta(E)$ for the measure space of E; in particular, if E is short then $\delta(E) = \text{crit}(E) + 1$.

Iteration trees: A fine structural iteration tree $\mathcal{T}$ consists of tree order $<^{\mathcal{T}}$, tree-predecessor function $\alpha + 1 \mapsto \text{pred}^{\mathcal{T}}(\alpha + 1)$, model-dropping node-set $\mathscr{D}^{\mathcal{T}} \subseteq \text{lh}(\mathcal{T})$, model-or-degree-dropping node-set $\mathscr{D}^{\mathcal{T}}_{\text{deg}} \subseteq \text{lh}(\mathcal{T})$, models $M^{\mathcal{T}}_\alpha$ and degrees $\deg^{\mathcal{T}}_\alpha$ (for $\alpha < \text{lh}(\mathcal{T})$), extenders $E^{\mathcal{T}}_\alpha \in \mathbb{E}_+(M^{\mathcal{T}}_\alpha)$ and model pre-images $M^{*\mathcal{T}}_{\alpha+1} \trianglelefteq M^{\mathcal{T}}_\beta$ where $\beta = \text{pred}^{\mathcal{T}}(\alpha + 1)$ (for $\alpha + 1 < \text{lh}(\mathcal{T})$), and here $M^{\mathcal{T}}_{\alpha+1} = \text{Ult}_d(M^{*\mathcal{T}}_{\alpha+1}, E^{\mathcal{T}}_\alpha)$ where $d = \deg^{\mathcal{T}}_\alpha$, and if $\alpha \leq^{\mathcal{T}} \beta$ and $(\alpha, \beta]_{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} = \emptyset$, iteration maps $i^{\mathcal{T}}_{\alpha\beta} : M^{\mathcal{T}}_\alpha \to M^{\mathcal{T}}_\beta$, and if α is also a successor ordinal, $i^{*\mathcal{T}}_{\alpha\beta} : M^{*\mathcal{T}}_\alpha \to M^{\mathcal{T}}_\beta$. (Note we are only indicating notation above; the definition of *iteration tree* has more demands.)

Let N be an n-sound premouse, where $n \leq \omega$, and $\mathcal{T}$ a fine structural iteration tree. Recall that $\mathcal{T}$ is *n-maximal on N* iff (i) $(M^{\mathcal{T}}_0, \deg^{\mathcal{T}}_0) = (N, n)$, (ii) $\text{lh}(E^{\mathcal{T}}_\alpha) < \text{lh}(E^{\mathcal{T}}_\beta)$ for $\alpha + 1 < \beta + 1 < \text{lh}(\mathcal{T})$, (iii) $\text{pred}^{\mathcal{T}}(\alpha + 1)$ is the least β such that $\text{crit}(E^{\mathcal{T}}_\alpha) < \lambda(E^{\mathcal{T}}_\beta)$, and (iv) $(M^{*\mathcal{T}}_{\alpha+1}, \deg^{\mathcal{T}}_{\alpha+1})$ is the lex-largest (P, p) such that $(M^{\mathcal{T}}_\beta || \text{lh}(E^{\mathcal{T}}_\beta), 0) \trianglelefteq (P, p) \trianglelefteq (M^{\mathcal{T}}_\beta, \deg^{\mathcal{T}}_\beta)$ and $\text{crit}(E^{\mathcal{T}}_\alpha) < \rho^P_p$. We say $\mathcal{T}$ is *normal* if it is n-maximal for some n. For δ an N-cardinal, we say $\mathcal{T}$ is *based on $N|\delta$* iff for all $\alpha + 1 < \text{lh}(\mathcal{T})$, if $[0, \alpha]_{\mathcal{T}}$ does not drop in model then $\text{lh}(E^{\mathcal{T}}_\alpha) \leq i^{\mathcal{T}}_{0\alpha}(\delta)$. We say $\mathcal{T}$ is *above κ* iff $\text{crit}(E^{\mathcal{T}}_\alpha) \geq \kappa$ for all $\alpha + 1 < \text{lh}(\mathcal{T})$, and *strictly above κ* iff $\text{crit}(E^{\mathcal{T}}_\alpha) > \kappa$ for all $\alpha + 1 < \text{lh}(\mathcal{T})$.

P-construction: Given a premouse M and $N \in M$, $\mathscr{P}^M(N)$, or just $\mathscr{P}(N)$ if M is understood, denotes the P-construction as computed in M over base set N. In [12, §1], this model would be denoted $\mathcal{P}(M, N, -)$. If $\mathcal{T}$ is a limit length iteration tree, $\mathscr{P}^N(\mathcal{T})$ abbreviates $\mathscr{P}^N(M(\mathcal{T}))$, and if $\mathcal{T}$ is the trivial tree (that is, uses no extenders) then $\mathscr{P}^N(\mathcal{T})$ denotes N. (The latter notation is just convenient when we set up indices for the direct limit systems, as then the trivial tree $\mathcal{T}$ on M indexes the base of the system computed in M.)

Remark 1.1. For our overall purposes Jensen indexing for premice is natural. However, genericity iterations are essential, which are somewhat cumbersome with Jensen indexing and Jensen iteration rules (as for *n-maximality* above).

The process for this is described in [22, Theorem 5.8]. We f also use *genericity inflation*, sketched in §4.2, and *minimal genericity inflation*, see [14, §5.2***].)

2 Ground generation

In this section we shall present an abstract version of the construction of a Varsovian model $\mathscr{V}$ derived from a given inner model M (satisfying the requirements below), and prove that $\mathscr{V}$ is a ground of M. It will take some time to lay out the required hypotheses ((ug1)–(ug24)); we will also collect some facts along the way.

Fix $M, (d, \preceq), (\mathcal{P}_p \colon p \in d)$ such that

(ug1) M is a proper class transitive model of ZFC,

(ug2) $(d, \preceq) \in M$ is a directed partial order,

(ug3) $(\mathcal{P}_p \colon p \in d)$ is an indexed system of transitive proper class inner models of M which is an M-class; that is, each $\mathcal{P}_p$ is a transitive proper class inner model of M, and $\{(p, x) \colon p \in d \wedge x \in \mathcal{P}_p\}$ is an M-class. [4]

Suppose that in V there is a system $(\pi_{pq} \colon p, q \in d \wedge p \preceq q)$ such that:

(ug4) $\pi_{pq} \colon \mathcal{P}_p \to \mathcal{P}_q$ is elementary whenever $p \preceq q$,

(ug5) the maps are commuting; that is, $\pi_{qr} \circ \pi_{pq} = \pi_{pr}$ for $p \preceq q \preceq r$.

Let
$$\mathscr{D}^{\mathrm{ext}} = \big(\langle \mathcal{P}_p \colon p \in d \rangle, \langle \pi_{pq} \colon p, q \in d \wedge p \preceq q \rangle \big)$$

be the directed system (the *ext* stands for *external*). Define the direct limit (model and maps)

$$(\mathcal{M}^{\mathrm{ext}}_\infty, \pi_{p\infty} \colon p \in d) = \ \mathrm{dir}\lim \mathscr{D}. \tag{1}$$

Suppose

(ug6) $\mathcal{M}^{\mathrm{ext}}_\infty$ is wellfounded; we take it transitive.

Note that the system $\mathscr{D}^{\mathrm{ext}}$ is not assumed to be an M-class, hence neither $\mathcal{M}^{\mathrm{ext}}_\infty$. But suppose that $\mathscr{D}^{\mathrm{ext}}$ is "covered" by an M-class, in the sense that there is an M-class $(d^+, \preceq)$ (we use the same symbol $\preceq$, since there will be no possibility of confusion) such that:

(ug7) $d^+ \subseteq d \times ([\mathrm{OR}]^{<\omega} \backslash \emptyset)$ and $\preceq$ is a directed partial order on d^+,

(ug8) if $(p, s), (q, t) \in d^+$ then $(p, s) \preceq (q, t)$ iff $p \preceq q$ and $s \subseteq t$

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[4]In practice, M and all $\mathcal{P}_p$, $p \in d$, will be (pure or strategic) premice, hence inner models constructed from a distinguished (class sized) predicate, in which case our definability hypothesis is supposed to mean that the collection of predicates constructing the $\mathcal{P}_p$, $p \in d$, is definable over M.

(ug9) if $(p,t) \in d^+$ and $\emptyset \neq s \subseteq t$ then $(p,s) \in d^+$,

(ug10) if $(p,s) \in d^+$, $q \in d$ and $p \preceq q$, then $(q,s) \in d^+$.

Further, there is a system

$$\mathscr{D} = \big(\langle H_s^p \colon (p,s) \in d^+ \rangle, \langle \pi_{ps,qt} \colon (p,s),(q,t) \in d^+ \wedge (p,s) \preceq (q,t) \rangle \big) \tag{2}$$

such that:

(ug11) $\mathscr{D}$ is an M-class,

(ug12) for all $(p,s) \in d^+$, H_s^p is an elementary substructure of $\mathcal{Q}_s^p = \mathcal{P}_p | \max(s)$,[5]

(ug13) for all $(p,s),(q,s) \in d^+$ with $p \preceq q$, the map $\pi_{ps,qs} : H_s^p \to H_s^q$ is elementary,

(ug14) for all $(p,t) \in d^+$ and $s \subseteq t$, we have $H_s^p \preceq_0 H_t^p$, and the map

$$\pi_{ps,pt} : H_s^p \to H_t^p$$

is the inclusion map (hence Σ_0-elementary),

(ug15) the maps $\pi_{ps,qt}$ commute, in that $\pi_{qt,ru} \circ \pi_{ps,qt} = \pi_{ps,ru}$,

Note if $(p,s),(q,t) \in d^+$ and $(p,s) \preceq (q,t)$, then $(p,s) \preceq (q,s) \preceq (q,t)$, so by (ug13), (ug14), (ug15),

$$\pi_{ps,qt} = \pi_{qs,qt} \circ \pi_{ps,qs} : H_s^p \to H_t^q$$

is Σ_0-elementary and has the same graph as has $\pi_{ps,qs}$, and in particular, the graph is independent of t. And note that $\pi_{ps,qs} \subseteq \pi_{pt,qt}$ whenever $s \subseteq t$ and $(p,t) \in d^+$ and $p \preceq q \in d$, because here $(p,s) \in d^+$ and

$$\pi_{pt,qt} \circ \pi_{ps,pt} = \pi_{ps,qt} = \pi_{qs,qt} \circ \pi_{ps,qs},$$

but $\pi_{ps,pt}$ and $\pi_{qs,qt}$ are just inclusion maps.

Definition 2.1. Given $\alpha \in \mathrm{OR}$ and $p \in d$, say α is *p-stable* iff $\pi_{pq}(\alpha) = \alpha$ for all $q \in d$ with $p \preceq q$. Say $s \in [\mathrm{OR}]^{<\omega}$ is *p-stable* iff α is p-stable for each $\alpha \in s$. Call $(p,s) \in d^+$ *true* iff s is p-stable and for all $q,r \in d$ with $p \preceq q \preceq r$, we have $\pi_{qs,rs} = \pi_{qr} \restriction H_s^q$. $\dashv$

Lemma 2.2. *For each $s \in [\mathrm{OR}]^{<\omega} \backslash \{\emptyset\}$, there is $p \in d$ such that s is p-stable.*

The proof is standard, using the wellfoundedness of $\mathcal{M}_\infty^{\mathrm{ext}}$. Assume further:

(ug16) for all $s \in [\mathrm{OR}]^{<\omega} \backslash \{\emptyset\}$, there is $p \in d$ with $(p,s) \in d^+$ and (p,s) true.

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[5]Here if $\mathcal{P}_p$ is a (possibly strategy) premouse, then this is precisely defined, and is passive (strategy) premouse; in general $\mathcal{P}_p$ should be stratified in an OR-indexed increasing chain of Σ_0-elementary substructures and $\mathcal{Q}_s^p$ should be the proper level of that hierarchy indexed at $\max(s)$.

(ug17) for all $p \in d$ and $x \in \mathcal{P}_p$, there exists s such that $(p,s) \in d^+$, (p,s) is true and $x \in H_s^p$.

Define the direct limit $(\mathcal{M}_\infty, \pi_{ps,\infty} \colon (p,s) \in d^+) = \text{dir} \lim \mathscr{D}$.

Lemma 2.3. $\mathcal{M}_\infty = \mathcal{M}_\infty^{\text{ext}}$ is M-definable.

Proof. Our assumptions immediately give that $\mathcal{M}_\infty$ is M-definable. Consider the equality. We proceed as in the proof of [8, Lemma 2.4] or the first few claims in [19]; the last few properties listed above been abstracted from those proofs. We will define a map $\chi \colon \mathcal{M}_\infty \to \mathcal{M}_\infty^{\text{ext}}$ and show that χ is the identity.

Let $(p,s) \in d^+$ and $x \in H_s^p$. By (ug10) and (ug16), we may fix q such that $(p,s) \preceq (q,s)$ and (q,s) is true. Define

$$\chi(\pi_{ps,\infty}(x)) = \pi_{q,\infty} \circ \pi_{ps,qs}(x).$$

By commutativity and truth (trueness), this does not depend on the choice of q, so χ is well-defined. Note that χ is Σ_0-elementary and cofinal, hence fully elementary, by [4, Theorem II.1, p. 54; Remark II.2, p. 55]. If $p \in d$ and $x \in \mathcal{P}_p$, then by (ug17) there is s with (p,s) true and $x \in H_s^p$. Hence $\pi_{p,\infty}(x) = \chi(\pi_{ps,\infty}(x)) \in \text{ran}(\chi)$. So χ is surjective, so $\chi = \text{id}$. $\quad\square$

Given $p \in d$, write $\mathcal{C}_p = \{q \in d \colon p \preceq q\}$, and given $(p,s) \in d^+$, write $\mathcal{C}_{(p,s)} = \{(q,t) \in d^+ \colon (p,s) \preceq (q,t)\}$. Call a set $\mathcal{C} \subseteq d$ a *cone* iff $\mathcal{C}_p \subseteq \mathcal{C}$ for some $p \in d$, and $\mathcal{C}' \subseteq d^+$ a *cone* iff $\mathcal{C}_{(p,s)} \subseteq \mathcal{C}'$ for some $(p,s) \in d^+$.

Lemma 2.4. *We have:*

1. *There is a cone $\mathcal{C} \subseteq d$ such that $\pi_{pq}(\rho) = \rho$ for all $p,q \in \mathcal{C}$ with $p \preceq q$.*

2. *There is a cone $\mathcal{C}' \subseteq d^+$ such that $\rho \in H_s^p$ and $\pi_{ps,qt}(\rho) = \rho$ for all $(p,s),(q,t) \in \mathcal{C}'$ with $(p,s) \preceq (q,t)$, and*

Proof. Let $s = \{\rho, \rho+1\}$ and (using (ug16)) let $p \in d$ be such that $(p,s) \in d^+$ and (p,s) is true (hence s is p-stable). Then $\mathcal{C}_p$ and $\mathcal{C}_{(p,s)}$ work. $\quad\square$

Note that part 2 of the previous lemma is understood by M. Using this, we can define the associated $*$-*map* OR $\to$ OR. For $\alpha \in$ OR write

$$\rho^* = \pi_{ps,\infty}(\rho), \tag{3}$$

where (p,s) is any element of any cone $\mathcal{C}'$ witnessing part 2 of the lemma. Note this is well-defined, and $*$ is a class of M.

Lemma 2.5. *Let $\rho \in$ OR, and $\mathcal{C}, \mathcal{C}'$ be cones witnessing Lemma 2.4. Then:*

1. *$\rho^* = \pi_{p\infty}(\rho)$ for all $p \in \mathcal{C}$,*

2. *$\rho \in H_s^p$ and $\rho^* = \pi_{ps,\infty}(\rho)$ for all $(p,s) \in \mathcal{C}'$,*

3. *$\rho^* = \min\{\pi_{p\infty}(\rho) \colon p \in d\}$.*

Proof. Parts 1 and 2 follow directly from Lemma 2.4 and the fact that the function χ defined in the proof of Lemma 2.3 is the identity.

Part 3: Let $\mathcal{C} \subseteq d$ witness Lemma 2.4. Let $p \in d$. Then there is $q \in \mathcal{C}$ with $p \preceq q$. Therefore $\pi_{pq}(\rho) \geq \rho$, so $\pi_{p\infty}(\rho) \geq \pi_{q\infty}(\rho)$, which suffices. $\qquad\square$

There is another important characterization of $\rho \mapsto \rho^*$, given some further properties. Assume:

(ug18) There is a unique $\preceq$-minimal $p_0 \in d$. Moreover, $M = \mathcal{P}_{p_0}$ and $\pi_{p_0 q}(p_0) = q$ for each $q \in d$.

So $\pi_{p_0 q} : M \to \mathcal{P}_q$ for all $q \in d$, and $\pi_{p_0 \infty} : M \to \mathcal{M}_\infty$. The next hypotheses guarantee a homogeneity property of the system, in that each $\mathcal{P}_p$ may equally serve as a base. Let $\mathscr{D}^{\mathcal{P}_p} = \pi_{p_0 p}(\mathscr{D})$, $\mathcal{M}_\infty^{\mathcal{P}_p} = \pi_{p_0 p}(\mathcal{M}_\infty)$, etc, for $p \in d$.[6] Let $c_p = d_p \cap d$ and $c_p^+ = d_p^+ \cap d^+$. Suppose:

(ug19) For all $p \in d$, c_p is dense in $(d_p, \preceq_p)$ and dense in $(d, \preceq)$, and $\preceq_p \restriction c_p = \preceq \restriction c_p$,

(ug20) For all $p \in d$ and all $(q, s), (r, t) \in c_p^+$ with $(q, s) \preceq (r, t)$, we have $(\mathcal{P}_q)^{\mathcal{P}_p} = \mathcal{P}_q$ and $(H_s^q)^{\mathcal{P}_p} = H_s^q$ and $(\pi_{qs,rt})^{\mathcal{P}_p} = \pi_{qs,rt}$.

(ug21) For all $s \in [\mathrm{OR}]^{<\omega} \setminus \{\emptyset\}$ there is $p \in d$ with (p, s) true and $(p, s) \in c_p^+$.

Using these properties, it is now straightforward to deduce:

Lemma 2.6. *For each $p \in d$, we have:*

1. *c_p^+ is dense in $(d_p^+, \preceq_p)$ and dense in $(d^+, \preceq)$, and $\preceq_p \restriction c_p^+ = \preceq_p \restriction c_p^+$.*

2. *The direct limit $\mathcal{M}_\infty^{\mathcal{P}_p}$ of $\mathscr{D}^{\mathcal{P}_p}$ is just $\mathcal{M}_\infty$, and the associated $*$-map $*^{\mathcal{P}_p}$ is just $*$, so $\pi_{p_0 p}(\mathcal{M}_\infty) = \mathcal{M}_\infty$ and $\pi_{p_0 p}(*) = *$.*

Definition 2.7. For $p \in d$ and $s, t \in [\mathrm{OR}]^{<\omega} \setminus \emptyset$, say (p, s, t) is *embedding-good* iff $(p, t) \in d^+$, $\pi_{ps,\infty} \in H_t^p$ and $\pi_{pt,qt}(\pi_{ps,\infty}) = \pi_{qs,\infty}$ for all $q \in \mathcal{C}_p$. $\dashv$

Note that *embedding-good* is an M-class.

Lemma 2.8.

1. *If (p, s) is as in (ug21) then (q, s) is true and $(q, s) \in d^+ \cap d_q^+$ for all $q \in \mathcal{C}_p$.*

2. *For each $s \in [\mathrm{OR}]^{<\omega} \setminus \emptyset$ there is $p \in d$ such that for each $q \in \mathcal{C}_p$, we have (q, s) true and $(q, s) \in d^+ \cap d_q^+$, and for each $x \in \mathcal{P}_q$ there is $t \in [\mathrm{OR}]^{<\omega}$ such that (q, t) is true and $x \in H_t^q$ and (q, s, t) is embedding-good.*

---

[6]Note that we write $\pi_{p_0 p}(\mathscr{D})$, not $\pi_{p_0 p}(\mathscr{D}^{\mathrm{ext}})$; of the course the latter does not make sense. We know $\mathcal{M}_\infty^{\mathrm{ext}} = \mathcal{M}_\infty$, but $\mathcal{M}_\infty^{\mathcal{M}_\infty}$ is of course computed in $\mathcal{M}_\infty$ as the direct limit of $\mathscr{D}^{\mathcal{M}_\infty}$. At this stage it is not relevant whether there is some external system of elementary embeddings associated with $\mathscr{D}^{\mathcal{M}_\infty}$ analogous to $\mathscr{D}^{\mathrm{ext}}$.

Proof. Part 1: We have $(q, s) \in d^+$ and (q, s) true because $(p, s) \in d^+$ and (p, s) is true, and $(q, s) \in d_q^+$ because $(p, s) \in d_p^+$ and $\pi_{pq}((p, s)) = (q, s)$ by (ug18) and because s is p-stable.

Part 2: Fix $p \in d$ with (p, s) true and $(p, s) \in d^+ \cap d_p^+$ (using (ug21)). Let $q \in \mathcal{C}_p$. By part 1 and (ug18)–(ug20), $(\pi_{rs,\infty})^{\mathcal{P}_r} = \pi_{rs,\infty}$ for all $r \in \mathcal{C}_q$. Since $\pi_{qr}(s) = s$ for such r, we get $\pi_{qr}(\pi_{qs,\infty}) = \pi_{rs,\infty}$. Let $x \in \mathcal{P}_q$. Using (ug17), let $t \in [\mathrm{OR}]^{<\omega}$ such that $(q, t) \in d^+$ is true and $x, \pi_{qs,\infty} \in H_t^q$. Then q, t works. $\square$

Let $\mathscr{G}$ be the class of all embedding-good tuples. Define $\mathscr{D}^{\mathcal{M}_\infty} = \pi_{0\infty}(\mathscr{D})$ and $\mathcal{M}_\infty^{\mathcal{M}_\infty} = \pi_{0\infty}(\mathcal{M}_\infty)$. Working in M, define $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ by

$$\pi_\infty = \bigcup_{(p,s,t) \in \mathscr{G}} \pi_{pt,\infty}(\pi_{ps,\infty}).$$

Lemma 2.9. $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ *is elementary and* $\pi_\infty(\rho) = \rho^*$ *for* $\rho \in \mathrm{OR}$. *Moreover,* $\pi_{p_0 q}(\pi_\infty) = \pi_\infty$ *for all* $q \in d$.

Proof. The well-definedness and elementarity of π_∞ is left to the reader. Fix $\rho \in \mathrm{OR}$. Let $(p, s) \in d^+$ with $\rho \in \mathrm{rg}(\pi_{ps,\infty})$, taking p as in Lemma 2.8 part 2 with respect to s. Note we may assume that $\pi_{pq}(\rho) = \rho$ for all $q \in \mathcal{C}_p$. Let $\pi_{ps,\infty}(\bar\rho) = \rho$. Let $t \in [\mathrm{OR}]^{<\omega} \backslash \{\emptyset\}$ be such that (p, t) is true, $\bar\rho, \rho, \pi_{ps,\infty} \in H_t^p$ and (p, s, t) is embedding-good. Then

$$
\begin{aligned}
\rho^* &= \pi_{p\infty}(\rho) \\
&= \pi_{pt,\infty}(\pi_{ps,\infty}(\bar\rho)) \\
&= \pi_{pt,\infty}(\pi_{ps,\infty})(\pi_{pt,\infty}(\bar\rho)) \\
&= \pi_\infty(\rho).
\end{aligned}
$$

The "moreover" clause is as in Lemma 2.6. $\square$

We now define the associated *Varsovian model* $\mathscr{V}$ as

$$\mathscr{V} = L[\mathcal{M}_\infty, \pi_\infty]. \tag{4}$$

So by Lemmas 2.6 and 2.9, $\mathscr{V}$ is a class of $\mathcal{P}_p$, for all $p \in d$, and $\pi_{p_0 p}(\mathscr{V}) = \mathscr{V}$. Let $\mathscr{V}^{\mathcal{M}_\infty} = \pi_{p_0\infty}(\mathscr{V})$, so $\mathscr{V}^{\mathcal{M}_\infty}$ is defined over $\mathcal{M}_\infty$ just as $\mathscr{V}$ over M. So letting $\pi_\infty^{\mathcal{M}_\infty} = \pi_{p_0\infty}(\pi_\infty)$,

$$\mathscr{V}^{\mathcal{M}_\infty} = L[\mathcal{M}_\infty^{\mathcal{M}_\infty}, \pi_\infty^{\mathcal{M}_\infty}].$$

Lemma 2.10. π_∞ *extends uniquely to an elementary*

$$\pi_\infty^+ : \mathscr{V} \to \mathscr{V}^{\mathcal{M}_\infty}$$

such that $\pi_\infty^+(\pi_\infty) = \pi_\infty^{\mathcal{M}_\infty}$ *(and* $\pi_\infty^+(\mathcal{M}_\infty) = \pi_\infty(\mathcal{M}_\infty) = \mathcal{M}_\infty^{\mathcal{M}_\infty}$*). Moreover,* π_∞^+ *is* $\mathscr{V}$-*definable from* $\mathcal{M}_\infty, \pi_\infty$.

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Proof. Since every element of $\mathscr{V}$ is definable over $\mathscr{V}$ from $\mathcal{M}_\infty$, π_∞ and some ordinal, it suffices to see that for all formulas φ and ordinals α, we have

$$\mathscr{V} \vDash \varphi(\mathcal{M}_\infty, \pi_\infty, \alpha) \iff \mathscr{V}^{\mathcal{M}_\infty} \vDash \varphi(\mathcal{M}_\infty^{\mathcal{M}_\infty}, \pi_\infty^{\mathcal{M}_\infty}, \pi_\infty(\alpha)).$$

But $\mathscr{V} \vDash \varphi(\mathcal{M}_\infty, \pi_\infty, \alpha)$ iff

$$\mathcal{P}_p \vDash \text{``}\mathscr{V} \vDash \varphi(\mathcal{M}_\infty, \pi_\infty, \alpha)\text{''}$$

for each $p \in d$. Taking p such that α is p-stable, and then applying $\pi_{p\infty}$, note that the latter holds iff $\mathcal{M}_\infty$ satisfies the corresponding formula regarding $\pi_\infty(\alpha)$; that is, iff

$$\mathscr{V}^{\mathcal{M}_\infty} \vDash \varphi(\mathcal{M}_\infty^{\mathcal{M}_\infty}, \pi_\infty^{\mathcal{M}_\infty}, \pi_\infty(\alpha)). \qquad \square$$

It therefore makes sense to define, for any $x \in \mathcal{M}_\infty$,

$$x^* = \pi_\infty(x), \tag{5}$$

and for $x \in \mathscr{V}$,

$$x^{*+} = \pi_\infty^+(x); \tag{6}$$

that is, $*$ and π_∞ denote the same function, as do $*+$ and π_∞^+.

We next formulate a few more assumptions which ensure that certain sets are generic over $\mathscr{V}$. Let $\delta \in \mathrm{OR}$ and $\mathbb{B} \in M$. Let $\delta_p = \pi_{p_0 p}(\delta)$ and $\delta_\infty = \pi_{p_0 \infty}(\delta)$, etc. Assume:

(ug22) $M \vDash$"δ is regular and $\mathbb{B}$ is a δ-cc complete Boolean algebra", and

(ug23) $\mathscr{V} \vDash$"δ_∞ is regular and $\mathbb{B}_\infty$ is δ_∞-cc".

Now work in $\mathscr{V}$. Let $\mathcal{L}$ be the infinitary propositional language, with propositional symbols P_ξ for each ordinal ξ, generated by closing under under negation and under conjunctions and disjunctions of length $< \delta_\infty$ (so if $\langle \varphi_\alpha \rangle_{\alpha < \theta} \subseteq \mathcal{L}_\mu$ where $\theta < \delta_\infty$, then $\bigwedge_{\alpha < \theta} \varphi_\alpha$ and $\bigvee_{\alpha < \theta} \varphi_\alpha$ are also in $\mathcal{L}_\mu$). (Note $\mathcal{L}$ is a proper class of $\mathscr{V}$.)

Working in any outer universe of $\mathscr{V}$, given a set B of ordinals, the satisfaction relation $B \vDash \varphi$ for $\varphi \in \mathcal{L}_\mu$ is defined recursively as usual; that is, $B \vDash P_\xi$ iff $\xi \in B$; $B \vDash \neg\varphi$ iff $B \nvDash \varphi$; $B \vDash \bigwedge_{\alpha < \theta} \varphi_\alpha$ iff $B \vDash \varphi_\alpha$ for all $\alpha < \theta$; and $B \vDash \bigvee_{\alpha < \theta} \varphi_\alpha$ iff $B \vDash \varphi_\alpha$ for some $\alpha < \theta$.

Fix some $\mathbb{B}_p$-name $\tau_p \in \mathcal{P}_p$ for a set of ordinals, for some $p \in d$, and let $\tau_j = \pi_{pq}(\tau_p)$ for $q \in \mathcal{C}_p$, and $\tau_\infty = \pi_{p\infty}(\tau_p)$.

Definition 2.11. Work in $\mathscr{V}$. Let $\mathbb{L}$ be the poset whose conditions are formulas $\varphi \in \mathcal{L}$ such that there is $p \in \mathbb{B}_\infty$ such that

$$p \Vdash_{\mathbb{B}_\infty}^{\mathcal{M}_\infty} \text{``}\tau_\infty \vDash \varphi^{*+}\text{''},$$

and with ordering $\varphi \leq \psi$ iff for every $p \in \mathbb{B}_\infty$, we have

$$p \Vdash_{\mathbb{B}_\infty}^{\mathcal{M}_\infty} \text{``}\tau_\infty \vDash (\varphi^{*+} \Rightarrow \psi^{*+})\text{''}. \qquad \dashv$$

Although $\mathcal{L}$ is proper class, it is easy to see that this is equivalent to a set forcing. Note here that since $\varphi \in \mathscr{V}$, φ^{*+} is well-defined and

$$\varphi^{*+} \in \mathscr{V}^{\mathcal{M}_\infty} \subseteq \mathcal{M}_\infty,$$

and the forcing assertions above make sense, as $\tau_\infty \in \mathcal{M}_\infty$ is a $\mathbb{B}_\infty$-name and $\varphi^{*+}, \psi^{*+} \in \mathcal{M}_\infty$. Note that by modding out by the equivalence relation

$$\varphi \approx \psi \iff \varphi \leq \psi \leq \varphi,$$

we get a forcing-equivalent poset defined as follows:

Definition 2.12. Work in $\mathscr{V}$. Let $\mathbb{L} \subseteq \mathbb{B}_\infty$ be the forcing whose conditions are those Boolean values in $\mathbb{B}_\infty$ of the form

$$||\text{``}\tau_\infty \vDash \varphi^{*+}\text{''}||_{\mathbb{B}_\infty}^{\mathcal{M}_\infty},$$

where $\varphi \in \mathcal{L}$, excluding the 0-condition of $\mathbb{B}_\infty$, and with ordering induced by $\mathbb{B}_\infty$. $\dashv$

Note that the forcing $\mathbb{L}$ depends on the name τ_∞; if we want to make this explicit, we will write $\mathbb{L}(\tau_\infty)$.

Lemma 2.13. $\mathscr{V} \vDash \text{``}\mathbb{L}$ *has the* δ_∞-*c.c.''*

Proof. Since $\mathscr{V} \vDash\text{``}\mathbb{B}_\infty$ has the δ_∞-cc" (by (ug23)), this is an immediate consequence of Definition 2.12. $\square$

Lemma 2.14. *Let* $A \in M$ *be a set of ordinals and* $p_1 \in d$ *and suppose that for all* $q \in \mathcal{C}_{p_1}$ *there is* $g \in M$ *such that* g *is* $(\mathcal{P}_q, \mathbb{B}_q)$-*generic and* $(\tau_q)_g = A$.[7] *Then the filter*

$$G_A = \{\varphi \in \mathbb{L} \mid A \vDash \varphi\};$$

or equivalently, if using Definition 2.12 to define $\mathbb{L}$, *the filter*

$$G_A = \left\{||\text{``}\tau_\infty \vDash \varphi^{*+}\text{''}||_{\mathbb{B}_\infty}^{\mathcal{M}_\infty} \mid \varphi \in \mathcal{L} \wedge A \vDash \varphi\right\},$$

is $(\mathscr{V}, \mathbb{L})$-*generic, and* $A \in \mathscr{V}[G_A]$ *(so note* $\mathscr{V}[A] = \mathscr{V}[G_A]$*).*

Proof. Easily, G_A is a filter. We verify genericity, and then clearly $A \in \mathscr{V}[G_A]$. Let $\langle \varphi_\alpha \rangle_{\alpha < \theta} \in \mathscr{V}$ be a maximal antichain of $\mathbb{L}$. We must see G_A meets $\langle \varphi_\alpha \rangle_{\alpha < \theta}$, or equivalently, that $A \vDash \varphi_\alpha$ for some α. Supposing otherwise, $A \vDash \psi$ where

$$\psi = \bigwedge_{\alpha < \theta} \neg \varphi_\alpha,$$

and by Lemma 2.13, $\theta < \delta_\infty$, so $\psi \in \mathcal{L}$.

---

[7]In our applications, where M will be a (pure or strategic) premouse, A will typically be a canonical code for $M|\mu$, and the name τ_q will provide a canonical translation of the pair $(\mathcal{P}_q|\mu, \dot{g})$ into $M|\mu$, where $\dot{g}$ is the generic filter.

For each $q \in \mathcal{C}_{p_1}$, we have $\mathscr{V} \subseteq \mathcal{P}_q$, so $\mathbb{L} \in \mathcal{P}_q$, and by hypothesis there is a $(\mathcal{P}_q, \mathbb{B}_q)$-generic $g \in M$ with $(\tau_q)_g = A$, so

$$\exists p \in \mathbb{B}_q \, [p \Vdash^{\mathcal{P}_q}_{\mathbb{B}_q} \text{``}\tau_q \vDash \psi\text{''}]. \tag{7}$$

Considering the definition of π_∞^+, note that we may take $q \in \mathcal{C}_p$ such that $\pi_{q\infty}(\psi) = \pi_\infty^+(\psi) = \psi^{*+}$, so that (7) implies

$$\exists p \in \mathbb{B}_\infty \, [p \Vdash^{\mathcal{M}_\infty}_{\mathbb{B}_\infty} \text{``}\tau_\infty \vDash \psi^{*+}\text{''}],$$

so $\psi \in \mathbb{L}$. But the elementarity of π_∞^+ easily gives that $\psi \perp \varphi_\alpha$ for every $\alpha < \theta$, so $\langle \varphi_\alpha \rangle_{\alpha < \theta}$ is not a maximal antichain. Contradiction! $\qquad\square$

Finally, suppose:

(ug24) For every ordinal μ with $\mathrm{card}^M(V_\mu^M) = \mu$, there is a set $A' \in \mathcal{P}(\mu)^M$ coding V_μ^M, and there is $p_1 \in d$ such that μ is p_1-stable, and there is a $\mathbb{B}_{p_1}$-name $\tau'_{p_1} \in \mathcal{P}_{p_1}$ such that the hypotheses of Lemma 2.14 hold for A', p_1, τ'_{p_1}.

So under these assumptions for a given μ, A', τ'_{p_1}, the conclusion of Lemma 2.14 holds with respect to $\mathbb{L}(\tau'_\infty)$. Assumption (ug24) basically says that the $\mathcal{P}_p$ form a system of grounds for M in a "uniform" manner.

Definition 2.15. For M, etc, as above, we say that $\mathscr{D}, \mathscr{D}^+$ provide *uniform grounds* for M iff conditions (ug1)–(ug24) hold. $\qquad\dashv$

Theorem 2.16. *Under the uniform grounds assumptions, $\mathscr{V}$ is a ground of M, via a forcing $\mathbb{P}$ such that $M \vDash$"$\mathbb{P}$ has cardinality $\leq 2^{\delta_\infty}$" and $\mathscr{V} \vDash$"$\mathbb{P}$ is δ_∞-cc". Therefore δ_∞ is a regular cardinal in M.*
However, $\mathcal{M}_\infty$ is not a ground of M.

Proof. We have that every set (of ordinals) in M is generic over $\mathscr{V}$ for some $\mathbb{L}(\tau'_\infty) \subseteq \mathbb{B}_\infty$. Since there are only set-many such forcings, $\mathscr{V}$ is in fact a ground of M for some such $\mathbb{L}(\tau'_\infty)$. Moreover, this forcing is δ_∞-cc in $\mathscr{V}$, by [11, Theorem 2.2], we can find a forcing $\mathbb{P} \in \mathscr{V}$ as desired.

The "therefore" clause now follows (recall δ_∞ is a regular cardinal of $\mathscr{V}$).

Now $\mathscr{V}^{\mathcal{M}_\infty}$ is a ground of $\mathcal{M}_\infty$ (since $\mathscr{V}$ is a ground of M). Suppose $\mathcal{M}_\infty$ is a ground of M. Then $\mathscr{V}^{\mathcal{M}_\infty}$ is also a ground of M. But M defines $\pi_\infty^+ : \mathscr{V} \to \mathscr{V}^{\mathcal{M}_\infty}$, which is then a non-trivial elementary embedding between two grounds of M, contradicting [5, Theorem 8]. $\qquad\square$

When we produce instances of uniform grounds later, we will actually know more: we will have $V_{\delta_\infty}^{\mathcal{M}_\infty} = V_{\delta_\infty}^{\mathscr{V}}$ and δ_∞ Woodin in $\mathscr{V}$ (hence also in $\mathcal{M}_\infty$, so δ Woodin in M, which will be an assumption), and $\mathbb{B} \subseteq \delta$, so $\mathbb{L} \subseteq \delta_\infty$, so some $\mathbb{L}(\tau'_\infty)$ will be a $\mathbb{P}$ as above, but in fact of cardinality δ_∞ in $\mathscr{V}$ and hence also in M.

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3   The model M_{swsw}

In this section introduce the mouse M_{swsw} we will be analyzing, and establish some of its basic properties, as well as some of those of its iteration strategy.

Definition 3.1. Let ψ_{swsw} be the statement, in the passive premouse language, asserting "There are ordinals $\delta_0 < \kappa_0 < \delta_1 < \kappa_1$ with δ_i Woodin and κ_i strong for $i \leq 1$, as witnessed by $\mathbb{E}$". Let $M^{\#}$ be the least active mouse such that $M^{\#}|\mu \vDash \psi_{\text{swsw}}$ where $\mu = \text{crit}(F^{M^{\#}})$.[8] Then M_{swsw} denotes the proper class model left behind by iterating $F^{M^{\#}}$ out of the universe. Note $\rho_\omega(M^{\#}) = \rho_1(M^{\#}) = \omega$, $p_1^{M^{\#}} = \emptyset$ and $M^{\#}$ is ω-sound. We assume throughout that $M^{\#}$ exists and is $(\omega, \text{OR}, \text{OR})$-iterable.[9] We usually write $M = M_{\text{swsw}}$.

Let Σ denote the (ω, OR)-iteration strategy (that is, for ω-maximal, hence normal, trees, of set length) for M which is induced by the unique (ω, OR)-strategy $\Sigma_{M^{\#}}$ for $M^{\#}$. Let $\Gamma = \Sigma^{\text{stk}}$ denote the optimal-$(\omega, \text{OR}, \text{OR})$-strategy for M which is induced by Σ via the normalization process of [14] (see Fact 3.4 below, especially item (Σ1) there). $\dashv$

Certain aspects of normalization, used to define $\Gamma = \Sigma^{\text{stk}}$ from Σ, will be used in the paper. The main features we need are the properties of Σ mentioned in Fact 3.4 below, which can be black-boxed. Some of the details of the normalization process will also come up to some extent later on, but the reader unfamiliar with those details should still be able to follow most of the arguments in the paper.

Remark 3.2. M knows enough of Σ that $M|\omega_1^M$ is definable over the universe of M (without parameters). Therefore by [20, Theorem 1.1], $\mathbb{E}^M$ is definable over the universe of M without parameters. Thus, when we talk about definability over M, it does not matter whether we are given $\mathbb{E}^M$ as a predicate or not. However, if g is M-generic, then $\text{HOD}^{M[g]}$ can differ from $\text{HOD}_{\mathbb{E}^M}^{M[g]}$, for example.

Definition 3.3. If $\vec{\mathcal{T}}$ is a stack on M via Γ, then $\Gamma_{\vec{\mathcal{T}}, N}$ denotes the tail stacks strategy for N induced by Γ, i.e. $\Gamma_{\vec{\mathcal{T}}, N}(\vec{\mathcal{U}}) = \Gamma(\vec{\mathcal{T}} \,^\frown \vec{\mathcal{U}})$. Also $\Sigma_{\vec{\mathcal{T}}, N}$ denotes the normal part of $\Gamma_{\vec{\mathcal{T}}, N}$. Actually by what follows below, we can and usually do write Γ_N and Σ_N. $\dashv$

Recall that $\Gamma = \Sigma^{\text{stk}}$ is the strategy for stacks induced by Σ.

Fact 3.4. *We have:*

(Σ1) *Σ is the unique (ω, OR)-strategy for M, so satisfies both strong hull condensation and minimal hull condensation, and therefore by [14]:*

  – every iterate of M via Γ is also an iterate via Σ,

---

[8] By [18], Woodinness and strength is automatically witnessed by $\mathbb{E}^{\mathcal{M}^{\#}}$, as a consequence of iterability, but we will also consider premice $N \vDash \psi_{\text{swsw}}$ which need not be iterable.

[9] We could probably just work with $(\omega, \omega_1 + 1)$-iterability. By [22, Theorems 9.1, 9.3], because $M^{\#}$ is ω-sound and projects to ω, $(\omega, \omega_1 + 1)$-iterability for $M^{\#}$ implies $(\omega, \omega_1, \omega_1 + 1)^*$-iterability, and similarly, (ω, OR)-iterability for $M^{\#}$ implies $(\omega, \text{OR}, \text{OR})$-iterability.

– if G is V-generic then Σ, Γ extend canonically to $V[G]$, with the same properties there; with an abuse of notation, we continue to write $\Sigma, \Gamma = \Sigma^{\text{stk}}$ for these extensions, or may write $\Sigma^{V[G]}$ or $\Gamma^{V[G]}$ to emphasize the distinction.

(Σ2) Γ is fully positional, in that whenever $\vec{\mathcal{T}}, \vec{\mathcal{U}}$ are two stacks via Γ with the same last model N, then $\Gamma_{\vec{\mathcal{T}}, N} = \Gamma_{\vec{\mathcal{U}}, N}$, irrespective of drops. However, positionality will only be relevant in the non-dropping case.

(Σ3) Γ is commuting, i.e., if $\vec{\mathcal{T}} \, \hat{} \, \vec{\mathcal{U}}$ and $\vec{\mathcal{T}} \, \hat{} \, \vec{\mathcal{V}}$ are non-dropping stacks via Γ with a common last model, then $i^{\vec{\mathcal{U}}} = i^{\vec{\mathcal{V}}}$; see [14, ***Theorem 10.4].

(Σ4) For all $\vec{\mathcal{T}}$ via Γ, with last model N, $\Sigma_{\vec{\mathcal{T}}, N}$ has minimal hull condensation and $\Gamma_{\vec{\mathcal{T}}, N} = (\Sigma_{\vec{\mathcal{T}}, N})^{\text{stk}}$; see [14, ***Theorem 10.2].[10] Thus, every iterate of N via $\Gamma_{\vec{\mathcal{T}}, N}$ is also an iterate via $\Sigma_{\vec{\mathcal{T}}, N}$, in a unique manner,

(Σ5) If $\mathcal{T}, \mathcal{U}$ are via Σ, of successor length, with non-dropping final branches, $P = M_\infty^{\mathcal{T}}$ and $Q = M_\infty^{\mathcal{U}}$, $\eta \in \text{OR}^M$, $\eta' = i^{\mathcal{T}}(\eta) = i^{\mathcal{U}}(\eta)$ and $P|\eta' = Q|\eta'$ then $\Sigma_{\mathcal{T}, M_\infty^{\mathcal{T}}}$ and $\Sigma_{\mathcal{U}, M_\infty^{\mathcal{U}}}$ agree with one another in their action on trees $\mathcal{V}$ based on $P|\eta'$. See [14, ***Theorem 10.5].[11]

Remark 3.5. Very strong hull condensation ([23]) implies minimal hull condensation ([14]), which implies minimal inflation condensation ([14]). For the normalization process of [14], minimal inflation condensation is sufficient, but for the generic absoluteness results, minimal hull condensation is used.

Definition 3.6. We say that a stack $\vec{\mathcal{T}}$ on M is *correct* if it is via Γ. We say that N is a Σ-*iterate* of M iff there is a correct stack $\vec{\mathcal{T}}$ on M with last model $N = M_\infty^{\vec{\mathcal{T}}}$. By the properties above, we may in fact take $\mathcal{T}$ via Σ (hence normal), and note that this $\mathcal{T}$ is uniquely determined by N (and Σ); we write $\mathcal{T}_N = \mathcal{T}$. A Σ-iterate is a *dropping iterate* iff $b^{\mathcal{T}}$ drops, and otherwise is *non-dropping*.

Let N be a non-dropping Σ-iterate. Then a Σ_N-*iterate* is similarly an iterate of N via Σ_N (equivalently, via Γ_N).[12] If P is a non-dropping Σ_N-iterate, let $i_{NP} : N \to P$ be the iteration map (via Σ_N). Given $\delta \leq \text{OR}^N$, we say that N is δ-*sound* iff, letting $\mathcal{T} = \mathcal{T}_N$, we have $N = \text{Hull}^N(\delta \cup \text{rg}(i^{\mathcal{T}}))$; equivalently, $\nu(E_\alpha^{\mathcal{T}}) \leq \delta$ for all $\alpha + 1 < \text{lh}(\mathcal{T})$. ⊣

Definition 3.7. Let $\mathscr{I}^M$ denote the class of critical points of the linear iteration of $F^{M^\#}$ which produces M. For N as above, let $\mathscr{I}^N = i_{MN} \text{``} \mathscr{I}^M$. ⊣

Definition 3.8 (M_{swsw}-like)**.** A premouse N is M_{swsw}-*like* iff N is proper class and satisfies a certain finite sub-theory T of the theory of M, including $\psi_{\text{swsw}} + $ "I have no active proper segment R such that $R|\text{crit}(F^R) \models \psi_{\text{swsw}}$". We will not

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[10]In order to define $(\Sigma_{\vec{\mathcal{T}}, N})^{\text{stk}}$, one also needs that N is n-standard, where $n = \deg_\infty^{\vec{\mathcal{T}}}$, but this follows from the fact that M is 0-standard, by [14, ***Remark 2.2].

[11]The precise version of this fact might be simplified by the fact that M is below superstrong.

[12]We don't need to iterate dropping iterates of M further.

spell T out exactly, but the reader should add statements to it as needed to make certain arguments work. For an M_{swsw}-like model N, write

$$\begin{cases} \delta_0^N = \text{ the least Woodin cardinal of } N \\ \kappa_0^N = \text{ the least strong cardinal of } N \\ \kappa_0^{+N} = (\kappa_0^N)^{+N} \\ \delta_1^N = \text{ the least Woodin cardinal of } N \text{ above } \kappa_0^N \\ \kappa_1^N = \text{ the least strong cardinal of } N \text{ above } \delta_0^N \\ \kappa_1^{+N} = (\kappa_1^N)^{+N} \end{cases} \tag{8}$$

If $N = M$, then we may suppress the superscript N, so $\delta_0 = \delta_0^M$, etc. $\dashv$

Fact 3.9. *Let N be a non-dropping Σ-iterate of M and $\mathcal{T} = \mathcal{T}_N$. Let $\delta = \delta(\mathcal{T}) = \sup_{\alpha+1<\text{lh}(\mathcal{T})} \nu(E_\alpha^{\mathcal{T}})$. Then $\delta \leq \kappa_1^N < \min(\mathscr{I}^N)$ and $\mathscr{I}^N$ is the unique club class of indiscernibles I such that $N = \text{Hull}_1^N(I \cup \delta)$, or alternatively such that $N = \text{Hull}_1^N(I \cup \kappa_1^P)$.*

The following two lemmas are instances of branch condensation (see [7]) and are simple variants of [8, Lemma 2.1]; we fill in a couple of key points which were omitted from that proof, however.

Lemma 3.10 (Branch condensation A)**.** *Let $\mathcal{U}_0$ be a successor length tree on M, via Σ, based on $M|\delta_0^M$, with $b^{\mathcal{U}_0}$ non-dropping. Let $\mathcal{T}, \mathcal{U}$ be on $N = M_\infty^{\mathcal{U}_0}$, via Σ_N, based on $N|\delta_0^N$, with $\mathcal{T}$ of limit length and $\mathcal{U}$ successor with $b^{\mathcal{U}}$ non-dropping. Let G be V-generic. Let $b, k \in V[G]$ where b is a non-dropping $\mathcal{T}$-cofinal branch and*
$$k : M_b^{\mathcal{T}}|\delta_0^{M_b^{\mathcal{T}}} \to M_\infty^{\mathcal{U}}|\delta_0^{M_\infty^{\mathcal{U}}}$$
is elementary with $k \circ i_b^{\mathcal{T}} \subseteq i_{0\infty}^{\mathcal{U}}$. Then $b = \Sigma_N(\mathcal{T})$.

Proof. Because Σ extends to $V[G]$, with corresponding properties there (cf. Fact 3.4(Σ1)), we may assume $G = \emptyset$. Let $c = \Sigma_N(\mathcal{T})$. Let $P_b = M_b^{\mathcal{T}}$ and $P_c = M_c^{\mathcal{T}}$.

Suppose first $\delta(\mathcal{T}) < \delta_0^{P_b}$. Then there is a Q-structure $Q' \lhd P_b$ for $\delta(\mathcal{T})$, and because $\delta(\mathcal{T})$ is a cardinal of P_b, $M(\mathcal{T})$ has no Woodin cardinals, so $\delta(\mathcal{T})$ is a strong cutpoint of Q'. Because we have k, Q' is iterable.

If c is non-dropping and $\delta(\mathcal{T}) = \delta_0^{P_c}$ then we can compare Q' versus P_c for a contradiction. So in any case, $Q = Q(\mathcal{T}, c)$ exists. Since $M(\mathcal{T})$ has no Woodins, $\delta(\mathcal{T})$ is also a strong cutpoint of Q, so we can compare and get $Q = Q'$, so $b = c$.

Now suppose $\delta(\mathcal{T}) = \delta_0^{P_b}$. Then we can argue as in the proof of [8, Lemma 2.1]; however, we fill in a seemingly key point: We extend k to
$$k^+ : P_b \to M_\infty^{\mathcal{U}}$$

with $k^+ \circ i_b^{\mathcal{T}} = i_{0\infty}^{\mathcal{U}}$ as in [8]. Now $P_b = P_c$ (this was not mentioned in [8]); for P_b is iterable and is $\delta_0^{P_b}$-sound, and likewise for P_c, but both are models of "I am M_{swsw}", so comparison gives $P_b = P_c$. And because $P_b = P_c$ and $i_b^{\mathcal{T}}, i_c^{\mathcal{T}}$ fix all sufficiently large indiscernibles, we can indeed conclude that $i_b^{\mathcal{T}} \restriction X = i_c^{\mathcal{T}} \restriction X$, where
$$X = \text{Hull}^N(I^N) \cap \delta_0^N,$$

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where I^N is the class of N-indiscernibles. So by the Zipper Lemma, we get $b = c$. $\qquad\square$

There is also a version at δ_1. We won't directly use this, but will use a variant, which will use a similar proof:

Lemma 3.11 (Branch condensation B). *Let $\mathcal{U}_0$ be successor length on M, via Σ, based on $M|\delta_1^M$, with $b^{\mathcal{U}_0}$ non-dropping. Let $\mathcal{T}, \mathcal{U}$ be on $N = M_\infty^{\mathcal{U}_0}$, via Σ_N, based on $N|\delta_1^N$, with $\mathcal{T}$ of limit length and above κ_0^{+N}, $\mathcal{U}$ successor length with $b^{\mathcal{U}}$ non-dropping. Let G be V-generic. Let $b, k \in V[G]$ where b is a non-dropping $\mathcal{T}$-cofinal branch and*
$$k : M_b^{\mathcal{T}}|\delta_1^{M_b^{\mathcal{T}}} \to M_\infty^{\mathcal{U}}|\delta_1^{M_\infty^{\mathcal{U}}}$$
is elementary with $k \circ i_b^{\mathcal{T}} = i_{0\infty}^{\mathcal{U}}$. Then $b = \Sigma_N(\mathcal{T})$.

Proof. Again we may assume $b, k \in V$. Let $\alpha \in b^{\mathcal{U}_0}$ be least with either $\alpha + 1 = \mathrm{lh}(\mathcal{U}_0)$ or $\kappa_0(M_\alpha^{\mathcal{U}_0}) < \mathrm{crit}(i_{\alpha\infty}^{\mathcal{U}_0})$. Let $\bar{N} = M_\alpha^{\mathcal{U}_0}$. So $\kappa_0^N = \kappa_0^{\bar{N}}$ and $\bar{N}$ is κ_0^N-sound and $\pi : \bar{N} \to N$ where $\pi = i_{\alpha\infty}^{\mathcal{U}_0}$ and $\mathrm{crit}(\pi) > \kappa_0^N$. Let $P_b = M_b^{\mathcal{T}}$, $c = \Sigma_N(\mathcal{T})$ and $P_c = M_c^{\mathcal{T}}$. If $\delta(\mathcal{T}) < \delta_1^{P_b}$ then let $Q_b \lhd P_b$ be the Q-structure for $\delta(\mathcal{T})$; this exists because δ_1^N is the least Woodin of N above κ_0^N, and $\mathcal{T}$ is above κ_0^{+N}. Otherwise let $Q_b = P_b$. This time, Q_b can have extenders E overlapping $\delta(\mathcal{T})$, but only with $\mathrm{crit}(E) = \kappa_0^N$. Let $Q_c \trianglelefteq P_c$ be likewise.

Define phalanxes $\bar{\mathfrak{P}} = ((\bar{N}, \kappa_0^N), Q_b, \delta(\mathcal{T}))$ and $\bar{\mathfrak{Q}} = ((\bar{N}, \kappa_0^N), Q_c, \delta(\mathcal{T}))$. We claim $\bar{\mathfrak{P}}, \bar{\mathfrak{Q}}$ are iterable.[13] Given this, we can compare $\bar{\mathfrak{P}}, \bar{\mathfrak{Q}}$, and because $\bar{N}$ is κ_0^N-sound and Q_b, Q_c are $\delta(\mathcal{T})$-sound, we get $Q_b = Q_c$, so if $b \neq c$ then $P_b = Q_b = Q_c = P_c$ and $\delta_1^{P_b} = \delta(\mathcal{T})$, and we reach a contradiction like before.

Define phalanxes $\mathfrak{P} = ((N, \kappa_0^N), P_b, \delta(\mathcal{T}))$ and $\mathfrak{Q} = ((N, \kappa_0^N), P_c, \delta(\mathcal{T}))$. It suffices to see $\mathfrak{P}, \mathfrak{Q}$ are iterable, because then we can reduce trees on $\bar{\mathfrak{P}}$ to trees on $\mathfrak{P}$, using $\pi : \bar{N} \to N$, and likewise $\bar{\mathfrak{Q}}$ to $\mathfrak{Q}$.

But $\mathfrak{Q}$ is iterable because $c = \Sigma(\mathcal{T})$. For $\mathfrak{P}$, we have $k^+ : P_b \to M_\infty^{\mathcal{U}}$ defined as before (note that the same definition still works in case $\delta(\mathcal{T}) < \delta_1^{P_b}$). So P_b is iterable. But $i_b^{\mathcal{T}} : N \to P_b$ with $\mathrm{crit}(i_b^{\mathcal{T}}) > \kappa_0^N$. So we can lift trees on $\mathfrak{P}$ to trees on P_b using the maps $(i_b^{\mathcal{T}}, \mathrm{id})$. So $\mathfrak{P}$ is iterable. $\qquad\square$

4 The first Varsovian model $\mathscr{V}_1$

We begin by identifying a natural direct limit system, giving uniform grounds for $M = M_{\mathrm{swsw}}$, in the sense Definition 2.15 in §2, hence yielding a Varsovian model, which we denote $\mathscr{V}_1$ (we will later define a second Varsovian model $\mathscr{V}_2$). The direct limit system will be defined analogously to that of [8, §2]. The main difference is in the increased large cardinal level. A smaller difference, one of approach, is that we use normalization, which means that we can focus on normal trees, instead of stacks.

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[13] The notation indicates that we start iterating the phalanx with extenders of index $> \delta(\mathcal{T})$, and extenders with critical point κ_0 apply to $\bar{N}$.

4.1 The models for the system

Definition 4.1. Let $\mathcal{T}$ be a limit length normal tree on M, based on $M|\delta_0^M$, via Σ.[14] Let $b = \Sigma(\mathcal{T})$. We say that $\mathcal{T}$ is *short* iff either b drops or $\delta(\mathcal{T}) < i_{0b}^{\mathcal{T}}(\delta_0^M) = \delta_0^{\mathcal{M}_b^{\mathcal{T}}}$; otherwise $\mathcal{T}$ is *maximal*. Let Σ_{sh} be the restriction of Σ to short trees.

If P is a non-dropping Σ-iterate of M and $\mathcal{T}$ is limit length normal on P and based on $P|\delta_0^P$, we define *short/maximal* for $\mathcal{T}$ analogously, and $\Sigma_{P,\mathrm{sh}}$ is the restriction of Σ_P to short trees. $\dashv$

Definition 4.2. Let $\mathbb{U}$ consist of all iteration trees $\mathcal{U} \in M|\kappa_0$ on M, such that either $\mathcal{U}$ is trivial, or

(a) $\mathcal{U}$ is based on $M|\delta_0$, via Σ_{sh} (hence is ω-maximal),

(b) $\mathcal{U}$ is maximal,

and for some strong cutpoint $\eta < \kappa_0$ of M, writing $\delta = \delta(\mathcal{U})$ and $R = M(\mathcal{U})$,

(c) $\mathrm{lh}(\mathcal{U}) = \eta^{+M} = \delta$,

(d) $\mathcal{U}$ is definable from parameters over $M|\delta$,

(e) $M|\delta$ is R-generic for $\mathbb{B}_\delta^R$,

(f) $P =_{\mathrm{def}} \mathscr{P}^M(R)$ is proper class;[15] [16] hence P is a ground of M via $\mathbb{B}_\delta^R$, and in fact $P[M|\delta] \hat{=} M$. We write here also $\mathscr{P}^M(\mathcal{U}) = P$. $\dashv$

The proof of [8, Lemma 2.2] or the first few claims in [19] give:

Lemma 4.3. Let $\mathcal{U} \in \mathbb{U}$, $b = \Sigma(\mathcal{U})$ and $P = \mathscr{P}^M(\mathcal{U})$. Then $\mathcal{M}_b^{\mathcal{U}} = P$, $\mathscr{I}^M = \mathscr{I}^P$ and $i_{0b}^{\mathcal{T}} \restriction \mathscr{I}^M = \mathrm{id}$.

We have $\mathbb{U} \subseteq M$ by definition, but because of the requirement that $\mathcal{U} \in \mathbb{U}$ be via Σ_{sh} (hence via Σ), it is not immediate that $\mathbb{U} \in M$. But we show in the next section that it is, and that $\mathbb{U}$ is rich, with the following properties: The restriction of Σ_{sh} to M is known to M, and whenever $P = \mathscr{P}^M(\mathcal{U})$ for some $\mathcal{U} \in \mathbb{U}$, the restriction of $\Sigma_{P,\mathrm{sh}}$ to M is known to M (and moreover, these are preserved by i_{MP}). Pseudo-genericity iterations can be formed using these strategies to produce trees in $\mathbb{U}$. Any two such models P_1, P_2 can be pseudo-compared with these strategies. Moreover, every maximal tree $\mathcal{T} \in M|\kappa_0$ via Σ_{sh} is "absorbed" by some $\mathcal{X} \in \mathbb{U}$.

4.2 The short tree strategy Σ_{sh} for M

We now show that M is closed under Σ_{sh} and $\Sigma_{\mathrm{sh}} \restriction M$ is a class of M, and that the same also holds with $M[g]$ replacing M, for any M-generic g (where g need not be in V); in fact when $g = \emptyset$ the class can be taken lightface.

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[14] Recall Σ is the $(0, \mathrm{OR})$-strategy (that is, for ω-maximal, hence normal, trees) for M.

[15] Recall the notation $\mathscr{P}^M$ from §1.1.

[16] Recall δ is Woodin in P, as witnessed by $\mathbb{E}^R$, and V_δ^P is the universe of R.

Let $\mathcal{T}$ be via Σ of limit length, and $b = \Sigma(\mathcal{T})$. Suppose we want to compute b. Since Σ has strong hull condensation, it suffices to find a tree $\mathcal{X}$ via Σ and $\mathcal{T}$-cofinal branch b' and a tree embedding $\Pi : \mathcal{T} \frown b' \to \mathcal{X}$, for then $b' = b$.

Suppose also $\mathcal{T} \in M[g]$ and is based on $M|\delta_0$. Working in $M[g]$ we want to (i) determine whether $\mathcal{T}$ is short, and (ii) if short, compute $\Sigma_{\mathrm{sh}}(\mathcal{T})$. If it happens that $\mathcal{T}$ incorporates, in an appropriate manner, a genericity iteration for making $\mathbb{E}^M$ generic, then we will be able to use P-constructions (combined with $*$-translation, discussed below) to achieve both of these goals. In the general case, we use the method of (*genericity*) *inflation* to reduce $\mathcal{T}$ to a tree $\mathcal{X}$ which does incorporate such a genericity iteration (see [22, §5.2], which adapts methods for tame mice from [12, §1]). We give here a sketch of the relevant methods from [22], restricted to our context; but the reader should consult [22] for details.

Suppose also that g is M-generic for some $\mathbb{P} \in V_\theta^M$, and $\mathcal{T} \in V_\theta^{M[g]}$. If $\theta < \kappa_0$ let $U = M$; otherwise let $E \in \mathbb{E}^M$ be M-total with $\mathrm{crit}(E) = \kappa_0$ and $V_\theta^M \subseteq U = \mathrm{Ult}(M, E)$ (so $\theta < \lambda(E) = \kappa_0^U$). Let η be a strong cutpoint of U with $\theta \leq \eta < \kappa_0^U$. Following [22], let $\mathcal{X}$ be the genericity inflation (explained further below) of $\mathcal{T}$ for making $U|\delta(\mathcal{X})$ generic for the $\delta(\mathcal{X})$-generator extender algebra, incorporating an initial linear iteration which moves the least measurable of $M(\mathcal{X})$ beyond η, and incorporating linear iterations past $*$-translations of Q-structures.[17] The $*$-translation is due to Steel, Neeman, Closson; see [1], together with an amendment in [16]. The $*$-translations of Q-structures are segments of U which compute the Q-structures which guide branch choices for $\mathcal{X}$.

Here is a sketch of the relevant material from [22]. We define $\mathcal{X}\!\restriction\!(\alpha+1)$ by induction on α. Suppose we have $\mathcal{X}\!\restriction\!(\alpha+1)$ defined, but have not yet succeeded in finding $\Sigma(\mathcal{T})$. We will have an ordinal $\eta_\alpha \leq \mathrm{OR}(M_\alpha^{\mathcal{X}})$ defined, and possibly have an ordinal $\beta_\alpha < \mathrm{lh}(\mathcal{T})$ and a lifting map

$$\sigma_\alpha : M_{\beta_\alpha}^{\mathcal{T}} ||\mathrm{lh}(E_{\beta_\alpha}^{\mathcal{T}}) \to M_\alpha^{\mathcal{X}}||\eta_\alpha$$

defined. (At $\alpha = 0$ we have $\beta_0 = 0$ and $\eta_0 = \mathrm{lh}(E_0^{\mathcal{T}})$ and $\sigma_0 = \mathrm{id}$.) We set $E_\alpha^{\mathcal{X}} = F^{M_\alpha^{\mathcal{X}}||\eta_\alpha}$, unless there is an extender $G \in \mathbb{E}^{M_\alpha^{\mathcal{X}}}$ with $\mathrm{lh}(G) < \eta_\alpha$ such that either (i) G induces an extender algebra axiom which is not satisfied by $\mathbb{E}^U$, and G satisfies some further conditions as explained in [22][18] or (ii) G is a measure to be used for one of the linear iterations mentioned above.[19] We say $E_\alpha^{\mathcal{X}}$ is either *copied* from $\mathcal{T}$ (when $\mathrm{lh}(E_\alpha^{\mathcal{X}}) = \eta_\alpha$ and $(\beta_\alpha, \sigma_\alpha)$ are defined) or is *inflationary* (otherwise). The stages α for which $(\beta_\alpha, \sigma_\alpha)$ is not defined correspond to a drop in model in $\mathcal{X}$, below the image of the relevant extender from $\mathcal{T}$, and arise because of the nature of genericity iteration with Jensen indexing. Let $\gamma = \mathrm{pred}^{\mathcal{X}}(\alpha+1)$. If $E_\alpha^{\mathcal{X}}$ is copied then $\beta_{\alpha+1} = \beta_\alpha + 1$ is defined,

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[17]The technique of inserting linear iterations past Q-structures comes from [17], where there are details of such a construction given.

[18]It suffices that ν_G is inaccessible in $M_\alpha^{\mathcal{X}}|\eta_\alpha$, but one must also consider other extenders, including partial ones, because of the nature of genericity iteration with Jensen indexing.

[19]The linear iterations need to be set up appropriately, to ensure that the process does not last too long; similar details are dealt with in the comparison arguments in [17].

and $\sigma_{\alpha+1}$ is the restriction of a map given by the Shift Lemma applied to σ_α and another map π (whose domain is $M^{*\mathcal{T}}_{\beta_\alpha+1}$; we have not specified π here). If $E^{\mathcal{X}}_\alpha$ is inflationary then $\beta_{\alpha+1}$ is defined just in case β_γ is defined and $E^{\mathcal{X}}_\alpha$ is total over $M^{\mathcal{X}}_\gamma||\eta_\gamma$, and in this case $\beta_{\alpha+1} = \beta_\gamma$ and $\sigma_{\alpha+1} = i^{*\mathcal{X}}_{\alpha+1} \circ \sigma_\gamma$.

Now consider a limit stage λ. The first thing to do is either compute $c = \Sigma(\mathcal{X}{\restriction}\lambda)$ (if $\lambda < (\eta^+)^U$), or declare $\mathcal{X}, \mathcal{T}$ maximal (if $\lambda = (\eta^+)^U$). Let $\delta = \delta(\mathcal{X}{\restriction}\lambda)$. If $M(\mathcal{X}{\restriction}\lambda)$ is a Q-structure for itself then b is trivial, and arguing as in [17] shows that in this case, $\lambda < (\eta^+)^U$ (the argument is mostly standard, but some variant details arise, which are discussed there). So suppose otherwise. Then $\lambda = \delta$ and $\mathcal{X}{\restriction}\delta$ is definable from parameters over $(U|\delta)[g]$. This is because $\eta < \delta$, $\mathcal{T} \in (U|\eta)[g]$, the process for determining $\eta_\alpha, E^{\mathcal{X}}_\alpha$ is locally definable, and the $*$-translations of Q-structures used to compute the branches of $\mathcal{X}{\restriction}\lambda$ are all proper segments of $U|\delta$, because of the linear iterations past these $*$-translations. Moreover, $U|\delta$ is generic for the δ-generator extender algebra of $M(\mathcal{X}{\restriction}\lambda)$.

From now on, let us assume that $g = \emptyset$ for simplicity; since η is a strong cut-point of U, the general case only involves shifting to $U[g]$. Let $c = \Sigma(\mathcal{X}{\restriction}\delta)$. Let $Q = Q(\mathcal{X}{\restriction}\delta, c)$, considering $\mathcal{X}$ as a tree on $M^\#$. (Maybe $\mathcal{X}{\restriction}\delta$ is not short.) Then Q could have extenders overlapping δ. But the $*$-translation Q^* of $(Q, \mathcal{X}{\restriction}\delta)$ is a premouse extending $U|\delta$ and having no overlaps of δ, and in fact, either (i) $\delta < (\eta^+)^U$ and $\mathcal{X}{\restriction}\delta$ is short and $Q^* \triangleleft U$, or (ii) $\lambda = (\eta^+)^U$ and $\mathcal{X}{\restriction}\lambda$ is maximal and $Q^* = U^\#$. So U can see which of case (i) and (ii) we are in, and in case (i), compute Q^*, Q, c (as Q^* is the unique segment of U whose inverse $*$-translation is well-defined and terminates with a Q-structure for $M(\mathcal{X}{\restriction}\lambda)$, which is then Q). Moreover, the branch c is determined by the $*$-translation Q^* of a Q-structure, as promised earlier.

Suppose $\lambda < (\eta^+)^U$. So we have computed $c = \Sigma(\mathcal{X}{\restriction}\lambda)$ in $U[g]$. By [22], this determines either (i) some $\beta_\lambda < \mathrm{lh}(\mathcal{T})$ (and possibly a σ_λ as before), in which case we continue the process; or (ii) a $\mathcal{T}$-cofinal branch b and a tree embedding

$$\Pi : \mathcal{T}\,{}^\frown b \hookrightarrow \mathcal{X}\,{}^\frown c$$

with b mapped cofinally into c, and b is encoded into c, in such a manner that $U[g]$ can compute b, Π from $(\mathcal{T}, \mathcal{X}, c)$.

Now suppose that the process reaches $\mathcal{X}$ of length $(\eta^+)^U$. So $\mathcal{X}$ is maximal and $Q^* = U^\#$. Let $c = \Sigma(\mathcal{X})$ and $b = \Sigma(\mathcal{T})$. So $i^{\mathcal{X}}_c(\delta^M_0) = \delta(\mathcal{X}{\restriction}\lambda) = (\eta^+)^U$. Again by [22], there is a tree embedding

$$\Pi : \mathcal{T}\,{}^\frown b \hookrightarrow \mathcal{X}\,{}^\frown c$$

which maps b cofinally in c, and since $\mathcal{T}, \mathcal{X}$ are based on $M|\delta^M_0$, then $\mathcal{T}$ is maximal. Also in this case, considering $\mathcal{T}, \mathcal{X}$ as trees on M, instead of on $M^\#$, we get that $M^{\mathcal{X}}_c = \mathscr{P}^U(\mathcal{X})$ (the P-construction of U above $\mathcal{M}(\mathcal{X})$, which is the analogue of the inverse $*$-translation of U in this case), so if $U = M$ (and still $g = \emptyset$) then $\mathcal{X} \in \mathbb{U}$.

This completes the sketch. For further details the reader should refer to [22], augmented by [1], [16] and [20].

Definition 4.4. For a non-dropping Σ-iterate P of M, $\Sigma_{P,\mathrm{sh}}$ (the *short tree strategy* for P) denotes the restriction of Σ_P to short trees. Also, $\Sigma_{P|\delta_0^P}$ denotes the (0-maximal) strategy for $P|\delta_0^P$ induced by Σ (including maximal trees) and $\Sigma_{P|\delta_0^P,\mathrm{sh}}$ denotes its restriction to short trees. ⊣

Note that by Fact 3.4($\Sigma 5$), the notations $\Sigma_{P|\delta_0^P}$ and $\Sigma_{P|\delta_0^P,\mathrm{sh}}$ are unambiguous; that is, if $P \neq Q$ are both non-dropping Σ-iterates of M with $P|\delta_0^P = Q|\delta_0^Q$, then Σ_P agrees with Σ_Q in terms of their action on trees based on $P|\delta_0^P$. Of course $\Sigma_{P,\mathrm{sh}}$ is equivalent to $\Sigma_{P|\delta_0^P,\mathrm{sh}}$, except that the two strategies have different base models. This is useful notationally below, where we can refer directly to $P|\delta_0^P$ but maybe not to P.

We summarize the main results of this section in the following two lemmas:

Lemma 4.5. *Let g be M-generic. Then:*

1. *$M[g]$ is closed under Σ_{sh}.*

2. *$\mathbb{E}^M$, $\Sigma_{\mathrm{sh}}{\upharpoonright}M[g]$ and $\mathrm{dom}(\Sigma_{\mathrm{sh}}{\upharpoonright}M[g])$ are classes of $M[g]$, definable over $M[g]$ (as a coarse structure) from the parameter $M|(\lambda^{+\omega})^M$ where $g \subseteq M|\lambda$, uniformly in λ.*

3. *If $g = \emptyset$ then these are in fact lightface classes of the universe $\lfloor M \rfloor$ of M.*

4. *Therefore $\mathbb{U}$ is lightface M-definable, as is $\left\langle \mathscr{P}^M(\mathcal{U}) \right\rangle_{\mathcal{U}\in\mathbb{U}}$ (recall $\mathscr{P}^N(\mathcal{U}) = N$ if $\mathcal{U}$ is the trivial tree on N),*

5. *For each non-dropping Σ-iterate P of M with $\bar{P} = P|\delta_0^P \in M$, M is closed under $\Sigma_{\bar{P},\mathrm{sh}}$, and $\Sigma_{\bar{P},\mathrm{sh}}{\upharpoonright}M$ is definable over M from $\bar{P}$, uniformly in $\bar{P}$. Therefore the function*
$$S : \bar{P} \mapsto \Sigma_{\bar{P},\mathrm{sh}}{\upharpoonright}M,$$
with domain the class of all such $\bar{P} \in M$, is lightface M-definable.

6. *The corresponding facts hold after replacing M by N and Σ_{sh} by $\Sigma_{N,\mathrm{sh}}$ and $\mathbb{U}$ by $\mathbb{U}^N$, for any non-dropping Σ-iterate N of M. Moreover,*
$$i_{MN}(\Sigma_{M,\mathrm{sh}}{\upharpoonright}M) = \Sigma_{N,\mathrm{sh}}{\upharpoonright}N,$$
and with S from part 5, $i_{MN}(S)$ has the corresponding domain in N, and $i_{MN}(S)(\bar{P}) = \Sigma_{\bar{P},\mathrm{sh}}{\upharpoonright}N$ for each $\bar{P} \in \mathrm{dom}(i_{MN}(S))$.

Proof. By the previous discussion, $M[g]$ is closed under Σ_{sh}. Moreover,

$$\Sigma_{\mathrm{sh}}{\upharpoonright}M[g] \text{ and } \mathrm{dom}(\Sigma_{\mathrm{sh}}{\upharpoonright}M[g])$$

are definable over $M[g]$ from the predicate $\mathbb{E}^M$. But the universe $\lfloor M \rfloor$ of M is definable over $M[g]$ from $M|(\lambda^{+\omega})^M$ by Woodin-Laver [6], [28]. By [20], $\mathbb{E}^M$ is definable over $\lfloor M \rfloor$ from $M{\upharpoonright}\omega_1^M$, but the latter is $(\omega, \omega_1 + 1)$-iterable in $\lfloor M \rfloor$ (via Σ_{sh}), and is therefore definable without parameters there (which is relevant to the case that $g = \emptyset$). Part 5 is a straightforward adaptation; in fact, note

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that trees via $\Sigma_{\bar{P},\mathrm{sh}}$ normalize to trees via Σ_{sh}. Part 6 is also straightforward, using the uniformity of the process.[20]　　　　　　　　　　　　　　　　□

Lemma 4.6. *Let g be M-generic for $\mathbb{P}$ and $\mathcal{T} \in M[g]$ be a limit length normal tree on M which is based on $M|\delta_0^M$ and via Σ. If $\mathcal{T} \in M|\kappa_0$ let $U = M$, and otherwise let $E \in \mathbb{E}^M$ be M-total with $\mathrm{crit}(E) = \kappa_0$ and*

$$\mathbb{P} \in V_\theta^M \subseteq U = \mathrm{Ult}(M, E)$$

and $\mathcal{T} \in V_\theta^{M[g]}$. Let $b = \Sigma(\mathcal{T})$. Let η be a strong cutpoint of U with $\theta \leq \eta < \kappa_0^U$. Then there is $\mathcal{X} = \mathcal{X}_{\mathcal{T},\eta} \in U[g] \subseteq M[g]$ such that:

(1) $\mathcal{X}$ is a limit length tree on U (but is equivalent to one on M), based on $M|\delta_0^M$, via Σ_U (hence ω-maximal); let $c = \Sigma(\mathcal{X})$ and $\delta = \delta(\mathcal{X})$,

(2) $\delta \leq (\eta^+)^U$,

(3) $U|\delta$ is $M(\mathcal{X})$-generic for $\mathbb{B}_\delta^{M(\mathcal{X})}$,

(4) If $\mathcal{T}$ is maximal then $\mathcal{X}$ is maximal, $\delta_0^{M_c^{\mathcal{X}}} = \delta = (\eta^+)^U$ and $M_c^{\mathcal{X}} = \mathscr{P}^U(\mathcal{X})$.

(5) Suppose $\mathcal{T}$ is short. Then $\mathcal{X}$ is short and $\eta < \delta(\mathcal{X}) < (\eta^+)^U$, and there is $R \triangleleft U|(\eta^+)^U$ which computes the Q-structure $Q(\mathcal{X}, c)$ via inverse $*$-translation above $M(\mathcal{X})$.

(6) There is a tree embedding $\Pi : \mathcal{T} \,\widehat{}\, b \hookrightarrow \mathcal{X} \,\widehat{}\, c$, and b, Π can be computed locally from $(\mathcal{T}, \mathcal{X}, c)$ (hence if $\mathcal{T}, \mathcal{X}$ are short then $b \in U[g]$).

(7) If $\mathcal{T} \in M|\kappa_0$ (so $U = M$) and $\mathcal{T}$ is maximal then $\mathcal{X} \in \mathbb{U}$.

Definition 4.7. We may also express the situation of the preceding lemma by saying that $\mathcal{T} \,\widehat{}\, b$ is *absorbed by* $\mathcal{X} \,\widehat{}\, c$, or $\mathcal{T}$ is *absorbed by* $\mathcal{X}$.　　　　　　⊣

4.3　The first direct limit system

4.3.1　The external direct limit system $\mathscr{D}^{\mathrm{ext}}$

We now define a system of uniform grounds for M. In the notation of §2, we use index set

$$d = \{M|\delta_0\} \cup \{\mathcal{M}(\mathcal{U}) \mid \mathcal{U} \in \mathbb{U} \text{ is non-trivial}\}.$$

For $p \in d$, the associated model is $\mathcal{P}_p = \mathscr{P}^M(p)$. Of course d and $\mathbb{U}$ are essentially equivalent. By Lemma 4.5, $(d, \langle \mathcal{P}_p \rangle_{p \in d})$ is lightface M-definable. Write $\mathscr{F} = \{\mathcal{P}_p \mid p \in d\}$.

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[20]However, working inside N, if $\mathcal{T}$ on $N|\delta_0^N$ is maximal and we minimally inflate $\mathcal{T}$ to produce $\mathcal{X}$, and build the proper class model $\mathscr{P}^N(M(\mathcal{X}))$ by P-construction, and $c = \Sigma_N(\mathcal{X})$, then it need not be that $M_c^{\mathcal{X}} = \mathscr{P}^N(M(\mathcal{X}))$. But in this case, $M_c^{\mathcal{X}}$ and $\mathscr{P}^N(M(\mathcal{X}))$ will still compare to a common model above $\delta(\mathcal{X})$. Related issues will be discussed further in §4.11.

We now define the partial order $\preceq$ on d, and maps π_{pq}. Let $\mathcal{T}, \mathcal{U} \in \mathbb{U}$ and $P = \mathscr{P}^M(\mathcal{T})$ and $Q = \mathscr{P}^M(\mathcal{U})$. Set

$$\mathcal{M}(\mathcal{T}) \preceq \mathcal{M}(\mathcal{U}) \Longleftrightarrow Q \text{ is a } \Sigma_P\text{-iterate of } P.$$

We also define the order $\preceq$ on $\mathscr{F}$ by $P \preceq Q$ iff $\mathcal{M}(\mathcal{T}) \preceq \mathcal{M}(\mathcal{U})$. The associated embedding $\pi_{\mathcal{M}(\mathcal{T})\mathcal{M}(\mathcal{U})}$ is just the iteration map i_{PQ}. We remark that if $P \neq Q$ then the tree witnessing that $P \preceq Q$ is of the form $\mathcal{V} \,\widehat{}\, \Sigma_P(\mathcal{V})$, with $\mathcal{V}$ via $\Sigma_{P,\mathrm{sh}}$ and $\mathcal{V} \in M$, but $\Sigma_P(\mathcal{V}) \notin M$.[21] We write $\mathcal{T}_{PQ}$ for $\mathcal{V} \,\widehat{}\, \Sigma_P(\mathcal{V})$, and $\mathcal{T}_{PQ,\mathrm{sh}} = \mathcal{V}$. While i_{PQ} is not amenable to M (if non-identity), we do have:

Lemma 4.8. $\preceq$ *is a directed partial order, is lightface M-definable, and the associated embeddings commute: if $P \preceq Q \preceq R$ then $i_{QR} \circ i_{PQ} = i_{PR}$.*

Proof Sketch. For the definability, note that $P \preceq Q$ iff the pseudo-comparison of P, Q, using $\Sigma_{P,\mathrm{sh}}$ to iterate P,[22] yields a limit length tree $\mathcal{T}$ on P with $\mathcal{M}(\mathcal{T}) = Q|\delta_0^Q$ (so Q does not move in the pseudo-comparison).

The fact that $\preceq$ is a partial order, and the commutativity, follows from the properties of Σ^{stk} in Fact 3.4.

For directedness, given $P, Q \in \mathscr{F}$, witnessed by trees $\mathcal{T}, \mathcal{U} \in \mathbb{U}$, form a simultaneous pseudo-comparison and $\mathbb{E}^M$-genericity iteration of P, Q in M, using $\Sigma_{P,\mathrm{sh}}, \Sigma_{Q,\mathrm{sh}}$, producing trees $\mathcal{V}, \mathcal{W}$ respectively, and $R \in \mathscr{F}$ with $P, Q \preceq R$; note that if we normalize the stack $(\mathcal{T}, \mathcal{V})$, or the stack $(\mathcal{U}, \mathcal{W})$, we get the same normal tree $\mathcal{X} \in \mathbb{U}$; here $R = \mathscr{P}^M(\mathcal{X})$. $\qquad\square$

Now define $\mathscr{D}^{\mathrm{ext}} = (P, Q, i_{PQ} \colon P \preceq Q \in \mathscr{F})$. By the lemma, $\mathscr{D}^{\mathrm{ext}}$ is a direct limit system with properties (ug1), (ug2), (ug3), (ug4), (ug5). Note that (ug18) holds, with $p_0 = M|\delta_0$. Define the direct limit model and maps

$$(\mathcal{M}_\infty^{\mathrm{ext}}, (i_{P\infty} \colon P \in \mathscr{F})) = \mathrm{dirlim}\, \mathscr{D}^{\mathrm{ext}}. \tag{9}$$

Notice that even though $\mathscr{F}$ is a definable collection of classes in M, this system is not "in" M, as the maps i_{NP} (when non-identity) are not amenable to M. As usual, $\mathcal{M}_\infty^{\mathrm{ext}}$ is wellfounded, giving (ug6).

Definition 4.9. For $P \in \mathscr{F}$, let τ^P be the canonical class $\mathbb{B}_{\delta_0^P}^P$-name for M; that is, τ^P is the name for the class "premouse" N such that $N|\delta_0^P$ is given by the extender algebra generic, and $\mathbb{E}^N\!\restriction\![\delta_0^P, \infty)$ is given by extending $\mathbb{E}^P$ via the usual extension to small generic extensions (equivalently, $\mathbb{E}^N\!\restriction\![\delta_0^P, \infty)$ agrees with $\mathbb{E}^P\!\restriction\![\delta_0^P, \infty)$ on the ordinals). (Of course, for some generics, this might not actually yield a premouse, but with g the generic for adding $M|\delta_0^P$, we have $(\tau^P)_g = M$.) Note that $i_{MP}(\tau^M) = \tau^P$. $\qquad\dashv$

Lemma 4.10. (ug19) *holds: for each $P \in \mathscr{F}$, $c^P = d \cap d^P$ is dense in $(\mathscr{F}^P, \preceq^P)$ and dense in $(\mathscr{F}, \preceq)$, and $\preceq^P\!\restriction c^P = \preceq\restriction c^P$.*

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[21] M cannot have an elementary embedding $P \to Q$, because P, Q are both grounds of M and by [5]. Therefore $\Sigma_P(\mathcal{V}) \notin M$. $\mathcal{V}$ is determined by $P, Q, \Sigma_{P,\mathrm{sh}}$ (in fact, just by P, Q, because all the relevant Q-structures are segments of Q), so $\mathcal{V} \in M$.

[22] Or just reading Q-structures from segments of Q to compute branches.

Proof. Let $P \in \mathscr{F}$. The fact that $\preceq^P \restriction c^P = \preceq \restriction c^P$ is by Lemma 4.5 part 6. So let $Q \in \mathscr{F}^P$ and $R \in \mathscr{F}$. We must find some $S \in \mathscr{F}^P \cap \mathscr{F}$ with $Q, R \preceq S$. Let σ be some $\mathbb{B}^P_{\delta^P_0}$-name such that $\sigma_g = R$. Let η be a strong cutpoint of P such that $\delta^P_0 < \eta$ and $Q|\delta^Q_0, \sigma \in P|\eta$. Let $p_1 \in \mathbb{B}^P_{\delta^P_0}$ be the Boolean value of the statement "τ^P is an M_{swsw}-like premouse and $\sigma \in \mathscr{F}^{\tau^P}$". Then working in P, we can form a tree $\mathcal{T}$ on P by Boolean-valued comparison of P, Q and all interpretations of σ below p_1, with $\mathbb{E}^P$-genericity and Boolean-valued τ^P-genericity iteration folded in, and using the short tree strategies to iterate. (The Boolean-valued genericity iteration means that we use extenders E under the usual circumstances as for genericity iteration, and given that there is some $\mathbb{B}^P_{\delta^P_0}$ condition forcing that E induces an axiom false for τ^P.) For each limit $\lambda \leq (\eta^+)^P$, if $M(\mathcal{T}\restriction\lambda)$ is not a Q-structure for itself then $\delta(\mathcal{T}\restriction\lambda) = \lambda$ and $\mathcal{T}\restriction\lambda$ is definable from parameters over $P|\lambda$, because (i) the segments of τ^P are determined level-by-level by $\mathbb{E}^P$ above δ^P_0, and (ii) for all limits $\xi < \lambda$, the Q-structures guiding branch choice at stage ξ do not overlap $\delta(\mathcal{T}\restriction\xi)$, and (iii) the Q-structures Q_ξ of all trees at stage ξ are identical, and hence $Q_\xi \lhd M(\mathcal{T}\restriction\lambda)$; this means that we can use P-construction to compute Q-structures, and obtain an iterate in $\mathscr{F} \cap \mathscr{F}^P$. (For (ii), suppose ξ is least such that a Q-structure overlaps $\delta(\mathcal{T}\restriction\xi)$. Then there are fatal drops passed before stage ξ. This has to originally arise from a disagreement between the extender sequences of some projecting structures, as opposed to extenders used for genericity iteration purposes (as the latter are only used when they are sufficiently total; cf. the process in [22]). But then we can find some mutual generics witnessing this disagreement, and because the short tree strategies extend to generic extensions (because τ^P is forced M_{swsw}-like), and given the fatal drop, these strategies suffice to complete the comparison between the conflicting projecting structures in the generic extension, which leads to the usual contradiction. Because Q_ξ does not overlap $\delta(\mathcal{T}\restriction\xi)$, note that no extenders in $\mathbb{E}^{Q_\xi}_+$ of length $> \delta(\mathcal{T}\restriction\xi)$ will be used for genericity iteration purposes.) $\qquad\square$

4.3.2 The internal direct limit system $\mathscr{D}$

Definition 4.11. Work in M. Given $P \in \mathscr{F}$ and $s \in [\mathrm{OR}]^{<\omega}\backslash\{\emptyset\}$, say P is *weakly s-iterable* iff for all $Q \in \mathscr{F}$ with $P \preceq Q$, letting $\mathcal{T} = \mathcal{T}_{PQ,\mathrm{sh}}$, there is $\lambda \in \mathrm{OR}$ such that $\mathrm{Coll}(\omega, \lambda)$ forces the existence of a $\mathcal{T}$-cofinal branch b such that

$$i^{\mathcal{T}}_b(s) = s \text{ and } i^{\mathcal{T}}_b(P|\max(s)) = Q|\max(s) \tag{10}$$

(in particular, s is in the wellfounded part of $M^{\mathcal{T}}_b$). We say that P is *s-iterable* iff every $Q \in \mathscr{F}$ with $P \preceq Q$ is weakly s-iterable.

Given an s-iterable P, define

$$\gamma^P_s = \sup(\mathrm{Hull}^{P|\max(s)}(s^-) \cap \delta^P_0),$$

$$H^P_s = \mathrm{Hull}^{P|\max(s)}(\gamma^P_s \cup s^-).$$

(Note that the hulls here are uncollapsed. Recall that $P|\max(s) = (P||\max(s))^{\mathrm{pv}}$ is passive by definition.) Given also a t-iterable Q with $s \subseteq t$ and $P \preceq Q$, define

$$\pi_{Ps,Qt} : H_s^P \to H_t^Q$$

as the common restriction of iteration maps $i_b^{\mathcal{T}}$ for b witnessing the weak s-iterability requirement (10). [23] (Those restrictions agree pairwise by the Zipper Lemma.) Say that P is *strongly s-iterable* iff P is s-iterable and whenever $Q, R \in \mathscr{F}$ and $P \preceq Q \preceq R$ (hence $P \preceq R$), then

$$\pi_{Ps,Rs} = \pi_{Qs,Rs} \circ \pi_{Ps,Qs}.$$

Let $\mathscr{F}^+ = \{(P,s) \mid P \in \mathscr{F} \text{ and } P \text{ is strongly } s\text{-iterable}\}$, and similarly let $d^+ = \{(P|\delta_0^P, s) \mid (P,s) \in \mathscr{F}^+\}$. The order $\preceq$ on d^+ is now determined by (ug8): for $(\bar{P},s), (\bar{Q},t) \in d^+$, we have $(\bar{P},s) \preceq (\bar{Q},t)$ iff $\bar{P} \preceq \bar{Q}$ and $s \subseteq t$. Define the order $\preceq$ on $\mathscr{F}^+$ in the same manner. Clearly

$$(P,s) \preceq (Q,t) \preceq (R,u) \implies \pi_{Ps,Ru} = \pi_{Qt,Ru} \circ \pi_{Ps,Qt}.$$

Define the system $\mathscr{D} = (H_s^P, H_t^Q, \pi_{Ps,Qt} \colon (P,s) \preceq (Q,t) \in \mathscr{F}^+)$.

Given $P \in \mathscr{F}$ and $s \in [\mathrm{OR}]^{<\omega}$, recall that s is *P-stable* iff $\pi_{PQ}(s) = s$ for every $Q \in \mathscr{F}$ with $P \preceq Q$. ⊣

Remark 4.12. Even though it is superfluous, we note that s-iterability actually implies strong s-iterability. This follows from calculations in [14]. For let $\mathcal{T} = \mathcal{T}_{PQ,\mathrm{sh}}$, $\mathcal{U} = \mathcal{U}_{QR,\mathrm{sh}}$ and $\mathcal{X} = \mathcal{X}_{PR,\mathrm{sh}}$. Say that a $\mathcal{T}$-cofinal branch is $\mathcal{T}$-*good* iff $M_b^{\mathcal{T}}$ is $\delta(\mathcal{T}) + 1$-wellfounded and $i_b^{\mathcal{T}}(\delta_0^P) = \delta(\mathcal{T})$; likewise for the other trees. Then the $\mathcal{X}$-good branches d correspond exactly to pairs (b,c) such that b is $\mathcal{T}$-good and c is $\mathcal{U}$-good; and moreover, the corresponding iteration maps $\restriction \delta_0^P$ then commute (see [14, ***Theorem 10.8]). Let b be $\mathcal{T}$-good and witness weak s-iterability. Let $\mathcal{T}'$ be the 0-maximal tree on $P|\max(s)$ given by $\mathcal{T}$. So $M_b^{\mathcal{T}'} = \mathrm{Ult}_0(P|\max(s), E_b^{\mathcal{T}})$, where $E_b^{\mathcal{T}}$ is the branch extender. By a condensation argument due to J. Steel, $M_b^{\mathcal{T}'} \trianglelefteq M_b^{\mathcal{T}}$, and since $i_b^{\mathcal{T}}(\max(s)) = \max(s)$, clearly $\mathrm{OR}(M_b^{\mathcal{T}'}) = \max(s)$, so $M_b^{\mathcal{T}'} = Q|\max(s)$, and similarly $i_b^{\mathcal{T}'}(s^-) = s^-$. Likewise for c being $\mathcal{U}$-good and witnessing weak s-iterability, and 0-maximal tree $\mathcal{U}'$ on $Q|\max(s)$. Let $d = d_{b,c}$ be the corresponding branch, and $\mathcal{X}'$ the 0-maximal tree on $P|\max(s)$. Then we get $M_d^{\mathcal{X}'} = M_c^{\mathcal{U}'} = R|\max(s)$, and $i_d^{\mathcal{X}'}(s^-) = i_c^{\mathcal{U}'}(i_b^{\mathcal{T}'}(s^-)) = s^-$, and commutativity in general. But note that these embeddings agree with the covering direct limit maps (consider the natural factor map $M_b^{\mathcal{T}'} \to M_b^{\mathcal{T}}|\max(s)$), and therefore we get strong s-iterability.

The following is proved by the usual arguments (recall that since $\mathcal{M}_\infty^{\mathrm{ext}}$ is wellfounded, for all $s \in [\mathrm{OR}]^{<\omega}$, there is $P \in \mathscr{F}$ such that s is P-stable):

Lemma 4.13. *We have:*

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[23]Notice that $\pi_{Ps,Qt}$ does not depend on t, because $\pi_{Ps,Qt}$ and $\pi_{Ps,Qs}$ have the same graph. In [8], the notation for the analogous map, π_{PQ}^s, does not mention t.

(a) *if $P \in \mathscr{F}$ and $s \in [\mathrm{OR}]^{<\omega} \backslash \{\emptyset\}$ and s is P-stable, then $(P,s) \in \mathscr{F}^+$ and (P,s) is true (see Definition 2.1).*

(b) *$(\mathscr{F}^+, \preceq)$ is directed – for (P,s), $(Q,t) \in \mathscr{F}^+$ there is $(R,u) \in \mathscr{F}^+$ with $(P,s) \preceq (R,u)$ and $(Q,t) \preceq (R,u)$ (note $u = s \cup t$ suffices).*

(c) *Therefore properties (ug7), (ug8), (ug9), (ug10), (ug16) hold.*

The following properties follow directly from the definitions; note that strong s-iterability is used for (ug15):

Lemma 4.14. *$\mathscr{D}$ is lightface M-definable, and properties (ug11), (ug12), (ug13), (ug14), (ug15) hold.*

For the following, recall $\mathscr{I}^P$ from Definition 3.7:

Lemma 4.15. *For each $P \in \mathscr{F}$, $\{\alpha\}$ is P-stable for every $\alpha \in \mathscr{I}^M = \mathscr{I}^P$. Therefore property (ug17) holds, as witnessed by some $s \in [\mathscr{I}^M]^{<\omega}$.*

Proof. The proof is standard, but we give a reminder. By Fact 3.9, $P = \mathrm{Hull}^P(\mathscr{I}^P \cup \delta_0^P)$. By Lemma 4.3, $\pi_{MP} \restriction \mathscr{I}^M = \mathrm{id}$, so $\mathscr{I}^P = \mathscr{I}^M$, and we have $\{\alpha\}$-stability for each $\alpha \in \mathscr{I}^M$. Now let $x \in P$. By these facts, we can fix $s \in [\mathscr{I}^M]^{<\omega}$ such that $x \in \mathrm{Hull}^{P|\max(s)}(\delta \cup \{s^-\})$. But we can also arrange that γ_s^P is as large below δ_0^P as desired. It follows we can get $x \in H_s^P$, and by Lemma 4.13, this works. $\qquad\square$

We now have enough properties from §2 to define (working in M) the direct limit

$$(\mathcal{M}_\infty, \pi_{Ps,\infty} \colon (P,s) \in \mathscr{F}^+) = \mathrm{dirlim}\ \mathscr{D}, \tag{11}$$

and the $*$-map, and Lemmas 2.3 and 2.5 hold, so in particular, $\chi \colon \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathrm{ext}}$ is the identity and $\mathcal{M}_\infty = \mathcal{M}_\infty^{\mathrm{ext}}$. Property (ug20) is straightforward (the main point is that if $P \in \mathscr{F}$ and $\bar{Q} \in d^P \cap d$ then $\mathcal{P}_{\bar{Q}}^P = \mathscr{P}^P(\bar{Q}) = \mathscr{P}^M(\bar{Q})$, because $\mathbb{E}^P, \mathbb{E}^M$ agree above δ_0^P). For property (ug21), given s, note that any $P \in \mathscr{F}$ such that s is P-stable works.

So far we have verified (ug1)–(ug21). For the remaining properties we use $\delta = \delta_0^M$ and $\mathbb{B} = \mathbb{B}_{\delta_0}^M$ (the δ_0-generator extender algebra of M at δ_0). This immediately secures (ug22). Recall we defined τ^P in Definition 4.9, and $\delta_\infty = i_{M\infty}(\delta_0) = \delta_0^{\mathcal{M}_\infty}$.

Lemma 4.16. *We have:*

1. *For each M-stable $\alpha \in \mathrm{OR}$ and each $P \in \mathscr{F}$, letting $\tau^P \restriction \alpha = i_{MP}(\tau^M \restriction \alpha)$ and g be the P-generic filter for $\mathbb{B}_{\delta_0^P}^P$ given by $M|\delta_0^P$, then $(\tau^P \restriction \alpha)_g = M|\alpha$. Moreover, $M \mathrel{\widehat{=}} P[g] \mathrel{\widehat{=}} P[M|\delta_0^P]$.*

2. *Property (ug24) holds.*

3. *κ_0^M is the least measurable cardinal of $\mathcal{M}_\infty$.*

4. *$\kappa_0^{+M} = \delta_\infty$.*

Proof. Part 1 is already clear, and part 2 is an easy consequence. Part 3 is also clear. Part 4 is by the proof of [8, Lemma 2.7(b)].[24]   □

Lemma 4.17. *Let $E \in \mathbb{E}^M$ be M-total with $\mathrm{crit}(E) = \kappa_0$ and let $\mathcal{N} = i_E(\mathcal{M}_\infty)$. We have:*

1. *$\mathscr{I}^{\mathcal{M}_\infty} = \mathscr{I}^M$, and so $i_{M\mathcal{M}_\infty}\!\restriction\!\mathscr{I}^M = \mathrm{id} = *\!\restriction\!\mathscr{I}^M$.*

2. *$\mathcal{N}$ is a $\delta_0^{\mathcal{N}}$-sound $\Sigma_{\mathcal{M}_\infty}$-iterate of $\mathcal{M}_\infty$.*

3. *$\mathscr{I}^{\mathcal{N}} = \mathscr{I}^{\mathrm{Ult}(M,E)}$.*

4. *$i_E^M\!\restriction\!\mathcal{M}_\infty : \mathcal{M}_\infty \to \mathcal{N}$ is the $\Sigma_{\mathcal{M}_\infty}$-iteration map.*

Proof. We delay the proof until Lemma 4.57, which is more general.[25]   □

Lemma 4.18. *The following are true.*

(a) *The restriction of $\Sigma_{\mathcal{M}_\infty}$ to trees in M and based on $\mathcal{M}_\infty|\delta_\infty$, is lightface definable over M.*

(b) *Let $\lambda \in \mathrm{OR}$ and g be in some generic extension of V, and be $\mathbb{P}$-generic over M for some $\mathbb{P} \in M|\lambda$. Let $\Sigma'_{\mathcal{M}_\infty}$ be the restriction of $\Sigma_{\mathcal{M}_\infty}$ to trees in $M[g]$ and based on $\mathcal{M}_\infty|\delta_\infty$. Then $\Sigma'_{\mathcal{M}_\infty}$ is definable over the universe of $M[g]$ from the parameter $x = M|(\lambda^{+\omega})^M$, uniformly in λ.[26]*

Proof. The short tree strategy for $\mathcal{M}_\infty$ is computed just like for M, and the definability is like in 4.5. The computation of branches at maximal stages is just like [8, Lemma 2.9(a),(b)], supplemented by Lemmas 3.10 and 4.17, and for the definability, use the definability of $\mathbb{E}^M$ from x in $M[g]$ (see 4.5). Here is a sketch for $g = \emptyset$. Let $E \in \mathbb{E}^M$ be M-total, with $\mathrm{crit}(E) = \kappa_0$ and $\mathcal{T} \in M|\lambda(E)$ a maximal tree on $\mathcal{M}_\infty|\delta_\infty$. Let $U = \mathrm{Ult}(M,E)$. By Lemma 4.17, $\mathcal{N} = i_E(\mathcal{M}_\infty)$ is a $\delta_0^{\mathcal{N}}$-sound iterate of $\mathcal{M}_\infty$ and $i_E^M\!\restriction\!\mathcal{M}_\infty : \mathcal{M}_\infty \to \mathcal{N}$ is the correct iteration map. Now let $P = M(\mathcal{T})$. Then $U \vDash$"P is a maximal Σ_{sh}-iterate of $M|\delta_0^M$", and therefore $U \vDash$"$\mathcal{N}|\delta_0^{\mathcal{N}}$ is a maximal $\Sigma_{P,\mathrm{sh}}$-iterate of P", considering statements satisfied by M regarding such iterates. But U is correct about this. Let $\mathcal{S}$ be the tree on P leading to $\mathcal{N}|\delta_0^{\mathcal{N}}$. Working in a generic extension of M, find a $\mathcal{T}$-cofinal branch b and $\mathcal{S}$-cofinal branch c such that $i_E^M\!\restriction\!(\mathcal{M}_\infty|\delta_\infty) = i_c^{\mathcal{S}} \circ i_b^{\mathcal{T}}$, and then verify that $b = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$, using Lemma 3.10.[27]   □

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[24]Actually, an easy cardinality calculation shows that $\delta_\infty \leq \kappa_0^{+M}$, and we will show directly later that δ_∞ is Woodin in $\mathscr{V}_1$ and $\mathscr{V}_1$ is a ground of M via a forcing which has the δ_∞-cc in $\mathscr{V}$, and hence δ_∞ is regular in M, so $\delta_\infty = \kappa_0^{+M}$, without using the proof of [8, Lemma 2.7(b)].

[25]It is fine to read 4.53–4.57 at this point, which covers what is needed.

[26]If $g \notin V$, we are extending Σ and Σ^{stk} canonically to $V[g]$ as in Fact 3.4(Σ1).

[27]There is an alternate proof which uses properties of normalization and is more direct. Let $\mathcal{W}\!\smallfrown\!d$ be the tree leading from $\mathcal{M}_\infty$ to $\mathcal{N}$ (with final branch d). We have $\mathcal{W}, d \in M$ by Lemma 4.17. But $\mathcal{W}$ (of limit length) is the normalization of the stack $(\mathcal{T}, \mathcal{S})$ (the trees in the first given proof). Letting b, c be the correct branches through $\mathcal{T}, \mathcal{S} \in M$, d determines (together with $\mathcal{T}, \mathcal{S}, \mathcal{W}$) the pair (b,c) uniquely via normalization.

Remark 4.19. The strategy $\Sigma_{\mathcal{M}_\infty}$ also has minimal hull condensation, so we get the canonical stacks strategy $(\Sigma_{\mathcal{M}_\infty})^{\mathrm{stk}}$ induced by $\Sigma_{\mathcal{M}_\infty}$, which agrees with the tail strategy $\Gamma_{\mathcal{M}_\infty} = (\Sigma^{\mathrm{stk}})_{\mathcal{M}_\infty}$, by Fact 3.4. It is easily definable from $\Sigma_{\mathcal{M}_\infty}$, and for stacks based on $\mathcal{M}_\infty|\delta_\infty$, we only need the normal strategy for trees based there. So $M[g]$ can also compute the restriction of $(\Sigma_{\mathcal{M}_\infty})^{\mathrm{stk}} = \Gamma_{\mathcal{M}_\infty}$ to stacks based on $\mathcal{M}_\infty|\delta_\infty$, which are in $M[g]$.

Note that we have now verified all of the requirements for uniform grounds, excluding (ug23). This will be established later in Lemma 4.30.

4.4 The first Varsovian model as $\mathcal{M}_\infty[*]$

In §2 we defined the elementary maps $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ and $\pi_\infty^+ : \mathscr{V} \to \mathscr{V}^{\mathcal{M}_\infty}$. We now want to show that $\mathcal{M}_\infty^{\mathcal{M}_\infty}$ is an iterate of $\mathcal{M}_\infty$ and π_∞ is a correct iteration map. We also want to generalize this to other iterates of M, but in general one must be a little careful.

Definition 4.20. Given an M_{swsw}-like premouse N, let $\mathscr{D}^N$ and $\mathcal{M}_\infty^N$ be defined over N just as how $\mathscr{D}$, $\mathcal{M}_\infty$ are defined over M. If N is a correct iterate of M_{swsw}, also define $(\mathcal{M}_\infty^{\mathrm{ext}})_N$ (the external direct limit) relative to N, as for M: given a maximal tree $\mathcal{T} \in \mathbb{U}^N$ (considered as a tree on N), let $b = \Sigma_N(\mathcal{T})$ and $M_{\mathcal{T}} = M_b^{\mathcal{T}}$, and let $(\mathcal{M}_\infty^{\mathrm{ext}})_N$ be the direct limit of these models $M_{\mathcal{T}}$ under the iteration maps (by Lemma 4.5, and in particular its part 6, these trees $\mathcal{T}$ are indeed according to Σ_N).[28] If in fact $M_{\mathcal{T}} = \mathscr{P}_{M(\mathcal{T})}^N$ (the model indexed by $M(\mathcal{T})$ in the covering system $\mathscr{D}^N$) for each such $\mathcal{T}$, then define $\chi_N : \mathcal{M}_\infty^N \to (\mathcal{M}_\infty^{\mathrm{ext}})_N$ as in §2. $\dashv$

Lemma 4.21. Let N be a δ_0^N-sound, non-dropping correct iterate of M. Then $M_{\mathcal{T}} = \mathscr{P}^N(M(\mathcal{T}))$ for each $\mathcal{T} \in \mathbb{U}^N$, $\mathcal{M}_\infty^N = (\mathcal{M}_\infty^{\mathrm{ext}})_N$ and $\chi^N = \mathrm{id}$, and $\mathcal{M}_\infty^N$ is a $\delta_0^{\mathcal{M}_\infty^N}$-sound, non-dropping correct iterate of both M and N.

Proof. The proof is just like for M, using the δ_0^N-soundness of N (and resulting $\delta_0^{\mathcal{M}_\infty^N}$-soundness of $\mathcal{M}_\infty^N$) as a substitute for the fact that $M = \mathrm{Hull}_1^M(\mathscr{I}^M)$, to see that the models of $\mathscr{D}^N$ really are iterates of N. $\square$

We will see later, however, that if N is a Σ-iterate of M which is κ_0^N-sound but non-δ_0^N-sound, then $(\mathcal{M}_\infty^{\mathrm{ext}})_N \neq \mathcal{M}_\infty^N$, so we need a little more care in this case.

By the lemma, $\mathcal{M}_\infty^{\mathcal{M}_\infty} = (\mathcal{M}_\infty^{\mathrm{ext}})_{\mathcal{M}_\infty}$ is an iterate of $\mathcal{M}_\infty$. Now recall that $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ is the union of all $\pi_{pt,\infty}(\pi_{ps,\infty})$ for embedding-good tuples (p, s, t), and that $\pi_\infty(x) = x^*$, and if $\rho \in \mathrm{OR}$ then

$$\pi_\infty(\rho) = \rho^* = \min(\{\pi_{N\infty}(\rho) : N \in \mathscr{F}\}) = \pi_{Ps,\infty}(\rho), \tag{12}$$

where $(P, s) \in \mathscr{F}^+$ is any pair with $\rho \in s$ and $\rho < \max(s)$. As usual we have:

---

[28]These models can in general be distinct from the models $\mathcal{P}_{M(\mathcal{T})}^N$ computed by N via P-construction, which is why the care mentioned above is needed.

Lemma 4.22. $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ *is the iteration map according to* $\Sigma_{\mathcal{M}_\infty}$.

Similarly, let N be as in Lemma 4.21 (so $\mathcal{M}_\infty^N$ a $\delta_0^{\mathcal{M}_\infty^N}$-*sound correct iterate,*
and likewise $\mathcal{M}_\infty^{\mathcal{M}_\infty^N}$). *Let* $\pi_\infty^N = i_{MN}(\pi_\infty)$. *Then* π_∞^N *is the iteration map*
$\mathcal{M}_\infty^N \to \mathcal{M}_\infty^{\mathcal{M}_\infty^N}$ *according to* $\Sigma_{\mathcal{M}_\infty^N}$.

Write $N = \mathcal{M}_\infty$. As in [8], since the tree from N to $\mathcal{M}_\infty^N$ is based on $N|\delta_0^N$,
$\pi_\infty : N \to \mathcal{M}_\infty^N$ is determined by $\pi = \pi_\infty{\restriction}(N|\delta_0^N)$ (as π_∞ is the ultrapower
map by the extender derived from π), which is in turn determined by the pair
$(\pi_\infty{\restriction}\delta_0^N, \mathcal{M}_\infty^N|\delta_0^{\mathcal{M}_\infty^N})$. Since $\mathcal{M}_\infty^N|\delta_0^{\mathcal{M}_\infty^N} \in N$, it follows that

$$L[\mathcal{M}_\infty, *] = L[\mathcal{M}_\infty, \pi_\infty] = L[\mathcal{M}_\infty, \pi_\infty{\restriction}\delta_\infty].$$

Moreover, π_∞ is definable over this universe from the predicate $N = \mathcal{M}_\infty$ (given
N, we can recover $\mathcal{M}_\infty^N$ and the maximal tree $\mathcal{T}$ leading from N to $\mathcal{M}_\infty^N$, but in
$L[N, \pi_\infty]$, there is a unique $\mathcal{T}$-cofinal branch b with $M_b^{\mathcal{T}} = \mathcal{M}_\infty^N$; but $i_b^{\mathcal{T}} = \pi_\infty$).

Definition 4.23. The *first Varsovian model* of M (cf. (4)) is the structure

$$\mathcal{M}_\infty[*] = (L[\mathcal{M}_\infty, *], \mathcal{M}_\infty, *); \tag{13}$$

that is, $\mathcal{M}_\infty[*]$ has universe $L[\mathcal{M}_\infty, *]$ and predicates $\mathcal{M}_\infty$ and $*$. ⊣

(By the preceding comments, it would suffice to just have the predicate
$\mathcal{M}_\infty$, but we include $*$ for convenience.) Later we will introduce a second
presentation $\mathscr{V}_1$ of $\mathcal{M}_\infty[*]$, constructed from a different predicate (but giving
the same universe). However, the two predicates will be inter-definable over
that universe.

Before we introduce that presentation, we first develop some properties of
$\mathcal{M}_\infty[*]$ using the presentation above. We may at times blur the distinction
between the universe $L[\mathcal{M}_\infty, *]$ and the structure $\mathcal{M}_\infty[*]$, but for definability
issues over $\mathcal{M}_\infty[*]$, we can by default use $(\mathcal{M}_\infty, *)$ as a predicate.

4.5 $\mathrm{HOD}_{\mathscr{E}}^{M[G]}$

Up until this point, the paper has covered material which is mostly a direct adap-
tation from that of [8]. But in this section we begin to see some new features.
In [8] it is shown that the Varsovian model has universe that of $\mathrm{HOD}^{M_{\mathrm{sw}}[g]}$,
where $g \subseteq \mathrm{Coll}(\omega, < \kappa)$ is M_{sw}-generic, for κ the strong cardinal of M_{sw}. In this
section we will establish an analogue of this fact.

Let G be $(M, \mathrm{Coll}(\omega, < \kappa_0))$-generic. Note that *if* $\mathrm{HOD}^{M[G]}$ is the universe
of $\mathcal{M}_\infty[*]$, then it would follow as in [8] that $\mathcal{M}_\infty[*]$ is closed under maximal
branches according to $\Sigma_{\mathcal{M}_\infty}$ (those branches are in M by 4.18, and have length
of uncountable cofinality in $M[G]$, and hence are unique there). Thus, such a
fact is at least useful in verifying that the first Varsovian model can iterate its
own least Woodin cardinal, which one would like to prove.

Also, in order to proceed to the next step of the mantle analysis, one might
want to consider iteration trees on $\mathcal{M}_\infty[*]$, based on $\mathcal{M}_\infty[*]|\delta_1^M$ (we will show

that δ_1^M is Woodin in $\mathcal{M}_\infty[*]$). But because $\mathcal{M}_\infty[*]$ is built from the somewhat cumbersome combination of $\mathcal{M}_\infty$ and $*$, the nature of its large cardinals above $\delta_0^{\mathcal{M}_\infty}$ (and so far also below there, though that is resolved by standard methods) is somewhat unclear, as is its fine structure.

Now if we are to expect $\mathcal{M}_\infty[*]$ to be closed under $\Sigma_{\mathcal{M}_\infty}$, a possible goal is to find a presentation of it as a strategy mouse, built from extenders and strategy for $\Sigma_{\mathcal{M}_\infty}$, with a fine structural hierarchy; with this target in mind, we write $\mathscr{V}_1$ for the desired model of this form (whatever its eventual presentation might be). The first two authors considered various candidates for such a presentation, with one possibility being a construction by level-by-level correspondence between $\mathscr{V}_1$ and M, via a modified kind of P-construction. This P-construction would result from restricting the extenders of M to segments of $\mathscr{V}_1$, starting above some point θ not too far above κ_0. (An early candidate was $\theta = (\kappa_0^{+3})^M$, but the second and third authors later reduced this to an optimal starting point, which we use.) Of course standard P-constructions build a ground of the outer model, and this feature was expected, via a Bukowsky-style forcing as in [8]. Here we use the forcing $\mathbb{L}$ from §2, which was eventually isolated by the second author. But note that a new feature in this P-construction would be that some extenders (those extenders E with $\mathrm{crit}(E) = \kappa_0$) overlap the forcing $\mathbb{L}$. This would cause a problem with a standard P-construction (where the base forcing is produced by genericity iteration). But such extenders E yield strategy information, via the process in the proof of 4.18 (whereas those with $\mathrm{crit}(E) > \kappa_0$ would be as usual). So it appears that one might construct $\mathscr{V}_1$ with such a P-construction, with extenders E having $\mathrm{crit}(E) = \kappa_0$ corresponding to strategy information, to be added fine structurally to the relevant segment of $\mathscr{V}_1$. This could then lead to a model closed under $\Sigma_{\mathcal{M}_\infty}$ for trees based on δ_∞, as is desired. The model $\mathscr{V}_1$ should also inherit iterability for itself, from the iterability of M, in a manner similar to standard P-constructions.

A first basic question is whether the model $\mathscr{V}_1$ constructed as above will end up $\subseteq \mathcal{M}_\infty[*]$. We now make the key observation which shows that it does. A useful first step is to restrict one's attention to the action of the extenders on the ordinals; this will be enough for the P-construction. For the purpose of the next lemma, let us write

$$\mathbb{F}_{>\kappa_0}^M = \{(\nu, \alpha, \beta) \in \mathrm{OR}^3 \mid \nu > \kappa_0^M, \ \mathbb{E}_\nu^M \neq \emptyset, \text{ and } \mathbb{E}_\nu^M(\alpha) = \beta\}, \qquad (14)$$

where, since we are using Jensen indexing, the M-extender $\mathbb{E}_\nu^M$ in (14) is an elementary embedding $j : M|\mu^{+M|\nu} \to M|\nu$ where $\mu = \mathrm{crit}(\mathbb{E}_\nu^M)$, so $(\mathbb{F}_{>\kappa_0}^M)_\nu = j{\restriction}\mu^{+M|\nu}$.

In the following lemma, recall that for definability over $\mathcal{M}_\infty[*]$, we by default have the predicate $(\mathcal{M}_\infty, *)$ available for free:

Lemma 4.24. $\mathbb{F}_{>\kappa_0}^M$ *is lightface definable over* $\mathcal{M}_\infty[*]$.

Proof. Let $(\nu, \alpha, \beta) \in \mathrm{OR}^3$ with $\nu > \kappa_0$. We claim that $(\nu, \alpha, \beta) \in \mathbb{F}_{>\kappa_0}^M$ iff

$$F \neq \emptyset \text{ and } F(\alpha^*) = \beta^* \text{ where } F = F^{\mathcal{M}_\infty || \nu^*},$$

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which suffices. This equivalence holds as for every $P \in \mathscr{F}$, we have $F^{P||\nu}\!\restriction\!\mathrm{OR} = F^{M||\nu}\!\restriction\!\mathrm{OR}$, since $\nu > \mathrm{crit}(F^{M||\nu}) \geq \kappa_0$ and P results from a P-construction of M above some point below κ_0. $\qquad\square$

One can now proceed directly with the P-construction, using $(\mathcal{M}_\infty | \kappa_0^{+\mathcal{M}_\infty}, * \restriction \delta_0^{\mathcal{M}_\infty})$ and $\mathbb{F}^M_{>\kappa_0}$ to define it. But we postpone this, and first establish some other characterizations of the universe of $\mathcal{M}_\infty[*]$.

The proof of the preceding lemma can be extended to show that $\mathcal{M}_\infty[*]$ has universe $\mathrm{HOD}^{M[G]}_{\mathscr{E}}$, for a certain collection $\mathscr{E}$ of premice in $M[G]$, and G as above. Thus, we use $\mathrm{HOD}_{\mathscr{E}}$ here in place of the use of HOD in [8]. We describe how this works next. What follows is slightly related to some methods from [17]; also cf. 4.34.

Definition 4.25. Let $L[\mathbb{E}]$ be a premouse, and let μ be a strong cutpoint of $L[\mathbb{E}]$. If g is $\mathrm{Col}(\omega, \mu)$-generic over $L[\mathbb{E}]$, then every extender $\mathbb{E}_\nu$ from $\mathbb{E}$ with $\nu > \mu$ (and hence $\mathrm{crit}(\mathbb{E}_\nu) > \mu$) lifts canonically to an extender $\mathbb{E}_\nu^g$ over $(L[\mathbb{E}]|\nu)[g]$. Let us write $\mathbb{E}^g = \langle \mathbb{E}_\nu^g \rangle_{\nu > \mu}$. Then $L[\mathbb{E}][g]$ gets reorganized as a premouse over $(L[\mathbb{E}]||\mu, g)$ with extender sequence $\mathbb{E}^g$; so $L[\mathbb{E}][g] \,\hat{=}\, L[\mathbb{E}^g](L[\mathbb{E}]||\mu, g)$.

Let $L[\mathbb{E}_0]$, $L[\mathbb{E}_1]$ be proper class premice and $\mu \in \mathrm{OR}$. Write

$$\mathbb{E}_0 \sim^\mu \mathbb{E}_1$$

iff μ is a strong cutpoint in both $L[\mathbb{E}_0]$ and $L[\mathbb{E}_1]$ and there are g_0, g_1 with g_i being $\mathrm{Col}(\omega, \mu)$-generic over $L[\mathbb{E}_i]$ and $L_{\mu+\omega}[\mathbb{E}_0][g_0] \,\hat{=}\, L_{\mu+\omega}[\mathbb{E}_1][g_1]$ and $(\mathbb{E}_0)^{g_0} = (\mathbb{E}_1)^{g_1}$. So "$\mathbb{E}_0 \sim^\mu \mathbb{E}_1$" expresses the fact that above μ, $\mathbb{E}_0$ and $\mathbb{E}_1$ are intertranslatable: for every $\nu > \mu$, $(\mathbb{E}_0)_\nu = (\mathbb{E}_1)_\nu^{g_1} \cap L[\mathbb{E}_0]|\nu$, and vice versa. Write

$$\mathbb{E}_0 \sim^{<\mu} \mathbb{E}_1$$

iff there is some $\bar{\mu} < \mu$ with $\mathbb{E}_0 \sim^{\bar{\mu}} \mathbb{E}_1$. Both $\sim^\mu$ and $\sim^{<\mu}$ are equivalence relations.

Let $L[\mathbb{E}]$ be a proper class premouse, let μ be inaccessible in $L[\mathbb{E}]$, and let H be $\mathrm{Col}(\omega, <\mu)$-generic over $L[\mathbb{E}]$. We denote by $\mathscr{E}_{\mathbb{E}}^{L[\mathbb{E}][H]}$ the collection of all $\mathbb{E}'$ such that $\mathbb{E}'|\mu \in L[\mathbb{E}][H]$ and $L[\mathbb{E}][H] \vDash \text{``}\mathbb{E}' \sim^{<\mu} \mathbb{E}\text{''}$. $\qquad\dashv$

Remark 4.26. Note that if $\mathbb{E}' \in \mathscr{E}_{\mathbb{E}}^{L[\mathbb{E}][H]}$ then

(i) There are $\bar{\mu} < \mu$ and generics $g, g' \in L[\mathbb{E}][H]$ witnessing that $\mathbb{E}' \sim^{\bar{\mu}} \mathbb{E}$.

(ii) $\mathbb{E}'$ is Σ_1-definable inside $L[\mathbb{E}][H]$ from the set parameter $\mathbb{E}'|\bar{\mu}$ and the class parameter $\mathbb{E}$, uniformly in $\mathbb{E}', \bar{\mu}$.

(iii) There is H' which is $\mathrm{Col}(\omega, <\mu)$-generic over $L[\mathbb{E}']$, with $L[\mathbb{E}'][H'] \,\hat{=}\, L[\mathbb{E}][H]$. For any such H', we have $\mathscr{E}_{\mathbb{E}}^{L[\mathbb{E}][H]} = \mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][H']}$.

(iv) $\mathscr{E}_{\mathbb{E}}^{L[\mathbb{E}][H]}$ is definable over $L[\mathbb{E}][H]$ from $\mathbb{E}, \mu$, uniformly in $\mathbb{E}, \mu, H$.

(v) $\mathscr{E}_{\mathbb{E}}^{L[\mathbb{E}][H]}$ is definable over $L[\mathbb{E}][H]$ from $\mathbb{E}', \mu$, for all $\mathbb{E}' \in \mathscr{E}_{\mathbb{E}}^{L[\mathbb{E}][H]}$, uniformly in $\mathbb{E}, \mu, H, \mathbb{E}'$.

Fix now $G \subseteq \mathrm{Col}(\omega, < \kappa_0)$ which is M-generic. We write $\mathscr{E} = \mathscr{E}^{L[\mathbb{E}^M][G]}$.

The following is now immediate.

Lemma 4.27. *If $N = L[\mathbb{E}'] \in \mathscr{F}$, then there is G' which is $\mathrm{Col}(\omega, < \kappa_0)$-generic over N such that $N[G'] \,\hat{=}\, M[G]$, and for any such G', we have $\mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][G']} = \mathscr{E}$.*

Theorem 4.28. *We have:*

(i) $\mathcal{M}_\infty[*]$ *(including its predicates) is definable over $M[G]$ from the parameter $\mathscr{E}$, and*

(ii) $\mathcal{M}_\infty[*]$ *has universe* $\mathrm{HOD}_{\mathscr{E}}^{M[G]}$.

Remark 4.29. This leaves the question: is $\mathrm{HOD}^{M[G]}$ the universe of $\mathcal{M}_\infty[*]$?

Proof. We first verify (i). Write $\mathbb{E} = \mathbb{E}^M$. Say that $\mathbb{E}'$ is M_{swsw}-*like* iff $L[\mathbb{E}']$ is M_{swsw}-like. Fix an M_{swsw}-like $\mathbb{E}' \in \mathscr{E}$. We claim that $\mathcal{M}_\infty$ and $*$ are defined over $L[\mathbb{E}']$ in the same manner as over M, which suffices. For the systems $\mathscr{F}^{L[\mathbb{E}']}$ and $\mathscr{F}^M$ have cofinally many points in common, which easily suffices. To see this fact, use a Boolean-valued comparison argument as in the last part of the proof of [8, Claim 2.11] (comparison with simultaneous genericity iteration, against $L[\mathbb{E}'']$ for various M_{swsw}-like $\mathbb{E}'' \in \mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][G']}$). Because "$M_{\mathrm{swsw}}$-like" includes short-tree iterability, etc, we can indeed form this iteration successfully.

Part (ii): By (i), $\mathcal{M}_\infty[*] \subseteq \mathrm{HOD}_{\mathscr{E}}^{M[G]}$. We now prove the converse. Let A be a set of ordinals, φ a formula, $\alpha \in \mathrm{OR}$ such that for every $\xi \in \mathrm{OR}$,

$$\xi \in A \iff M[G] \vDash \varphi(\xi, \alpha, \mathscr{E}).$$

So for every $\mathbb{E}' \in \mathscr{E}$ and G' with $L[\mathbb{E}'][G'] \,\hat{=}\, M[G]$, and every $\xi \in \mathrm{OR}$, we have

$$\xi \in A \iff L[\mathbb{E}'][G'] \vDash \varphi(\xi, \alpha, \mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][G']})$$

(since $\mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][G']} = \mathscr{E}$, by Remark 4.26).

Given an M_{swsw}-like $\mathbb{E}'$, write $\dot{\mathscr{E}}^{L[\mathbb{E}']}$ for the natural $\mathrm{Col}(\omega, < \kappa_0^{L[\mathbb{E}']})$-name for $\mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][G']}$ (for G' the $\mathrm{Col}(\omega, < \kappa_0^{L[\mathbb{E}']})$-generic filter; cf. the remarks on the uniform definability of $\mathscr{E}_{\mathbb{E}'}^{L[\mathbb{E}'][G']}$ above).

Let $\xi \in \mathrm{OR}$. Pick $N = L[\mathbb{E}'] \in \mathscr{F}$ with $\xi^* = \pi_{N \mathcal{M}_\infty}(\xi)$ and $\alpha^* = \pi_{N \mathcal{M}_\infty}(\alpha)$, and let G' be as in Lemma 4.27 for N. Then

$$
\begin{aligned}
\xi \in A \quad &\iff \quad M[G] \vDash \varphi(\xi, \alpha, \mathscr{E}) \\
&\iff \quad N[G'] \vDash \varphi(\xi, \alpha, \mathscr{E}_{\mathbb{E}'}^{N[G']}) \\
&\iff \quad \Vdash_{\mathrm{Col}(\omega, < \kappa_0)}^{N} \varphi(\check{\xi}, \check{\alpha}, \dot{\mathscr{E}}^N) \\
&\iff \quad \Vdash_{\mathrm{Col}(\omega, < \kappa_0^{\mathcal{M}_\infty})}^{\mathcal{M}_\infty} \varphi(\check{\xi}^*, \check{\alpha}^*, \dot{\mathscr{E}}^{\mathcal{M}_\infty}).
\end{aligned}
$$

Therefore $A \in \mathcal{M}_\infty[*]$. $\qquad\square$

4.6 Uniform grounds

Recall that $\delta_\infty = \delta^{\mathcal{M}_\infty}$ is the least Woodin of $\mathcal{M}_\infty$. The following lemma completes the proof that the first direct limit system for M provides uniform grounds (§2):

Lemma 4.30. *We have:*

(a) $V_{\delta_\infty}^{\mathcal{M}_\infty[*]} = V_{\delta_\infty}^{\mathcal{M}_\infty}$.

(b) δ_∞ *is (the least) Woodin in* $\mathcal{M}_\infty[*]$.

(c) *Property* (ug23) *of uniform grounds holds for* $\mathscr{V}_1, \delta_\infty$; *that is,* $\mathscr{V}_1 \vDash$ "δ_∞ *is regular and* $\mathbb{B}_\infty$ *is* δ_∞-*cc". Moreover,* $\mathscr{V}_1 \vDash$ "$\mathbb{B}_\infty$ *is a complete Boolean algebra".*

Proof. (a): The usual considerations show that $* \restriction \eta \in \mathcal{M}_\infty$ for every $\eta < \delta_\infty$. Combined with the proof of Theorem 4.28, this suffices. (Cf. [8, Claim 2.12 (b)].)

(b): By the proof of [8, Theorem 2.19] or of [19, Claim 13].

(c): Property (ug23) holds because δ_∞ is Woodin in $\mathcal{M}_\infty[*]$ and $\mathbb{B}_\infty$ is the extender algebra. The "moreover" clause follows from this and (a). $\square$

So by Theorem 2.16, $\mathcal{M}_\infty[*]$ is a ground for M, and in fact as in its proof, there is some M-stable $\mu \in \mathrm{OR}$ such that $M|\mu$ is $(\mathcal{M}_\infty[*], \mathbb{L})$-generic, where $\mathbb{L} = \mathbb{L}(\mu^*)$, and $\mathcal{M}_\infty[*][M|\mu] \mathrel{\hat{=}} M$. We will actually refine this result in Lemma 4.38.

We can immediately deduce the following corollary, which however will be extended in Lemma 4.46:

Corollary 4.31. *For all maximal trees* $\mathcal{T} \in \mathcal{M}_\infty[*]$ *via* $\Sigma_{\mathcal{M}_\infty}$, *based on* $\mathcal{M}_\infty|\delta_\infty$, *we have* $b = \Sigma_{\mathcal{M}_\infty}(\mathcal{T}) \in \mathcal{M}_\infty[*]$, *and* b *is the unique* $\mathcal{T}$-*cofinal branch in* $\mathcal{M}_\infty[*]$.

Proof. By Lemma 4.18, $b \in M$. And by Lemma 4.16, δ_∞ is regular in M and in $M[G]$, whenever $G \subseteq \mathrm{Coll}(\omega, <\kappa_0)$ is M-generic. Therefore $M[G]$ contains no other $\mathcal{T}$-cofinal branch. Since $\mathcal{M}_\infty[*]$ is the universe of $\mathrm{HOD}_{\mathscr{E}}^{M[G]}$ (with notation as in Theorem 4.28), and $\mathcal{T} \in \mathcal{M}_\infty[*]$, it follows that $b \in \mathcal{M}_\infty[*]$ also. $\square$

4.7 The structure $\mathcal{Q}$

Let $U = E_\nu^{\mathcal{M}_\infty}$ be the least total measure on the $\mathcal{M}_\infty$-sequence with critical point $\kappa_0^{\mathcal{M}_\infty}$. Fix a natural tree $\mathcal{T} \in \mathbb{U}^{\mathrm{ult}(\mathcal{M}_\infty; U)}$ with $\delta(\mathcal{T}) = \kappa_0^{+\mathcal{M}_\infty}$ and which makes $\mathcal{M}_\infty|\kappa_0^{+\mathcal{M}_\infty}$ generic, after iterating the least measurable out to $\kappa_0^{\mathcal{M}_\infty}$. As $\mathcal{T}$ lives on $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$, we may and shall construe $\mathcal{T}$ as a tree on $\mathcal{M}_\infty$ rather than on $\mathrm{ult}(\mathcal{M}_\infty; U)$. If $b = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$, then

$$\mathscr{P}^{\mathrm{ult}(\mathcal{M}_\infty; U)}(\mathcal{M}(\mathcal{T})) = \mathcal{M}_b^{\mathcal{T}}$$

and $\delta(\mathcal{T}) = \pi_{0,b}^{\mathcal{T}}(\delta_0^{\mathcal{M}_\infty})$. Also,

$$\mathscr{P}^{\mathrm{ult}(\mathcal{M}_\infty; U)}(\mathcal{M}(\mathcal{T}))[\mathcal{M}_\infty|\theta_0^{\mathcal{M}_\infty}] = \mathrm{ult}(\mathcal{M}_\infty; U). \tag{15}$$

Let us write

$$\iota = \pi_{0,b}^{\mathcal{T}} \restriction \delta_0^{\mathcal{M}_\infty}. \tag{16}$$

By Corollary 4.31, $\iota \in \mathcal{M}_\infty[*]$. Hence

$$\mathcal{Q} = (\mathcal{M}_\infty | \kappa_0^{+\mathcal{M}_\infty}; \iota) \tag{17}$$

is an amenable structure and is an element of $\mathcal{M}_\infty[*]$.

Lemma 4.32 (Soundness of $\mathcal{Q}$). $\mathcal{Q} = \mathrm{Hull}^{\mathcal{Q}}(\iota)$.

Proof. Let

$$\sigma \colon \bar{\mathcal{Q}} = (\bar{M}; \bar{\iota}) \cong X = \mathrm{Hull}^{\mathcal{Q}}(\iota) \prec \mathcal{Q}.$$

As $\iota = \{(\xi,\eta) \colon \xi < \delta_0^{\mathcal{M}_\infty} \wedge \eta = \iota(\xi)\}$, $\delta_0^{\mathcal{M}_\infty} \subset X$, so that $\sigma \restriction \delta_0^{\mathcal{M}_\infty} = \mathrm{id}$ and $\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty} \lhd \bar{M}$. Let $\bar{\mathcal{T}}$ be defined over $\bar{M}$ as $\mathcal{T}$ was defined over $\mathcal{M}_\infty | \theta_0^{\mathcal{M}_\infty}$. The tree $\bar{T}$ is on $\bar{M}$ which lives on $\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty}$, but we may and shall construe $\bar{T}$ as a tree on $\mathcal{M}_\infty$, and as such $\bar{\mathcal{T}}$ is according to $\Sigma_{\mathcal{M}_\infty}$.

Let $\bar{b} = \Sigma_{\mathcal{M}_\infty}(\bar{\mathcal{T}})$. By branch condensation Lemma 3.10, $\bar{b}$ is the pullback of b via σ. We will have that $\bar{\iota} = \pi_{0,\bar{b}}^{\bar{\mathcal{T}}} \restriction \delta_0^{\mathcal{M}_\infty}$ and there is a canonical elementary embedding

$$\hat{\sigma} \colon \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}} \to \mathcal{M}_b^{\mathcal{T}}$$

defined by

$$\hat{\sigma}(\pi_{0,\bar{b}}^{\bar{\mathcal{T}}}(f)(a)) = \pi_{0,b}^{\mathcal{T}}(f)(\sigma(a)),$$

where $f \in \mathcal{M}_\infty$ and $a \in [\mathrm{OR} \cap \bar{M}]^{<\omega}$. We will have that $\mathcal{M}(\bar{\mathcal{T}}) = \mathrm{dom}(\sigma) \cap \mathrm{dom}(\hat{\sigma})$ and σ and $\hat{\sigma}$ agree on this common part of their domains.

If $E_\nu^{\mathcal{M}(\bar{\mathcal{T}})}$ is a total extender from the $\mathcal{M}(\bar{\mathcal{T}})$-sequence, then by the elementarity of σ, $\bar{M}$ satisfies all the axioms of the extender algebra of $\mathcal{M}(\bar{\mathcal{T}})$ given by $E_\nu^{\mathcal{M}(\bar{\mathcal{T}})}$, as $\mathcal{M}_\infty | \theta_0^{\mathcal{M}_\infty}$ satisfies all the axioms of the extender algebra of $\mathcal{M}(\mathcal{T})$ given by $E_{\sigma(\nu)}^{\mathcal{M}_\infty}$. We may conclude that $\bar{M}$ is generic over $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$. If $\bar{g}$ is the associated generic over $\mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}$ and g that over $\mathcal{M}_b^{\mathcal{T}}$, then $\hat{\sigma}``\bar{g} = \sigma``\bar{g} \subset g$, and hence we may lift $\hat{\sigma}$ to an elementary embedding

$$\hat{\sigma}^* \colon \mathcal{M}_{\bar{b}}^{\bar{\mathcal{T}}}[\bar{M}] \to \mathcal{M}_b^{\mathcal{T}}[\mathcal{M}_\infty | \kappa_0^{+\mathcal{M}_\infty}] = \mathrm{ult}(\mathcal{M}_\infty; U),$$

defined by $\hat{\sigma}^*(\tau^{\bar{g}}) = (\hat{\sigma}(\tau))^g$.

But notice that $\delta_0^{\mathcal{M}_\infty} \cup \{\mathrm{crit}(U)\} \subset \mathrm{ran}(\hat{\sigma}^*)$, so that by the $\delta_0^{\mathcal{M}_\infty}$-soundness of $\mathcal{M}_\infty$ and by the choice of U as a measure, $\hat{\sigma}^*$ must be the identity, and therefore so is σ. $\qquad\square$

Corollary 4.33.

(a) $\kappa_0^{+\mathcal{M}_\infty}$ has size $\delta_0^{\mathscr{V}_1}$ inside $\mathcal{M}_\infty[*]$.

(b) $\kappa_0^{++\mathcal{M}_\infty} = \kappa_0^{++M}$.

Proof. (a) follows from Lemma 4.32 together with (17). (b) is then immediate by $\kappa_0^{++M} \in \mathrm{Card}^{\mathcal{M}_\infty}$. $\qquad\square$

This corollary should be compared with Lemma 4.41, to come.

4.8 The κ_0-mantle of M

We now give another characterization of the universe of $\mathcal{M}_\infty[*]$, though no results outside of this subsection actually depend on it.

The following definitions are essentially taken from [26], though the notation and terminology is different. If W is an inner model and λ is a cardinal of W, then $\bar{W} \subset W$ is a λ *ground of* W iff $\bar{W}$ is an inner model of W and there is some poset $\mathbb{P} \in \bar{W}$ which has size λ in $\bar{W}$ and some g which is $\mathbb{P}$-generic over $\bar{W}$ such that $W = \bar{W}[g]$. $\bar{W}$ is called a $< \lambda$ *ground of* W iff there is some $\bar{\lambda} < \lambda$ such that $\bar{W}$ is a $\bar{\lambda}$ ground of W. The $< \lambda$ *mantle of* W is the intersection of all $< \lambda$ grounds of W. We write $\mathbb{M}_{<\lambda}^W$ for the $< \lambda$-mantle of W.

The main result of this subsection is that $\mathcal{M}_\infty[*]$ has universe $\mathbb{M}_{<\kappa_0}^M$; see 4.36. The following fact and its proof are similar to parts of [20, ***Lemmas 5.11 and 5.16] and [17, ***Lemma 5.2]; in particular, we make use of the condensation stacks from [20].

Let $\mu < \kappa_0$ be a cardinal strong cutpoint of M, and $\eta = \mu^{+M}$. Let g be $\mathrm{Coll}(\omega, \mu)$-generic over M. Let $\mathscr{P}_M^{M[g]}$ denote the set of all $N \in M[g]$ such that:

– N is a premouse of ordinal height η,

– N has a largest cardinal $\lambda < \eta$, and λ is a strong cutpoint of N,

– $M[g] \vDash$ "N is fully iterable above λ"

– there is a proper class premouse N' with $N \lhd N'$ and $M[g] \vDash$ "$\mathbb{E}^{N'} \sim^{<\eta} \mathbb{E}^M$".

Note that because of the restriction on $\mathbb{E}^{N'}$ above η, we don't actually need to quantify over proper classes here, and clearly $\mathscr{P}_M^{M[g]}$ is definable over $M[g]$ from the class M. We now refine this fact:

Lemma 4.34. *Let g be $\mathrm{Coll}(\omega, \mu)$-generic over M. Then (i) $\mathscr{P}_M^{M[g]}$ is definable over $M[g]$ from no parameters, uniformly in μ, g. Further, (ii) for all N, N' as in the definition of $\mathscr{P}_M^{M[g]}$, N' is definable over $M[g]$ from the parameter N, uniformly in μ, g, N, N'.*

Proof. Note that $\eta = \omega_1^{M[g]}$ and $\mathrm{HC}^{M[g]}$ is the universe of $(M|\eta)[g]$. Now working in $M[g]$, let $\mathscr{P}'$ be the set of all premice N of height η such that for some h,

– N has a largest cardinal, λ, which is a strong cutpoint of N,

– N is fully iterable above λ,

– h is $\mathrm{Coll}(\omega, \lambda)$-generic over N and $N[h]$ has universe HC,

– the condensation stack $(N[h])^+$ above $N[h]$ (relativized as a premouse over $(N|\lambda, h)$) is well-defined, hence is proper class with universe V.[29]

---

[29] Recall we are working in $M[g]$, so "V" refers to $M[g]$ here. Because we relativize over $(N|\lambda, h)$, $N[h]$ plays the role that $P|\omega_1^P$ plays in the unrelativized condensation stack for a premouse P.

For $N \in \mathscr{P}'$, we write λ^N for the largest cardinal of N, and with h as above, we consider $N[h]$ as an $(N|\lambda^N, h)$-premouse. Now $\mathscr{P}' \neq \emptyset$, and in fact $M|\eta \in \mathscr{P}'$, as witnessed by g. For by [20], the condensation stack above $(M|\eta)[g]$, as computed in $M[g]$, is just $M[g]$ (arranged as a premouse over $(M|\mu, g)$).

We will show that $\mathscr{P}_M^{M[g]} = \mathscr{P}'$, which gives part (i). The first direction is basically as in [17, Lemma 5.2 part 1]:

CLAIM 1. $\mathscr{P}_M^{M[g]} \subseteq \mathscr{P}'$.

Proof. Let $N \in \mathscr{P}_M^{M[g]}$, as witnessed by N'. Everything is clear enough except for the fact that, in $M[g]$, the condensation stack $N[h]^+$ is well-defined. But $N'[h]$ has universe that of $M[g]$, and there is $\alpha < \eta$ such that $N'[h]$ is iterable above α (in V), and we can take α here such that $N|\alpha$ projects to λ^N. By this iterability, $N'[h]$, *when considered as an $(N|\alpha, h)$-premouse*, therefore is just the condensation stack above $N[h]$. But $N'[h]|\eta = N[h]$ is also iterable in $M[g]$ (above λ^N; that is, as an $(N|\lambda^N, h)$-premouse). So $N[h]$ satisfies all standard condensation facts (as an $(N|\lambda^N, h)$-premouse). So we can argue as in the proof of [17, Lemma 5.2 part 1] to see that $N'[h]$, *when considered as an $(N|\lambda^N, h)$-premouse*, is also the condensation stack above $N[h]$, as computed in $M[g]$, as desired. $\square$

CLAIM 2. Let $N \in \mathscr{P}'$, witnessed by h. Then there is α such that $\mu, \lambda^N < \alpha < \eta$ and $M[g]||\alpha$ and $N[h]||\alpha$ have the same universe, with largest cardinal $\omega_1^{M[g]||\alpha} < \alpha$, and the two structures $M[g]||\alpha$ and $N[h]||\alpha$ are inter-definable from parameters and project $\leq \omega_1^{M[g]||\alpha}$.

Proof. We basically follow the proof of [20, Lemma 5.11***]. Let M' be $M[g]$ arranged as a premouse over $(M|\mu, g)$, and N' be the condensation stack over $N[h]$ as computed in $M[g]$.

So M', N' both have universe $M[g]$, and in particular, $\eta = \omega_1^{M'} = \omega_1^{N'}$ and $(\mathcal{H}_{\omega_2})^{M'} = (\mathcal{H}_{\omega_2})^{N'}$. We first find $\alpha' < \omega_2^{M[g]}$ such that $M'||\alpha'$ and $N'||\alpha'$ have the same universe and are inter-definable from parameters (and some more). Define $\langle M_n, N_n \rangle_{n<\omega}$ as follows. Let $M_0 = M'||\eta$ and $N_0 = N'||\eta$. Now given M_n, N_n, let N_{n+1} be the least $S \lhd N'$ such that $N_n \lhd S \lhd N'$ and $M_n \in S$ and $\rho_\omega^S = \eta$. Define M_{n+1} symmetrically.

Let $\widetilde{M} = \text{stack}_{n<\omega} M_n$ and $\widetilde{N} = \text{stack}_{n<\omega} N_n$. Note that $\widetilde{M}, \widetilde{N}$ have the same universe U, which has largest cardinal η, and therefore $\widetilde{M} \lhd M'$ and $\widetilde{N} \lhd N'$ (that is, $M'||\text{OR}^U$ and $N'||\text{OR}^U$ are both passive, as η is a strong cutpoint of M', N'). Note that $\widetilde{M}$ is definable from the parameter M_0 over U; in fact, $\widetilde{M}$ is the Jensen stack above M_0 there; cf. [20]. It follows that $\widetilde{M}$ is $\Sigma_1^U(\{M_0\})$. Likewise for $\widetilde{N}$ and N_0. Using this, note that also $\langle M_n, N_n \rangle_{n<\omega}$ is $\Sigma_1^U(\{(M_0, N_0)\})$, and so $\rho_1^{\widetilde{M}} = \rho_1^{\widetilde{N}} = \eta$.

Now as in [20], we can find $\bar{\eta} < \eta$ and $x \in U$ such that $\mu, \lambda^N < \bar{\eta}$ and the hulls $\text{Hull}_1^{\widetilde{M}}(\bar{\eta} \cup \{x\})$ and $\text{Hull}_1^{\widetilde{N}}(\bar{\eta} \cup \{x\})$ have the same elements, and letting

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$\bar{M}, \bar{N}$ be the transitive collapses and $\pi : \bar{M} \to \widetilde{M}$ and $\sigma : \bar{N} \to \widetilde{N}$ the uncollapse maps (so $\bar{M}, \bar{N}$ have the same universe $\bar{U}$ and π, σ the same graphs), and such that $M_0, N_0 \in \mathrm{rg}(\pi)$ and $\mathrm{crit}(\pi) = \bar{\eta}$ and $\pi(\bar{\eta}) = \eta$ and $\bar{M}, \bar{N}$ are 1-sound with $\rho_1^{\bar{M}} = \bar{\eta} = \rho_1^{\bar{N}}$, so $\bar{M} \lhd M'$ and $\bar{N} \lhd N'$. Note that the $\Sigma_1^U(\{N_0\})$ definition of $\widetilde{N}$ reflects down to a $\Sigma_1^{\bar{U}}(\{\bar{N}_0\})$ definition of $\bar{N}$, where $\pi(\bar{N}_0) = N_0$. Likewise vice versa. Therefore $\bar{M}, \bar{N}$ have the same universe and are inter-definable from parameters. So letting $\alpha = \mathrm{OR}^{\bar{M}} = \mathrm{OR}^{\bar{N}}$, we are done. $\square$

CLAIM 3. $\mathscr{P}' \subseteq \mathscr{P}_M^{M[g]}$.

Proof. Let N, h, α be as in Claim 2. Then note that α is a strong cutpoint of M and of N and of the condensation stack $N[h]^+$ above $N[h]$ (as an $(N|\lambda^N, h)$-premouse, as computed in $M[g]$). So the iterability of $M|\eta$ and N above α (in $M[g]$) implies that $(\mathbb{E}_\nu^M)^g = (\mathbb{E}_\nu^N)^h$ for each $\nu > \alpha$. Thus, the extender sequences of $M|\eta$ and N are intertranslatable (modulo a generic) above α. So by a proof almost identical to [17, ***Lemma 5.2 part 1], $N[h]^+$ and $M[g]$ (as an $(M|\mu, g)$-premouse) have the same extender sequence above α. Now let N^+ be the result of the P-construction of $N[h]^+$ above N. Because $\eta = \omega_1^{N[h]^+}$, this works fine structurally, giving a proper class premouse N^+ extending N. But since $N[h]^+$ and $M[g]$ agree above α, it follows that $\mathbb{E}^{N^+} \sim^{<\eta} \mathbb{E}^M$. So $N \in \mathscr{P}$. $\square$

This proves part (i). For (ii), working in $M[g]$, given $N \in \mathscr{P}_M^{M[g]} = \mathscr{P}'$, we first define the condensation stack $N[h]^+$ above $N[h]$, and then the P-construction N of $N[h]^+$ above N, which gives the desired N'. $\square$

Lemma 4.35. $\mathscr{F}$ is dense in the $< \kappa_0$-grounds of M, so $\bigcap \mathscr{F} = \mathbb{M}_{<\kappa_0}^M$.

Proof. Let $\mu < \kappa_0$ be a regular cardinal strong cutpoint of M, and let W be a $< \mu$-ground of M, via a generic filter k (so $W[k] \cong M$). Let g be $(M, \mathrm{Coll}(\omega, \mu))$-generic and h be $(W, \mathrm{Coll}(\omega, \mu))$-generic, with $W[h] \cong M[g]$. Let $\widetilde{W}$ be the model produced by the P-construction of M over $(\mathcal{H}_{\mu^+})^W$. By Lemma 4.34, $\widetilde{W} \subseteq W$ and $\widetilde{W}$ is definable from parameters over W. And $\widetilde{W}$ is a ground of M; in fact $\widetilde{W}[k] \cong M$, as $M|(\mu^+)^M$ is definable over $(\mathcal{H}_{\mu^+})^M = (\mathcal{H}_{\mu^+})^W[k]$ from the parameter $M|\mu$, via the Jensen stack.[30]

Now working in $\widetilde{W}$, we can compute some $N \in \mathscr{F}$ by forming a Boolean-valued comparison/genericity iteration above $(\mu^+)^W$, to compute $N|\delta_0^N$, and then using P-construction to compute the rest of N. $\square$

Theorem 4.36. $\mathcal{M}_\infty[*]$ has universe $\bigcap \mathscr{F} = \mathbb{M}_{<\kappa_0}^M$.

Proof. We already know $\mathcal{M}_\infty[*] \subseteq \bigcap \mathscr{F} = \mathbb{M}_{<\kappa_0}^M$ by Lemma 4.35.

---

[30]It follows by the standard forcing argument that W is the actually the universe of $\widetilde{W}$.

Let us verify $\mathbb{M}^M_{<\kappa_0} \subseteq \mathcal{M}_\infty[*]$.[31] Let $U \in \mathbb{E}^M$ be the order 0 total measure on κ_0. Let

$$j \colon M \to M' = \mathrm{Ult}(M, U)$$

be the ultrapower map. For $P \in \mathscr{F}$, $i_{MP}(U) = U \cap P \in \mathbb{E}^P$ is the order 0 total measure on κ_0 in P. Let $j^P \colon P \to \mathrm{Ult}(P, i_{MP}(U))$ be the ultrapower map. Note $j{\restriction}\mathrm{OR} = j^P{\restriction}\mathrm{OR}$. Let $U_\infty = i_{M\infty}(U)$ and $j_\infty \colon \mathcal{M}_\infty \to \mathcal{M}'_\infty = \mathrm{Ult}(\mathcal{M}_\infty, U_\infty)$ be the ultrapower map.

Arguing much as in the proof of Lemma 4.24, $j \restriction \mathrm{OR}$ is definable over $\mathcal{M}_\infty[*]$: for all $\eta, \xi \in \mathrm{OR}$ and $P \in \mathscr{F}$, we have

$$\eta = j(\xi) \Longleftrightarrow \eta = j^P(\xi) \Longleftrightarrow \eta^* = j_\infty(\xi^*).$$

(For the second equivalence, just take P such that η, ξ are P-stable.)

Now let $X \in \mathbb{M}^M_{<\kappa_0}$ be a set of ordinals. By the preceding paragraph,

$$j \restriction \sup(X) \in \mathcal{M}_\infty[*]. \tag{18}$$

By elementarity, $j(X) \in \mathbb{M}^{M'}_{<j(\kappa_0)}$. It is straightforward to verify that $\mathcal{M}'_\infty$ is the direct limit of a system $\mathscr{F}'$ of uniform grounds of M' in much the same way as $\mathcal{M}_\infty$ is the direct limit of the system $\mathscr{F}$ of uniform grounds of M; here the models $P' \in \mathscr{F}'$ are exactly those of the form $P' = \mathrm{Ult}(P, i_{MP}(U))$ for some $P \in \mathscr{F}$. So by Theorem 2.16,

$$Q = \mathcal{M}'_\infty[*]$$

is a $< j(\kappa_0)$-ground of M'. So $j(X) \in Q$, but note $Q \subseteq \mathcal{M}_\infty[*]$, so $j(X) \in \mathcal{M}_\infty[*]$, but then using line (18) we get $X \in \mathcal{M}_\infty[*]$, since

$$\alpha \in X \Longleftrightarrow j(\alpha) \in j(X). \qquad \square$$

4.9 The first Varsovian model as the strategy mouse $\mathscr{V}_1$

We will now give another presentation of $\mathcal{M}_\infty[*]$, as a strategy mouse $\mathscr{V}_1$ in a fine structural hierarchy $\langle \mathscr{V}_1 || \nu \rangle_{\nu \in \mathrm{OR}}$, as sketched at the beginning of §4.5. To motivate this, first notice the following.

A routine first observation is:

Lemma 4.37. *Let P be an active (Jensen indexed) premouse. Then F^P is Σ_1-definable over $(P^{\mathrm{pv}}, F^P{\restriction}\mathrm{OR})$, uniformly in P.*

We remark that it is important here that we have the universe and (internal) extender sequence $\mathbb{E}^P$ of P available.

Let $\mathbb{L} = \mathbb{L}^{\mathcal{M}_\infty[*]}(\kappa_0)$ for the poset of Definition 2.11, for adding generic subset of κ_0.

---

[31]The original argument used for the proof that $\mathbb{M}^M_{<\kappa_0} \subseteq \mathcal{M}_\infty[*]$, found by the 3rd author, was slightly different; that argument is sketched for the analogous result Corollary 5.80. The 2nd author then adapted that one to yield the one presented here. In either form, the argument is related to Usuba's ZFC proof of the fact that if κ is extendible then $\mathbb{M}_{<\kappa} = \mathbb{M}$. Related arguments have since been used by the second author in [9] and the third author in [21], [15].

Lemma 4.38. $M|\kappa_0$ is $\mathbb{L}$-generic over $\mathcal{M}_\infty[*]$ and $\mathcal{M}_\infty[*][M|\kappa_0] \cong M$.

Proof. Since we have verified the properties for uniform grounds, that $M|\kappa_0$ is $\mathbb{L}$-generic over $\mathcal{M}_\infty[*]$ follows from §2. We aim to show that $\mathcal{M}_\infty[*][M|\kappa_0] \cong M$ by performing a "P-construction."

We identify the sequence $\langle M|\nu, M||\nu\rangle_{\nu \in \mathrm{OR}}$ inside $\mathcal{M}_\infty[*][M|\kappa_0]$, from the parameter $M|\kappa_0$ and classes $\mathcal{M}_\infty, *$, as follows. We start with $M|\kappa_0$ given. Fix ν with $\kappa_0 < \nu \in \mathrm{OR}$. The sequence $\langle M||\beta\rangle_{\beta < \nu}$ determines $M|\nu$. By Lemma 4.24, $\mathcal{M}_\infty[*]$ knows whether $\mathbb{E}_\nu^M \neq \emptyset$, and if $\mathbb{E}_\nu^M \neq \emptyset$, knows $\mathbb{E}_\nu^M \upharpoonright \mathrm{OR}$, uniformly in ν (definably from $\mathcal{M}_\infty, *$). But from the pair $(M|\nu, \mathbb{E}_\nu^M \upharpoonright \mathrm{OR})$ we can compute $\mathbb{E}_\nu^M$, also uniformly in ν, by Lemma 4.37. $\square$

Definition 4.39. We now define a class structure $\mathscr{V}_1$, structured analogously to a premouse, built from a sequence $\mathbb{E}^{\mathscr{V}_1} = \langle \mathbb{E}_\nu^{\mathscr{V}_1} \rangle_{\nu \in \mathrm{OR}}$ of extenders. However, some of the extenders will be (properly) long, and will not cohere the sequence. We write $\mathscr{V}_1||\nu$ and $\mathscr{V}_1|\nu$ with the usual meaning. For those segments $\mathscr{V}_1||\nu$ active with long extenders, some of the premouse axioms will fail (in particular, coherence).[32] Write $\gamma^{\mathscr{V}_1} = \kappa_0^{+\mathcal{M}_\infty}$.

The map $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ (see (5) and the preceding discussion) is an iteration map. We define (recursively on ν):

$$
\mathbb{E}_\nu^{\mathscr{V}_1} = \begin{cases} \mathbb{E}_\nu^{\mathcal{M}_\infty} & \text{if } \nu < \gamma^{\mathscr{V}_1} \\ \pi_\infty \upharpoonright (\mathcal{M}_\infty | \delta_0^{\mathcal{M}_\infty}) & \text{if } \nu = \gamma^{\mathscr{V}_1} \\ \mathbb{E}_\nu^M \upharpoonright (\mathscr{V}_1|\nu) & \text{if } \nu > \gamma^{\mathscr{V}_1}. \end{cases} \tag{19}
$$

(We will verify in Lemma 4.42 that this definition makes sense; in particular, that $\mathbb{E}_\nu^M \text{``}(\mathscr{V}_1|\nu) \subseteq \mathscr{V}_1|\nu$ when $\nu > \gamma^{\mathscr{V}_1}$.) Let us also write

$$
\begin{align}
\mathbb{E}^{\mathscr{V}_1} &= \{(\nu, x, y) : \mathbb{E}_\nu^{\mathscr{V}_1} \neq \emptyset \text{ and } y = \mathbb{E}_\nu^{\mathscr{V}_1}(x)\}, \tag{20} \\
\mathbb{E}^{\mathscr{V}_1} \upharpoonright \nu &= \{(\bar{\nu}, x, y) \in \mathbb{E}^{\mathscr{V}_1} : \bar{\nu} < \nu\}. \tag{21}
\end{align}
$$

We define the structure

$$
\mathscr{V}_1 = (L[\mathbb{E}^{\mathscr{V}_1}]; \in, \mathbb{E}^{\mathscr{V}_1}).
$$

Like with $\mathcal{M}_\infty[*]$, when we discuss definability or write an equation with $\mathscr{V}_1$, or some similar structure, then we refer to the structure itself, not just its universe $\lfloor \mathscr{V}_1 \rfloor = L[\mathbb{E}^{\mathscr{V}_1}]$. In particular, definability over $\mathscr{V}_1$ has the class predicate $\mathbb{E}^{\mathscr{V}_1}$ available by default, just like for premice. (But when no confusion can arise, such as with expressions like "$x \in \mathscr{V}_1$" or "$x \subseteq \mathscr{V}_1$", we really mean "$x \in \lfloor \mathscr{V}_1 \rfloor$" or "$x \subseteq \lfloor \mathscr{V}_1 \rfloor$", etc.)

Write $e^{\mathscr{V}_1} = \mathbb{E}_{\gamma^{\mathscr{V}_1}}^{\mathscr{V}_1}$, so $e^{\mathscr{V}_1}$ is the least (properly) long extender of $\mathbb{E}^{\mathscr{V}}$, which is just the extender induced by $*$. $\dashv$

---

[32]The segments $\mathscr{V}_1||\nu$ where $\nu = \gamma^{\mathscr{V}_1}$ or $[\nu > \gamma^{\mathscr{V}_1}$ and $\mathrm{crit}(\mathbb{E}_\nu^M) = \kappa_0]$ do not satisfy the usual premouse axioms with respect to their active extender.

Definition 4.40 (Fine structure of $\mathscr{V}_1$)**.** The fine structural concepts ($r\Sigma_{n+1}$, ρ_{n+1}, p_{n+1}, $(n+1)$-solidity, etc, for $n < \omega$) are defined for segments $P \trianglelefteq \mathscr{V}_1$, and also for P^{pv}, just as for standard premice with Jensen indexing below superstrong: $r\Sigma_1^P = \Sigma_1^P$ (without any constant symbols in the fine structural language), which determines everything else via the usual recursion. $\dashv$

We will show that all segments of $\mathscr{V}_1$ are well-defined, sound, and establish a fine structural correspondence between segments of $\mathscr{V}_1$ and segments of M, above a certain starting point. The first non-trivial instance of these facts is given by 4.38 together with the next lemma; it uses techniques reminiscent of those in [20]. It results in a tighter bound on $\kappa_0^{+\mathcal{M}_\infty} = \gamma^{\mathscr{V}_1}$ than that given by Corollary 4.33.

Lemma 4.41. *Let $\bar{\mathscr{V}} = \mathscr{V}_1 || \gamma^{\mathscr{V}_1}$ (so $\bar{\mathscr{V}}$ has $e^{\mathscr{V}_1}$ active). Then:*

(a) $\mathbb{L}$ *is Σ_1-definable over $\bar{\mathscr{V}}$.*

(b) $\bar{\mathscr{V}}$ *is isomorphic to a structure which is definable without parameters over $M|\kappa_0^{+M}$.*

(c) $\bar{\mathscr{V}}$ *is sound, with $\rho_\omega^{\bar{\mathscr{V}}} = \rho_1^{\bar{\mathscr{V}}} = \delta_\infty$ and $p_1^{\bar{\mathscr{V}}} = \emptyset$.*

(d) $\mathrm{OR}^{\bar{\mathscr{V}}} < \xi_0$*, where ξ_0 is the least $\xi > \kappa_0^{+M}$ such that $M|\xi$ is admissible. Therefore $\mathscr{V}_1 || \nu$ is passive for every $\nu \in (\gamma^{\mathscr{V}_1}, \xi_0]$.*

Proof. Part (a) follows from an inspection of §2.

Part (b): Let $\overline{\mathscr{F}}^+$ be like $\mathscr{F}^+$, but consisting of pairs (P, s) such that there is P' with $(P', s) \in \mathscr{F}^+$ and $P = P'|\kappa_0^{+M}$ and $s \subseteq \kappa_0^{+M}$. (Let the associated ordering, models and embeddings be the corresponding ones of $\mathscr{F}^+$.) Let $\overline{\mathcal{M}}_\infty$ be the direct limit of $\overline{\mathscr{F}}^+$, and $\bar{*}$ the associated $*$-map. Let

$$\bar{\sigma} : \overline{\mathcal{M}}_\infty \to \mathcal{M}_\infty|(\kappa_0^{\mathcal{M}_\infty})^{+\mathcal{M}_\infty}$$

be the natural map determined by how $\overline{\mathscr{F}}^+$ sits within $\mathscr{F}^+$. Noting that $\overline{\mathscr{F}}^+$ is definable over $M|\kappa_0^{+M}$, it suffices to see that $\bar{\sigma}$ is the identity.

Fix $s \in [\mathscr{I}^M]^{<\omega}$ with $s \neq \emptyset$. For $P \in \mathscr{F}$, let K^P be the transitive collapse of

$$H^P = \mathrm{Hull}^{P|\max(s)}(s^- \cup \kappa_0^M),$$

and let $\tau^P : K^P \to H^P$ be the uncollapse map, and $\bar{s}^P = t \cup \{\mathrm{OR}(K^P)\}$ where $\tau^P(t) = s^-$.

We claim that $\pi_{MP}(\bar{s}^M) = \bar{s}^M$, and therefore $(P|\kappa_0^{+M}, \bar{s}^P) \in \overline{\mathscr{F}}^+$. For this, note $\pi_{MP}(s, \kappa_0) = (s, \kappa_0)$, so $\pi_{MP}(\bar{s}^M) = \bar{s}^P$. But M is a small (of size $< \kappa_0$) forcing extension of P, which implies $H^M \cap \mathrm{OR} = H^P \cap \mathrm{OR}$, so $\bar{s}^P = \bar{s}^M$, as required.

So write $\bar{s}$ for the common value of $\bar{s}^P$. One can now use the argument of the proof of Lemma 2.3 (which showed that the natural map $\chi : \mathcal{M}_\infty^+ \to \mathcal{M}_\infty$ is the identity), but replacing the use of s there with $\bar{s}$. It follows that $\bar{\sigma} = \mathrm{id}$.

Part (c): Since $\bar{\mathscr{V}} \in \mathcal{M}_\infty[*]$, it suffices that $\bar{\mathscr{V}} = \mathrm{Hull}_1^{\bar{\mathscr{V}}}(\delta_\infty)$.[33] Let $\alpha < \gamma^{\mathscr{V}_1} = \mathrm{OR}^{\bar{\mathscr{V}}}$; we want to see that $\gamma \in \mathrm{Hull}_1^{\bar{\mathscr{V}}}(\delta_\infty)$. Fix a non-empty $s \in [\mathscr{I}^M]^{<\omega}$ and $N \in \mathscr{F}$ such that $\alpha \in \mathrm{rg}(\pi_{Ns,\infty})$. Let $\bar{s}$ be as above. As before, $\bar{s}$ is N-stable, and note that $\alpha \in \mathrm{rg}(\pi_{N\bar{s},\infty})$ (because $\alpha < \kappa_0^{+M}$, if $\pi_{Ns,\infty}(\bar{\alpha}) = \alpha$ then $\pi_{N\bar{s},\infty}(\bar{\alpha}) = \alpha$). But then, as desired, we have

$$\alpha \in \mathrm{Hull}^{\mathcal{M}_\infty | \max(\bar{s})^*}((\bar{s}^-)^* \cup \delta_\infty) \subseteq \mathrm{Hull}_1^{\bar{\mathscr{V}}}(\delta_\infty).$$

Part (d) follows from (b) and the definition of $\mathbb{E}^{\mathscr{V}_1}$ above $\gamma^{\mathscr{V}_1}$. $\qquad\square$

The levels of $\mathscr{V}_1$ correspond tightly to the levels of M, as follows.

Lemma 4.42. *Let $g = g_{M|\kappa_0}$ be the $(\mathcal{M}_\infty[*], \mathbb{L})$-generic determined by $M|\kappa_0$. (so $\mathcal{M}_\infty[*][g] \cong M$). For every $\nu \in \mathrm{OR}$ we have:*

1. *$\mathscr{V}_1|\nu$ and $\mathscr{V}_1||\nu$ are in $\mathcal{M}_\infty[*]$,*

2. *$\mathscr{V}_1|\nu$ and $\mathscr{V}_1||\nu$ are sound,*

3. *Suppose $\nu \geq \xi_0$.[34] Then*

   (a) *$\mathbb{L} \in \mathscr{V}_1|\nu$ and g is $(\mathscr{V}_1|\nu, \mathbb{L})$-generic,*

   (b) *$(\mathscr{V}_1|\nu)[g] =^* M|\nu$,[35]*

   (c) *$(\mathscr{V}_1||\nu)[g] =^* M||\nu$,*

   (d) *if $\mathbb{E}_\nu^M \neq \emptyset$ and $\mathrm{crit}(\mathbb{E}_\nu^M) > \kappa_0$ then $\mathscr{V}_1||\nu$ satisfies the usual premouse axioms with respect to its active predicate (with Jensen indexing; in particular, $\mathbb{E}_\nu^{\mathscr{V}_1}$ is an extender over $\mathscr{V}_1|\nu$), and*

   (e) *if $E = \mathbb{E}_\nu^M \neq \emptyset$ and $\mathrm{crit}(E) = \kappa_0$ then $\mathbb{E}_\nu^{\mathscr{V}_1}$ is a long (δ_∞, ν)-extender over $\mathcal{M}_\infty$ and*

   $$\mathrm{Ult}(\mathcal{M}_\infty|\delta_\infty, \mathbb{E}_\nu^{\mathscr{V}_1}) = i_E^M(\mathcal{M}_\infty|\delta_\infty) = \mathcal{M}_\infty^{\mathrm{Ult}(M,E)}|i_E^M(\delta_\infty)$$

   is a lightface proper class of $\mathscr{V}_1|\nu$, uniformly in ν.

Remark 4.43. Here the notation $=^*$ is the usual one in this context, meaning that (i) the two structures have the same universe, (ii) for each $\alpha \in [\xi_0, \nu)$ (or $[\xi_0, \nu]$), $\mathbb{E}_\alpha^{\mathscr{V}_1} = \mathbb{E}_\alpha^M \restriction (\mathscr{V}_1|\alpha)$ (which is already true by definition), and conversely, if $\mathrm{crit}(\mathbb{E}_\alpha^M) > \kappa_0$ then $\mathbb{E}_\alpha^M$ is the canonical small forcing extension of $\mathbb{E}_\alpha^{\mathscr{V}_1}$ to $M|\alpha$ and if $\mathrm{crit}(\mathbb{E}_\alpha^M) = \kappa_0$ then $\mathbb{E}_\alpha^M$ is determined by $\mathbb{E}_\alpha^M \restriction \mathrm{OR} = \mathbb{E}_\alpha^{\mathscr{V}_1} \restriction \mathrm{OR}$ and $M|\alpha$ as usual for a premouse, and (iii) the structures $\mathscr{V}_1|\nu$ and $M|\nu$ (or $\mathscr{V}_1||\nu$ and $M||\nu$) have corresponding fine structure in the manner usual for P-constructions as in [12], with matching projecta and parameters, etc.

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[33] Let $\mathcal{Q}$ be the structure defined in §4.7. We already know $\mathcal{Q} = \mathrm{Hull}_1^{\mathcal{Q}}(\delta)$, but it seems that the branch through the genericity tree involved there might not be computable from $*$. So the soundness of $\bar{\mathscr{V}}$ is not an obvious corollary.

[34] Also, $M|\theta_0^M$ and $\mathscr{V}_1||\gamma^{\mathscr{V}_1}$ are "generically equivalent in the codes", and letting

$$f : (\theta_0^M, \xi_0) \to (\gamma^{\mathscr{V}_1}, \xi_0)$$

be the unique surjective order-preserving map, then $M|\alpha = M||\alpha$ are likewise equivalent with $\mathscr{V}_1|f(\alpha) = \mathscr{V}_1||f(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$, but we will not need this.

[35] The notation is explained in 4.43.

Proof. The lemma holds for $\nu < \gamma^{\mathscr{V}_1}$ directly by definition, for $\nu = \gamma^{\mathscr{V}_1}$ by 4.41. For $\gamma^{\mathscr{V}_1} < \nu \le \xi_0$ it is a straightforward consequence: we have $\mathscr{V}_1||\xi_0 \in \mathcal{M}_\infty[*]$ since $A \in \mathcal{M}_\infty[*]$, and therefore $\mathscr{V}_1||\nu$ cannot project $< \delta_\infty$. Note that $\gamma^{\mathscr{V}_1} \in \mathrm{Hull}_1^{\mathscr{V}_1|\nu}(\emptyset)$, just because $\mathscr{V}_1||\gamma^{\mathscr{V}_1}$ is the least segment with an active extender of its kind. Using 4.41, therefore $\gamma^{\mathscr{V}_1} + 1 \subseteq \mathrm{Hull}_1^{\mathscr{V}_1|\nu}(\delta_\infty)$. If $\nu < \omega \cdot \gamma^{\mathscr{V}_1}$ then we therefore get $\mathscr{V}_1|\nu \subseteq \mathrm{Hull}_1^{\mathscr{V}_1|\nu}(\delta_\infty)$, and hence $\mathscr{V}_1|\nu$ is sound. If instead $\omega \cdot \gamma^{\mathscr{V}_1} \le \nu$ then note that $(\mathscr{V}_1|\nu)[g]$ has universe that of $M|\nu$, and the $\Sigma_0^{M|\nu}$ forcing relation is $\Delta_1^{\mathscr{V}_1|\nu}$ (lightface, as $\gamma^{\mathscr{V}_1} \in \mathrm{Hull}_1^{\mathscr{V}_1|\nu}(\emptyset)$). But $M|\nu = \mathrm{Hull}_1^{M|\nu}(\kappa_0^{+M} + 1)$, and since all of the Σ_1 facts witnessing this get forced, it follows that $\mathscr{V}_1|\nu = \mathrm{Hull}_1^{\mathscr{V}_1|\nu}(\delta_\infty)$, so $\mathscr{V}_1|\nu$ is sound.

For $\nu > \xi_0$ we discuss parts 1 and 3(d),(e); for the rest one mostly uses standard calculations as for P-constructions. Suppose we have established that $(\mathscr{V}_1|\nu)[g] =^* M|\nu \in \mathcal{M}_\infty[*]$, and $\mathbb{E}_\nu^M \ne \emptyset$, so $\mathbb{E}_\nu^{\mathscr{V}_1} = \mathbb{E}_\nu^M \restriction (\mathscr{V}_1|\nu) \ne \emptyset$. We already know that $\mathbb{E}_\nu^{\mathscr{V}_1} \restriction \mathrm{OR} \in \mathcal{M}_\infty[*]$ (by Lemma 4.24). We must verify that $\mathbb{E}_\nu^{\mathscr{V}_1} \in \mathcal{M}_\infty[*]$ (and uniformly so), and hence $\mathscr{V}_1||\nu \in \mathcal{M}_\infty[*]$, and that $\mathscr{V}_1||\nu$ has the right properties.

Suppose $\kappa_0 < \mathrm{crit}(\mathbb{E}_\nu^M)$. Then $\mathbb{E}_\nu^{\mathscr{V}_1}$ is an extender over $\mathscr{V}_1|\nu$ satisfying the usual requirements for premice, by the usual proof (using induction and that $M|\nu$ is a small forcing extension of $\mathscr{V}_1|\nu$). It follows that $\mathbb{E}_\nu^{\mathscr{V}_1}$ can be computed from the pair $(\mathscr{V}_1|\nu, \mathbb{E}_\nu^{\mathscr{V}_1} \restriction \mathrm{OR})$, as usual. But $\mathscr{V}_1|\nu$ is in $\mathcal{M}_\infty[*]$ by induction.

Now suppose that $\kappa_0 = \mathrm{crit}(\mathbb{E}_\nu^M)$. Then $E = \mathbb{E}_\nu^{\mathscr{V}_1}$ is a (long) (δ_∞, ν)-extender over $\mathcal{M}_\infty|\delta_\infty$, but this time E does not cohere $\mathscr{V}_1|\nu$. In order to compute the full E from $E \restriction \mathrm{OR}$, one also needs the target model

$$U = \mathrm{Ult}(\mathcal{M}_\infty|\delta_\infty, E) = \mathcal{M}_\infty^{\mathrm{Ult}(M, \mathbb{E}_\nu^M)}|\nu.$$

By 4.41, this is computed by the local covering system of $M|\nu$ (as defined in that proof, but over $M|\nu$, not $M|(\kappa_0^{+M})$). But since $\mathscr{V}_1|\nu =^* M|\nu$ and the forcing $\mathbb{L} \in \mathscr{V}_1|\lambda$ where λ is the largest cardinal of $\mathscr{V}_1|\nu$, and $\mathcal{M}_\infty|\delta_\infty \in \mathscr{V}_1|\lambda$, there is a canonical definition of U over $\mathscr{V}_1|\nu$ (uniform in all such ν). That is, although we don't have the full $M|\nu$ directly available, the agreement between $\mathbb{E}^M$ and $\mathbb{E}^{\mathscr{V}_1}$ ensures that the short tree strategy for $\mathcal{M}_\infty$ is computed almost like when we do have $M|\nu$: Given a strong cutpoint γ of $\mathscr{V}_1|\nu$ with $\delta_\infty^{+(\mathscr{V}_1|\nu)} < \gamma$, let G be $(\mathscr{V}_1|\nu, \mathrm{Coll}(\omega, \gamma))$-generic. Then $(\mathscr{V}_1|\nu)[G]$ can be arranged as a premouse over $(\mathscr{V}_1||\gamma, G)$, and note that we can also take G such that there is an $(M, \mathrm{Coll}(\omega, \gamma))$-generic G' such that $(M|\nu)[G'] =^* (\mathscr{V}_1|\nu)[G]$ (with $(M||\gamma, G')$ equivalent intercomputable with $(\mathscr{V}_1||\gamma, G)$). Therefore we can use $(\mathscr{V}_1|\nu)[G]$ to compute short tree strategy for $\mathcal{M}_\infty|\delta_\infty$ in the same manner we use $(M|\nu)[G']$ (working above γ), and by homogeneity, this computation restricted to trees in $\mathscr{V}_1|\nu$ is independent of the choice of G. The computation of maximal trees (above γ) is similarly absolute, and note that the P-constructions determined by these trees also agree between $M|\nu$ and $\mathscr{V}_1|\nu$. The system computed in $\mathscr{V}_1|\nu$ is also easily dense in that of $M|\nu$. Therefore $\mathscr{V}_1|\nu$ computes $\mathcal{M}_\infty^{\mathrm{Ult}(M, E)}|\nu$, as desired.[36] $\qquad\square$

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[36] Note that the foregoing proof did not use Lemmas 4.17 or 4.18, which we are yet to actually

Lemma 4.44. $\mathcal{M}_\infty[*]$ and $\mathscr{V}_1$ (as defined in 4.23 and 4.39) have the same universe.

Proof. We have $\mathscr{V}_1 \subseteq \mathcal{M}_\infty[*]$ by Lemma 4.42 part 1.

Let us show $\mathcal{M}_\infty[*] \subseteq \mathscr{V}_1$. Write $\mathbb{L} = \mathbb{L}_{\theta_0^M}^{\mathscr{V}_1}$. By 4.42 part 3, $\mathbb{L} \in \mathscr{V}_1$, and since $\mathscr{V}_1 \subseteq \mathcal{M}_\infty[*]$, therefore $g = g_{M|\theta_0^M}$ is $\mathbb{L}$-generic over both $\mathcal{M}_\infty[*]$ and $\mathscr{V}_1$. But then as in the proof of Lemma 4.38, $\mathscr{V}_1[g]$ has universe $\lfloor M \rfloor$ (as does $\mathcal{M}_\infty[*][g]$).

Now let $x \in \mathcal{M}_\infty[*]$ be a set of ordinals. We show that $x \in \mathscr{V}_1$. Let τ be an $\mathbb{L}$-name in $\mathscr{V}_1$ such that $x = \tau^g$. Let $p \in g$ be such that $p \Vdash_{\mathcal{M}_\infty[*]}^{\mathbb{L}} \tau = \check{x}$. It is easy to see that then
$$x = \{\xi \colon p \Vdash_{\mathscr{V}_1}^{\mathbb{L}} \check{\xi} \in \tau\} \in \mathscr{V}_1.$$

So the two models have the same universe. $\square$

The preceding fact will be refined later in Lemma 4.47.

Remark 4.45. We may also reorganize $\mathscr{V}_1$ as a strategy premouse, by representing the information contained in the long extenders in $\mathbb{E}^{\mathscr{V}_1}$ differently. These extenders are easily seen to be intertranslatable with fragments of $\Sigma_{\mathcal{M}_\infty}$ for trees based on $\mathcal{M}_\infty|\delta_\infty$. Namely, let us define a sequence $(\mathbb{F}_\nu^{\mathscr{V}_1} \colon \nu \in \mathrm{OR})$ as follows. Except for those ν where $\mathbb{E}_\nu^{\mathscr{V}_1}$ is long, we set $\mathbb{F}_\nu^{\mathscr{V}_1} = \mathbb{E}_\nu^{\mathscr{V}_1}$. If $\mathbb{E}_\nu^{\mathscr{V}_1}$ is long, then $\mathbb{F}_\nu^{\mathscr{V}_1} = \Sigma_{\mathcal{M}_\infty}(\mathcal{T})$, where $\mathcal{T}$ is the normal tree on $\mathcal{M}_\infty$ leading from $\mathcal{M}_\infty|\delta_\infty$ to $i_E^M(\mathcal{M}_\infty|\delta_\infty)$. Then easily, Lemma 4.42 holds also after replacing $\mathbb{E}^{\mathscr{V}_1}$ with $\mathbb{F}^{\mathscr{V}_1}$, and $\mathbb{E}^{\mathscr{V}_1}, \mathbb{F}^{\mathscr{V}_1}$ are level-by-level intertranslatable. So $L[\mathbb{F}^{\mathscr{V}_1}] \cong \mathscr{V}_1$.[37]

Lemma 4.46. *We have:*

(a) *The restriction of $\Sigma_{\mathcal{M}_\infty}$ to trees in $\mathscr{V}_1$ and based on $\mathcal{M}_\infty|\delta_\infty$, is lightface definable over $\mathscr{V}_1$ (so by Lemma 4.47 below, it is also lightface definable over $\mathcal{M}_\infty[*]$).*

(b) *Let g be in some set generic extension of V and be set-generic over $\mathscr{V}_1$. Let $\Sigma'_{\mathcal{M}_\infty}$ be the restriction of $\Sigma_{\mathcal{M}_\infty}$ to trees in $\mathscr{V}_1[g]$ and based on $\mathcal{M}_\infty|\delta_\infty$. Then $\Sigma'_{\mathcal{M}_\infty}$ is definable over the universe of $\mathscr{V}_1[g]$ from the predicate $\mathscr{V}_1$.[38]*

Proof. This is much like the proof of Lemma 4.18 (whose complete proof will rely on Lemma 4.57, still to come), although now we don't have M itself available. The computation of short tree strategy is as in the proof of Lemma 4.42 part 3(e). The computation of branches at maximal stages is like in the proof of Lemma 4.18. $\square$

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prove; it does not matter here whether $\mathcal{M}_\infty^{\mathrm{Ult}(M,\mathbb{E}_\nu^M)}|\nu$ is a correct iterate of $\mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$. But in any case, we could have proved those lemmas at the point they appeared in the text.

[37] However, it is not clear whether $\mathscr{V}_1$ can be arranged as a strategy mouse in one of the more traditional hierarchies, like those used for HOD mice, or the least (tree) branch hierarchy.

[38] Regarding trees $\notin V$, cf. Footnote 26.

To summarize, we have isolated several representations of the universe of $\mathscr{V}_1$, indicating that $\mathscr{V}_1$ is a natural object:

$$
\begin{aligned}
\mathscr{V}_1 \ &\mathrel{\hat{=}}\ L[\mathbb{E}^{\mathscr{V}_1}] \text{ (Definition 4.39)} \\
&\mathrel{\hat{=}}\ L[\mathbb{F}^{\mathscr{V}_1}] \text{ (Remark 4.45)} \\
&\mathrel{\hat{=}}\ \mathcal{M}_\infty[*] \text{ (see (13), Lemma 4.44)} \\
&\mathrel{\hat{=}}\ \mathrm{HOD}_{\mathscr{E}}^{M[G]} \text{ (Lemma 4.28)} \\
&\mathrel{\hat{=}}\ \bigcap \mathscr{F} = \text{the } < \kappa_0 \text{ mantle of } M \text{ (Proposition 4.36)}
\end{aligned}
$$

4.10   Varsovian strategy premice

In this section we will introduce an axiomatization for premice in the hierarchy of $\mathscr{V}_1$. But first, we refine Lemma 4.44 as follows:

Lemma 4.47.

1. $\mathrm{Ult}(\mathscr{V}_1, e^{\mathscr{V}_1}) = \mathscr{V}_1^{\mathcal{M}_\infty}$.

2. $\mathscr{V}_1$ is a lightface class of $\mathcal{M}_\infty[*]$,

3. $\mathcal{M}_\infty[*]$ is a lightface class of $\mathscr{V}_1$.

Proof. Part 2: Argue as in the proof of Lemma 4.42, using again Lemma 4.24.

Part 1: Let $e = e^{\mathscr{V}_1}$. We have $\mathcal{M}_\infty^{\mathcal{M}_\infty} = \mathrm{Ult}(\mathcal{M}_\infty, e)$ and $\pi_\infty : \mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathcal{M}_\infty}$ is the ultrapower map, so $e \subseteq \pi_\infty$. By Lemma 2.10, π_∞ extends elementarily to

$$\pi_\infty^+ : \mathcal{M}_\infty[*] \to \mathcal{M}_\infty^{\mathcal{M}_\infty}[*^{\mathcal{M}_\infty}],$$

so considering how $\mathscr{V}_1$ is defined over $\mathcal{M}_\infty[*]$, and $\mathscr{V}_1^{\mathcal{M}_\infty}$ over $\mathcal{M}_\infty^{\mathcal{M}_\infty}[*^{\mathcal{M}_\infty}]$,

$$\pi_\infty^+ : \mathscr{V}_1 \to \mathscr{V}_1^{\mathcal{M}_\infty}$$

is also elementary. Since $V_{\delta_\infty}^{\mathcal{M}_\infty} = V_{\delta_\infty}^{\mathscr{V}_1}$, it follows that e is also derived from π_∞^+. Let $\mathscr{V}' = \mathrm{Ult}(\mathscr{V}_1, e)$ and $j^+ : \mathscr{V}_1 \to \mathscr{V}'$ be the ultrapower map and $k^+ : \mathscr{V}' \to \mathscr{V}_1^{\mathcal{M}_\infty}$ the factor map, so $k^+ \circ j^+ = \pi_\infty^+$. Since $\pi_\infty \subseteq \pi_\infty^+$ and π_∞ is the ultrapower map, we have $k^+\!\restriction\!\mathrm{OR} = \mathrm{id}$ (and $j^+\!\restriction\!\mathrm{OR} \subseteq \pi_\infty^+$). So in fact $k^+ = \mathrm{id}$ and $\mathscr{V}' = \mathscr{V}_1^{\mathcal{M}_\infty}$.[39]

---

[39]Here is a slightly alternate argument. We have $\mathcal{M}_\infty^{\mathcal{M}_\infty} = \mathrm{Ult}(\mathcal{M}_\infty, e)$ and π_∞ is the ultrapower map. Let $N[e'] = \mathrm{Ult}(\mathcal{M}_\infty[e], e)$ and $j : \mathcal{M}_\infty[e] \to N[e']$ be the ultrapower map. Then as before, in fact $N = \mathcal{M}_\infty^{\mathcal{M}_\infty}$ and $\pi_\infty \subseteq j = \pi_\infty^+$. Moreover, by considering some fixed indiscernibles,

$$e' = \pi_\infty^+(e) = \bigcup_{\alpha < \delta_\infty} \pi_\infty(e\!\restriction\!\alpha)$$

is the correct extender of the iteration map $\mathcal{M}_\infty^{\mathcal{M}_\infty} \to (\mathcal{M}_\infty)^{\mathcal{M}_\infty^{\mathcal{M}_\infty}}$. Now $\mathscr{V}_1$ and $\mathcal{M}_\infty[e]$ have the same universe, and $\mathscr{V}_1$ is defined over $\mathcal{M}_\infty[e]$ via the procedure mentioned for part 2 of the lemma. So $\mathrm{Ult}(\mathscr{V}_1, e)$ is defined over $\mathcal{M}_\infty^{\mathcal{M}_\infty}[e']$ in the same manner. But e' agrees with $*^{\mathcal{M}_\infty}$, so this definition actually yields $\mathscr{V}_1^{\mathcal{M}_\infty}$.

Part 3: It suffices to define $\mathcal{M}_\infty$ and $*\!\restriction\!\delta_0^{\mathcal{M}_\infty}$. But $*\!\restriction\!\delta_0^{\mathcal{M}_\infty}$ is just the active extender of $\mathscr{V}_1||\gamma^{\mathscr{V}_1}$. To define $\mathcal{M}_\infty$, we first have $\mathcal{M}_\infty||\kappa_0^{+\mathcal{M}_\infty} = \mathscr{V}_1|\gamma^{\mathscr{V}_1}$. But $\mathscr{V}_1^{\mathcal{M}_\infty}$ is a lightface class of $\mathscr{V}_1$ by part 1. And $\mathbb{E}^{\mathcal{M}_\infty}\!\restriction\![\kappa_0^{+\mathcal{M}_\infty},\infty)$ is determined by $M||\kappa_0^{+\mathcal{M}_\infty}$ and $\mathbb{E}^{\mathscr{V}_1^{\mathcal{M}_\infty}}\!\restriction\!(\gamma^{\mathscr{V}_1^{\mathcal{M}_\infty}},\infty)$, since the two sequences agree over the ordinals. $\qquad\square$

Definition 4.48. For an M_{swsw}-like N, $\mathcal{M}_\infty^N$, $*^N$, $\mathcal{M}_\infty[*]^N$ and $\mathscr{V}_1^N$ denote the lightface N-classes defined over N just as the corresponding classes are defined over M.

Also given an M_{swsw}-like N and $\bar{N} \trianglelefteq N$ with $\kappa_0^{+N} \leq \mathrm{OR}^{\bar{N}}$, we define $\mathscr{V}_1^{\bar{N}}$ by recursion on $\mathrm{OR}^{\bar{N}}$ by setting $\mathscr{V}_1^{N||(\kappa_0^{+N}+\alpha)} = \mathscr{V}_1^N||(\gamma+\alpha)$, where $\gamma = \gamma^{\mathscr{V}_1^N}$. Noting that this definition is level-by-level, we similarly define $\mathscr{V}_1^{\bar{N}}(\kappa)$ whenever $\bar{N}$ is M_{swsw}-small and κ is an inaccessible limit of cutpoints and Woodins of $\bar{N}$ and $\kappa < \mathrm{OR}^{\bar{N}}$, level-by-level (starting by defining $\mathscr{V}_1^{\bar{N}|\kappa^{+\bar{N}}}$ as $\mathscr{V}_1||\gamma^{\mathscr{V}_1}$ is defined (in the codes) over $M|\kappa_0^{+M}$). We will often suppress the κ from the notation, writing just $\mathscr{V}_1^{\bar{N}}$. $\qquad\dashv$

We now want to axiomatize structures in the hierarchy of $\mathscr{V}_1$ to some extent:

Definition 4.49. A *base Vsp* is an amenable transitive structure $\mathscr{V} = (P_\infty, F)$ such that in some forcing extension there is P such that:

1. P, P_∞ are premice which model ZFC^- and are M_{swsw}-small; that is, they have no active segments which satisfy "There are $\kappa_0 < \delta_1 < \kappa_1$ with δ_1 Woodin and κ_0, κ_1 strong".

2. P has a least Woodin cardinal δ_0^P and a largest cardinal $\kappa_0^P > \delta_0^P$, and κ_0^P is inaccessible in P and a limit of cutpoints of P; likewise for $P_\infty, \delta_0^{P_\infty}, \kappa_0^{P_\infty}$,

3. $\mathrm{OR}^P = \delta_0^{P_\infty}$, κ_0^P is the least measurable of P_∞, and $\mathscr{V}_1^P = \mathrm{cHull}_1^{\mathscr{V}}(\delta_0^{P_\infty})$,

4. $\mathcal{M}_\infty^{P_\infty}$ (defined over P_∞ like $\mathcal{M}_\infty|\gamma^{\mathscr{V}_1}$ is defined over $M|\kappa_0^{+M}$) is well-defined, and has least measurable $\kappa_0^{P_\infty}$ and least Woodin $\delta_0^{\mathcal{M}_\infty^{P_\infty}} = \mathrm{OR}^{P_\infty}$,

5. $\mathcal{M}_\infty^{P_\infty}|\delta_0^{\mathcal{M}_\infty^{P_\infty}}$ is obtained by iterating $P_\infty|\delta_0^{P_\infty}$, via a normal tree $\mathcal{T}$ of length $\delta_0^{\mathcal{M}_\infty^{P_\infty}}$,

6. F is a cofinal Σ_1-elementary (hence fully elementary, by ZFC^-) embedding

$$P_\infty|\delta_0^{P_\infty} \to \mathcal{M}_\infty^{P_\infty}|\delta_0^{\mathcal{M}_\infty^{P_\infty}},$$

and there is a $\mathcal{T}$-cofinal branch b such that $F \subseteq i_b^{\mathcal{T}}$, and $i_b^{\mathcal{T}}(\delta_0^{P_\infty}) = \delta_0^{\mathcal{M}_\infty^{P_\infty}}$ (so b is intercomputable with F, and note that by amenability of $\mathscr{V}$, F is amenable to P_∞, and hence so is b),

7. $\rho_1^{\mathscr{V}} = \delta_0^{P_\infty}$ and $p_1^{\mathscr{V}} = \emptyset$ and $\delta_0^{P_\infty}$ is Woodin in $\mathcal{J}(\mathfrak{C}_1(\mathscr{V}))$, as witnessed by $\mathbb{E}^{P_\infty}$,

8. P is $(\mathcal{J}(\mathfrak{C}_1(\mathscr{V})), \mathbb{L}^{\mathscr{V}})$-generic, where $\mathbb{L}^{\mathscr{V}}$ is defined over $\mathscr{V}$ as $\mathbb{L}$ above was defined over $\mathscr{V}_1 || \gamma^{\mathscr{V}_1}$. $\dashv$

Remark 4.50. Let $C = \mathfrak{C}_1(\mathscr{V})$ and $\pi : C \to \mathscr{V}$ be the core map. Then

$$\mathrm{rg}(\pi) = \mathrm{Hull}_1^{P_\infty}(\delta_0^{P_\infty} \cup F``\delta_0^{P_\infty}). \tag{22}$$

For we have $\rho_1^{\mathscr{V}} = \delta_0^{P_\infty}$ and $p_1^{\mathscr{V}} = \emptyset$ by hypothesis. So $\supseteq$ is clear, and $\subseteq$ is because for each $\alpha < \delta_0^{P_\infty}$, $F \restriction \alpha$ is in the hull on the right, by calculations like with the Zipper Lemma.

Now because $\delta_0^{P_\infty}$ is Woodin in $\mathcal{J}(\mathfrak{C}_1(\mathscr{V}))$, and in particular regular, we have that $\mathfrak{C}_1(\mathscr{V})$ is sound with $\rho_\omega^{\mathfrak{C}_1(\mathscr{V})} = \delta_0^{P_\infty}$, and in particular $V_{\delta_0^{P_\infty}}^{\mathcal{J}(\mathfrak{C}_1(\mathscr{V}))} = V_{\delta_0^{P_\infty}}^{P_\infty}$.

Note also that $\mathbb{L}$ is (lightface) $\Sigma_1^{\mathscr{V}_1 || \gamma^{\mathscr{V}_1}}$, and we use the natural Σ_1 definition above to define $\mathbb{L}^{\mathscr{V}}$, so $\mathbb{L}^{\mathscr{V}} = \mathbb{L}^{\mathfrak{C}_1(\mathscr{V})}$. Moreover, like for $\mathbb{L}$, $\mathbb{L}^{\mathscr{V}}$ is a sub-algebra of the extender algebra of P_∞ at $\delta_0^{P_\infty}$ and $\mathcal{J}(\mathfrak{C}_1(\mathscr{V})) \vDash ``\mathbb{L}^{\mathscr{V}}$ is a $\delta_0^{P_\infty}$-cc complete Boolean algebra''.

The definition is actually specified by an (infinite) first-order theory satisfied by $\mathscr{V}$, modulo the wellfoundedness of $\mathscr{V}$. (The theory needs to be infinite because of the assertion of Woodinness in $\mathcal{J}(\mathfrak{C}_1(\mathscr{V}))$ in condition 7.) To see this, observe that the (generic) existence of P is first-order: Working in $\mathscr{V}$, we can say that there is some condition of $\mathbb{L}^{\mathscr{V}}$ forcing over $\mathfrak{C}_1(\mathscr{V})$ that the generic object is a premouse P such that the conditions above hold (and by the preceding discussion, all relevant antichains are in P_∞); a small subtlety here is that we need to refer to F and the hull in (22) in order to talk about $\mathfrak{C}_1(\mathscr{V})$ and assert that it is isomorphic to $\mathscr{V}^P$; note that we can just talk about the relevant theories to assert this. (We don't demand that $\mathcal{M}_\infty^{P_\infty}$ be wellfounded, but only what was asserted above, which gives that it is wellfounded through $\delta_0^{\mathcal{M}_\infty^{P_\infty}} + 1$.)

Unsound base Vsps naturally arise from iterating sound ones.

Definition 4.51. A *Varsovian strategy premouse (Vsp)* is a structure

$$\mathscr{V} = (\mathcal{J}_\alpha^{\mathbb{E}}, \mathbb{E}, F)$$

for some sequence $\mathbb{E}$ of extenders, where either $\mathscr{V}$ is a premouse, or:

1. $\alpha \leq \mathrm{OR}$ and $\mathscr{V}$ is an amenable acceptable J-structure,

2. $\mathscr{V}$ has a least Woodin cardinal $\delta_0^{\mathscr{V}}$, and an initial segment $\mathscr{V}||\gamma$ which is a base Vsp,

3. $\delta_0^{\mathscr{V}} < \gamma$, so $\delta_0^{\mathscr{V}}$ is the least Woodin of $\mathscr{V}||\gamma$,

4. if $F \neq \emptyset$ and $\gamma < \mathrm{OR}^{\mathscr{V}}$ then either:

   (a) $\mathscr{V}$ satisfies the premouse axioms (for Jensen indexing) with respect to F, and $\gamma < \mathrm{crit}(F)$, or

   (b) We have:

      i. $\mathscr{V}^{\mathrm{pv}} = (\mathcal{J}_\alpha^{\mathbb{E}}, \mathbb{E}, \emptyset) \vDash \mathrm{ZFC}^-$,

     ii. $\mathscr{V}$ has largest cardinal μ, which is inaccessible in $\mathscr{V}$ and a limit of cutpoints of $\mathscr{V}$ (where *cutpoint* is with regard to all kinds of extenders),

    iii. $\mathcal{N} = \mathcal{M}_\infty^{\mathscr{V}^{\mathrm{pv}}}$ is well-defined, and satisfies the axioms of a premouse (but is possibly illfounded) with a Woodin cardinal $\delta_0^{\mathcal{N}}$, and is $(\mathrm{OR}^{\mathscr{V}} + 1)$-wellfounded with $\delta_0^{\mathcal{N}} = \mathrm{OR}^{\mathscr{V}}$,

    iv. $\mathcal{N}|\delta_0^{\mathcal{N}}$ is a proper class of $\mathscr{V}^{\mathrm{pv}}$ and has least measurable μ,

     v. F is a cofinal Σ_1-elementary embedding $F : \mathscr{V}|\delta_0^{\mathscr{V}} \to \mathcal{N}|\delta_0^{\mathcal{N}}$,

    vi. $\mathcal{N}|\delta_0^{\mathcal{N}}$ is pseudo-normal iterate of $\mathscr{V}|\delta_0^{\mathscr{V}}$, via tree $\mathcal{T}$, and there is a $\mathcal{T}$-cofinal branch b such that $F \subseteq i_b^{\mathcal{T}}$ (hence b is amenable to $\mathscr{V}$ and inter-definable with F over $\mathscr{V}^{\mathrm{pv}}$),

5. each proper segment of $\mathscr{V}$ is a sound Vsp (defining *Vsp* recursively), where the fine structural language for base Vsps and segments as in 4b is just that with symbols for $\in, \mathbb{E}, F$,

6. some $p \in \mathbb{L}^{\mathscr{V}} = \mathbb{L}^{\mathscr{V}||\gamma}$ forces that the generic object is a premouse N of height $\delta_0^{\mathscr{V}}$ with $\mathscr{V}^N = \mathscr{V}||\gamma$, and there is an extension of N to a premouse N^+ such that $\mathscr{V}^{N^+} = \mathscr{V}$ (and note then that N^+ is level-by-level definable over $\mathscr{V}[N]$, via inverse P-construction).

We write $\gamma^{\mathscr{V}} = \gamma$ above (if $\mathscr{V}$ is not a premouse).   ⊣

Definition 4.52. A Vsp $\mathscr{V}$ is $\mathscr{V}_1$-*like* iff it is proper class and in some set-generic extension, $\mathscr{V} = \mathscr{V}^N$ for some M_{swsw}-like premouse N. (Note this is first-order over $\mathscr{V}$.)

When talking about the extenders $E \in \mathbb{E}_+^{\mathscr{V}}$, for a Vsp $\mathscr{V}$, we say that E is *short* if $\mathscr{V}||\mathrm{lh}(E)$ satisfies the usual premouse axioms with respect to E, and *long* otherwise; likewise for the corresponding segments. So $\mathscr{V}||\gamma^{\mathscr{V}}$ is the least long segment.

We write $\mathcal{M}_\infty = \mathscr{V}_1 \downarrow 0$. Let $\mathscr{V}$ be $\mathscr{V}_1$-like. We define $\mathscr{V} \downarrow 0$ analogously (first-order over $\mathscr{V}$ as in the proof of Lemma 4.47 part 3). In fact, let us define $\mathscr{V} \downarrow 0$ more generally, including the case that $\mathscr{V}$ is illfounded, but satisfies the first order properties of a $\mathscr{V}_1$-like structure. Also if N is a premouse, let $N \downarrow 0 = N$. We write $\mathscr{V}^-$ for (the premouse) $\mathscr{V}|\delta_0^{\mathscr{V}}$. (So $\mathscr{V}_1^- = \mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$.) We write $\Lambda^{\mathscr{V}}$ for the strategy for $\mathscr{V} \downarrow 0$ (for trees based on $\mathscr{V}^-$) defined over $\mathscr{V}$ just as the corresponding restriction of $\Sigma_{\mathcal{M}_\infty}$ is defined over $\mathscr{V}_1$, via the proof of Lemma 4.46.

We write $\mathscr{V}_1 = \mathscr{V}(\mathcal{M}_\infty, *) = \mathscr{V}(\mathcal{M}_\infty, *{\restriction}\delta_\infty) = \mathscr{V}(\mathcal{M}_\infty, e^{\mathscr{V}_1})$. Given a pair $(N, *')$ or $(N, *'{\restriction}\delta)$ or (N, e) where N is M_{swsw}-like and the pair has similar first-order properties as does $(\mathcal{M}_\infty, *)$ or $(\mathcal{M}_\infty, *{\restriction}\delta_\infty)$ or $(\mathcal{M}_\infty, e^{\mathscr{V}_1})$ respectively, we define $\mathscr{V}(N, *')$ or $\mathscr{V}(N, *'{\restriction}\delta)$ or $\mathscr{V}(N, e)$ analogously (via the proof of Lemma 4.47 part 2).   ⊣

4.11 The action of M-iteration on $\mathcal{M}_\infty$

We now aim to extend Lemma 4.46, analyzing the nature of $\mathcal{M}_\infty^N$ for iterates N of M, and the partial iterability of $\mathcal{M}_\infty^N$ in N.

Lemma 4.53. *Let N be any non-dropping Σ-iterate of M. Let $P \in \mathscr{F}^N$. Let*

$$\bar{P} = \text{cHull}^P(\delta_0^P \cup \mathscr{I}^N).$$

Then $\bar{P}$ is a $\delta_0^{\overline{P}}$-sound non-dropping Σ-iterate of M, $\bar{P}|\delta_0^{\overline{P}} = P|\delta_0^P$, and letting $\pi : \bar{P} \to P$ be the uncollapse map, then $\pi``\mathscr{I}^{\bar{P}} = \mathscr{I}^N$.[40]

Proof. We have $P|\delta_0^P = M_\alpha^{\mathcal{T}}|\delta_0^{M_\alpha^{\mathcal{T}}}$ for some tree $\mathcal{T}$ via Σ, where $[0,\alpha]_{\mathcal{T}}$ does not drop; this is because the Q-structures used to guide the short tree strategy computing P are correct. But $\bar{P}|\delta_0^{\bar{P}} = P|\delta_0^P$ and $\bar{P}$ is an iterable, $\delta_0^{\bar{P}}$-sound, M_{swsw}-like premouse, and with α above minimal, comparison gives $\bar{P} = M_\alpha^{\mathcal{T}}$. Finally note that $\mathscr{I} = \pi^{-1}``\mathscr{I}^N$ is a club class of generating indiscernibles for $\bar{P}$, so $\mathscr{I} = \mathscr{I}^{\bar{P}}$. $\square$

Definition 4.54. Let N be a non-dropping Σ-iterate of M. Define $(\mathcal{M}_\infty^{\overline{\text{ext}}})_N$ as the direct limit of the iterates $\bar{P}$, for $P \in \mathscr{F}^N$ (the notation $\bar{P}$ as in Lemma 4.53). (Cf. Definition 4.20.) For $P \in \mathscr{F}^N$ let $\pi_{\bar{P}P} : \bar{P} \to P$ be the uncollapse map and

$$H^P = \text{Hull}_1^P(\delta_0^P \cup \mathscr{I}^N) = \text{rg}(\pi_{\bar{P}P}).$$

Suppose further that N is κ_0^N-sound. Let $\alpha \in \text{OR}$ and $P \in \mathscr{F}^N$. We say that α is $(P, \mathscr{F}^N)$-*stable* iff whenever $P \preceq Q \in \mathscr{F}^N$, we have $\alpha \in H^Q$ and

$$\pi_{\bar{Q}Q} \circ i_{\bar{P}\bar{Q}} \circ \pi_{\bar{P}P}^{-1}(\alpha) = \alpha. \qquad \dashv$$

The definition of stability above is more complicated than the version for M, because it can be that $P \leq Q \in \mathscr{F}^N$ but Q is not actually an iterate of P (although $Q|\delta_0^Q$ is an iterate of $P|\delta_0^P$).

Lemma 4.55. *Let N be a non-dropping Σ-iterate of M. Then:*

1. *$P = (\mathcal{M}_\infty^{\overline{\text{ext}}})_N$ is a δ_0^P-sound non-dropping Σ-iterate of M.*

2. *$\mathcal{M}_\infty^N|\delta_\infty^N = P|\delta_0^P$.*

3. *If $M|\kappa_0^M \lhd N$ then $\mathcal{M}_\infty|\delta_\infty \in N$ and P is a $\Sigma_{\mathcal{M}_\infty}$-iterate of $\mathcal{M}_\infty$.*

Proof. Part 1 is immediate from the definitions. Part 2 follows from Lemma 4.53, since infinitely many indiscernibles are fixed by the iteration maps. For part 3, note that because $M|\kappa_0^M \lhd N$ and N is a non-dropping iterate, in fact $M|\kappa_0^{+M} \lhd N$, so $\mathcal{M}_\infty|\delta_\infty \in N$, and then it is an easy consequence of part 1. $\square$

Lemma 4.56. *Let N be a κ_0^N-sound non-dropping Σ-iterate of M. Then:*

1. *For each $P \preceq Q \in \mathscr{F}^N$, we have $H^P \cap \text{OR} \subseteq H^Q \cap \text{OR}$.*

---

[40]But if N is not δ_0^N-sound then $\bar{P}$ is not a Σ_N-iterate of N.

2. For each $\alpha \in \mathrm{OR}$ there is $P \in \mathscr{F}^N$ such that α is $(P, \mathscr{F}^N)$-stable.

Proof. Part 1: We in fact that $H^P \cap \mathrm{OR} = \mathrm{Hull}_1^N(\delta_0^P \cup \mathscr{I}^N)$ (which immediately gives $H^P \subseteq H^Q$). This is just by extender algebra genericity and definability of $P|\delta_0^P$ over $N|\delta_0^P$.

Part 2: Since N is κ_0^N-sound, we can fix $s \in \vec{\mathscr{I}}^N$ and $\beta < \kappa_0^N$ and a term t such that $\alpha = t^N(s, \beta)$. Then taking $P \in \mathscr{F}^N$ with $\beta < \delta_0^P$, we get $\alpha \in H^P$. Now since $(\mathcal{M}_\infty^{\overline{\mathrm{ext}}})^N$ is wellfounded, it suffices to see that

$$\pi_{\bar{Q}Q} \circ i_{\bar{P}\bar{Q}} \circ \pi_{\bar{P}P}^{-1}(\alpha) \geq \alpha$$

whenever $P \preceq Q \in \mathscr{F}^N$ and $\alpha \in H^P$. So let $s \in \vec{\mathscr{I}}^N$ with $\alpha \in H_s^P$. Then $\pi_{Ps,Qs}(\alpha) \geq \alpha$ whenever $P \preceq Q \in \mathscr{F}^N$, because M satisfies the same about $i_{MN}^{-1}(s)$ (because whenever $R \preceq S \in \mathscr{F}^M$, S is actually an iterate of R, and these iterates are δ_0-sound, etc). But note that

$$\pi_{Ps,Qs}(\alpha) = \pi_{\bar{Q}Q} \circ i_{\bar{P}\bar{Q}} \circ \pi_{\bar{P}P}^{-1}(\alpha)$$

(because any generic branch witnessing the definition of $\pi_{Ps,Qs}^N$ must move the relevant theory of indiscernibles and elements $< \gamma_s^P$ correctly, since these agree appropriately between $P, \bar{P}$ and $Q, \bar{Q}$). This gives the desired conclusion. $\square$

The following lemma is proved like a similar fact in [15], integrated with part of the argument for [8, Lemma 2.9]:

Lemma 4.57. *Let N be a κ_0^N-sound non-dropping Σ-iterate of M and N' a $\kappa_0^{N'}$-sound non-dropping Σ_N-iterate of N. Let $\bar{N}$ be the δ_0^N-core of N. Then:*

1. $\mathscr{I}^{\mathcal{M}_\infty} = \mathscr{I}^M$ and $i_{M\mathcal{M}_\infty} \upharpoonright \mathscr{I}^M = \mathrm{id} = * \upharpoonright \mathscr{I}^M$,

2. $\mathscr{I}^{\mathcal{M}_\infty^N} = \mathscr{I}^N$ and $*^N \upharpoonright \mathscr{I}^N = \mathrm{id}$,

3. $\mathcal{M}_\infty^N = i_{MN}(\mathcal{M}_\infty) = (\mathcal{M}_\infty^{\overline{\mathrm{ext}}})_N$ is a $\delta_0^{\mathcal{M}_\infty^N}$-sound $\Sigma_{\bar{N}}$-iterate of $\bar{N}$.

4. If $N|\kappa_0^N \lhd N'$ then

    (a) $\mathcal{M}_\infty^{N'}$ is a $\Sigma_{\mathcal{M}_\infty^N}$-iterate of $\mathcal{M}_\infty^N$, and

    (b) $i_{NN'} \upharpoonright \mathcal{M}_\infty^N$ is just the $\Sigma_{\mathcal{M}_\infty^N}$-iteration map $\mathcal{M}_\infty^N \to \mathcal{M}_\infty^{N'}$.

5. If $\bar{N} \neq N$ then $\mathcal{M}_\infty^N$ is not a Σ_N-iterate of N.

Proof. Part 1: We have $\mathscr{I}^{\mathcal{M}_\infty} = i_{M\mathcal{M}_\infty} \text{``} \mathscr{I}^M$, and $\mathcal{M}_\infty$ is $\delta_0^{\mathcal{M}_\infty}$-sound. Suppose $\kappa \in \mathscr{I}^M$ is least such that $i_{M\mathcal{M}_\infty}(\kappa) > \kappa$, fix a tuple $\vec{\kappa} \in \mathscr{I}^{\mathcal{M}_\infty}$ and a term t and $\alpha < \delta_0^{\mathcal{M}_\infty}$ such that $\kappa = t^{\mathcal{M}_\infty}(\vec{\kappa}, \alpha)$, and note we may assume that $\vec{\kappa} \backslash \kappa \in \mathscr{I}^M \backslash (\kappa + 1)$ by shifting this part up, but since $\vec{\kappa} \cap \kappa \subseteq \mathscr{I}^M$ and $\mathcal{M}_\infty$ is a lightface M-class, this gives a contradiction.

Part 3: Note that Lemma 4.55 applies. Like in §2, we will define an elementary $\chi : \mathcal{M}_\infty^N \to (\mathcal{M}_\infty^{\overline{\mathrm{ext}}})_N$ and show that $\chi = \mathrm{id}$. We can cover $\mathscr{D}^N$ by with indices of the form (P, u) with $u \in [\mathscr{I}^N]^{<\omega}$. For given any $(Q, s) \in \mathscr{D}^N$,

by Lemma 4.56, we can fix $P \in \mathscr{F}^N$ such that s is $(P, \mathscr{F}^N)$-stable (see Definition 4.54) and $Q \preceq P$, and with δ_0^P large enough that there is $u \in [\mathscr{I}^N]^{<\omega}$ such that $s \in H_u^P$, which suffices. Because of this covering, we can define $\chi : \mathcal{M}_\infty^N \to (\mathcal{M}_\infty^{\overline{\text{ext}}})_N$ in the natural way; i.e. for each such (P, u) and $x \in H_u^P$, set $\chi(\pi_{Pu,\infty}^N(x)) = i_{\bar{P}\mathcal{M}_\infty^N}(\pi_{\bar{P}P}^{-1}(x))$; note we have $u, x \in \text{rg}(\pi_{\bar{P}P})$. It is now easy to see that $\mathcal{M}_\infty^N = (\mathcal{M}_\infty^{\overline{\text{ext}}})_N$ and $\chi = \text{id}$.

Part 2: By part 1 and since $i_{MN} \text{``} \mathscr{I}^M = \mathscr{I}^N$, we get $*^N {\upharpoonright} \mathscr{I}^N = \text{id}$. And by part 3 and its proof, $\mathcal{M}_\infty^N = \text{Hull}_1^{\mathcal{M}_\infty^N}(\delta_0^{\mathcal{M}_\infty^N} \cup (*^N \text{``} \mathscr{I}^N))$. Since $\mathcal{M}_\infty^N$ is also a lightface N-class, $\mathscr{I}^N$ are model theoretic indiscernibles for $\mathcal{M}_\infty^N$. Therefore $\mathscr{I}^{\mathcal{M}_\infty^N} = \mathscr{I}^N$.

Part 4: 4a is an easy consequence of the fact that $\mathcal{M}_\infty^N$ and $\mathcal{M}_\infty^{N'}$ are δ_∞^N- and $\delta_\infty^{N'}$-sound respectively. For 4b, we argue partly like in [8, Lemma 2.9(a)], but somewhat differently.[41] So, note that by the preceding parts, $\mathcal{M}_\infty^{N'}$ is indeed an iterate of $\mathcal{M}_\infty^N$, and $i_{NN'} {\upharpoonright} \mathscr{I}^{\mathcal{M}_\infty^N} = i_{\mathcal{M}_\infty^N \mathcal{M}_\infty^{N'}} {\upharpoonright} \mathscr{I}^{\mathcal{M}_\infty^N}$, so it just remains to see that

$$i_{NN'} {\upharpoonright} \delta_0^{\mathcal{M}_\infty^N} = i_{\mathcal{M}_\infty^N \mathcal{M}_\infty^{N'}} {\upharpoonright} \delta_0^{\mathcal{M}_\infty^N}.$$

So let $\alpha < \delta_0^{\mathcal{M}_\infty^N}$. Let $s \in [\mathscr{I}^{\mathcal{M}_\infty^N}]^{<\omega} = [\mathscr{I}^N]^{<\omega}$ be such that $\alpha < \gamma_s^{\mathcal{M}_\infty^N}$. Fix $P \in \mathscr{F}^N$ with some $\bar{\alpha} < \gamma_s^P$ such that $\pi_{Ps,\infty}^N(\bar{\alpha}) = \alpha$. Note that $\mathcal{M}_\infty^N$ is a $\Sigma_{\bar{P}}$-iterate of $\bar{P}$ and

$$i_{\bar{P}\mathcal{M}_\infty^N}(\bar{\alpha}) = \alpha. \tag{23}$$

Now

$$i_{NN'}(\alpha) = \pi_{P's',\infty}^{N'}(\bar{\alpha})$$

where $P' = i_{NN'}(P) \in \mathscr{F}^{N'}$ and $s' = i_{NN'}(s) \in \vec{\mathscr{I}}^{N'}$. But then with the map χ defined as earlier, but for N' instead of N,

$$i_{NN'}(\alpha) = \chi(\pi_{P's',\infty}^{N'}(\bar{\alpha})) = i_{\overline{P'}\mathcal{M}_\infty^{N'}}(\pi_{\overline{P'}P'}^{-1}(\bar{\alpha})) = i_{\bar{P}\mathcal{M}_\infty^{N'}}(\bar{\alpha}) = i_{\mathcal{M}_\infty^N \mathcal{M}_\infty^{N'}}(\alpha),$$

using that $\chi = \text{id}$, $\overline{P'} = \bar{P}$, $\bar{\alpha} < \text{crit}(\pi_{\overline{P'}P'})$, and line (23).

Part 5: If $\bar{N} \neq N$ then N is not δ_0^N-sound, but then any non-dropping iterate O of N is non δ_0^O-sound, so $O \neq (\mathcal{M}_\infty^{\overline{\text{ext}}})_N = \mathcal{M}_\infty^N$. $\qquad\square$

Note that with the preceding lemma, we have completed the proofs of Lemmas 4.17, 4.18 and 4.46.

4.12 Iterability of $\mathscr{V}_1$

In this subsection we will define a normal iteration strategy $\Sigma_{\mathscr{V}_1}$ for $\mathscr{V}_1$ in V. We will first define and analyze the action of $\Sigma_{\mathscr{V}_1}$ for trees based on $\mathcal{M}_\infty | \delta_\infty$.

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[41]Moreover, the proof of [8, Lemma 2.9(a)] has a bug: with notation as there, it talks about iteration maps $\pi_{j(N),N^*}$ and $\pi_{N^*,j(\mathcal{M}_\infty)}$, with the implication that $j(\mathcal{M}_\infty)$ is in fact an iterate of $j(N)$, but this is not true, as $j(N)$ is not $\delta_0^{j(N)}$-sound, whereas $j(\mathcal{M}_\infty)$ is $\delta_0^{j(\mathcal{M}_\infty)}$-sound.

4.12.1 Tree translation from M to $\mathscr{V}_1$

The iteration strategy for $\mathscr{V}_1$ will be tightly connected to that for M, as we describe now. But first the basic notion under consideration:

Definition 4.58. Let N be a $\mathscr{V}_1$-like Vsp. A 0-*maximal iteration tree* $\mathcal{T}$ *on* N *of length* $\lambda \geq 1$ is a system

$$\left(<_{\mathcal{T}}, \langle M_\alpha, m_\alpha \rangle_{\alpha < \lambda}, \langle E_\alpha \rangle_{\alpha + 1 < \lambda} \right)$$

with the usual properties, except that when E_α is a long extender (which is allowed), then $\mathrm{pred}^{\mathcal{T}}(\alpha + 1)$ is the least $\beta \leq \alpha$ such that $[0, \beta]_{\mathcal{T}}$ does not drop and $\delta_0^{M_\beta^{\mathcal{T}}} < \mathrm{lh}(E_\alpha^{\mathcal{T}})$.

We say that $\mathcal{T}$ is *short-normal* iff $\mathcal{T}$ uses no long extenders.

Iteration strategies and *iterability* for N are now defined in the obvious manner. $\dashv$

Definition 4.59. A *short-normal tree* on a $\mathscr{V}_1$-like Vsp $\mathscr{V}$ is a 0-maximal tree that uses no long extenders. Note that a short-normal tree is of the form $\mathcal{T} ^{\frown} \mathcal{S}$, where $\mathcal{T}$ is based on $\mathscr{V} | \delta_0^{\mathscr{V}}$, and either

(i) [$\mathcal{T}$ has limit length or $b^{\mathcal{T}}$ drops] and $\mathcal{S} = \emptyset$, or

(ii) $\mathcal{T}$ has successor length, $b^{\mathcal{T}}$ does not drop and $\mathcal{S}$ is above $\delta_0^{M_\infty^{\mathcal{T}_0}}$.

Say that $\mathcal{T}, \mathcal{S}$ are the *lower, upper components* respectively. $\dashv$

Definition 4.60. Let N be M_{swsw}-like. An iteration tree $\mathcal{T}$ on $\mathscr{V}_1^N$ is $\mathscr{V}_1^N$-*translatable* iff:

1. $\mathcal{T}$ is 0-maximal, and

2. $\kappa_0^{+M_\alpha^{\mathcal{T}}} < \mathrm{lh}(E_\alpha^{\mathcal{T}})$ for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ such that $[0, \alpha]_{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} = \emptyset$. $\dashv$

Remark 4.61. Under 0-maximality, condition 2 holds iff $\kappa_0^{+M_\eta^{\mathcal{T}}} < \mathrm{lh}(E_\eta^{\mathcal{T}})$ for $\eta = 0$ and for all limits η such that $\eta + 1 < \mathrm{lh}(\mathcal{T})$ and $[0, \eta]_{\mathcal{T}} \cap \mathscr{D}^{\mathcal{T}} = \emptyset$ and $i_{0\eta}^{\mathcal{T}}(\kappa_0^N) = \delta(\mathcal{T} \restriction \eta)$. This follows easily from the fact that $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \mathrm{lh}(E_\beta^{\mathcal{T}})$ for $\alpha < \beta$ (using Jensen indexing).

Definition 4.62. Let N be M_{swsw}-like. Let $\mathcal{T}$ on N be $\mathscr{V}_1^N$-translatable. The $\mathscr{V}_1^N$-*translation* of $\mathcal{T}$ is the 0-maximal tree $\mathcal{U}$ on $\mathscr{V}_1^N$ such that:

1. $\mathrm{lh}(\mathcal{U}) = \mathrm{lh}(\mathcal{T})$ and $\mathcal{U}, \mathcal{T}$ have the same tree, drop and degree structure,

2. $\mathrm{lh}(E_\alpha^{\mathcal{U}}) = \mathrm{lh}(E_\alpha^{\mathcal{T}})$ for all $\alpha + 1 < \mathrm{lh}(\mathcal{T})$. $\dashv$

Remark 4.63. Let N be M_{swsw}-like. Let $\mathcal{T}$ be a tree on N and let $\alpha <_{\mathcal{T}} \varepsilon + 1 \leq_{\mathcal{T}} \beta$ be such that $[0, \alpha]_{\mathcal{T}}$ does not drop, $\varepsilon + 1 \in \mathscr{D}^{\mathcal{T}}$, $\mathrm{pred}^{\mathcal{T}}(\varepsilon + 1) = \alpha$ and $\kappa_0^{+M_\alpha^{\mathcal{T}}} < \mathrm{crit}(E_\varepsilon^{\mathcal{T}})$. Note that

$$\kappa_0^{+M_\alpha^{\mathcal{T}}} < \gamma = \gamma^{\mathscr{V}_1(M_\alpha^{\mathcal{T}})} < \xi < \mathrm{crit}(E_\varepsilon^{\mathcal{T}})$$

where ξ is the least $M_\alpha^{\mathcal{T}}|\kappa_0^{+M_\alpha^{\mathcal{T}}}$-admissible. Note here that $\mathscr{V}_1^{M_\beta^{\mathcal{T}}} = \mathscr{V}_1(M_\beta^{\mathcal{T}})$ is the set-sized model $\mathscr{V}$ such that $\mathscr{V}||\gamma = \mathscr{V}_1^{M_\alpha^{\mathcal{T}}}||\gamma$ and above γ, $\mathbb{E}_+^{\mathscr{V}}$ is the level-by-level translation of $\mathbb{E}_+(M_\beta^{\mathcal{T}})$. Note that because $\gamma < \mathrm{OR}^{\mathscr{V}_1(M_\beta^{\mathcal{T}})}$, $M_\beta^{\mathcal{T}}$ is a $\kappa_0^{+M_\alpha^{\mathcal{T}}}$-cc forcing extension of $\mathscr{V}_1(M_\beta^{\mathcal{T}})$.

Lemma 4.64. *Let $\mathcal{T}$ on N be $\mathscr{V}_1^N$-translatable, where N is M_{swsw}-like. Then:*

1. *The $\mathscr{V}_1^N$-translation $\mathcal{U}$ of $\mathcal{T}$ exists and is unique.*

2. $M_\alpha^{\mathcal{U}} = \mathscr{V}_1^{M_\alpha^{\mathcal{T}}}$ *and* $\deg_\alpha^{\mathcal{U}} = \deg_\alpha^{\mathcal{T}}$ *and* $\gamma^{M_\alpha^{\mathcal{U}}} < \mathrm{OR}(M_\alpha^{\mathcal{U}})$ *for all* $\alpha < \mathrm{lh}(\mathcal{T})$.

3. $i_{\alpha\beta}^{\mathcal{U}} = i_{\alpha\beta}^{\mathcal{T}} \!\restriction\! M_\alpha^{\mathcal{U}}$ *for all* $\alpha <_{\mathcal{T}} \beta$ *such that* $(\alpha, \beta]_{\mathcal{T}}$ *does not drop.*

4. $M_{\alpha+1}^{*\mathcal{U}} = \mathscr{V}_1^{M_{\alpha+1}^{*\mathcal{T}}}$ *for all* $\alpha + 1 < \mathrm{lh}(\mathcal{T})$.

5. $i_{\alpha+1}^{*\mathcal{U}} = i_{\alpha+1}^{*\mathcal{T}} \!\restriction\! M_{\alpha+1}^{*\mathcal{U}}$ *for all* $\alpha + 1 < \mathrm{lh}(\mathcal{T})$.

Proof. This is partly via the usual translation of iteration trees between models and P-constructions thereof. However, there is a new feature here, when $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ and is such that $[0, \alpha + 1]_{\mathcal{T}}$ does not drop and letting $\beta = \mathrm{pred}^{\mathcal{T}}(\alpha + 1)$, then $\mathrm{crit}(E_\alpha^{\mathcal{T}}) = \kappa = \kappa_0(M_\beta^{\mathcal{T}})$, so consider this situation.

Then $E_\alpha^{\mathcal{U}}$ is long with space $\delta = \delta_0(M_\beta^{\mathcal{U}}) = \kappa_0^{+M_\beta^{\mathcal{T}}}$, and $E_\alpha^{\mathcal{U}} = E_\alpha^{\mathcal{T}} \!\restriction\! (M_\beta^{\mathcal{U}}|\delta)$ and $[0, \beta]_{\mathcal{U}}$ does not drop, and $M_\beta^{\mathcal{U}} = \mathscr{V}_1(M_\beta^{\mathcal{T}})$. Let (a, f) be such that $f \in M_\beta^{\mathcal{T}}$ and $a \in [\nu(E_\alpha^{\mathcal{T}})]^{<\omega}$ and

$$f : [\kappa]^{|a|} \to M_\beta^{\mathcal{U}} = \mathscr{V}_1(M_\beta^{\mathcal{T}}).$$

We need some (b, g) with $g \in M_\beta^{\mathcal{U}}$ and $b \in [\mathrm{lh}(E_\alpha^{\mathcal{U}})]^{<\omega}$ such that $a \subseteq b$ and $f^{ab}(u) = g(u)$ for $(E_\alpha^{\mathcal{T}})_b$-measure one many u. We may assume $\mathrm{rg}(f) \subseteq \mathrm{OR}$.

If $\mathrm{rg}(f) \subseteq \delta$, the existence of (b, g) is just because $E_\alpha^{\mathcal{U}}$ is the restriction of $E_\alpha^{\mathcal{T}}$, and this restriction is cofinal in $\mathrm{lh}(E_\alpha^{\mathcal{T}})$. In general we will reduce to this case.

Now $M_\beta^{\mathcal{T}}$ is a δ-cc forcing extension of $M_\beta^{\mathcal{U}}$, so $\mathrm{rg}(f) \subseteq X$ for some $X \in M_\beta^{\mathcal{U}}$, where X has cardinality $< \delta$ in $M_\beta^{\mathcal{U}}$. Let η be the ordertype of X, so $\eta < \delta$, let $\pi : X \to \eta$ be the collapse, and let $\widetilde{f} = \pi \circ f$. So $\widetilde{f} \in M_\beta^{\mathcal{T}}$ and $\mathrm{rg}(\widetilde{f}) \subseteq \delta$, so we get a corresponding pair $(b, \widetilde{g})$, with $\widetilde{g} \in M_\beta^{\mathcal{U}}$. Letting $g = \pi^{-1} \circ \widetilde{g}$, then $g \in M_\beta^{\mathcal{U}}$ and (b, g) works. $\square$

4.12.2 Trees based on $\mathcal{M}_\infty|\delta_\infty$

We now transfer trees on $\mathcal{M}_\infty$, based on $\mathcal{M}_\infty|\delta_\infty$, to trees on $\mathscr{V}_1$.

Definition 4.65. Write $\Sigma_{\mathcal{M}_\infty, \mathscr{V}_1^-}$ for the normal strategy for $\mathcal{M}_\infty$ for trees based on $\mathscr{V}_1^-$, induced by $\Sigma_{\mathcal{M}_\infty}$. We use analogous notation $\Sigma_{N, N|\alpha}$ more generally. Let $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$ denote the putative normal strategy for trees on $\mathscr{V}_1$ based on $\mathscr{V}_1^-$, induced by $\Sigma_{\mathcal{M}_\infty, \mathscr{V}_1^-}$. This makes sense by Lemma 4.30. $\dashv$

Remark 4.66. Let $\mathcal{U}$ be a putative tree on $\mathcal{V}_1$, based on $\mathcal{V}_1^-$, via $\Psi_{\mathcal{V}_1,\mathcal{V}_1^-}$. Let $\alpha < \mathrm{lh}(\mathcal{U})$. Suppose $[0,\alpha]_{\mathcal{U}}$ does not drop. Then $M_\alpha^{\mathcal{U}} \downarrow 0 = i_{0\alpha}^{\mathcal{U}}(\mathcal{M}_\infty)$, and if $M_\alpha^{\mathcal{U}}$ is wellfounded then it is $\mathcal{V}_1$-like. If instead $[0,\alpha]_{\mathcal{U}}$ drops, note that it drops below the image of $\delta_0^{\mathcal{V}_1}$ and $M_\alpha^{\mathcal{U}}$ is a premouse (note that it is wellfounded in this case), so $M_\alpha^{\mathcal{U}} \downarrow 0 = M_\alpha^{\mathcal{U}}$.

Definition 4.67. Let $\mathcal{V}$ be a Vsp. Then $\Lambda^{\mathcal{V}}$ denotes the partial putative strategy for $\mathcal{V}|\delta_0^{\mathcal{V}}$ determined by the long extenders of $\mathcal{V}$. That is, $\Lambda^{\mathcal{V}}(\mathcal{T}) = b$ iff $\mathcal{T} \in \mathcal{V}$, $\mathcal{T}$ is on $\mathcal{V}|\delta_0^{\mathcal{V}}$, $\mathcal{V} \vDash$ "$\mathcal{T}$ is via Σ_{sh}", and either

- $\mathcal{V} \vDash$ "$\mathcal{T}$ is short and $b = \Sigma_{\mathrm{sh}}(\mathcal{T})$", or

- $\mathcal{V} \vDash$ "$\mathcal{T}$ is maximal" and there is a long $E \in \mathbb{E}_+(\mathcal{V})$ such that $\mathcal{T} \in \mathcal{V}|\lambda(E)$ and b is computed via factoring through c as in Footnote 27, where c is the cofinal branch through the tree from $\mathcal{V}|\delta_0^{\mathcal{V}}$ to $j(\mathcal{V}|\delta_0^{\mathcal{V}})$ determined by $j = i_E^{\mathcal{V}|\mathrm{lh}(E)}$. $\dashv$

The following lemma, which is the main point of this subsubsection, is the analogue of [8, Lemma 2.17] and [19, Claim 12].

Lemma 4.68. $\Psi_{\mathcal{V}_1,\mathcal{V}_1^-}$ *yields wellfounded models. Moreover, let* $\mathcal{T}$ *be on* $\mathcal{M}_\infty$, *via* $\Sigma_{\mathcal{M}_\infty,\mathcal{V}_1^-}$, *and let* $\mathcal{U}$ *be the corresponding tree on* $\mathcal{V}_1$ *(so via* $\Psi_{\mathcal{V}_1,\mathcal{V}_1^-}$*). Let*

$$\pi_\alpha : M_\alpha^{\mathcal{T}} \to M_\alpha^{\mathcal{U}} \downarrow 0 \subseteq M_\alpha^{\mathcal{U}}$$

be the natural copy map (where $\pi_0 = \mathrm{id}$*). Then:*

- $[0,\alpha]_{\mathcal{T}}$ *drops iff* $[0,\alpha]_{\mathcal{U}}$ *drops.*

- *If* $[0,\alpha]_{\mathcal{T}}$ *drops then* $M_\alpha^{\mathcal{T}} = M_\alpha^{\mathcal{U}} = M_\alpha^{\mathcal{U}} \downarrow 0$.

- *If* $[0,\alpha]_{\mathcal{T}}$ *does not drop then* $M_\alpha^{\mathcal{T}} = M_\alpha^{\mathcal{U}} \downarrow 0$ *and* $M_\alpha^{\mathcal{U}} = \mathcal{V}(M_\alpha^{\mathcal{T}}, \ell)$ *where* $\ell : M_\alpha^{\mathcal{T}} \to \mathcal{M}_\infty^{M_\alpha^{\mathcal{T}}}$ *is the* $\Sigma_{M_\alpha^{\mathcal{T}}}$*-iteration map, and in fact,* $\Lambda^{M_\alpha^{\mathcal{U}}} \subseteq \Sigma_{N,N|\delta_0^N}$ *where* $N = M_\alpha^{\mathcal{T}}$.

- $\pi_\alpha = \mathrm{id}$; *therefore,* $i_\alpha^{\mathcal{T}} \subseteq i_\alpha^{\mathcal{U}}$.

Proof. We include the proof, mostly following that of [19]. Let $\mathcal{T}, \mathcal{U}$ be as above, of length $\alpha + 1$. The interesting case is the non-dropping one, so consider this.

Let $P = M_\alpha^{\mathcal{T}}$ and $P_\infty = \mathcal{M}_\infty^P$. Let $i_{\mathcal{M}_\infty P}$, i_{PP_∞} and $i_{\mathcal{M}_\infty P_\infty}$ be the iteration maps. We have

$$\mathcal{M}_\infty = \mathrm{Hull}_1^{\mathcal{M}_\infty}(\delta_0^{\mathcal{M}_\infty} \cup \mathscr{I}^{\mathcal{M}_\infty});$$

likewise for P, P_∞ (maybe $\mathscr{I}^P \neq \mathscr{I}^{\mathcal{M}_\infty}$, but $\mathscr{I}^{P_\infty} = \mathscr{I}^P = i_{\mathcal{M}_\infty P}$"$\mathscr{I}^{\mathcal{M}_\infty}$). We have

$$P_\infty = \mathcal{V}_1^P \downarrow 0 \subseteq \mathcal{V}_1^P.$$

The analogue of the following claim was used in the proof of [19, Claim 12], where it was implicitly asserted but the proof not explicitly given. We give the proof here. It is just a slight generalization of the proofs of [19, Claims 8–10]

(or see [8, Lemma 2.15]), the main conclusion of which is that if $j : M \to \mathcal{M}_\infty$ is the iteration map, then

$$\mathcal{M}_\infty \cap \text{Hull}^{\mathcal{M}_\infty[*]}(\text{rg}(j)) = \text{Hull}^{\mathcal{M}_\infty}(\text{rg}(j)) = \text{rg}(j).$$

CLAIM. We have:

(i) $P_\infty \cap \text{Hull}^{P_\infty[*^P]}(\text{rg}(i_{\mathcal{M}_\infty P_\infty})) = \text{rg}(i_{\mathcal{M}_\infty P_\infty})$ and

(ii) $P_\infty \cap \text{Hull}^{P_\infty[*^P]}(\text{rg}(i_{PP_\infty})) = \text{rg}(i_{PP_\infty})$.

Proof. Consider (i). Fix $\alpha \in \text{OR}$ and $s \in [\mathscr{I}^{\mathcal{M}_\infty}]^{<\omega} \backslash \{\emptyset\}$ and $\beta < \delta_0^{\mathcal{M}_\infty}$ and a term t such that

$$\alpha = t^{P_\infty[*^P]}(i_{\mathcal{M}_\infty P_\infty}(s^-, \beta))$$

and $\beta < \gamma_s^{\mathcal{M}_\infty}$. We need to see $\alpha \in \text{rg}(i_{\mathcal{M}_\infty P_\infty})$. It suffices to see $*^P(\alpha) \in \text{rg}(i_{\mathcal{M}_\infty P_\infty})$, by arguments in [19]. But this holds just as in [19], except that we have a fixed term u such that for each $N \in \mathscr{F}^P$,

$$\alpha = u^N(i_{\mathcal{M}_\infty N}(s, \beta)).$$

This suffices. Part (ii) is analogous. $\qquad\square$

Let $\widetilde{P} = \text{cHull}^{P_\infty[*^P]}(\text{rg}(i_{PP_\infty}))$ and $\widetilde{\mathcal{M}_\infty} = \text{cHull}^{P_\infty[*^P]}(\text{rg}(i_{\mathcal{M}_\infty P_\infty}))$, let $i_{PP_\infty}^+$ and $i_{\mathcal{M}_\infty P_\infty}^+$ the uncollapse maps, and $i_{\mathcal{M}_\infty P}^+ = (i_{PP_\infty}^+)^{-1} \circ i_{MP_\infty}^+$. By the claim, $i_{\mathcal{M}_\infty P} \subseteq i_{\mathcal{M}_\infty P}^+$.

Also, $*^P \subseteq h$ where $h : P_\infty \to \mathcal{M}_\infty^{P_\infty}$ is the iteration map. Letting

$$i_{\mathcal{M}_\infty P_\infty}^+(*') = i_{PP_\infty}^+(*'') = *^P{\restriction}\delta_0^{P_\infty},$$

it easily follows that $*'$ and $*''$ agree with the iteration maps $\mathcal{M}_\infty \to (\mathcal{M}_\infty)^{\mathcal{M}_\infty}$ and $P \to P_\infty$ respectively. Therefore $\widetilde{\mathcal{M}_\infty} = \mathcal{M}_\infty[*]$.

Let E be the $(\delta_0^{\mathcal{M}_\infty}, \delta_0^P)$-extender derived from $i_{\mathcal{M}_\infty P}$, or equivalently from $i_{\mathcal{M}_\infty P}^+$, also equivalently the $[0, \alpha]_{\mathcal{T}}$-branch extender of $\mathcal{T}$. So (recalling $\mathcal{U}$ is the corresponding tree on $\mathcal{M}_\infty[*]$)

$$M_\alpha^{\mathcal{U}} = \text{Ult}(\mathcal{M}_\infty[*], E)$$

and $i_\alpha^{\mathcal{U}} = i_E^{\mathcal{M}_\infty[*]}$. We also have $P = M_\alpha^{\mathcal{T}} = \text{Ult}(\mathcal{M}_\infty, E)$ and $i_{\mathcal{M}_\infty P} = i_\alpha^{\mathcal{T}}$. Let $\pi : M_\alpha^{\mathcal{U}} \to \widetilde{P}$ be the natural factor map, i.e.

$$\pi(i_\alpha^{\mathcal{U}}(f)(a)) = i_{\mathcal{M}_\infty P}^+(f)(a)$$

whenever $f \in \mathcal{M}_\infty[*]$ and $a \in [\delta_0^P]^{<\omega}$. Then π is surjective, because if $\alpha \in \text{OR}$ then there is $f \in \mathcal{M}_\infty$ and $a \in [\delta_0^P]^{<\omega}$ such that $i_{\mathcal{M}_\infty P}(f)(a) = \alpha$, and since $i_{\mathcal{M}_\infty P} \subseteq i_{\mathcal{M}_\infty P}^+$, therefore $i_{\mathcal{M}_\infty P}^+(f)(a) = \alpha = \pi(\alpha)$. So in fact $M_\alpha^{\mathcal{U}} = \widetilde{P}$ and $\pi = \text{id}$, so $i_\alpha^{\mathcal{U}} = i_{\mathcal{M}_\infty P}^+$, so $i_\alpha^{\mathcal{T}} \subseteq i_\alpha^{\mathcal{U}}$, and letting $\pi_\alpha : M_\alpha^{\mathcal{T}} \to (M_\alpha^{\mathcal{U}} \downarrow 0)$ be the natural copy map, then $\pi_\alpha \subseteq \pi = \text{id}$.

It just remains to see that $\Lambda^{M_\alpha^{\mathcal{U}}} \subseteq \Sigma_P$ (still with $P = M_\alpha^{\mathcal{T}}$). First consider the case that for some correct normal above-κ_0 tree $\mathcal{V}$ on M and $E = E_\alpha^{\mathcal{V}}$, we have $\mathrm{crit}(E) = \kappa_0$ and E is M-total, and $P = \mathcal{M}_\infty^U$ where $U = \mathrm{Ult}(M, E)$. Here by Lemma 4.57, $\mathcal{M}_\infty^U$ is indeed a $\delta_0^{\mathcal{M}_\infty^U}$-sound iterate of $\mathcal{M}_\infty$, and $i_E {\restriction} \mathcal{M}_\infty$ is just the iteration map. Moreover, by Lemma 4.64, $\mathcal{V}_1^U = i_E(\mathcal{V}_1)$ is the corresponding iterate of $\mathcal{V}_1$. But now the calculations that work for $\Lambda^{\mathcal{V}_1}$ (the proof of Lemma 4.46, using Lemma 4.57) also work for $\Lambda^{\mathcal{V}_1^U}$.

Now consider the general case. We will reduce this to the special case above via Lemma 3.10. Let $E \in \mathbb{E}^M$ be M-total with $\mathrm{crit}(F) = \kappa_0$, and $\bar{\delta}$ the least Woodin of $M | \mathrm{lh}(F)$ such that $\kappa_0 < \bar{\delta}$. Form a genericity iteration at $\bar{\delta}$, above κ_0, making $P | \delta_0^P$ etc generic. Let E be the eventual image of F. Then E is as in the previous case; let $U = \mathrm{Ult}(M, E)$ and let $\mathcal{M}_\infty^U$. Recall $\widetilde{P} = M_\alpha^{\mathcal{U}}$ is an iterate of $\mathcal{V}_1$, and note $\mathcal{V}_1^U$ is the corresponding iterate of $\widetilde{P}$; let $k : \widetilde{P} \to \mathcal{V}_1$ be the iteration map. Let β be such that $\widetilde{P} | \beta$ is active with a long extender G, $\beta' = k(\beta)$, and $G' = F^{\mathcal{V}_1^U | \beta'}$. Note that k is continuous at δ_0^P and β (as $\mathrm{cof}^P(\beta) = \delta_0^P$). Let $j : P \to \mathrm{Ult}(P, G)$ and $j' : \mathcal{M}_\infty^U \to \mathrm{Ult}(\mathcal{M}_\infty^U, G')$ be the ultrapower maps. Let $\mathcal{T}$ be the length β tree from P to $\mathrm{Ult}(P, G)$ and $\mathcal{T}'$ the length β' tree from $\mathcal{M}_\infty^U$ to $\mathrm{Ult}(\mathcal{M}_\infty^U, G')$; note that by first order considerations, these exist, and G determines a $\mathcal{T}$-cofinal branch b such that $M_b^{\mathcal{T}} = \mathrm{Ult}(P, G)$ and $j = i_b^{\mathcal{T}}$, and likewise for $G', \mathcal{T}', b', \mathcal{M}_\infty^U, j'$. We know that $\mathcal{T}' {}^\frown b'$ is via $\Sigma_{\mathcal{M}_\infty^U}$, by the previous case. Note also that $\mathcal{T}$ is via Σ_P, because the Q-structure for each $\mathcal{T} {\restriction} \lambda$ (for limit $\lambda < \beta$) does not overlap $\delta(\mathcal{T} {\restriction} \lambda)$, and is embedded into the Q-structure for $\mathcal{T}' {\restriction} k(\lambda)$. But

$$k \circ j {\restriction} (P | \delta_0^P) = j' \circ k {\restriction} (P | \delta_0^P)$$

and $j(\delta_0^P) = \beta$ and $j'(\delta_\infty^U) = \beta'$. So by Lemma 3.10, $b = \Sigma_P(\mathcal{T})$, as desired. $\quad\square$

Definition 4.69. Given a non-dropping $\Psi_{\mathcal{V}_1, \mathcal{V}_1^-}$-iterate $\mathcal{V}$ of $\mathcal{V}_1$, let $\Psi_{\mathcal{V}, \mathcal{V}^-}$ be induced by $\Sigma_{\mathcal{V} \downarrow 0}$ just as $\Psi_{\mathcal{V}_1, \mathcal{V}_1^-}$ is induced by $\Sigma_{\mathcal{M}_\infty}$ (this makes sense by Lemma 4.68). $\quad\dashv$

4.12.3   Condensation properties for full normalization

The strategy $\Sigma_{\mathcal{V}_1}$ (together with $\mathcal{V}_1$) will have the properties required for extending to a strategy for stacks with full normalization. We now lay out the properties of $\mathcal{V}_1$ needed for this. Recall the notions n-standard, $(n+1)$-relevantly condensing and $(n + 1)$-sub-condensing from [14, ***Definition 2.1]. We adapt these in an obvious manner to Vsps.

Definition 4.70. Let $m < \omega$ and let $\mathcal{V}$ be an $(m + 1)$-sound Vsp. We say that $\mathcal{V}$ is $(m + 1)$-relevantly condensing iff either $\mathcal{V}$ is a premouse which is $(m + 1)$-relevantly condensing, or $\mathcal{V}$ is a sound base Vsp, or $\gamma^{\mathcal{V}} < \mathrm{OR}^{\mathcal{V}}$ and $\mathcal{V}$ satisfies the requirements of $(m + 1)$-relevantly condensing from [14, Definition 2.1] for $\pi : P \to \mathcal{V}$ such that P is an $(m + 1)$-sound Vsp, $\gamma^P < \mathrm{OR}^P$ and $\mathrm{crit}(\pi) > \delta_0^{\mathcal{V}}$ (so $\mathrm{crit}(\pi) > \gamma^{\mathcal{V}}$). Likewise for $(m + 1)$-sub-condensing.

For $n < \omega$, a Vsp $\mathscr{V}$ is *n-standard* iff $\mathscr{V}$ is n-sound and either $\mathscr{V}$ is an n-standard premouse, or $\mathscr{V}$ is a base Vsp and $\mathscr{V}^{\mathrm{pv}}$ is ω-standard, or $\gamma^{\mathscr{V}} < \mathrm{OR}^{\mathscr{V}}$ and $\mathscr{V}$ is $(m+1)$-relevantly condensing for each $m < n$, and every $M \lhd \mathscr{V}$ is $(m+1)$-relevantly-condensing and $(m+1)$-sub-condensing for each $m < \omega$.

A Vsp is *ω-standard* iff n-standard for each $n < \omega$. ⊣

Lemma 4.71. *$\mathscr{V}_1$ is ω-standard. (Thus, we take $\mathscr{V}_1$-like to include ω-standard.)*

Proof. Let $\alpha > \gamma^{\mathscr{V}}$ and P be a Vsp and $\pi : P \to \mathscr{V}|\alpha$ be an embedding as in the definition of $(m+1)$-relevantly- or $(m+1)$-sub-condensing. We want to know that $P \lhd \mathscr{V}$. But note that there is a premouse N such that $M|\kappa_0^{+M} \lhd N$ and $\mathscr{V}^N = P$ and π extends to $\pi^+ : N \to M|\alpha$, which also satisfies the conditions of $(m+1)$-relevantly- or $(m+1)$-sub-condensing, respectively. So $N \lhd M$, so $P \lhd \mathscr{V}$. □

Remark 4.72. Let $\mathscr{V}$ be $\mathscr{V}_1$-like. Then as for premice, if $\mathcal{T}$ is a 0-maximal tree on $\mathscr{V}$ then $M_\alpha^{\mathcal{T}}$ is $\deg_\alpha^{\mathcal{T}}$-standard (see [14, ***Remark 2.2]).

4.12.4 Short-normal trees on $\mathscr{V}_1$

Recall that *short-normal* trees on $\mathscr{V}_1$-like Vsps were defined in Definition 4.59.

Definition 4.73. Let $\mathscr{V}$ be a (possibly dropping, putative) iterate of $\mathscr{V}_1$, via a short-normal tree $\mathcal{T} ^\frown \mathcal{S}$ with lower and upper components $\mathcal{T}, \mathcal{S}$. We say that $\mathscr{V}$ is *good* iff $\mathcal{T}$ is via $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$ and if $b^{\mathcal{T}}$ does not drop then $\mathscr{V}$ is wellfounded and for every long $E \in \mathbb{E}_+^{\mathscr{V}}$, $\mathcal{M}_\infty^{\mathscr{V}|\mathrm{lh}(E)} = N|\delta_0^N$ for some $\Sigma_{\mathscr{V}\downarrow 0}$-iterate N of $\mathscr{V} \downarrow 0$, and E is the corresponding iteration map. Say that a (partial) iteration strategy Ψ is *good* iff all putative iterates via Ψ are good. ⊣

Note we have already shown that $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$ is good. We now want to extend $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$ to a good short-normal 0-maximal strategy Ψ_{sn} for $\mathscr{V}_1$. So we start by setting $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-} \subseteq \Psi_{\mathrm{sn}}$. As an easy next step, we deal with trees based on $\mathscr{V}_1 || \gamma^{\mathscr{V}_1}$.

Definition 4.74. Write $\Psi_{\mathscr{V}_1, \gamma^{\mathscr{V}_1}}$ for the putative strategy Ψ for $\mathscr{V}_1$, for short-normal 0-maximal trees based on $\mathscr{V}_1 || \gamma^{\mathscr{V}_1}$, as follows:

1. $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-} \subseteq \Psi$, and

2. given $\mathcal{T}$ via $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$, of successor length $\alpha + 1$, where $[0, \alpha]_{\mathcal{T}}$ does not drop, and given a putative 0-maximal tree $\mathcal{U}$ on $M_\alpha^{\mathcal{T}} || \gamma^{M_\alpha^{\mathcal{T}}}$, which is above $\delta_0^{M_\alpha^{\mathcal{T}}}$, then $\mathcal{T} ^\frown \mathcal{U}$ is via Ψ iff there is a tree $\mathcal{U}'$ on $M_\alpha^{\mathcal{T}} \downarrow 0$, via $\Sigma_{M_\alpha^{\mathcal{T}}\downarrow 0}$, with the same extenders and tree order as $\mathcal{U}$. ⊣

Note here that by Lemma 4.41, $\rho_1^{M_\alpha^{\mathcal{T}} || \gamma^{M_\alpha^{\mathcal{T}}}} = \delta_0^{M_\alpha^{\mathcal{T}}}$, a strong cutpoint of $(M_\alpha^{\mathcal{T}} || \gamma^{M_\alpha^{\mathcal{T}}})^{\mathrm{pv}}$, so $\mathcal{T} ^\frown \mathcal{U}$ is indeed a putative 0-maximal tree on $\mathscr{V}_1$.

Lemma 4.75. $\Psi_{\mathscr{V}_1,\gamma^{\mathscr{V}_1}}$ *is a short-normal 0-maximal strategy (hence yields well-founded models). Moreover, let $\mathcal{T} \frown \mathcal{U}$ and $\mathcal{U}'$ be as in Definition* 4.74, *with $\mathcal{U} \neq \emptyset$. Then:*

1. $M_0^{\mathcal{U}} = M_\alpha^{\mathcal{T}} || \gamma^{M_\alpha^{\mathcal{T}}}$ *and* $\deg_0^{\mathcal{U}} = 0$,

2. $M_0^{\mathcal{U}'} = M_\alpha^{\mathcal{T}} \downarrow 0$ *and* $\deg_0^{\mathcal{U}'} = 0$, *so* $(M_0^{\mathcal{U}})^{\mathrm{pv}} = M_0^{\mathcal{U}'} | \kappa_0^{+M_0^{\mathcal{U}'}}$,

3. *for* $0 < \beta < \mathrm{lh}(\mathcal{U})$, $\beta \in \mathscr{D}_{\deg}^{\mathcal{U}} \Leftrightarrow \beta \in \mathscr{D}_{\deg}^{\mathcal{U}'}$, *and* $\deg_\beta^{\mathcal{U}} = \deg_\beta^{\mathcal{U}'}$,

4. *if* $0 < \beta < \mathrm{lh}(\mathcal{U})$ *and* $[0,\beta]_{\mathcal{U}}$ *drops then* $M_\beta^{\mathcal{U}} = M_{\beta'}^{\mathcal{U}}$,

5. *if* $0 < \beta < \mathrm{lh}(\mathcal{U})$ *and* $[0,\beta]_{\mathcal{U}}$ *does not drop then* $(M_\beta^{\mathcal{U}})^{\mathrm{pv}} = M_\beta^{\mathcal{U}'} | \kappa_0^{+M_\beta^{\mathcal{U}'}}$,

6. *if* $0 < \beta + 1 < \mathrm{lh}(\mathcal{U})$ *and* $[0,\beta+1]_{\mathcal{U}}$ *drops then* $M_{\beta+1}^{*\mathcal{U}} = M_{\beta+1}^{*\mathcal{U}'}$ *and* $i_{\beta+1}^{*\mathcal{U}} = i_{\beta+1}^{*\mathcal{U}'}$,

7. *if* $0 < \beta + 1 < \mathrm{lh}(\mathcal{U})$ *and* $[0,\beta+1]_{\mathcal{U}'}$ *does not drop then* $i_{\beta+1}^{*\mathcal{U}} \subseteq i_{\beta+1}^{*\mathcal{U}'}$,

8. *if* $0 \leq \beta < \mathrm{lh}(\mathcal{U})$ *and* $[0,\beta]_{\mathcal{U}}$ *does not drop then* $M_\beta^{\mathcal{U}'}$ *is a* $\kappa_0^{M_\beta^{\mathcal{U}'}}$-*sound* $\Sigma_{M_\alpha^{\mathcal{T}} \downarrow 0}$-*iterate of* $M_\alpha^{\mathcal{T}} \downarrow 0$, $\mathcal{M}_\infty^{M_\beta^{\mathcal{U}'}}$ *is a* $\delta_0^{\mathcal{M}_\infty^{M_\beta^{\mathcal{U}'}}}$-*sound* $\Sigma_{M_\alpha^{\mathcal{T}} \downarrow 0}$-*iterate of* $M_\alpha^{\mathcal{T}} \downarrow 0$,

$$\mathrm{Ult}(M_\alpha^{\mathcal{T}} \downarrow 0, F(M_\beta^{\mathcal{T}})) = \mathcal{M}_\infty^{M_\beta^{\mathcal{U}'}}$$

and $F(M_\beta^{\mathcal{T}})$ *is the extender of the* $\Sigma_{M_\alpha^{\mathcal{T}} \downarrow 0}$-*iteration map.*

Therefore $\Psi_{\mathscr{V}_1,\gamma^{\mathscr{V}_1}}$ *is good.*

Proof. We omit most of the proof, as it follows from the usual calculations. However, for part 8, just note that the action of the $\mathcal{U}$- and $\mathcal{U}'$-iteration maps j and j' on $M_0^{\mathcal{U}} = M_\alpha^{\mathcal{T}} || \gamma^{M_\alpha^{\mathcal{T}}}$ are identical (i.e. $j \subseteq j'$), since $(M_0^{\mathcal{U}})^{\mathrm{pv}} = M_0^{\mathcal{U}'} | \kappa_0^{+M_0^{\mathcal{U}'}}$, and so

$$j(\mathcal{M}_\infty^{M_0^{\mathcal{U}}}) = j'(\mathcal{M}_\infty^{M_0^{\mathcal{U}'}} | \delta_\infty^{M_0^{\mathcal{U}'}}) = \mathcal{M}_\infty^{M_\beta^{\mathcal{U}'}} | \delta_\infty^{M_\beta^{\mathcal{U}'}};$$

but then the fact that $F(M_\beta^{\mathcal{T}})$ agrees with the $\Sigma_{M_\alpha^{\mathcal{T}} \downarrow 0}$-iteration map

$$(M_\alpha^{\mathcal{T}} \downarrow 0) | \delta_0^{M_\alpha^{\mathcal{T}} \downarrow 0} \to \mathcal{M}_\infty^{M_\beta^{\mathcal{U}'}}$$

is a consequence, since $F^{\mathscr{V} || \gamma^{\mathscr{V}}}$ is likewise correct, where $\mathscr{V} = M_\alpha^{\mathcal{T}}$ (by Lemma 4.68) and j' preserves indiscernibles. $\square$

Note that if $\mathcal{T} \frown \mathcal{U}$ is as above, with last model $M_\beta^{\mathcal{U}}$, then applying $F(M_\beta^{\mathcal{U}})$ as the next extender (giving a non-short-normal tree), the next model is again an iterate of $\mathscr{V}_1$ via $\Psi_{\mathscr{V}_1,\mathscr{V}_1^-}$.

Definition 4.76. Say that $(\mathcal{W}, U, \mathcal{T})$ is M-*standard* iff $\mathcal{W}$ is a Σ-tree on M which is above κ_0, $\mathrm{lh}(\mathcal{W}) = \xi + 2$ for some ξ, and letting $E = E_\xi^{\mathcal{W}}$, then $\mathrm{crit}(E) = \kappa_0$ and E is M-total, $U = \mathrm{Ult}(M, E) = M_{\xi+1}^{\mathcal{W}}$, and $\mathcal{T}$ is the tree leading from $\mathscr{V}_1$ to $\mathscr{V}_1^U$; so

$$M_\infty^{\mathcal{T}} = \mathscr{V}_1^U = \mathrm{Ult}(\mathscr{V}_1, E{\restriction}\mathscr{V}_1)$$

and $i^{\mathcal{T}} \subseteq i^{\mathcal{W}}$.

Suppose $(\mathcal{W}, U, \mathcal{T})$ is M-standard. We define a strategy $\Gamma_{\mathcal{W}}$ for above-$\delta_0^{\mathscr{V}_1^U}$ short-normal trees $\mathcal{S}$ on $\mathscr{V}_1^U$. [42] Let $\mathscr{V} = \mathscr{V}_1^U$ and $N = \mathscr{V} \downarrow 0 = \mathcal{M}_\infty^U$.

If $\mathrm{lh}(E_0^{\mathcal{S}}) < \gamma^{\mathscr{V}}$, then $\Gamma_{\mathcal{W}}$ follows $\Psi_{\mathscr{V}_1, \gamma^{\mathscr{V}_1}}$ (recalling that $\mathcal{T}$ is via $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$).

Suppose $\mathrm{lh}(E_0^{\mathcal{S}}) > \gamma^{\mathscr{V}}$. Let Γ be the above-κ_0^{+U}-strategy for U given by Σ_U. Then since $\mathscr{V} = \mathscr{V}_1^U$ is defined by P-construction, Γ induces a short-normal above-$\gamma^{\mathscr{V}}$-strategy for $\mathscr{V}$, which $\Gamma_{\mathcal{W}}$ follows in this case.

We extend Γ to $\Gamma^{V[G]}$, for set-generic extensions $V[G]$ of V, using that Σ extends canonically to $\Sigma^{V[G]}$. $\dashv$

Lemma 4.77. $\Gamma_{\mathcal{W}}$ *is good, and hence so is each* $\Gamma_{\mathcal{W}}^{V[G]}$.

Proof. Clearly $\Gamma_{\mathcal{W}}$ yields wellfounded models, and we already saw that $\Psi_{\mathscr{V}_1, \gamma^{\mathscr{V}_1}}$ is good. So with notation as in Definition 4.76, suppose $\mathrm{lh}(E_0^{\mathcal{S}}) > \gamma^{\mathscr{V}}$ and $\mathcal{S}$ has successor length, and let $\mathcal{S}'$ be the corresponding above-κ_0^{+U} tree on U. Let $\beta \leq \mathrm{OR}(M_\infty^{\mathcal{S}})$ with $F = F(M_\infty^{\mathcal{S}} || \beta)$ long, and $\beta > \gamma^{\mathscr{V}}$. Let $F' = F(M_\infty^{\mathcal{S}'} || \beta)$. Then $\mathrm{crit}(F') = \kappa_0^U$ and F' is U-total, and $F \subseteq F'$. So goodness with respect to F is an easy consequence of Lemma 4.57 (note that U and $U' = \mathrm{Ult}(U, F')$ are κ_0^U- and $\kappa_0^{U'}$-sound Σ-iterates of M). $\square$

Lemma 4.78. *Let A be a set of ordinals. Then there is an M-standard $(\mathcal{W}, U, \mathcal{X})$ such that A is $(U, \mathbb{P})$-generic for some $\mathbb{P} \in U | \kappa_0^U$. Moreover, if $\mathscr{V}$ is a non-dropping $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$-iterate of $\mathscr{V}_1$, via maximal tree $\mathcal{T} = A$, then $\mathscr{V}_1^U$ is a $\Psi_{\mathscr{V}, \mathscr{V}^-}$-iterate of $\mathscr{V}$ (see Definition 4.69).*

Likewise if $V[G]$ is a set-generic extension of V and $A \in \mathcal{P}(\mathrm{OR}) \cap V[G]$.

Proof. Let $F \in \mathbb{E}^M$ be M-total with $\mathrm{crit}(F) = \kappa_0$. Then λ_F is a limit of Woodin cardinals of $M | \mathrm{lh}(F)$, and $\nu_F < \lambda_F$. Let $\delta \in (\nu_F, \lambda_F)$ be Woodin in $M | \mathrm{lh}(F)$. Let $\mathcal{W}'$ be an above-ν_F genericity iteration of M, for the extender algebra of $M | \mathrm{lh}(F)$ at δ, making A generic. Note that $\rho_1^{M || \mathrm{lh}(F)} \leq \nu_F$, so $\mathcal{W}'$ drops immediately to $M || \mathrm{lh}(F)$, at degree 0. Set $\mathcal{W} = \mathcal{W}' ^\frown \left\langle F(M_\infty^{\mathcal{W}'}) \right\rangle$. The "moreover" clause is a routine consequence, and the extension to $V[G]$ is similar. $\square$

We can now define the full short-normal strategy Ψ_{sn} for $\mathscr{V}_1$. In the end the method used to define the corresponding strategy for $\mathscr{V}_2$, in §5.6.4 (especially

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[42] We use subscript $\mathcal{W}$, not $\mathcal{T}$, as one can have M-standard $(\mathcal{W}', U', \mathcal{T}') \neq (\mathcal{W}, U, \mathcal{T})$ with $\mathcal{T}' = \mathcal{T}$. We will later see that, in this case however, $\Gamma_{\mathcal{W}} = \Gamma_{\mathcal{W}'}$.

Definition 5.67), will be somewhat different, instead of being a direct general-ization. One could probably use the methods of §5.6.4 to define a strategy for $\mathscr{V}_1$ below, which would have benefit of making the construction more uniform. But historically, the approach below was found earlier, and the verification that it works involves ideas that do not come up for the methods of §5.6.4, which and seem of interest. So in order to record more information, we use the two different methods, as opposed to aiming for succinctness through uniformity.

Definition 4.79. Let $\mathscr{T}$ be the class of all trees $\mathcal{T}$ on $\mathscr{V}_1$ via $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$, of successor length, with $b^{\mathcal{T}}$ non-dropping. For $\mathcal{T} \in \mathscr{T}$, letting $\mathscr{V} = M_\infty^{\mathcal{T}}$, we will define a good above-$\delta_0^{\mathscr{V}}$ short-normal strategy $\Psi_{\mathcal{T}}$ for $\mathscr{V}$. We will then define

$$\Psi_{\mathrm{sn}} = \Psi_{\mathscr{V}_1, \mathscr{V}_1^-} \cup \bigcup_{\mathcal{T} \in \mathscr{T}} \Psi_{\mathcal{T}}.$$

So fix $\mathcal{T}$. Let $(\mathcal{W}, U, \mathcal{X})$ be M-standard and such that $\mathcal{T}$ is $(U, \mathbb{P})$-generic for some $\mathbb{P} \in U|\kappa_0^U$, which exists by Lemma 4.78. Let $j : \mathscr{V} \to \mathscr{V}_1^U$ be the correct iteration map. By 4.71, $\mathscr{V}_1^U$ is ω-standard. We define

$$\Psi_{\mathcal{T}} = \text{ the minimal } j\text{-pullback of } \Gamma_{\mathcal{W}}$$

(see [14, 10.3, 10.4] and Remark 4.80 below; by ω-standardness, the minimal j-pullback is well-defined, but we verify below that $\Psi_{\mathcal{T}}$ is independent of the choice of $\mathcal{W}$[43]).

We also generalize this to set-generic extensions $V[G]$ of V. Let $\mathscr{T}^{V[G]}$ be the class of all trees $\mathcal{T}$ on $\mathscr{V}_1$ via $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}^{V[G]}$ (determined by $\Sigma^{V[G]}$ just as $\Psi_{\mathscr{V}_1, \mathscr{V}_1^-}$ is from Σ), of successor length, with $b^{\mathcal{T}}$ non-dropping. Fix $\mathcal{T} \in \mathscr{T}^{V[G]}$. Let $(\mathcal{W}, U, \mathcal{X})$ be as above with respect to $\mathcal{T}$ (but still with $\mathcal{W} \in V$, so $U \subseteq V$ and $\mathcal{X} \in V$ also). Let $\mathscr{V} = M_\infty^{\mathcal{T}}$. and $j : \mathscr{V} \to \mathscr{V}_1^U$ be the correct iteration map. Define

$$\psi_{\mathcal{T}}^{V[G]} = \text{ the minimal } j\text{-pullback of } \Gamma_{\mathcal{W}}^{V[G]}. \qquad \dashv$$

Remark 4.80. Let us summarize how the minimal j-pullback determining $\Psi_{\mathcal{T}}$ is defined. It is like a standard copying construction, except that the method for copying extenders is different. Let E_j be the $(\delta_0^{\mathscr{V}}, \delta_0^{\mathscr{V}_1^U})$-extender derived from j. For $\mathcal{S}$ via $\Psi_{\mathcal{T}}$, we will have a tree $\mathcal{S}'$ via $\Gamma_{\mathcal{W}}$, with the same length, tree order, drop and degree structure, and for $\alpha < \mathrm{lh}(\mathcal{S})$, a $d = \deg_\alpha^{\mathcal{S}}$-embedding

$$\pi_\alpha : M_\alpha^{\mathcal{S}} \to M_\alpha^{\mathcal{S}'},$$

and moreover,

$$M_\alpha^{\mathcal{S}'} = \mathrm{Ult}_d(M_\alpha^{\mathcal{S}}, E_j)$$

and π_α is the associated ultrapower map, and if $\alpha + 1 < \mathrm{lh}(\mathcal{T})$ then

$$M_\alpha^{\mathcal{S}'} || \mathrm{lh}(E_\alpha^{\mathcal{S}'}) = \mathrm{Ult}_0(M_\alpha^{\mathcal{S}} || \mathrm{lh}(E_\alpha^{\mathcal{S}}), E_j),$$

---

[43]Of course, we could have simply chosen $\mathcal{W}$ in a canonical fashion, and then $\Psi_{\mathcal{T}}$ would be trivially well-defined. But the independence from $\mathcal{W}$ will be important later.

and letting $d^* = \deg^{\mathcal{S}}(\alpha + 1)$, then

$$M_{\alpha+1}^{*\mathcal{S}'} = \mathrm{Ult}_{d^*}(M_{\alpha+1}^{*\mathcal{S}}, E_j).$$

The remaining details are essentially as in [24] and [14], using ω-standardness for Vsps (the latter ensures that, for example, when $E_\alpha^{\mathcal{S}} \neq F(M_\alpha^{\mathcal{S}})$ or $\deg_\alpha^{\mathcal{S}} > 0$, the ultrapower above determining $E_\alpha^{\mathcal{S}'}$ does indeed produce a segment of $M_\alpha^{\mathcal{S}'}$).

Lemma 4.81. [44] *We have:*

1. *$\Psi_\mathcal{T}$ is well-defined for each $\mathcal{T} \in \mathscr{T}^{V[G]}$.*

2. *$\Psi_{\mathrm{sn}}^{V[G]}$ is good.*

Proof. Since G doesn't make a significant difference, we assume $G = \emptyset$.

Part 2: Let $\mathcal{S}$ on $\mathcal{V}$ be via $\Psi_\mathcal{T}$, and $\mathcal{S}'$ the minimal j-copy, as in Remark 4.80. Using the copy maps $\pi_\alpha : M_\alpha^{\mathcal{S}} \to M_\alpha^{\mathcal{S}'}$, we can argue just like in the last part of the proof of Lemma 4.68 to see that the long extenders in $\mathbb{E}_+(M_\alpha^{\mathcal{S}})$ are correct.

Part 1: Let $(\mathcal{W}, U, \mathcal{X})$ and $(\mathcal{W}', U', \mathcal{X}')$ be as in the definition, and let Ψ, Ψ' respectively be the induced strategies for $\mathcal{V} = M_\infty^{\mathcal{T}}$.

Roughly, we would like to compare U with U', producing a common iterate U'' and corresponding $\mathcal{W}'', \mathcal{X}''$, and show that Ψ, Ψ' both agree with $\Psi_{\mathcal{W}''}$, and hence are equal. However, a standard comparison of U, U' doesn't work for this, as the resulting iteration map could have critical point κ_0^M on one side, which would cause problems. Instead, we form a modified kind of comparison, as follows.

Let $D \in \mathbb{E}^U$ be the U-total order 0 measure on κ_0^U. Let δ be the least Woodin of $\mathrm{Ult}(U, D)$ such that $\delta > \kappa_0^U$. Let D', δ' be likewise for U'.

Recall from [22] that the *meas-lim extender algebra* of a premouse N is like the usual extender algebra, except that we only induce axioms with extenders $E \in \mathbb{E}^N$ such that ν_E is a limit of measurable cardinals of N. We will form a *simultaneous genericity iteration* $(\mathcal{Y}, \mathcal{Y}')$ of $\mathrm{Ult}(U, D)$ and $\mathrm{Ult}(U', D')$ for the meas-lim extender algebras at δ, δ', above $\kappa_0^U + 1$ and $\kappa_0^{U'} + 1$ respectively, arranging that $M_\infty^{\mathcal{Y}}$ and $M_\infty^{\mathcal{Y}'}$ are generic over one another, and $\delta(\mathcal{Y}) = i_{0\infty}^{\mathcal{Y}}(\delta) = i_{0\infty}^{\mathcal{Y}'}(\delta') = \delta(\mathcal{Y}')$. To help ensure the latter, we also (i) arrange genericity of $(U|\delta, U'|\delta')$, which will allow $\mathcal{Y}, \mathcal{Y}'$ to be recovered in the generic extensions, and (ii) insert short linear iterations which ensure that every measurable of $M_\infty^{\mathcal{Y}}$ below $i_{0\infty}^{\mathcal{Y}}(\delta)$ is a cardinal of $M_\infty^{\mathcal{Y}}$, and vice versa; this is like similar arguments in [17] and [15]. However, executing this process in the most obvious manner, using the process for genericity iteration with Jensen indexing on both sides (as described in [22, Theorem 5.8]) seems to lead to the possibility of the trees $\mathcal{Y}, \mathcal{Y}'$ being non-normal. Thus, instead of this, we produce a sequence $\langle \mathcal{Y}_\alpha, \mathcal{Y}_\alpha' \rangle_{\alpha \leq \iota}$ of normal trees approximating the eventual desired trees $\mathcal{Y} = \mathcal{Y}_\iota$ and $\mathcal{Y}' = \mathcal{Y}_\iota'$, with the sequence converging in a natural way.

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[44]Although this lemma logically precedes later parts of the paper, its necessity and proof were actually found later, in particular after the proof of Theorem 5.79, which was found after the proof of Lemma 5.8.

Here are the details. We initially iterate linearly with the least measurable of $\mathrm{Ult}(U, D)$ which is $> \kappa_0^U$ (hence $< \delta$), and likewise $> \kappa_0^{U'}$ for $\mathrm{Ult}(U', D')$, until they reach some common closure point $> \max(\delta, \delta')$. Suppose we have defined $\mathcal{X} = \mathcal{Y}_\alpha, \mathcal{X}' = \mathcal{Y}'_\alpha$, at some point after this initial phase. These trees will be padded 0-maximal, of successor length, and if $E_\beta^{\mathcal{X}} \neq \emptyset \neq E_\beta^{\mathcal{X}'}$ then $\mathrm{lh}(E_\beta^{\mathcal{X}}) = \mathrm{lh}(E_\beta^{\mathcal{X}'})$. We will determine some extenders E_α, E'_α, or stop the process. First let $G_\alpha \in \mathbb{E}_+(M_\alpha^{\mathcal{X}})$ be the extender selected for the purposes of genericity iteration, for making $(\mathbb{E}^{M_\infty^{\mathcal{X}'}}, U|\delta, U'|\delta')$ generic, and given current tree $\mathcal{X}$ (but not demanding that $\mathrm{lh}(G_\alpha) > \mathrm{lh}(E_\beta^{\mathcal{X}})$ for all $\beta + 1 < \mathrm{lh}(\mathcal{X})$; if $\mathrm{lh}(G_\alpha) < \mathrm{lh}(E_\beta^{\mathcal{X}})$ for some β, that is okay); we follow the extender selection procedure for genericity iteration for Jensen indexing here (see [22, Theorem 5.8]), and if $b^{\mathcal{X}}$ drops then $M_\infty^{\mathcal{X}}$ will be active, and in this case $F(M_\infty^{\mathcal{X}})$ is automatically set as G_α, if no extender with lower index is. (If there is no such extender, set $G_\alpha = \emptyset$.) Define G'_α symmetrically, for making $(\mathbb{E}^{M_\infty^{\mathcal{X}}}, U|\delta, U'|\delta')$ generic. If $G_\alpha \neq \emptyset$ then let $\gamma = \mathrm{lh}(G_\alpha)$; if otherwise and $b^{\mathcal{X}}$ does not drop then let $\gamma = i_{0\infty}^{\mathcal{X}}(\delta)$, and otherwise let $\gamma = \mathrm{OR}(M_\infty^{\mathcal{X}})$. Define γ' symmetrically. If there is $F \in \mathbb{E}(M_\infty^{\mathcal{X}}|\gamma)$ which is $M_\infty^{\mathcal{X}}$-total and $\kappa_0^U < \mathrm{crit}(F)$ and $\mathrm{crit}(F)$ is not a cardinal of $M_\infty^{\mathcal{X}'}$, then let $F_\alpha =$ the least such F, and otherwise let $F_\alpha = G_\alpha$. (If $\mathrm{crit}(F)$ is not a cardinal of $M_\infty^{\mathcal{X}'}$ for the trivial reason that $\mathrm{OR}(M_\infty^{\mathcal{X}'}) \leq \mathrm{crit}(F)$, and hence $b^{\mathcal{X}'}$ drops, then it will follow from Claim 1 below that $M_\infty^{\mathcal{X}'}$ is active, $G'_\alpha \neq \emptyset$ and so $\mathrm{lh}(G'_\alpha) \leq \mathrm{lh}(F)$, and in this case, F will actually be irrelevant.) Define F'_α symmetrically. If $F_\alpha \neq \emptyset$ and either $F'_\alpha = \emptyset$ or $\mathrm{lh}(F_\alpha) \leq \mathrm{lh}(F'_\alpha)$ then set $E_\alpha = F_\alpha$, and otherwise set $E'_\alpha = \emptyset$; define E'_α symmetrically.

If $E_\alpha = \emptyset = E'_\alpha$ then we stop the process (setting $\iota = \alpha$). If $E_\alpha \neq \emptyset$ then let β be least such that $E_\alpha \in \mathbb{E}_+(M_\beta^{\mathcal{X}})$, and set $\mathcal{Y}_{\alpha+1} = \mathcal{Y}_\alpha \restriction (\beta + 1) \,\widehat{}\, E_\alpha$ (as a 0-maximal tree). If $E'_\alpha \neq \emptyset$, define $\mathcal{Y}'_{\alpha+1}$ symmetrically. If $E_\alpha \neq \emptyset = E'_\alpha$ then set $\mathcal{Y}'_{\alpha+1} = \mathcal{Y}_\alpha \restriction (\beta + 1)$, where β is least such that either $\mathcal{Y}'_\alpha = \mathcal{Y}'_\alpha \restriction (\beta + 1)$ or $\mathrm{lh}(E_\beta^{\mathcal{Y}'_\alpha}) > \mathrm{lh}(E_\alpha)$. And if $E_\alpha = \emptyset \neq E'_\alpha$, proceed symmetrically.

Finally, given $\mathcal{Y}_\alpha, \mathcal{Y}'_\alpha$ for all $\alpha < \eta$, where η is a limit, define $\mathcal{Y}_\eta$ as the natural lim inf of the $\mathcal{Y}_\alpha$ for $\alpha < \eta$, extended with the relevant iteration strategy Ψ as necessary. That is, $\mathcal{Y}_\eta$ is via Ψ, and $\mathcal{Y}_\eta$ uses an extender E iff $\mathcal{Y}_\alpha$ uses E for eventually all $\alpha < \eta$, and if this yields a limit length tree, then we extend it using Ψ to successor length.

This determines the mutual genericity iteration. The first claim below is much as in [22, Theorem 5.8] and related arguments in [17]:

CLAIM 1.

1. $\mathcal{Y}_\alpha$ is 0-maximal and if $b^{\mathcal{Y}_\alpha}$ drops then $M_\infty^{\mathcal{Y}_\alpha}$ is active; likewise for $\mathcal{Y}'_\alpha$,

2. the process terminates at some $\iota < \infty$, giving $\mathcal{Y} = \mathcal{Y}_\iota$ and $\mathcal{Y}' = \mathcal{Y}'_\iota$, and

3. $b^{\mathcal{Y}}, b^{\mathcal{Y}'}$ do not drop (although there *can* be $\alpha < \mathrm{lh}(\mathcal{Y}, \mathcal{Y}')$ such that $[0, \alpha]_{\mathcal{Y}}$ or $[0, \alpha]_{\mathcal{Y}'}$ drops, because of Jensen indexing).

Proof. Part 1: The 0-maximality is directly by definition; the rest is as in [22].

Part 2: Suppose not. Fix some large enough regular cardinal χ. Assume for the sake of illustration that for all $\alpha \in \mathrm{OR}$ there are $\beta, \beta' > \alpha$ such that $E_\beta \neq \emptyset$ and $E'_{\beta'} \neq \emptyset$; the other case is similar. Let $\mathcal{Y} = \mathcal{Y}_\chi$ and $\mathcal{Y}' = \mathcal{Y}'_\chi$, noting that $\chi + 1 = \mathrm{lh}(\mathcal{Y})$ and $\chi = \delta(\mathcal{Y})$ and χ is a limit cardinal of $M_\infty^{\mathcal{Y}}$, and likewise for $\mathcal{Y}'$. Let $J \preccurlyeq \mathcal{H}_{\chi^+}$ with everything relevant in J and $\kappa = J \cap \chi \in \chi$, let H be the transitive collapse of J and $\pi : H \to \mathcal{H}_{\chi^+}$ the uncollapse map. So $\pi(\kappa) = \chi$ and we have $\bar{\mathcal{Y}}, \bar{\mathcal{Y}}' \in H$ with $\pi(\bar{\mathcal{Y}}, \bar{\mathcal{Y}}') = (\mathcal{Y}, \mathcal{Y}')$. Note that $\bar{\mathcal{Y}} = \mathcal{Y}_\kappa$ and $\mathrm{lh}(\bar{\mathcal{Y}}) = \kappa + 1$, $\delta(\bar{\mathcal{Y}}) = \kappa$ is a limit cardinal of $M_\infty^{\bar{\mathcal{Y}}}$, $\kappa \in b^{\bar{\mathcal{Y}}} = b^{\mathcal{Y}} \cap (\kappa + 1)$, and for all $\alpha \in [\kappa, \chi]$, we have $\bar{\mathcal{Y}} = \mathcal{Y}_\alpha{\restriction}(\kappa + 1)$. Likewise for $\bar{\mathcal{Y}}'$.

As usual, $i_{\kappa\chi}^{\mathcal{Y}}{\restriction}\mathcal{P}(\kappa) \subseteq \pi$. Let E be the first extender used forming $i_{\kappa\chi}^{\mathcal{Y}}$, so $\mathrm{crit}(E) = \kappa$. Let $\alpha \in [\kappa, \chi]$ with $E = E_\alpha$. Since κ is a limit cardinal of $M_\infty^{\bar{\mathcal{Y}}'}$, and hence of $M_\infty^{\mathcal{Y}'_\alpha}$, either (i) $E = G_\alpha$ was chosen for genericity iteration purposes, and let $\beta = \alpha$, or (ii) $b^{\mathcal{Y}_\alpha}$ drops and $E_\alpha = F(M_\infty^{\mathcal{Y}_\alpha})$. But if (ii) holds then as is usual for genericity iteration with λ-indexing (see [22]), there is $\beta \in [\kappa, \alpha)$ such that $E_\beta = G_\beta$ is used in $\mathcal{Y}$ (but not along $b^{\mathcal{Y}}$), and used in $\mathcal{Y}_\alpha$, and $G_\beta{\restriction}\nu(G_\beta) = E_\alpha{\restriction}\nu(G_\beta)$. So in either case (i) or (ii), G_β is used in $\mathcal{Y}$ and $G_\beta{\restriction}\nu(G_\beta)$ is derived from $i_{\kappa\chi}^{\mathcal{Y}}$. But note then that $M_\infty^{\mathcal{Y}'_\beta}|\nu(G_\beta) = M_\infty^{\mathcal{Y}'}|\nu(G_\beta)$, and therefore we obtain a contradiction like in the usual proof that genericity iteration terminates.

Part 3 is by construction and part 1. $\qquad\square$

Let $\delta^{\mathcal{Y}} = i_{0\infty}^{\mathcal{Y}}(\delta)$ and $\delta^{\mathcal{Y}'} = i_{0\infty}^{\mathcal{Y}'}(\delta')$. Note that $\delta^{\mathcal{Y}}$ is a strong cutpoint of $M_\infty^{\mathcal{Y}}$, and $M_\infty^{\mathcal{Y}}$ is $\delta^{\mathcal{Y}}$-sound; likewise for $\delta^{\mathcal{Y}'}$ and $M_\infty^{\mathcal{Y}'}$.

CLAIM 2. $\delta^{\mathcal{Y}} = \delta^{\mathcal{Y}'}$.

Proof. We may assume $\delta^{\mathcal{Y}} < \delta^{\mathcal{Y}'}$. By minimality of δ', $\delta^{\mathcal{Y}}$ is not Woodin in $M_\infty^{\mathcal{Y}'}$. Let $Q' \lhd M_\infty^{\mathcal{Y}'}$ be the Q-structure for $\delta^{\mathcal{Y}}$. Because of the inserted linear iterations at measurables, $\delta^{\mathcal{Y}}$ is a cardinal of $M_\infty^{\mathcal{Y}'}$. Note then that by choice of D', δ' (and smallness), $\delta^{\mathcal{Y}}$ is a strong cutpoint of Q'. But $(M_\infty^{\mathcal{Y}}|\delta^{\mathcal{Y}}, U|\delta, U'|\delta')$ is meas-lim extender algebra generic over $M_\infty^{\mathcal{Y}}$ at $\delta^{\mathcal{Y}}$, and $(M_\infty^{\mathcal{Y}}|\delta^{\mathcal{Y}}, U|\delta, U'|\delta')$ is likewise over Q', also at $\delta^{\mathcal{Y}}$, because $\delta^{\mathcal{Y}}$ is a cardinal of $M_\infty^{\mathcal{Y}'}$. Therefore these two premice can be lifted to premice $(M_\infty^{\mathcal{Y}})^+$ and $(Q')^+$ over $(M_\infty^{\mathcal{Y}}|\delta^{\mathcal{Y}}, M_\infty^{\mathcal{Y}'}|\delta^{\mathcal{Y}}, U|\delta, U'|\delta')$. Comparing $(M_\infty^{\mathcal{Y}})^+$ with $(Q')^+$ (so the comparison is above $\delta^{\mathcal{Y}}$) and considering smallness and $\delta^{\mathcal{Y}}$-soundness, we have $(Q')^+ \lhd (M_\infty^{\mathcal{Y}})^+$.

Now working in $(M_\infty^{\mathcal{Y}})^+$, where we have $U|\delta$, $M_\infty^{\mathcal{Y}}|\delta^{\mathcal{Y}}$, $U'|\delta'$, $M_\infty^{\mathcal{Y}'}|\delta^{\mathcal{Y}}$ and Q', we can recover $\bar{\mathcal{Y}}{\restriction}\delta^{\mathcal{Y}}$ and $\bar{\mathcal{Y}}'{\restriction}(\delta^{\mathcal{Y}'} + 1)$, where $\bar{\mathcal{Y}}$ is just like $\mathcal{Y}$ but as a tree on $U|\delta$, and $\bar{\mathcal{Y}}'$ likewise. For (i) we have $U|\delta$ and $U'|\delta'$, and we just proceed by comparing $U|\delta$ with $M_\infty^{\mathcal{Y}}|\delta^{Y}$ and $U'|\delta'$ with Q', (ii) Q' determines the branch of $\bar{\mathcal{Y}}'$ at stage $\delta^{\mathcal{Y}}$, and (iii) the intermediate Q-structures used to guide $\bar{\mathcal{Y}}, \bar{\mathcal{Y}}'$ are segments of $M_\infty^{\mathcal{Y}}|\delta^{\mathcal{Y}}$ and $M_\infty^{\mathcal{Y}'}|\delta^{\mathcal{Y}}$ (for example if $\xi = \delta(\bar{\mathcal{Y}}'{\restriction}\xi)$ and the Q-structure for $\bar{\mathcal{Y}}'{\restriction}\xi$ is non-trivial, then ξ is a limit of measurable cardinals of $M_\infty^{\mathcal{Y}'}$, hence a limit cardinal of $M_\infty^{\mathcal{Y}}$, so the Q-structures on both sides do not overlap ξ, which implies that the next extenders used have index beyond the Q-structures (and likewise at all stages after ξ), so the Q-structures are retained).

So in $(M_\infty^{\mathcal{Y}})^+$, where $\delta^{\mathcal{Y}}$ is a regular cardinal, we can execute a slight variant of the termination-of-genericity-iteration proof used above for Claim 1 (with $\delta^{\mathcal{Y}}$ in the role of χ there). (We may not have the sequence of trees $\langle \mathcal{Y}_\alpha, \mathcal{Y}'_\alpha \rangle_{\alpha < \delta^{\mathcal{Y}}}$ in $(M_\infty^{\mathcal{Y}})^+$, but note that the trees we do have are enough.) $\qquad\square$

Let $\delta^* = \delta^{\mathcal{Y}} = \delta^{\mathcal{Y}'}$ and $Q = M_\infty^{\mathcal{Y}}$ and $Q' = M_\infty^{\mathcal{Y}'}$. Note now that (by δ^*-soundness and as δ^* is a strong cutpoint on both sides), Q, Q' are equivalent modulo a generic above δ^*, i.e. letting Q^+ be $Q[Q'|\delta^*]$, considered as a premouse over $(Q|\delta^*, Q'|\delta^*)$, and $(Q')^+$ likewise (we no longer need $U|\delta$ and $U'|\delta'$), then $Q^+ = (Q')^+$. It follows that $\kappa_0^Q = \kappa_0^{Q'}$ and $\mathcal{V}_1^Q = \mathcal{V}_1^{Q'}$, and by uniqueness of iteration strategies, the above-δ^* strategy for Q translates to that of Q'. Finally let $\mathcal{Z}$ be the normal tree equivalent to the stack $(\mathcal{W}, D, \mathcal{Y})$ and $\mathcal{Z}'$ that to $(\mathcal{W}', D', \mathcal{Y}')$; it follows that $\Gamma_{\mathcal{Z}} = \Gamma_{\mathcal{Z}'}$.

Let $j : \mathcal{V} \to \mathcal{V}_1^U$ and $j' : \mathcal{V} \to \mathcal{V}_1^{U'}$ be the correct iteration maps. Now $i^{D,\mathcal{Y}} \!\restriction\! \mathcal{V}_1^U$ is the correct iteration map $k : \mathcal{V}_1^U \to \mathcal{V}_1^Q$ (because we wanted this, we couldn't just compare U with U' in the usual manner). Likewise $k' = i^{D',\mathcal{Y}'} \!\restriction\! \mathcal{V}_1^{U'}$. So

$$ k \circ j = k' \circ j' : \mathcal{V} \to \mathcal{V}_1^Q = \mathcal{V}_1^{Q'} $$

is also the correct iteration map. So the minimal $k \circ j$-pullback of $\Gamma_{\mathcal{Z}}$ equals the minimal $k' \circ j'$-pullback of $\Gamma_{\mathcal{Z}'}$ (a strategy for $\mathcal{V}$); denote this by Ψ^*. Recall that Ψ is the minimal j-pullback of $\Gamma_{\mathcal{W}}$, and Ψ the minimal j'-pullback of $\Gamma_{\mathcal{W}'}$. It suffices to see that $\Psi = \Psi^*$, since then by symmetry, $\Psi' = \Psi^*$ also.

Let Σ^Q be the above-κ_0^Q strategy for Q given by Σ_Q. Let Σ^U be likewise for U. Then by κ_0^U-soundness and since κ_0^U is a cutpoint of U, Σ^U is the minimal $i^{D,\mathcal{Y}}$-pullback of Σ^Q. But then $\Gamma_{\mathcal{W}}$ is the minimal k-pullback of $\Gamma_{\mathcal{Z}}$; for trees $\mathcal{S}$ on $\mathcal{V}_1^U$ with $\mathrm{lh}(E_0^{\mathcal{S}}) > \gamma^{\mathcal{V}_1^U}$, this uses that $k \subseteq i^{D,\mathcal{Y}}$ and the fine structural translation of $\mathcal{S}$ to a tree on U; for trees with $\mathrm{lh}(E_0^{\mathcal{S}}) < \gamma^{\mathcal{V}_1^U}$, it uses minimal hull condensation for Σ_M, and the fact that the action of $\Gamma_{\mathcal{W}}$ and $\Gamma_{\mathcal{Z}}$ on such trees in induced by Σ_M. Therefore $\Psi = \Psi^*$, as desired. $\qquad\square$

4.12.5 Normal trees on $\mathcal{V}_1$

So we have a good short-normal strategy Ψ_{sn}, extending $\Psi_{\mathcal{V}_1, \mathcal{V}_1^-}$. This extends easily to a normal strategy $\Sigma_{\mathcal{V}_1}$.

Definition 4.82. We define a 0-maximal iteration strategy $\Sigma_{\mathcal{V}_1}$ for $\mathcal{V}_1$, determined by the following properties:

1. $\Psi_{\mathrm{sn}} \subseteq \Sigma_{\mathcal{V}_1}$.

2. Let $\mathcal{T}$ be on $\mathcal{V}_1$, 0-maximal, of length $\eta + 2$, with $\mathcal{T} \!\restriction\! \eta + 1$ short-normal and via Ψ_{sn}, and $E_\eta^{\mathcal{T}}$ long. Then (by goodness) $M_{\eta+1}^{\mathcal{T}}$ is a non-dropping iterate of $\mathcal{V}_1$ via a tree $\mathcal{U}$ according to $\Psi_{\mathcal{V}_1, \mathcal{V}_1^-}$, and $\mathrm{lh}(E_\eta^{\mathcal{T}}) = \delta_0^{M_{\eta+1}^{\mathcal{T}}}$. Then $\Sigma_{\mathcal{V}_1}$ acts on trees normally extending $\mathcal{T} \!\restriction\! \eta + 2$ by following $\Psi_{\mathcal{U}}$, until another long extender is used.

3. Likewise, whenever $\mathcal{T}$ is on $\mathscr{V}_1$, via $\Sigma_{\mathscr{V}_1}$, and $E_\eta^\mathcal{T}$ is long, then $M_{\eta+1}^\mathcal{T}$ is a non-dropping $\Psi_{\mathscr{V}_1,\mathscr{V}_1^-}$-iterate, via a tree $\mathcal{U}$, and $\Sigma_{\mathscr{V}_1}$ extends $\mathcal{T}\!\restriction\!(\eta+2)$ by following $\Psi_\mathcal{U}$, until another long extender is used.

4. If λ is a limit and there are long extenders used cofinally in λ, there is a unique $\mathcal{T}\!\restriction\!\lambda$-cofinal branch, and $M_\lambda^\mathcal{T}$ is again an iterate via $\Psi_{\mathscr{V}_1,\mathscr{V}_1^-}$ (by normalization for transfinite stacks). In this case, $\delta(\mathcal{T}\!\restriction\!\lambda)$ is the least measurable of $M_\eta^\mathcal{T}$, so we can have $\mathcal{T}\!\restriction\![\lambda,\alpha)$ based on $M_\lambda^\mathcal{T}|\delta_0^{M_\lambda^\mathcal{T}}$, with $\lambda < \alpha$. This interval is formed using $\Psi_{\mathscr{V},\mathscr{V}^-}$ (see Definition 4.69). Letting α be least such that $(\lambda,\alpha]_\mathcal{T}$ does not drop and $\delta_0^{M_\alpha^\mathcal{T}} < \mathrm{lh}(E_\alpha^\mathcal{T})$, then (by normalization) there is a short-normal tree $\mathcal{U}$ via $\Psi_{\mathscr{V}_1,\mathscr{V}_1^-}$ with last model $M_\alpha^\mathcal{T}$, and $\Sigma_{\mathscr{V}_1}$ extends $\mathcal{T}\!\restriction\!(\alpha+1)$ by following $\Psi_\mathcal{U}$, until the next long extender. $\dashv$

The following lemma is now easy to see:

Lemma 4.83. $\Sigma_{\mathscr{V}_1}$ *is a good 0-maximal strategy for* $\mathscr{V}_1$. *Moreover, for every successor length tree* $\mathcal{T}$ *via* $\Sigma_{\mathscr{V}_1}$ *there is a unique short-normal tree via* Ψ_{sn} *with the same last model.*

Lemma 4.84. $\Sigma_{\mathscr{V}_1}$ *has minimal inflation condensation (mic).*

Proof. We just consider short-normal trees; it is easy to extend this to arbitrary normal trees, and we leave this extension to the reader.

Let $\mathcal{T},\mathcal{U}$ be short-normal trees on $\mathscr{V}_1$, via Ψ_{sn}, such that $\mathcal{U}$ has length $\lambda+1$ for some limit λ, $\mathcal{T}$ has successor length, and $\mathcal{U}\!\restriction\!\lambda$ is a minimal inflation of $\mathcal{T}$; we must show that $\mathcal{U}$ is also a minimal inflation of $\mathcal{T}$. Let $\mathcal{T} = \mathcal{T}_0 \,^\frown \mathcal{T}_1$ and $\mathcal{U} = \mathcal{U}_0 \,^\frown \mathcal{U}_1$ with lower components $\mathcal{T}_0,\mathcal{U}_0$ and upper $\mathcal{T}_1,\mathcal{U}_1$.

Now $\Psi_{\mathscr{V}_1,\mathscr{V}_1^-}$ has mic, because it follows $\Sigma_{\mathcal{M}_\infty}$, which has mic, since Σ_M does, and by [14, ***Theorem 10.2]. So we may assume $\mathcal{T}_1 \neq \emptyset$. Therefore, $\mathcal{T}_0$ has successor length $\alpha+1$, $[0,\alpha]_{\mathcal{T}_0}$ does not drop, and $\mathcal{T}_1$ is based on $M_\alpha^{\mathcal{T}_0}$ and is above $\delta_0^{M_\alpha^{\mathcal{T}_0}}$ and uses only short extenders. And $\mathcal{U}_0,\mathcal{U}_1,\beta$ are likewise, and note that $\beta \in I_\alpha^{\mathcal{T} \leadsto_{\min} \mathcal{U}}$. Let

$$\Pi_0 = \Pi_\beta^{\mathcal{T} \leadsto_{\min} \mathcal{U}} : \mathcal{T}_0 \hookrightarrow_{\min} \mathcal{U}_0$$

(the minimal tree embedding at stage β of the inflation), and $j : M_\alpha^{\mathcal{T}_0} \to M_\beta^{\mathcal{U}_0}$ be the copy map determined by Π_0. Then in fact, $M_\beta^{\mathcal{U}_0}$ is a $\Sigma_{\mathscr{V},\mathscr{V}^-}$-iterate of $\mathscr{V}$ where $\mathscr{V} = M_\alpha^{\mathcal{T}_0}$, and j is the iteration map (see [14, ***Lemma 4.5]).

Now it suffices to see that Ψ_{sn} has minimal hull condensation (mhc) with respect to extensions of Π_0 "above δ_0"; that is, whenever $\mathcal{T}_2,\mathcal{U}_2$ are trees on $M_\alpha^{\mathcal{T}_0}$ and $M_\beta^{\mathcal{U}_0}$, above $\delta_0^{M_\alpha^{\mathcal{T}_0}}$ and $\delta_0^{M_\beta^{\mathcal{U}_0}}$ respectively, with $\mathcal{U}_0 \,^\frown \mathcal{U}_2$ via Ψ_{sn}, and

$$\Pi : \mathcal{T}_0 \,^\frown \mathcal{T}_2 \hookrightarrow_{\min} \mathcal{U}_0 \,^\frown \mathcal{U}_2$$

is a minimal tree embedding with $\Pi_0 \subseteq \Pi$, then $\mathcal{T}_0 \,^\frown \mathcal{T}_2$ is also via Ψ_{sn}.

Let $\mathcal{T}_2' = j``\mathcal{T}_2$, a (putative) tree on $M_\beta^{\mathcal{U}_0}$. Then $\mathcal{T}_2'$ has wellfounded models, and in fact, there is a minimal tree embedding

$$\Pi' : \mathcal{U}_0 \,\widehat{}\, \mathcal{T}_2' \hookrightarrow_{\min} \mathcal{U}_0 \,\widehat{}\, \mathcal{U}_2$$

determined in the obvious manner: for $\alpha < \mathrm{lh}(\mathcal{U}_0)$, we have $I_\alpha^{\Pi'} = [\alpha, \alpha]$, and for $\alpha < \mathrm{lh}(\mathcal{T}_2)$, we have $I_{\mathrm{lh}(\mathcal{U}_0)+\alpha}^{\Pi'} = \gamma_{\mathrm{lh}(\mathcal{T}_0)+\alpha}^{\Pi}$; this determines Π'.

CLAIM 1. $\mathcal{U}_0 \,\widehat{}\, \mathcal{T}_2'$ is via Ψ_{sn}.

Proof. If $\mathrm{lh}(E_0^{\mathcal{T}_2'}) < \gamma^{M_\beta^{\mathcal{U}_0}}$ then this is just because $\Psi_{\mathcal{U}_0}$ follows the strategy induced by $\Sigma_{M_\beta^{\mathcal{U}_0}\downarrow 0}$ in this case, which has mhc.

So suppose $\gamma^{M_\beta^{\mathcal{U}_0}} < \mathrm{lh}(E_0^{\mathcal{T}_2'})$. Let $(\mathcal{W}, U, \mathcal{Y})$ be M-standard for $\mathcal{U}_0$. Let $\ell : M_\beta^{\mathcal{U}_0} \to \mathscr{V}_1^U$ be the iteration map. Since $\mathcal{U}_2$ follows $\Psi_{\mathcal{U}_0}$, the minimal ℓ-copy $\widetilde{\mathcal{U}_2}$ of $\mathcal{U}_2$ (a tree on $\mathscr{V}_1^U$) follows $\Gamma_\mathcal{W}$. Let $\widetilde{\mathcal{T}_2'}$ be the minimal ℓ-copy of $\mathcal{T}_2'$ (see [14, ***10.3, 10.4]). Then $\widetilde{\mathcal{T}_2'}$ has wellfounded models, and in fact there is a minimal tree embedding

$$\widetilde{\Pi'} : \widetilde{\mathcal{T}_2'} \hookrightarrow_{\min} \widetilde{\mathcal{U}_2},$$

determined in the obvious manner. But since $\widetilde{\mathcal{U}_2}$ is via $\Gamma_\mathcal{W}$, and this strategy has mhc, because Σ_U does, therefore $\widetilde{\mathcal{T}_2'}$ is also via $\Gamma_\mathcal{W}$, and therefore $\mathcal{T}_2'$ is via $\Psi_{\mathcal{U}_0}$, as desired. $\square$

CLAIM 2. $\mathcal{T}_0 \,\widehat{}\, \mathcal{T}_2$ is via Ψ_{sn}.

Proof. If $\mathrm{lh}(E_0^{\mathcal{T}_2}) < \gamma^{M_\alpha^{\mathcal{T}_0}}$, this is again easy, using mhc for $\Sigma_{M_\alpha^{\mathcal{T}_0}\downarrow 0}$. So suppose otherwise. Let $(\mathcal{W}, U, \mathcal{Y})$ be simultaneously M-standard for $\mathcal{T}_0$ and for $\mathcal{U}_0$. Let ℓ be as before, and $k : M_\alpha^{\mathcal{T}_0} \to \mathscr{V}_1^U$ be the correct iteration map. So $\Psi_{\mathcal{T}_0}$ and $\Psi_{\mathcal{U}_0}$ are the minimal k-pullback and ℓ-pullback of $\Gamma_\mathcal{W}$ respectively. But $\ell \circ j = k$, since these are correct iteration maps, and therefore $\Psi_{\mathcal{T}_0}$ is the minimal j-pullback of $\Psi_{\mathcal{U}_0}$, which, since $\mathcal{T}_2'$ is via $\Psi_{\mathcal{U}_0}$, proves the claim. $\square$

This completes the proof. $\square$

The preceding proof does not seem to give that $\Sigma_{\mathscr{V}_1}$ has mhc, because it relies heavily on the fact that $\ell \circ j = k$, and if j were instead just a copy map arising from an arbitrary minimal tree embedding, then it need not be an iteration map (and in fact $M_\beta^{\mathcal{U}_0}$ need not be an iterate of $M_\alpha^{\mathcal{T}_0}$).

5 The second Varsovian model $\mathscr{V}_2$

5.1 The δ_1-short tree strategy for $\mathscr{V}_1$

Definition 5.1. Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$.

Let $\mathcal{T}$ be a short-normal tree on $\mathscr{V}$ via $\Sigma_\mathscr{V}$, based on $\mathscr{V}|\delta_1^\mathscr{V}$, of limit length. Let $b = \Sigma_\mathscr{V}(\mathcal{T})$. Say that $\mathcal{T}$ is δ_1-*short* iff either b drops or $\delta(\mathcal{T}) < i_b^\mathcal{T}(\delta_1^\mathscr{V})$, and δ_1-*maximal* otherwise.

We define δ_1-*short* and δ_1-*maximal* analogously for trees on M. $\dashv$

It will be shown in [16] that $\mathscr{V}_1$ knows its own strategy for δ_1-short trees, and, moreover, has a *modified P-construction* which also computes the correct branch model $M_b^{\mathcal{T}}$ for δ_1-maximal $\mathcal{T}$, given that $\mathcal{T}$ is appropriate for forming a P-construction.

In this paper we explain the main new idea needed to prove this, illustrated with a restricted class of trees $\mathcal{T}$ (*P-illustrative* trees) which suffice, for example, for genericity iterations at $\delta_1^{\mathscr{V}_1}$. This restriction will ensure that for such $\mathcal{T}$ and limits λ with $\mathcal{T}{\restriction}\lambda$ being δ_1-short and Q be the correct Q-structure for $\mathcal{T}{\restriction}\lambda$, the only overlaps of $\delta(\mathcal{T}{\restriction}\lambda)$ in Q are long extenders, and this will mean that we have no need for $*$-translation. We will compute Q via a modified P-construction; a key issue is that the P-construction has a new feature, due to the long extenders on the $\mathcal{T}$-side, and the extenders with critical point κ_0^M on the M-side. Similarly, for δ_1-maximal P-illustrative trees $\mathcal{T}$, $M_b^{\mathcal{T}}$ will also be computed by a modified P-construction. We will need a new argument (5.8) to see that the P-construction does indeed compute the correct model. We will actually first consider analogous P-illustrative trees on M, and then transfer these results to trees on $\mathscr{V}_1$. We will then adapt the results to $\delta_1^{\mathscr{V}}$-sound non-dropping $\Sigma_{\mathscr{V}_1}$-iterates $\mathscr{V}$ of $\mathscr{V}_1$.

P-illustrative trees suffice to construct $\mathscr{V}_2$, and prove a significant amount about it. However, in order to prove that it fully knows how to iterate itself, and related facts, we need to consider arbitrary trees, including the full δ_1-short tree strategy. Such facts are moreover used our proof that $\mathscr{V}_2$ is the mantle (because it uses a comparison argument, which seems needs iterability with respect to arbitrary trees). In order to deal with arbitrary trees, we need to deal with trees having overlapped Q-structures, and therefore need $*$-translation, adapted to incorporate the modified P-construction. This material is deferred to [16]; at certain points we summarize results from there we need.

5.1.1 P-illustrative trees on M

Definition 5.2. Given a (strategy) premouse N and $\kappa \leq \eta \in \mathrm{OR}^N$, we say that η is a *κ-cutpoint of N* iff for all $E \in \mathbb{E}_+^N$, if $\mathrm{crit}(E) < \eta < \mathrm{lh}(E)$ then $\mathrm{crit}(E) = \kappa$, and a *strong κ-cutpoint* iff for all $E \in \mathbb{E}_+^N$, if $\mathrm{crit}(E) \leq \eta < \mathrm{lh}(E)$ then $\mathrm{crit}(E) = \kappa$. $\dashv$

Definition 5.3. Let $\mathcal{T}$ be an iteration tree on an M_{swsw}-like premouse N and Σ a partial strategy for N. We say that $\mathcal{T}$ is *P-illustrative* iff there are $E_1, U, \lambda, \eta, \alpha_0, \mu$ such that:

1. either

   (i) $E_1 = \emptyset$, $U = N$ and $\lambda = \kappa_1^N$, or

   (ii) $E_1 \in \mathbb{E}^N$, E_1 is N-total, $\mathrm{crit}(E_1) = \kappa_1^N$, $U = \mathrm{Ult}(N, E_1)$ and $\lambda = \lambda(E_1) = \kappa_1^U$,

2. $\mathcal{T} \in N$, $\mathcal{T}$ is normal, of limit length, is above κ_0^N and based on $N|\delta_1^N$, and letting $\mathcal{T}'$ be the corresponding tree on $N|\delta_1$, we have $\mathcal{T}' \in U|\lambda$,

3. $\kappa_0^N < \eta < \lambda$ and η is a strong κ_0^N-cutpoint of U, and η is a U-cardinal, and if $E_1 \neq \emptyset$ then $\kappa_1^N < \eta$[45]

4. $0 \leq \alpha_0 < \eta$ and $\alpha_0 < \mathrm{lh}(\mathcal{T})$ and $[0, \alpha_0]_{\mathcal{T}}$ does not drop and $\mu = \kappa_0^{M_{\alpha_0}^{\mathcal{T}}} < \eta$,

5. $\mathcal{T}{\upharpoonright}[\alpha_0, \mathrm{lh}(\mathcal{T}))$ is above $(\mu^+)^{M_{\alpha_0}^{\mathcal{T}}}$,

6. for each $\beta + 1 \in (\alpha_0, \mathrm{lh}(\mathcal{T}))$, every Woodin of $M_\beta^{\mathcal{T}}|\mathrm{lh}(E_\beta^{\mathcal{T}})$ is $< \mu$,

7. $\eta < \delta = \delta(\mathcal{T})$, η is the largest cardinal of $U|\delta$, $\mathcal{T}$ is definable from parameters over $U|\delta$, and $U|\delta$ is extender algebra generic over $M(\mathcal{T})$.   ⊣

Condition 6 prevents us from requiring $*$-translation.

Definition 5.4. Let $\mathcal{T}$ be P-illustrative and N, U as in 5.3. We define the *P-construction* $\mathscr{P}^U(M(\mathcal{T}))$ of U over $M(\mathcal{T})$. This is the largest premouse P such that:

1. $M(\mathcal{T}) \trianglelefteq P \vDash$ "δ is Woodin", where $\delta = \delta(\mathcal{T})$,

2. for $\alpha \in [\delta, \mathrm{OR}^P]$, $P||\alpha$ is active iff $U||\alpha$ is active,[46]

3. Let α_0 be as in 5.3, let $\alpha > \delta$ be such that $U||\alpha$ is active and let $E = F^{P||\alpha}$ and $F = F^{U||\alpha}$. Then either:

   (a) $\mathrm{crit}(F) > \kappa_0^M$ (so $\mathrm{crit}(F) > \delta$) and $E{\upharpoonright}\mathrm{OR} = F{\upharpoonright}\mathrm{OR}$, or

   (b) $\mathrm{crit}(F) = \kappa_0^M$ and $E \circ i_{0\alpha_0}^{\mathcal{T}}{\upharpoonright}\kappa_0^{+M} = F{\upharpoonright}\kappa_0^{+M}$.   ⊣

Remark 5.5. A key point in the above definition is that in condition 3b, with $j = i_{0\alpha_0}^{\mathcal{T}}$, we only require $E \circ j$ and F to agree over *ordinals*, not the full model $N|\kappa_0^{+N}$ (although $N|\kappa_0^{+N}$ is an initial segment of both sides). In fact (as we require that P is a premouse),

$$E \circ j{\upharpoonright}(N|\kappa_0^{+N}) \neq F{\upharpoonright}(N|\kappa_0^{+N}),$$

because $P|\kappa_0 = N|\kappa_0 = U|\kappa_0$, but $P|\lambda(E) \neq N|\lambda(F)$, and because E must cohere $P|\alpha$, therefore $E(j(N|\kappa_0)) \neq F(N|\kappa_0)$. However, recall the following fact, which we leave as an exercise for the reader.

Fact 5.6. *Let N be a passive premouse, κ be a cardinal of N, $X \subseteq \kappa^{+N}$ be unbounded in κ^{+N} and $f : X \to \mathrm{OR}^N$. Then there is at most one active premouse N' whose reduct is N and $F^N{\upharpoonright}\mathrm{OR} = f$, and in fact, N' is Σ_1-definable over (N, f).*

(Note though that it is important that N is given; there can actually be another active N_1' with $N \neq N_1$, but $F^{N_1'}{\upharpoonright}\mathrm{OR} = f$.)

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[45]Note there is no $F \in \mathbb{E}^U$ with $\mathrm{crit}(F) = \kappa_0^N$ and $\lambda(F) = \eta$, since otherwise N is past superstrong.

[46] $U||\delta$ is passive, because η is the largest cardinal of $U||\delta$ and by Footnote 45.

Remark 5.7. It is not immediate that the P-construction P is well-defined, as we have defined it directly as the largest premouse with the above properties, and one also needs one small observation to see that, if well-defined, then P is unique (and appropriately locally definable).

Consider instead defining $P|\alpha$ and $P||\alpha$ by recursion on α. Note that $\bar{j} = j \restriction \kappa_0^{+N} \in U|\eta \subseteq U|\delta$, so $\bar{j}$ is available as a parameter when making definitions over $U||\beta$ for some $\beta \geq \delta$, and $\bar{j}$ is also in the generic extension $(P||\beta)[N|\delta]$.

Now given $P||\beta$ for all $\beta < \alpha$, α a limit, (all $P||\beta$ sound), we get a premouse $P|\alpha$ satisfying "δ is Woodin". Suppose $U||\alpha$ is active with F. We need to see that we get a premouse $P||\alpha$, fine structurally equivalent with $U||\alpha$ (modulo the generic). We need to in particular see that there is a unique premouse $P||\alpha$ with the right properties. If $\mathrm{crit}(F^{U||\alpha}) = \kappa_0$, existence is not immediately clear, and will be verified in Lemma 5.8. Uniqueness and the manner in which $F^{P||\alpha}$ is determined, requires a short argument. We have $P|\alpha$ and can compute $F^{P||\alpha} \restriction \mathrm{rg}(\bar{j})$, and $\mathrm{rg}(\bar{j})$ is cofinal in $\mu^{+(P|\alpha)}$ (where $\mu = \kappa_0^{M_{\alpha_0}^{\mathcal{T}}} = \mathrm{crit}(F^{P||\alpha})$). By Fact 5.6 this (very locally) determines $F^{P||\alpha}$. For the case that $\mathrm{crit}(F^{U||\alpha}) > \delta$, one makes the usual P-construction observations, although the generic equivalence here involves the parameter $\bar{j}$.

Overall we maintain level by level that

$$P|\beta \text{ is } \Delta_1^{U|\beta}(\{P|\delta, \bar{j}\}),$$

$$P||\beta \text{ is } \Delta_1^{U||\beta}(\{P|\delta, \bar{j}\}),$$

uniformly in $\beta > \delta$ (and recalling $P|\delta = M(\mathcal{T})$), and also that

$$U|\beta =_\delta^* (P|\beta)[U|\delta] \text{ and } U|\beta \text{ is } \Delta_1^{(P|\beta)[U|\delta]}(\{U|\delta, \bar{j}\}),$$

$$U||\beta =_\delta^* (P||\beta)[U|\delta] \text{ and } U||\beta \text{ is } \Delta_1^{(P||\beta)[U|\delta]}(\{U|\delta, \bar{j}\}),$$

uniformly in β, where $(P|\beta)[U|\delta]$ is a generic extension, which for definability purposes has has $P|\beta$ available as a predicate (and similarly for $P||\beta$), and where $=^*$ means that $U|\beta$ is the premouse which extends $U|\delta$, followed by the small forcing extension of extenders E in $\mathbb{E}^{P|\beta}$ or $\mathbb{E}_+^{P||\beta}$ when $\mathrm{crit}(E) > \delta$, and the extender determined by $E \circ \bar{j}$ otherwise, and also that the usual fine structural correspondence holds between the two sides (employing an extender algebra name for $\bar{j}$).

Lemma 5.8. *Let $\mathcal{T}$ be P-illustrative on M, via Σ_M, and $b = \Sigma_M(\mathcal{T})$, and let U be as in Definition 5.3. If $\mathcal{T}$ is short let $Q = Q(\mathcal{T}, b)$, and otherwise let $Q = M_b^{\mathcal{T}}$. Then $\mathscr{P}^U(M(\mathcal{T})) = Q$.*

Proof. Let $\delta = \delta(\mathcal{T})$. Working in V, we "compare the phalanx $\Phi(\mathcal{T}, Q)$ with the phalanx $((M, \delta), U)$, modulo the generic at δ". That is, in the "comparison", we *only use extenders indexed above δ*, with least disagreements determined by the restrictions of extenders to the ordinals, and after composing extenders E overlapping δ on the $\Phi(\mathcal{T}, Q)$ side with $\bar{j}$ (notation as above). That is, we

define normal padded trees $\mathcal{U}$ on $\Phi(\mathcal{T}, Q)$ and $\mathcal{V}$ on $((M, \delta), U)$, with both corresponding to trees on M via Σ_M,[47] and such that, given $(\mathcal{U}, \mathcal{V}) \restriction (\alpha + 1)$, letting $\gamma > \delta$ be least such that $E = F^{M_\alpha^{\mathcal{U}} || \gamma} \neq \emptyset$ or $F = F^{M_\alpha^{\mathcal{V}} || \gamma} \neq \emptyset$ and either:

- $\text{crit}(E) = \mu$ and $E \circ j \restriction \kappa_0^{+M} \neq F \restriction \kappa_0^{+M}$, or

- $\text{crit}(E) > \mu$ (so $\text{crit}(E) > \delta$) and $E \restriction \text{OR} \neq F \restriction \text{OR}$, or

- $E = \emptyset \neq F$,

then $E_\alpha^{\mathcal{U}} = E$ and $E_\alpha^{\mathcal{V}} = F$ (and the comparison terminates if there is no such $\gamma > \delta$). We need to see that this comparison is trivial, i.e. no extenders are used. So suppose otherwise.

CLAIM 1. The comparison terminates in set length.

Proof. Note that if $E_\beta^{\mathcal{U}}$ overlaps δ then $M_{\beta+1}^{\mathcal{U}}$ is proper class and

$$\kappa_0^{M_{\beta+1}^{\mathcal{U}}} = \lambda(E_\beta^{\mathcal{U}})$$

is a cutpoint of $M_{\beta+1}^{\mathcal{U}}$, so $\mathcal{U} \restriction [\beta+1, \infty)$ is above $\lambda(E_\beta^{\mathcal{U}})$. Therefore there is at most one such β. Likewise for $\mathcal{V}$. But the comparison after these overlaps becomes standard comparison modulo the small generic at δ, so the usual argument then shows that the comparison terminates. $\square$

So say we get $\text{lh}(\mathcal{U}, \mathcal{V}) = \alpha + 1$.

CLAIM 2. At least one of the trees $\mathcal{U}, \mathcal{V}$ uses an extender overlapping δ.

Proof. Suppose otherwise. Suppose $\text{OR}(M_\alpha^{\mathcal{U}}) = \text{OR}(M_\alpha^{\mathcal{V}})$; the other case is similar. Then the level-by-level translation process described in 5.7 works between $M_\alpha^{\mathcal{U}}$ and $M_\alpha^{\mathcal{V}}$, and we get fine structural correspondence above δ. Suppose that $\text{OR}(M_\alpha^{\mathcal{U}}) < \text{OR}$. Let A be either the core of $M_\alpha^{\mathcal{U}}$ if $\rho_\omega(M_\alpha^{\mathcal{U}}) > \delta$, and otherwise

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[47]For $((M, \delta), U)$, the notation means that the exchange ordinal associated to M is δ, so in fact, since $\delta < \text{lh}(E_0^{\mathcal{V}})$, and δ is a strong κ_0-cutpoint of U, if $\text{crit}(E_\alpha^{\mathcal{V}}) < \delta$ then $\text{crit}(E_\alpha^{\mathcal{V}}) = \kappa_0$. If $U = M$ then we $\mathcal{V}$ is directly equivalent to a tree on M via Σ_M. If $U = \text{Ult}(M, E)$ where $E \in \mathbb{E}^M$ and $\text{crit}(E) = \kappa_1$, then a simple instance of normalization produces the tree $\mathcal{V}'$ on M, via Σ_M, corresponding to $\mathcal{V}$: If $\mathcal{V} \restriction \beta$ is above δ but based on $U | \lambda^{+U} = U | \kappa_1^{+U}$, then $\mathcal{V}' \restriction \beta$ is the tree on M, via Σ_M, which uses the same extenders and has the same tree structure as does $\mathcal{V} \restriction \beta$. Note that because η, δ are strong κ_0-cutpoints of U, and by the Mitchell-Steel ISC, E's natural length $\nu(E) \leq \eta$, so $\rho_1^{M || \text{lh}(E)} \leq \eta$, so for each $\alpha + 1 < \beta$ such that $\text{pred}^{\mathcal{V}}(\alpha + 1) = 0$ and $\alpha + 1 \notin \mathscr{D}^{\mathcal{U}}$, we have that $\mathcal{V}'$ drops in model and degree at $\alpha + 1$ to $(M_{\alpha+1}^{*\mathcal{V}'}, \text{deg}_{\alpha+1}^{\mathcal{V}'}) = (M || \text{lh}(E), 0)$; $\mathcal{V} \restriction \beta$ and $\mathcal{V}' \restriction \beta$ otherwise agree in drop and degree structure, and so their models and embeddings agree in a simple manner. If $\mathcal{V} \restriction \beta$ is as above but $[0, \beta]_{\mathcal{V}}$ does not drop and $\kappa_1^{+M_\beta^{\mathcal{V}}} < \text{lh}(E_\beta^{\mathcal{U}})$, then $\mathcal{V}'$ uses first $F(M_\beta^{\mathcal{V}'})$ as an extra extender, and note then that $M_\beta^{\mathcal{V}} = M_{\beta+1}^{\mathcal{V}'}$. Since $\kappa_1^{M_\beta^{\mathcal{V}}}$ is a κ_0-cutpoint of $M_\beta^{\mathcal{V}}$, if $\mathcal{V} \restriction [\beta, \gamma)$ is above $\kappa_1^{M_\beta^{\mathcal{V}}}$ then $\mathcal{V} \restriction [\beta, \gamma)$ is directly equivalent to $\mathcal{V}' \restriction [\beta + 1, \gamma')$ where $\gamma' = \gamma + 1$ or $\gamma = \gamma$ in the obvious manner. Finally if ξ least such that $\text{crit}(E_\xi^{\mathcal{V}}) = \kappa_0$ (either with $\xi < \beta$ or $\beta < \xi$ as above) then $\mathcal{V}, \mathcal{V}'$ are again directly equivalent thereafter.

the δ-core of $M_\alpha^{\mathcal{U}}$. Let B be likewise from $M_\alpha^{\mathcal{V}}$. Let π, σ be the core maps respectively. Then by the fine structural correspondence and forcing calculations,

$$\mathrm{rg}(\pi) \cap \mathrm{OR} = \mathrm{rg}(\sigma) \cap \mathrm{OR},$$

so $\mathrm{OR}^A = \mathrm{OR}^B$. But also, the core maps preserve the fact that the level-by-level translation works, so $B =_\delta^* A[M(\mathcal{T})]$ etc (with fine structural agreement up to the relevant level). But we had $A \trianglelefteq M_\gamma^{\mathcal{U}}$ and $B \trianglelefteq M_\gamma^{\mathcal{V}}$ for some γ, and either $E_\gamma^{\mathcal{U}}$ or $E_\gamma^{\mathcal{V}}$ came from $\mathbb{E}_+^A$ or $\mathbb{E}_+^B$ respectively, a contradiction. So $\mathrm{OR}(M_\alpha^{\mathcal{U}}) = \mathrm{OR}$, so there is no dropping on main branches, and $\mathcal{T}$ is maximal. But now we can just replace "δ-core" with the hull of $\delta \cup \mathscr{I}$, where $\mathscr{I}$ is the class of indiscernibles of $M_\alpha^{\mathcal{U}}$, or equivalently, $M_\alpha^{\mathcal{V}}$, and run the analogous argument, using that Q, U are δ-sound (if $U \neq M$, this is because η is a strong κ_0-cutpoint of U, and hence if $U = \mathrm{Ult}(M, E)$ where $E \in \mathbb{E}^M$ with $\mathrm{crit}(E) = \kappa_1$, then $\nu(E) \leq \eta$, where $\nu(E)$ is the natural length of E). $\qquad \square$

Now let β be least such that $E_\beta^{\mathcal{U}}$ or $E_\beta^{\mathcal{V}}$ overlaps δ. The following claim is the most central issue:

CLAIM 3. Not both of $E_\beta^{\mathcal{U}}, E_\beta^{\mathcal{V}}$ overlap δ.

Proof. Suppose $F = E_\beta^{\mathcal{V}}$ overlaps δ. Then $F \!\restriction\! \kappa_0^{+M}$ is a restriction of the iteration map

$$\mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathrm{Ult}(M,F)}.$$

Similarly, supposing $E = E_\beta^{\mathcal{U}}$ overlaps δ, $E \!\restriction\! \mathrm{OR}$ is a restriction of the iteration map

$$\mathcal{M}_\infty^{M_{\alpha_0}^{\mathcal{T}}} \to \mathcal{M}_\infty^{\mathrm{Ult}(M_{\alpha_0}^{\mathcal{T}}, E)},$$

so $E \circ j \!\restriction\! \kappa_0^{+M}$ is the restriction of the iteration map

$$\mathcal{M}_\infty \to \mathcal{M}_\infty^{\mathrm{Ult}(M_{\alpha_0}^{\mathcal{T}}, E)}.$$

So it suffices to see that

$$\mathcal{M}_\infty^{\mathrm{Ult}(M_{\alpha_0}^{\mathcal{T}}, E)} = \mathcal{M}_\infty^{\mathrm{Ult}(M,F)}, \tag{24}$$

as then $E \circ j \!\restriction\! \kappa_0^{+M} = F \!\restriction\! \kappa_0^{+M}$, contradicting the disagreement of extenders. But (24) holds because

$$(M_\beta^{\mathcal{U}} | \mathrm{lh}(E))[M|\delta] =_\delta^* M_\beta^{\mathcal{V}} | \mathrm{lh}(F),$$

since $\delta < \lambda(E) = \lambda(F)$ and E, F constitute the least disagreement. $\qquad \square$

Now assume for notational simplicity that $E_\beta^{\mathcal{U}}$ overlaps δ; the other case is very similar. Note that:

– $[0, \beta+1]_{\mathcal{U}}$ does not drop in model, and

– $\mathcal{U} \!\restriction\! [\beta+1, \mathrm{lh}(\mathcal{U}))$ is above $\lambda(E_\beta^{\mathcal{U}})$.

As in the proof of Claim 2 we get:

CLAIM 4. Neither $(\mathcal{T}, b)^\frown\mathcal{U}$ nor $\mathcal{V}$ drops on its main branch.

It easily follows that there is $\gamma > \beta$ such that $E_\gamma^\mathcal{V}$ overlaps δ. We have $\beta + 1 \in b^\mathcal{U}$ and $\gamma + 1 \in b^\mathcal{V}$. Let $\mathscr{I} = \mathscr{I}^{M_\alpha^\mathcal{U}} = \mathscr{I}^{M_\alpha^\mathcal{V}}$. Let $\lambda = \lambda(E_\beta^\mathcal{U})$, so

$$\kappa_0^{+M} < \delta < \lambda < \lambda(E_\gamma^\mathcal{V}),$$

so $F_\lambda = E_\gamma^\mathcal{V}{\upharpoonright}\lambda$ is a non-whole segment of $E_\gamma^\mathcal{V}$, by the ISC and smallness of M. Let

$$H^\mathcal{U} = \mathrm{cHull}^{M_\alpha^\mathcal{U}}(\lambda \cup \mathscr{I}) \text{ and } H^\mathcal{V} = \mathrm{cHull}^{M_\alpha^\mathcal{V}}(\lambda \cup \mathscr{I}),$$

and $\pi^\mathcal{U}, \pi^\mathcal{V}$ the uncollapses. Then $\kappa_0^{H^\mathcal{U}} = \lambda$ because $H^\mathcal{U} = M_{\beta+1}^\mathcal{U}$ (and note $i_{\beta+1,\alpha}^\mathcal{U} = \pi^\mathcal{U}$). Similarly, $H^\mathcal{V} = \mathrm{Ult}(M, F_\lambda)$, but $\lambda < i_{F_\lambda}^M(\kappa_0) = \kappa_0^{H^\mathcal{V}}$ since F_λ is not whole. But since $\delta < \lambda$, we also have $\pi^\mathcal{U}{\upharpoonright}\mathrm{OR} = \pi^\mathcal{V}{\upharpoonright}\mathrm{OR}$, so $\pi^\mathcal{V}(\lambda) = \pi^\mathcal{U}(\lambda) = \kappa_0^{M_\alpha^\mathcal{U}} = \kappa_0^{M_\alpha^\mathcal{V}}$, so $\lambda = \kappa_0^{H^\mathcal{V}}$, contradiction. $\square$

We now want to consider similar P-constructions internal to $\mathscr{V}_1$, and also iterates of $\mathscr{V}_1$ and their generic extensions.

Definition 5.9. Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let $\mathcal{T}$ be an iteration tree on $\mathscr{V}$. We say that $\mathcal{T}$ is *dl-somewhat-relevant (dsr)* iff there are $\mathcal{T}_0, \mathscr{V}', \mathcal{T}_1$ such that:

1. $\mathcal{T}$ is short-normal,

2. $\mathcal{T}$ has lower and upper components $\mathcal{T}_0, \mathcal{T}_1$ respectively, $\mathcal{T}_0$ has successor length, $b^{\mathcal{T}_0}$ does not drop, $\mathscr{V}' = M_\infty^{\mathcal{T}_0}$, and $\mathcal{T}_1$ (on $\mathscr{V}'$) is above $\gamma^{\mathscr{V}'}$,

3. for each $\beta + 1 < \mathrm{lh}(\mathcal{T}_1)$, $\delta_0^{\mathscr{V}'}$ is the unique Woodin of $M_\beta^{\mathcal{T}_1}|\mathrm{lh}(E_\beta^{\mathcal{T}_1})$.

Note that every dl-somewhat-relevant tree on $\mathscr{V}$ is based on $\mathscr{V}|\delta_1^{\mathscr{V}}$. $\dashv$

Easily:

Lemma 5.10. Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let $\mathcal{T}$ on $\mathscr{V}$ be via $\Sigma_\mathscr{V}$ and δ_1-maximal. Then $\mathcal{T}$ is dsr.

Definition 5.11. For a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate $\mathscr{V}$ of $\mathscr{V}_1$, write $\Sigma_{\mathscr{V},\mathrm{sh}}$ for the restriction of $\Sigma_\mathscr{V}$ to δ_1-short trees, and $\Sigma_{\mathscr{V},\mathrm{sh}}^{\mathrm{dsr}}$ for the restriction of $\Sigma_{\mathscr{V},\mathrm{sh}}$ to dsr trees. $\dashv$

Definition 5.12. Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let $\mathcal{T}$ be an iteration tree on $\mathscr{V}$. Let $\mathbb{P} \in \mathscr{V}$ and g be $(\mathscr{V}, \mathbb{P})$-generic. Say $\mathcal{T}$ is *P-suitable for* $\mathscr{V}[g]$ iff there are $\mathcal{T}_0, \mathscr{V}', \mathcal{T}_1, E_1, U, E_0, \eta, \delta, \lambda$ such that:

(a) g is $(\mathscr{V}, \mathbb{P})$-generic and $\mathcal{T} \in \mathscr{V}[g]$,

(b) $\mathcal{T}$ is short-normal with lower and upper components $\mathcal{T}_0, \mathcal{T}_1$ respectively, $\mathscr{V}' = M_\infty^{\mathcal{T}_0}$ exists and $b^{\mathcal{T}_0}$ is non-dropping, and $\mathcal{T}_1$ is based on $\mathscr{V}'|\delta_1^{\mathscr{V}'}$,

(c) $\mathcal{T}$ is via $\Sigma_{\mathscr{V}}$,

(d) either

    (i) $E_1 = \emptyset$ and $U = \mathscr{V}$, or

    (ii) $E_1 \in \mathbb{E}^{\mathscr{V}}$ is short and $\mathscr{V}$-total, $\mathrm{crit}(E_1) = \kappa_1^{\mathscr{V}}$ and $U = \mathrm{Ult}(\mathscr{V}, E_1)$,

(e) $\mathbb{P} \in U|\kappa_1^U$ and $\mathcal{T}' \in (U|\kappa_1^U)[g]$ where $\mathcal{T}'$ on $\mathscr{V}|\delta_1^{\mathscr{V}}$ is equivalent to $\mathcal{T}$,

(f) either

    (i) $E_0 = \emptyset$ and $\mathcal{T}_0$ is trivial (so $\mathscr{V}' = \mathscr{V}$), or

    (ii) $E_0 \in \mathbb{E}^{\mathscr{V}}$ is long, $\gamma^{\mathscr{V}} < \mathrm{lh}(E_0)$ and the lower component $\mathcal{T}_0$ of $\mathcal{T}$ is just the (successor length) short-normal tree corresponding to E_0 (so $\mathscr{V}' = \mathrm{Ult}(\mathscr{V}, E_0)$),

(g) $\delta_0^{\mathscr{V}'} < \eta < \kappa_1^U$ and η is a strong $\delta_0^{\mathscr{V}}$-cutpoint of U, η is a U-cardinal, $\mathbb{P} \in U|\eta$, and if $E_1 \neq \emptyset$ then $\kappa_1^{\mathscr{V}} < \eta$,

(h) $\mathcal{T}$ has limit length, $\eta < \delta = \delta(\mathcal{T}) < \kappa_1^U$, η is the largest cardinal of $U|\delta$, $\mathcal{T}$ is definable from parameters over $(U|\delta)[g]$, and $(U|\delta, g)$ is $\mathbb{B}_{\delta\delta_0^{\mathscr{V}'}}^{M(\mathcal{T})}$-generic over $M(\mathcal{T})$.

Say $\mathcal{T}$ is *dl-relevant (for $\mathscr{V}$)* iff $\mathcal{T} \in \mathscr{V}$ is δ_1-maximal (hence dsr) P-suitable, as witnessed by $E_1 = \emptyset$.   ⊣

Definition 5.13. Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let g be set-generic over $\mathscr{V}$. Let $\mathcal{T} \in \mathscr{V}[g]$ be P-suitable for $\mathscr{V}[g]$, as witnessed by $U, \mathcal{T}_0$. Then $\mathscr{P}^{U,g}(M(\mathcal{T}))$ denotes the P-construction P of $U[g]$ over $M(\mathcal{T})$, using $\mathbb{E}^U$, computed analogously to that in Definition 5.4; so when $F = F^{U||\alpha} \neq \emptyset$ and F is long, then $E = F^{P||\alpha}$ is determined by demanding $E \circ \bar{j} \subseteq F$, where $\bar{j} = i_{0\infty}^{\mathcal{T}_0}|\delta_0^{\mathscr{V}}$.   ⊣

It is now straightforward to deduce a version of Lemma 5.8 for dsr P-suitable trees $\mathcal{T} \in \mathscr{V}_1$ on $\mathscr{V}_1$, by translating them to trees on M and applying 5.8. Combined with the minimal inflation method used for M, this allows us to compute the δ_1-short tree strategy for $\mathscr{V}_1$ inside $\mathscr{V}_1$, and also the models for forming the second direct limit system. However, before we proceed to this, we want to also consider the analogous issues for iterates $\mathscr{V}$ of $\mathscr{V}_1$ (and generic extensions $\mathscr{V}[g]$ thereof). The argument given above does not immediately adapt to such iterates $\mathscr{V}$ in general, because (i) $\mathscr{V}$ need not be as sound as $\mathscr{V}_1$, and (ii) $\mathscr{V}$ need not correspond appropriately to an iterate of M. To deal with these possibilities, we will adjust somewhat the conclusion and argument for Lemma 5.8, in Lemma 5.16 below. Also note that it is not immediate that P-suitability and dl-relevance are first-order over $\mathscr{V}$ (or $\mathscr{V}_1$), because of the demand that $\mathcal{T}$ be via $\Sigma_{\mathscr{V}}$. We will address this issue also, in a manner similar to that for M.

5.1.2 DSR trees on iterates of $\mathscr{V}_1$

To deal with issue (ii) mentioned above, it turns out we can replace the use of M (or some iterate thereof) with an M_{swsw}-like *generic extension* N of $\mathscr{V}$ (together with such an extension of a related iterate $\mathscr{V}_1$; see below). It inherits iterability (above κ_0^N) from the corresponding iterate of $\mathscr{V}_1$:

Lemma 5.14. *Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let g be $\mathbb{L}^{\mathscr{V}}$-generic and N an M_{swsw}-like premouse such that $N =^* \mathscr{V}[g]$ and $\mathscr{V} = \mathscr{V}_1^N$. Let $\mathscr{I} = \mathscr{I}^{\mathscr{V}}$. Then:*

1. *N is $(0, \mathrm{OR})$-iterable with respect to trees $\mathcal{T}$ with $\mathrm{lh}(E_0^{\mathcal{T}}) > \kappa_0^N$.*

2. *$\mathscr{I}$ is the class $\mathscr{I}^N$ of Silver indiscernibles for N (with respect to the generator set $\kappa_1^{\mathscr{V}} = \kappa_1^N$). If $\mathscr{V}$ is η-sound where $\delta_0^{\mathscr{V}} \leq \eta$ then N is η-sound. If $\mathscr{V}$ is $\delta_0^{\mathscr{V}}$-sound then N is κ_0^N-sound.*

Proof. Part 1: Let $P \lhd N$ with $\rho_\omega^P = \kappa_0^N$. Then P is $(0, \mathrm{OR})$-iterable with respect to trees $\mathcal{T}$ with $\mathrm{lh}(E_0^{\mathcal{T}}) > \kappa_0^N$, because κ_0^N is a cutpoint of P, and letting $E \in \mathbb{E}^{\mathscr{V}}$ be long, E extends to $E^+ \in \mathbb{E}^N$, and $E^+(P)$ is above-$\lambda(E)$-$(0, \mathrm{OR})$-iterable, since iterating $E^+(P)$ above $\lambda(E)$ is equivalent to iterating $\mathscr{V} \| \mathrm{OR}(E^+(P))$ above $\lambda(E)$.

So we may assume that $\kappa_0^{+N} = \delta_0^{\mathscr{V}} < \mathrm{lh}(E_0^{\mathcal{T}})$, and so $\gamma^{\mathscr{V}} < \mathrm{lh}(E_0^{\mathcal{T}})$. Consider translatable trees $\mathcal{T}$ on N. We get an iteration strategy for such trees induced by $\Sigma_{\mathscr{V}}$, and the resulting iterates of $N, \mathscr{V}$ are related according to Lemma 4.64. Now suppose $\mathcal{T} {\upharpoonright} (\alpha + 1)$ is translatable, and let $\mathcal{U}$ on $\mathscr{V}$ be its translation to $\mathscr{V}$, but $\mathcal{T} {\upharpoonright} (\alpha + 2)$ is not translatable. Then α is a limit ordinal and a limit of stages when $\mathcal{U}$ uses a long extender, and $\mathrm{lh}(E_\alpha^{\mathcal{T}}) < \delta_0^{M_\alpha^{\mathcal{U}}}$. But then $\mathcal{T} {\upharpoonright} [\alpha, \infty)$ is just a tree on some $P \lhd M_\alpha^{\mathcal{T}}$ where $\rho_\omega^P = \kappa_0^{M_\alpha^{\mathcal{T}}}$, and $\kappa_0^{M_\alpha^{\mathcal{T}}}$ is a cutpoint of P, and this P is also iterable above $\kappa_0^{M_\alpha^{\mathcal{T}}}$, like in the previous paragraph.

Part 2: By genericity, $\mathscr{I}$ form indiscernibles for N. Note that $\mathscr{V}$ is $\kappa_1^{\mathscr{V}}$-sound. Let $\eta \in [\delta_0^{\mathscr{V}}, \kappa_1^{\mathscr{V}}]$ be least such that $\mathscr{V}$ is η-sound. Note that $N = \mathrm{Hull}_1^N(\mathscr{I} \cup \eta)$. So $\mathscr{I}^N = \mathscr{I}$. Finally suppose $\eta = \delta_0^{\mathscr{V}}$, so $N = \mathrm{Hull}_1^N(\kappa_0^{+N} \cup \mathscr{I})$. But also $\mathrm{Hull}_1^{\mathscr{V}}(\mathscr{I})$ is cofinal in $\delta_0^{\mathscr{V}}$ and $\mathscr{V}$ is definable over N, so $\mathrm{Hull}_1^N(\kappa_0^N \cup \mathscr{I})$ is cofinal in κ_0^{+N} and transitive below κ_0^{+N}, hence contains all of κ_0^{+N}, and hence all of N. $\qquad\square$

Definition 5.15. Under the hypotheses of Lemma 5.14, let $\Phi_{N \supseteq \mathscr{V}}$ be the above-κ_0^N-strategy for N induced by $\Sigma_{\mathscr{V}}$ as in the proof of the lemma. (Note that the components of this strategy which do not translate to a tree on $\mathscr{V}$, i.e. on one of the projecting structures P in the proof, are uniquely determined by P.) $\dashv$

Lemma 5.16. *Let $\mathscr{V}$ be a $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let $\mathcal{S}$ be the short-normal tree on $\mathscr{V}_1$, via $\Sigma_{\mathscr{V}_1}$, with last model $\mathscr{V}$. Let $\bar{\mathscr{V}} = \mathrm{cHull}_1^{\mathscr{V}}(\mathscr{I}^{\mathscr{V}} \cup \delta_1^{\mathscr{V}})$ and $\bar{\bar{\mathscr{V}}} = \mathrm{cHull}_1^{\mathscr{V}}(\mathscr{I}^{\mathscr{V}} \cup \delta_0^{\mathscr{V}})$. Then*

1. *$\bar{\mathscr{V}} = M_\alpha^{\mathcal{S}}$ for some $\alpha \in b^{\mathcal{S}}$, and $\mathcal{S} {\upharpoonright} [\alpha, \infty)$ is equivalent to a tree on $\bar{\mathscr{V}}$ via $\Sigma_{\bar{\mathscr{V}}}$, $\bar{\mathscr{V}} | \delta_1^{\bar{\mathscr{V}}} = \mathscr{V} | \delta_1^{\mathscr{V}}$, and $\bar{\mathscr{V}}$ is $\delta_1^{\bar{\mathscr{V}}}$-sound. Likewise for $\bar{\bar{\mathscr{V}}}$ and $\delta_0^{\mathscr{V}}$.*

Let $\mathbb{P} \in \mathscr{V}$ and g be $(\mathscr{V}, \mathbb{P})$-generic. Let $\mathcal{T} \in \mathscr{V}[g]$, on $\mathscr{V}$, be dsr-P-suitable and $b = \Sigma_{\mathscr{V}}(\mathcal{T})$, and U, η, δ as in Definition 5.12. Let $\bar{\mathcal{T}}$ be the tree on $\bar{\mathscr{V}}$ equivalent to $\mathcal{T}$. If $\mathcal{T}$ is δ_1-short let $\bar{Q} = Q = Q(\mathcal{T}, b)$, and otherwise let $Q = M_b^{\mathcal{T}}$ and

$$\bar{Q} = M_b^{\bar{\mathcal{T}}} = \mathrm{cHull}_1^Q(\mathscr{I}^Q \cup \delta).$$

Let $P = \mathscr{P}^{U,g}(M(\mathcal{T}))$. If P is set-sized let $\bar{P} = P$ and otherwise let $\bar{P} = \mathrm{Hull}_1^P(\mathscr{I}^U \cup \delta)$ and $\pi : \bar{P} \to P$ be the uncollapse. Then:

2. $\bar{P} = \bar{Q}$

3. if $\mathcal{T}$ is δ_1-maximal then $\delta = (\eta^+)^U$ (and $P, \bar{P}$ are proper class),

4. if $\bar{P} \neq P$ then P is an above-δ, $\Sigma_{\bar{Q}}$-iterate of $\bar{Q}$ (and hence a $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$) and π is the iteration map, so $\pi\,\text{``}\mathscr{I}^{\bar{Q}} = \mathscr{I}^P = \mathscr{I}^{\mathscr{V}}$.

Proof (sketch). Part 1: This is straightforward and left to the reader.

Part 2: We will prove this by considering a comparison of two phalanxes. It will take a little work to define the phalanxes and describe their relevant properties.

Let $\mathcal{S} = \mathcal{S}_0 \,\widehat{}\, \mathcal{S}_1 \,\widehat{}\, \mathcal{S}_2$ where $\mathcal{S}_0$ is the lower component of $\mathcal{S}$, $\mathcal{S}_1 \,\widehat{}\, \mathcal{S}_2$ the upper component, with $\mathcal{S}_1$ based on $M_\infty^{\mathcal{S}_0}|\delta_1^{M_\infty^{\mathcal{S}_0}}$ and $\mathcal{S}_2$ above $\delta_1^{M_\infty^{\mathcal{S}_1}}$. Then $\bar{\bar{\mathscr{V}}} = M_\infty^{\mathcal{S}_0}$ is $\delta_0^{\bar{\bar{\mathscr{V}}}}$-sound and $\mathcal{S}_1$ is above $\gamma^{\bar{\bar{\mathscr{V}}}}$, and is via $\Sigma_{\bar{\bar{\mathscr{V}}}}$, and $\bar{\mathscr{V}} = M_\infty^{\mathcal{S}_1}$ is $\delta_1^{\bar{\mathscr{V}}}$-sound and $\mathcal{S}_2$ is via $\Sigma_{\bar{\mathscr{V}}}$.

Let $\mathscr{V}^+$ be an M_{swsw}-like generic extension of $\mathscr{V}$ with $\mathscr{V} = \mathscr{V}_1^{\mathscr{V}^+}$, and such that $\mathscr{V}^+, g$ are mutually $\mathscr{V}$-generic. Let $g_{\mathscr{V}^+} \subseteq \mathbb{L}^{\mathscr{V}}$ be the generic filter. Since $\mathbb{L}^{\mathscr{V}} = \mathbb{L}^{\bar{\bar{\mathscr{V}}}}$ is below $\mathrm{crit}(i_{0\infty}^{\mathcal{S}_1 \,\widehat{}\, \mathcal{S}_2})$, we have $\mathbb{L}^{\bar{\bar{\mathscr{V}}}} = \mathbb{L}^{\mathscr{V}}$ and $g_{\mathscr{V}^+}$ is also $(\bar{\bar{\mathscr{V}}}, \mathbb{L}^{\bar{\bar{\mathscr{V}}}})$-generic, and extends uniquely to an M_{swsw}-like generic extension $\bar{\bar{\mathscr{V}}}^+$ of $\bar{\bar{\mathscr{V}}}$ with $\bar{\bar{\mathscr{V}}} = \mathscr{V}_1^{\bar{\bar{\mathscr{V}}}^+}$; note $\bar{\bar{\mathscr{V}}}^+|\kappa_0^{+\bar{\bar{\mathscr{V}}}^+} = \mathscr{V}^+|\kappa_0^{+\mathscr{V}^+}$. Lemma 5.14 applies to $(\bar{\bar{\mathscr{V}}}^+, \bar{\bar{\mathscr{V}}})$, so $\mathcal{S}_1 \,\widehat{}\, \mathcal{S}_2$ translates to a tree $\mathcal{S}_1^+ \,\widehat{}\, \mathcal{S}_2^+$ on $\bar{\bar{\mathscr{V}}}^+$, via $\Phi_{\bar{\bar{\mathscr{V}}}^+ \supseteq \bar{\bar{\mathscr{V}}}}$, and note that $\mathscr{V}^+ = M_\infty^{\mathcal{S}_1^+ \,\widehat{}\, \mathcal{S}_2^+}$.

Let $E_0, E_1 \in \mathbb{E}^{\mathscr{V}}$ be as in Definition 5.12. Let $E_i^+ \in \mathbb{E}^{\mathscr{V}^+}$ with $\mathrm{lh}(E_i^+) = \mathrm{lh}(E_i)$ (or $E_i^+ = \emptyset = E_i$). Let $\bar{\bar{U}}_0^+ = \mathrm{Ult}(\bar{\bar{\mathscr{V}}}^+, E_0^+)$. Since $\bar{\bar{\mathscr{V}}}^+$ is $\kappa_0^{\bar{\bar{\mathscr{V}}}^+}$-sound, $\bar{\bar{U}}_0^+$ is $\kappa_0^{\bar{N}'}$-sound. (Note $\bar{\bar{U}}_0^+$ need not be $\subseteq \mathscr{V}^+$.)

Let $U^+ = \mathrm{Ult}(\mathscr{V}^+, E_1^+)$, if $E_1 \neq \emptyset$, and $U^+ = \mathscr{V}^+$ otherwise. Since η is a δ_0^U-strong cutpoint of U and $U|\delta$ has largest cardinal η, so is δ, and δ is also a $\kappa_0^{U^+}$-strong cutpoint of U^+. Let $\bar{U}^+ = \mathrm{Hull}_1^{U^+}(\mathscr{I}^{U^+} \cup \delta)$. Letting $\pi : \bar{U}^+ \to U^+$ be the uncollapse, note that $\delta < \mathrm{crit}(\pi)$ (as $\delta \leq (\eta^+)^{U^+}$ and δ is a strong $\kappa_0^{U^+}$-cutpoint), and δ is a $\kappa_0^{\bar{U}^+} = \kappa_0^{U^+}$-strong cutpoint of $\bar{U}^+$, and $\bar{U}^+$ is δ-sound. Note that if P is proper class then $\delta = \delta_1^P = \delta_1^{\bar{P}}$, P is a ground of $U^+[g]$ via the extender algebra at δ (to reach $U[g]$) followed by some smaller forcing (to reach $U^+[g]$), $\delta = \eta^{+U}$, $\mathscr{I}^P = \mathscr{I}^U = \mathscr{I}^{U^+}$,

$$\mathrm{OR} \cap \mathrm{rg}(\pi) = \mathrm{OR} \cap \mathrm{Hull}_1^R(\mathscr{I}^R \cup \delta) \text{ is independent of } R \in \{P, U, U^+\},$$

$\bar{P}$ is δ-sound, and letting $\bar{U} = \mathrm{cHull}_1^U(\mathscr{I}^U \cup \delta)$, then $\bar{P} \subseteq \bar{U}[g] \subseteq \bar{U}^+[g]$ are related as are $P \subseteq U[g] \subseteq U^+[g]$, so $\bar{P} = \mathscr{P}^{\bar{U}}(M(\mathcal{T}))$, etc.

Let $Q^+ = M_b^{\mathcal{T}^+}||\mathrm{OR}^Q$ where $\mathcal{T}^+ \,\widehat{\ }\, b$ is the translation of $\mathcal{T} \,\widehat{\ }\, b$ to a tree on $\mathscr{V}^+$. Let $\bar{Q}^+ = Q^+$ if Q has set size, and $\bar{Q}^+ = \mathrm{Hull}_1^{Q^+}(\mathscr{I}^Q \cup \delta)$ otherwise. Write $\mathcal{T} = \mathcal{T}_0 \,\widehat{\ }\, \mathcal{T}_1$ for the lower and upper components of $\mathcal{T}$, and $\mathcal{T}^+ = \mathcal{T}_0^+ \,\widehat{\ }\, \mathcal{T}_1^+$ correspondingly. Note that we can rearrange $\mathcal{T}^+$ as a tree $\bar{\bar{\mathcal{T}}}^+ = \bar{\bar{\mathcal{T}}}_0^+ \,\widehat{\ }\, \bar{\bar{\mathcal{T}}}_1^+$ on $\bar{\bar{\mathscr{V}}}^+$ with $\bar{\bar{\mathcal{T}}}_0^+$ equivalent to $\mathcal{T}_0^+$ (so $M_\infty^{\bar{\bar{\mathcal{T}}}_0^+} = \bar{\bar{U}}_0^+$), and $\bar{\bar{\mathcal{T}}}_1^+$ given by normalizing a stack equivalent to $(j\text{``}\mathcal{S}_1, \mathcal{T}_1^+)$, where $j : \bar{\bar{\mathscr{V}}}^+ \to \bar{\bar{U}}_0^+$ is the iteration map and $j\text{``}\mathcal{S}_1$ is the minimal j-copy of $\mathcal{S}_1$. If Q is set size then $\bar{Q}^+ = Q(\bar{\bar{\mathcal{T}}}^+, b)$ and otherwise $\bar{Q}^+ = M_b^{\bar{\bar{\mathcal{T}}}^+}$. Note that $\bar{Q}^+$ is δ-sound.

Now compare the phalanx $((\bar{\bar{U}}_0^+, \delta), \bar{Q}^+)$ versus the phalanx $((\bar{\bar{\mathscr{V}}}^+, \delta), \bar{U}^+)$, "above δ, modulo the generic at δ and translation for overlapping extenders", just like in the proof of Lemma 5.8, using $\Phi_{\bar{\bar{\mathscr{V}}}^+ \supseteq \bar{\bar{\mathscr{V}}}}$ to iterate the phalanxes (a little bit of normalization shows this works). Because $\bar{\bar{U}}_0^+$ is $\kappa_0^{\bar{\bar{U}}_0^+}$-sound and $\bar{\bar{\mathscr{V}}}^+$ is $\kappa_0^{\bar{\bar{\mathscr{V}}}^+}$-sound and $\bar{Q}^+, \bar{U}^+$ are δ-sound, essentially the same proof as before shows that the comparison is trivial, which gives $\bar{P} = \bar{Q}$.

We leave part 3 to the reader.

Part 4, sketch:[48] Suppose $\bar{P} \neq P$; then P is proper class and $\bar{U} \neq U$. Letting $\mathcal{S}' = \mathcal{S} \,\widehat{\ }\, E_1$ if $E_1 \neq \emptyset$, and $\mathcal{S}' = \mathcal{S}$ otherwise, we have $U = M_\infty^{\mathcal{S}'}$. Now it need not be that $\bar{U}$ is $M_\alpha^{\mathcal{S}'}$ for some α, but this is almost the case. In fact, because δ is a δ_0^U-strong cutpoint of U, we get the following: Let α be least such that $\mathrm{lh}(E_\alpha^{\mathcal{S}'}) > \delta$. Then there is a unique tree $\mathcal{S}^*$ extending $\mathcal{S}'\restriction(\alpha+1)$ and such that $E_{\alpha+i}^{\mathcal{S}^*} \neq \emptyset$ iff $[0, \alpha+i]_{\mathcal{S}^*}$ drops, and if non-empty, $E_{\alpha+i}^{\mathcal{S}^*} = F(M_{\alpha+i}^{\mathcal{S}^*})$, and $M_\infty^{\mathcal{S}^*} = \bar{U}$. Moreover, U is a $\Sigma_{\bar{U}}$-iterate of $\bar{U}$, via a tree $\widetilde{\mathcal{S}}$ which is a straightforward translation of $\mathcal{S}\restriction[\alpha, \infty)$ via a little normalization (in [13] there are similar kinds of calculations, though here it is easier). But $\widetilde{\mathcal{S}}$ is above $\delta+1$. Therefore it translates to a tree $\mathcal{S}^\dagger$ on $\bar{P}$ whose last model is P. We have $\bar{Q} = \bar{P}$ is an iterate of $\mathscr{V}_1$ and $\delta = \delta_1^{\bar{Q}}$. But by the smallness of M, and since $\mathcal{S}^\dagger$ is above $\delta_1^{\bar{Q}}$ and does not drop on its main branch, it must be via $\Sigma_{\bar{Q}}$ (that is, P has nothing remotely resembling a Woodin cardinal $> \delta_1^{\bar{Q}}$, so the Q-structures at limit stages are of $\mathcal{S}^\dagger$ are trivial). This completes the sketch. □

5.1.3 Definability of $\Sigma_{\mathscr{V}, \mathrm{sh}}$ and (variants of) $M_b^{\mathcal{T}}$

We consider first the question of whether $\mathscr{V}_1$ can define its own extender sequence over its universe. We don't know whether this is the case or not, but in this direction:

Lemma 5.17. *We have:*

1. $\mathscr{V}_1|\delta_0^{\mathscr{V}_1}$ *is definable over the universe of* $\mathscr{V}_1$.

2. $\mathscr{V}_1$ *is definable over its universe from the parameter* $\mathcal{M}_\infty|\kappa_0^{\mathcal{M}_\infty}$.

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[48] We don't really need this part of the lemma, but it is convenient to have it.

Proof. Part 1: Since $\mathscr{V}_1|\delta_0^{\mathscr{V}_1} = \mathcal{M}_\infty|\delta_0^{\mathcal{M}_\infty}$ and these have the same $V_{\delta_0^{\mathscr{V}_1}}$, this is an easy corollary of Remark 3.2 (and its proof).

Part 2: Let U be the universe of $\mathscr{V}_1$. Recall that U is closed under $\Sigma_{\mathcal{M}_\infty}$ for maximal trees $\mathcal{T}$ via $\Sigma_{\mathcal{M}_\infty,\mathrm{sh}}$. Since δ_∞ is Woodin in $\mathscr{V}_1$, $\Sigma_{\mathcal{M}_\infty}(\mathcal{T})$ is in fact the unique $\mathcal{T}$-cofinal branch in $\mathscr{V}_1$, for such $\mathcal{T}$. Moreover, by the (local) definability of the short tree strategy and of maximality, $\mathcal{M}_\infty|\kappa_0^{\mathcal{M}_\infty}$ can define the collection of trees in $\mathcal{M}_\infty|\kappa_0^{\mathcal{M}_\infty}$ which are maximal via $\Sigma_{\mathcal{M}_\infty,\mathrm{sh}}$. Therefore working in U, from parameter $\mathcal{M}_\infty|\kappa_0^{\mathcal{M}_\infty}$, $\mathcal{N} = \mathcal{M}_\infty^{\mathcal{M}_\infty}|\delta_\infty^{\mathcal{M}_\infty}$ can be computed. But then the branch through the tree from $\mathcal{M}_\infty|\delta_\infty$ to $\mathcal{N}$ can be computed, and hence also $e^{\mathscr{V}_1}$ also. Therefore we can compute $\mathrm{Ult}(U, e^{\mathscr{V}_1})$, which is the universe of $\mathrm{Ult}(\mathscr{V}_1, e^{\mathscr{V}_1}) = \mathscr{V}_1^{\mathcal{M}_\infty}$ (by Lemma 4.47). But $\mathscr{V}_1^{\mathcal{M}_\infty}[\mathcal{M}_\infty|\kappa_0^{\mathcal{M}_\infty}] \,\hat{=}\, \mathcal{M}_\infty$, so we can identify the universe of $\mathcal{M}_\infty$, so by Remark 3.2, we can identify $\mathcal{M}_\infty$ itself. But from e, we therefore compute $*$, hence $\mathcal{M}_\infty[*]$, and hence $\mathscr{V}_1$, by Lemma 4.47. $\square$

Lemma 5.18. *Let $\mathscr{V}$ be a non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let $\lambda \in \mathrm{OR}$ with $\lambda \geq \delta_0^{\mathscr{V}}$ and $\mathbb{P} \in \mathscr{V}|\lambda^{+\mathscr{V}}$ and g be $(\mathcal{V}, \mathbb{P})$-generic.[49] Then $\mathscr{V}$ is definable over the universe of $\mathscr{V}[g]$ from the parameter $x = \mathscr{V}|\lambda^{+\mathscr{V}}$.*

Proof. This is an immediate corollary of Lemma 5.17 and ground definability (from the parameter $\mathcal{P}(\mathbb{P})$). $\square$

Lemma 5.19. *Let $\mathscr{V}$ be a $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Let $\lambda \geq \delta_0^{\mathscr{V}}$, let $\mathbb{P} \in \mathscr{V}|\lambda$ and g be $(\mathscr{V}, \mathbb{P})$-generic. Let $x = \mathscr{V}|\lambda^{+\mathscr{V}}$. Then:*

1. *$\mathscr{V}[g]$ is closed under $\Sigma_{\mathscr{V},\mathrm{sh}}^{\mathrm{dsr}}$ and $\Sigma_{\mathscr{V},\mathrm{sh}}^{\mathrm{dsr}}{\restriction}\mathscr{V}[g]$ is definable over the universe of $\mathscr{V}[g]$ from the parameter x (hence lightface $\mathscr{V}$-definable if $g = \emptyset$).*

2. *The notions*

   (a) *dsr,*

   (b) *δ_1-short/δ_1-maximal dsr via $\Sigma_{\mathscr{V}}$,*

   (c) *dsr-P-suitable, and*

   (d) *dl-relevant,*

   are each definable over $\mathscr{V}[g]$ from x (hence lightface $\mathscr{V}$-definable if $g = \emptyset$).

3. *For each δ_1-maximal (hence dsr) P-suitable tree $\mathcal{T} \in \mathscr{V}[g]$, as witnessed by U, letting $\bar{\mathcal{T}}$ be the equivalent tree on $\bar{\mathscr{V}} = \mathrm{cHull}_1^{\mathscr{V}}(\mathscr{I}^{\mathscr{V}} \cup \delta_1^{\mathscr{V}})$ (which is via $\Sigma_{\bar{\mathscr{V}}}$) and $b = \Sigma_{\bar{\mathscr{V}}}(\bar{\mathcal{T}})$, we have*

   $$M_b^{\bar{\mathcal{T}}} = \mathrm{cHull}_1^{\mathscr{P}^{U,g}(M(\mathcal{T}))}(\mathscr{I}^U \cup \delta(\mathcal{T})).$$

   Therefore letting η be least such that $\mathscr{V}$ is η-sound (so $\eta \leq \kappa_1^{\mathscr{V}}$), if $\eta \leq \delta = \delta(\mathcal{T})$ then $M_b^{\bar{\mathcal{T}}} = \mathscr{P}^{U,g}(M(\mathcal{T}))$ and the function $\mathcal{T} \mapsto M_b^{\bar{\mathcal{T}}}$ (with domain all such $\mathcal{T}$ with $\eta \leq \delta(\mathcal{T})$) is definable over the universe of $\mathscr{V}[g]$ from the parameter (x, η) (hence from η over $\mathscr{V}$ if $g = \emptyset$).

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[49] When we deal with such generic extensions of such $\mathscr{V}$ (here and later), we allow g to appear in some set-generic extension of V, as opposed to demanding $g \in V$.

4. Suppose $g = \emptyset$. Then for each δ_1-maximal tree $\mathcal{T} \in \mathcal{V}$ via $\Sigma_{\mathcal{V}}$, with $\mathrm{lh}(\mathcal{T}) < \kappa_1^{\mathcal{V}}$, letting $\bar{\mathcal{T}}$ be as in part 3, there is a dl-relevant tree $\mathcal{X} \in \mathcal{V}$, on $\mathcal{V}$, and such that, letting $\bar{\mathcal{X}}$ be likewise, then $M_c^{\bar{\mathcal{X}}}$ is a $\Sigma_{M_b^{\bar{\mathcal{T}}}}$-iterate of $M_b^{\bar{\mathcal{T}}}$, where $b = \Sigma_{\bar{\mathcal{V}}}(\bar{\mathcal{T}})$ and $c = \Sigma_{\bar{\mathcal{V}}}(\bar{\mathcal{X}})$.

Moreover, the definability is uniform in $\mathcal{V}, x$, and hence preserved by iteration maps.

Proof Sketch. Part 3 is an immediate consequence of Lemma 5.16.

Parts 1, 2: For simplicity we assume $g = \emptyset$, but the general case is very similar. Let $\mathcal{T} \in \mathcal{V}$ be dsr of limit length and via $\Sigma_{\mathcal{V}}$; we will determine whether $\mathcal{T}$ is δ_1-short or δ_1-maximal, and if δ_1-short, compute $\Sigma_{\mathcal{V}}(\mathcal{T})$. Let $\mathcal{T} = \mathcal{T}_0 \,\widehat{\ }\, \mathcal{T}_1$ with lower and upper components $\mathcal{T}_0, \mathcal{T}_1$ respectively. Let $E_0 \in \mathbb{E}^{\mathcal{V}}$ be long with $\gamma^{\mathcal{V}} < \mathrm{lh}(E_0)$ and $\mathcal{T}_0 \in \mathcal{V}|\lambda'$ where $\lambda' = \mathrm{lgcd}(\mathcal{V}||\mathrm{lh}(E_0))$. Let $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_0 \,\widehat{\ }\, \widetilde{\mathcal{T}}_1$ where $\widetilde{\mathcal{T}}_0$ is the successor length tree corresponding to E_0, and letting $k : M_\infty^{\mathcal{T}_0} \to M_\infty^{\widetilde{\mathcal{T}}_0}$ be the iteration map (recall this is known to $\mathcal{V}$), $\widetilde{\mathcal{T}}_1$ is the minimal k-copy of $\mathcal{T}_1$ (so $\widetilde{\mathcal{T}}$ is also via $\Sigma_{\mathcal{V}}$, by [14, ***10.3, 10.4]). It suffices to compute $\Sigma_{\mathcal{V}}(\widetilde{\mathcal{T}})$. So instead assume that $\mathcal{T}$ is itself in the form of $\widetilde{\mathcal{T}}$.

Let $E_1 \in \mathbb{E}^{\mathcal{V}}$ be $\mathcal{V}$-total with $\mathrm{crit}(E_1) = \kappa_1^{\mathcal{V}}$ and $\mathcal{T}' \in \mathcal{V}_1|\lambda$ where $\lambda = \lambda(E_1)$ and and $\mathcal{T}'$ is the tree on $\mathcal{V}|\delta_1$ equivalent to $\mathcal{T}$. Let $U = \mathrm{Ult}(\mathcal{V}, E_1)$. Let $\eta < \lambda$ be a strong $\delta_0^{\mathcal{V}}$-cutpoint and cardinal of U with $\mathcal{T}' \in U|\eta$. Now working in U, form a minimal inflation $\mathcal{X}$ of $\mathcal{T}_1'$, first iterating the least measurable $> \delta_0^{M_\infty^{\mathcal{T}_0}}$ out to η, and then folding in $\mathbb{E}^U$-genericity iteration. Now $\mathcal{X}$ is dsr (the issue being that we do not introduce new Woodin cardinals below the index of some $E_\beta^{\mathcal{X}}$, condition 3 in the definition of *dsr* (see 5.9)), because $\mathcal{T}$ is dsr and the inflationary extenders are only being used for genericity iteration (and the linear iteration at the start). The remaining details of the minimal inflation and overall process are as sketched in §4.2 (but the minimal variant, which is essentially the same), using that $\Sigma_{\mathcal{V}}$ has minimal inflation condensation, by Lemma 4.84 and [14, ***Theorem 10.2].

Part 4 follows from the proof of part 2 in the case that $\mathcal{T}$ is δ_1-maximal, since by [14], both the conversion from $\mathcal{T}$ to $\widetilde{\mathcal{T}}$ and minimal inflation yields a correct iterate. $\qquad\square$

Definition 5.20. Let $\mathcal{V}$ be a non-dropping $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$, and g be $\mathcal{V}$-generic. Let $\mathcal{V}^- = \mathcal{V}|\delta_1^{\mathcal{V}}$. Then $\Sigma_{\mathcal{V}^-}$ denotes the strategy for $\mathcal{V}^-$ induced by $\Sigma_{\mathcal{V}}$, and $\Sigma_{\mathcal{V}^-,\mathrm{sh}}$ denotes its restriction to δ_1-short trees, and $\Sigma_{\mathcal{V}^-,\mathrm{sh}}^{\mathrm{dsr}}$ its restriction dsr-δ_1-short trees.[50] Also if R is a non-dropping $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$, and $R^- \in \mathcal{V}[g]$ is a $\Sigma_{\mathcal{V}^-}$-iterate of $\mathcal{V}^-$, then $\Sigma_{R^-,\mathrm{sh}}^{\mathcal{V}[g]}$ and $(\Sigma_{R^-,\mathrm{sh}}^{\mathrm{dsr}})^{\mathcal{V}[g]}$ denote the restrictions of $\Sigma_{R^-,\mathrm{sh}}$ and $\Sigma_{R^-,\mathrm{sh}}^{\mathrm{dsr}}$ to trees in $\mathcal{V}[g]$.

Let $\mathcal{T} \in \mathcal{V}[g]$ be P-suitable for $\mathcal{V}[g]$, as witnessed by U. Let $R = \mathscr{P}^{U,g}(M(\mathcal{T}))$ (so R is a $\mathcal{V}[g]$-class and is a $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$). Then $\Sigma_{R,\mathrm{sh}}^{\mathcal{V}[g]}$ and $(\Sigma_{R,\mathrm{sh}}^{\mathrm{dsr}})^{\mathcal{V}[g]}$

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[50]Note that if $\mathcal{V}, \mathcal{V}'$ are both such and $\mathcal{V}^- = (\mathcal{V}')^-$, then we get the same strategy for $\mathcal{V}^-$ induced by $\Sigma_{\mathcal{V}}$ and $\Sigma_{\mathcal{V}'}$.

denote the restriction of Σ_R to δ_1-short and dsr-δ_1-short trees in $\mathscr{V}[g]$, respectively.[51] $\dashv$

Lemma 5.21. *Let $\mathscr{V}$ be a $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$, $\lambda \geq \delta_0^{\mathscr{V}}$, $\mathbb{P} \in \mathscr{V}|\lambda$ and g be $(\mathscr{V}, \mathbb{P})$-generic. Let $x = \mathscr{V}|\lambda^{+\mathscr{V}}$. Let $\mathcal{T} \in \mathscr{V}[g]$ be a δ_1-maximal tree on $\mathscr{V}^-$, via $\Sigma_{\mathscr{V}^-}$, and if $\mathcal{T}$ is P-suitable for $\mathscr{V}[g]$ then let $R = \mathscr{P}^{U,g}(M(\mathcal{T}))$, where U is as above. Then:*

1. *$\mathscr{V}[g]$ is closed under $(\Sigma_{M(\mathcal{T}),\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$ and $(\Sigma_{M(\mathcal{T}),\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$ is definable over $\mathscr{V}[g]$ from $(\mathcal{T}, x)$, uniformly in $\mathcal{T}$; hence likewise for $(\Sigma_{R,\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$ for δ_1-maximal P-suitable trees $\mathcal{T} \in \mathscr{V}[g]$.*

2. *The notions*

   – *dsr, and*

   – *δ_1-short/δ_1-maximal dsr via $\Sigma_{M(\mathcal{T})}$,*

   applied to trees in $\mathscr{V}[g]$ on $M(\mathcal{T})$, are definable over $\mathscr{V}[g]$ from $(\mathcal{T}, x)$, uniformly in $\mathcal{T}$.

3. *Suppose $\mathcal{T}$ is P-suitable. Let $\mathcal{T}'$ be the tree on $\mathscr{V}^-$ iterating out to $M(\mathcal{T}) = R^-$. Let $\mathcal{U}' \in \mathscr{V}[g]$ be on R^- and via $(\Sigma_{R^-,\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$. Then the stack $(\mathcal{T}', \mathcal{U}')$ normalizes to a tree on $\mathscr{V}^-$ via $(\Sigma_{\mathscr{V}^-,\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$.*

   Moreover, the definability is uniform in $\mathscr{V}, x$, and so preserved by the iteration maps.

Lemma 5.16 suffers from a significant drawback, which is that it is restricted to dsr trees. In [16] there is a generalization of this to arbitrary trees, but this involves a further modification of the P-construction, given by merging the preceding methods with ∗-translation. We now summarize the key consequences of this, also proven in [16], which we will need later.

Lemma 5.22. *Lemmas 5.19 and 5.21 both remain true after striking out every instance of the term* dsr.

Note that that parts parts 3 and 4 of Lemma 5.19 are not actually modified by striking out *dsr*, because all δ_1-maximal trees are dsr anyway.

5.2 The second direct limit system

We now define a system of uniform grounds for $\mathscr{V}_1$, and the associated Varsovian model $\mathscr{V}_2$. This is analogous to the construction of $\mathscr{V}_1$ in §4, albeit slightly more involved. For the most part it is similar, and so we omit details and remarks which are like before. We use the results of §5.1, and in particular the modified P-construction, dsr δ_1-short tree strategies, etc.

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[51]So in the case that $\mathcal{T}$ is P-suitable, $\Sigma_{M(\mathcal{T}),\mathrm{sh}}^{\mathscr{V}[g]}$ and $\Sigma_{R,\mathrm{sh}}^{\mathscr{V}[g]}$ are equivalent, as are $(\Sigma_{M(\mathcal{T}),\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$ and $(\Sigma_{R,\mathrm{sh}}^{\mathrm{dsr}})^{\mathscr{V}[g]}$.

5.2.1 The external direct limit system $\mathscr{D}_1^{\text{ext}}$

Definition 5.23. Let $\mathbb{U}_1$ be the $\mathscr{V}_1$-class of all dl-relevant iteration trees (Definition 5.9). Define

$$d_1 = \{\mathscr{V}_1|\delta_1^{\mathscr{V}_1}\} \cup \{\mathcal{M}(\mathcal{U}) \mid \mathcal{U} \in \mathbb{U}_1 \text{ is non-trivial}\}.$$

For $p \in d_1$, set $\mathcal{P}_p = \mathscr{P}^{\mathscr{V}_1}(p)$ (P-construction as in Definition 5.13). Write $\mathscr{F}_1 = \{\mathcal{P}_p \mid p \in d_1\}$. Define $\preceq$ on d_1 and on $\mathscr{F}_1$ and maps $\pi_{pq} = i_{\mathcal{P}_p \mathcal{P}_q}$ for $p \preceq q$ as in §4. $\dashv$

By Lemma 5.19, $(d_1, \langle \mathcal{P}_p \rangle_{p \in d_1})$ is lightface $\mathscr{V}_1$-definable, as are $\mathbb{U}_1$ and $\mathscr{F}_1$.

Lemma 5.24. $\preceq$ *is a directed partial order, is lightface $\mathscr{V}_1$-definable, and the associated embeddings commute: if $P \preceq Q \preceq R$ then $i_{QR} \circ i_{PQ} = i_{PR}$.*

Proof Sketch. For the definability, that $\preceq$ is partial order, and the commutativity, see the proof of Lemma 4.8. For directedness, let $\mathcal{T}, \mathcal{U} \in \mathbb{U}_1$, with lower and upper components $\mathcal{T}_0, \mathcal{T}_1$ and $\mathcal{U}_0, \mathcal{U}_1$ respectively. Let $E_0, F_0 \in \mathbb{E}^{\mathscr{V}_1}$ with be $\mathscr{V}_1$-total with $\text{crit}(E_0) = \text{crit}(F_0) = \kappa_0$ and such that $\mathcal{T}_0, \mathcal{U}_0$ correspond to E_0, F_0 respectively. We may assume $\text{lh}(E_0) \leq \text{lh}(F_0)$, so if $E_0 \neq F_0$ then in fact $\text{lh}(E_0) < \lambda(F_0)$. Therefore $M_\infty^{\mathcal{U}_0}$ is a (possibly trivial) iterate of $M_\infty^{\mathcal{T}_0}$. Let $j : M_\infty^{\mathcal{T}_0} \to M_\infty^{\mathcal{U}_0}$ be the iteration map. Let $\mathcal{T}_1'$ be the minimal j-copy of $\mathcal{T}_1$. Now proceed with a pseudo-comparison of $\mathcal{U}_0 {}^\frown \mathcal{T}_0'$ and $\mathcal{U}$ intertwined with pseudo-genericity-iteration, as in Lemma 4.8. $\square$

Define the external direct limit system $\mathscr{D}_1^{\text{ext}} = (P, Q, i_{PQ} \colon P \preceq Q \in \mathscr{F}_1)$. We have (ug1), (ug2), (ug3), (ug4), (ug5), (ug6), (ug18), and write

$$(\mathcal{M}_{\infty 1}^{\text{ext}}, (i_{P\infty} \colon P \in \mathscr{F}_1)) = \text{dirlim } \mathscr{D}_1^{\text{ext}}. \tag{25}$$

Let $P \in \mathscr{F}_1$. Then $\mathscr{V}_1|\delta_1^P$ is $(P, \mathbb{B}_{\delta_1^P \delta_0^P}^P)$-generic and hence $P[\mathscr{V}_1|\delta] =_\delta^* \mathscr{V}_1$, so P is a ground for $\mathscr{V}_1$ via the extender algebra $\mathbb{B}_{\delta_1^P \delta_0^P}$ (at δ_1^P, using extenders with critical point $\geq \delta_0^P$ (hence $> \delta_0^P$)). Thus:

Definition 5.25. For $P \in \mathscr{F}_1$, let τ_1^P be the canonical class $\mathbb{B}_{\delta_1^P \delta_0^P}$-name for $\mathscr{V}_1$, like in Definition 4.9, but incorporating the appropriate conversion for the overlapping extenders (note the generic filter determines $\mathscr{V}_1|\delta_0^P$, which in turn determines the "key" to this conversion). $\dashv$

Lemma 5.26. *(ug19) holds: for each $P \in \mathscr{F}_1$, $c_1^P = d_1 \cap d_1^P$ is dense in $(d_1^P, \preceq^P)$ and dense in $(d_1, \preceq)$, and $\preceq^P \restriction c_1^P = \preceq \restriction c_1^P$.*

Proof. Let $P \in \mathscr{F}_1$. That $\preceq^P \restriction c_1^P = \preceq \restriction c_1^P$ is by Lemma 5.21. So let $Q \in \mathscr{F}_1$ and $R \in \mathscr{F}_1^P$. We must find $S \in \mathscr{F}_1 \cap \mathscr{F}_1^P$ with $Q, R \preceq S$. Let $\mathcal{T}_P = \mathcal{T}_{P0} {}^\frown \mathcal{T}_{P1}$ be the maximal tree leading from $\mathscr{V}_1$ to P, with lower and upper components $\mathcal{T}_{P0}, \mathcal{T}_{P1}$ respectively, and likewise for R, and let $\mathcal{T}_{PQ}, \mathcal{T}_{PQ0}, \mathcal{T}_{PQ1}$ be likewise for Q in P. Let $E_0 \in \mathbb{E}^{\mathscr{V}_1}$ be long with $\text{lh}(E_0) < \kappa_1$ be $\text{lh}(E_0)$ sufficiently large that P, Q, R are all translations of one another above some $\gamma < \lambda(E_0)$

and the various trees are in $\mathscr{V}_1|\lambda(E_0)$. Letting $E_0^P \in \mathbb{E}^P$ and $E_0^R \in \mathbb{E}^R$ with $\mathrm{lh}(E_0^P) = \mathrm{lh}(E_0^R) = \mathrm{lh}(E_0)$, then E_0^R, E_0^P are translations of E_0, so $\mathrm{Ult}(\mathscr{V}_1, E_0)$, $\mathrm{Ult}(P, E_0^P)$ and $\mathrm{Ult}(R, E_0^R)$ agree through their common least Woodin $\mathrm{lh}(E_0)$ (but not above there if $P \neq \mathscr{V}_1$, as $\mathrm{Ult}(\mathscr{V}_1, E_0)$ is $\delta_0^{\mathrm{Ult}(\mathscr{V}_1, E_0)}$-sound, whereas then the others are not). Let $\sigma \in P$ be a $\mathbb{B}_{\delta_1^P \delta_0^P}$-name for Q, and let $p_1 \in \mathbb{B}_{\delta_1^P \delta_0^P}$ be the Boolean value of the statement "τ^P is $\mathscr{V}_1$-like and $\sigma \in \mathscr{F}_1^{\tau^P}$". Working in P, we will form a Boolean-valued comparison/genericity iteration of $\mathrm{Ult}(P, E_0^P)$, $\mathrm{Ult}(R, E_0^R)$ and all interpretations of $\mathrm{Ult}(\sigma, F^{\sigma||\mathrm{lh}(E_0)})$ below p_1, much as in the proof of Lemma 4.10 (in particular incorporating Boolean-valued τ_1^P-genericity). However, because we have not yet established that $\mathscr{V}_1$ knows its own δ_1-short tree strategy, we cannot quite argue as for Lemma 4.10. Thus, we tweak the comparison as in the following sketch (the process will be use an idea from [22, §7]; see especially [22, Corollary 7.5 and Theorem 7.3 (Claim 8)]).

We define a $\mathbb{B}_{\delta_1^P \delta_0^P}$-name for a padded tree $\dot{\mathcal{U}}$ on σ, and define padded trees $\mathcal{T}$ on $\mathrm{Ult}(P, E_0^P)$ and $\mathcal{V}$ on $\mathrm{Ult}(R, E_0^R)$, recursively on length in the usual manner for comparison. Given (names for) the trees up to length $\alpha + 1$, we will also have some condition q_α, with $q_\alpha \leq q_0 = p_1$. Let $q_{\alpha+1}$ be the Boolean value, below q_α, of the statement "the least disagreement between $M_\alpha^{\dot{\mathcal{U}}}$ and $M_\alpha^{\mathcal{T}}$ and $M_\alpha^{\mathcal{V}}$, if it exists, involves a dsr extender" (that is, satisfying condition 3 of Definition 5.9). We then take the least forced disagreement working below $q_{\alpha+1}$, and use this index and genericity iteration considerations to determine the next extender, etc. Given everything through some limit stage γ, which is short, the strategies $\Sigma_{\cdot,\mathrm{sh}}^{\mathrm{dsr}}$ determine branches (as required), and set q_γ to be the infimum of $\langle q_\alpha \rangle_{\alpha < \gamma}$. The rest is as usual. The conditions q_α are always non-zero, and in fact $q_\alpha \in g$ where g is the generic adding $\mathscr{V}_1|\delta_0^P$, because $\dot{\mathcal{U}}_g, \mathcal{T}, \mathcal{V}$ are then correct trees on R, P, Q, which were themselves iterates via dsr trees, and by the analysis of comparison in [14, ***§8], the least disagreement must be an ultrapower-image of one of the extenders used in those dsr trees, and hence be itself appropriate for dsr. Because we rule out the use of non-dsr extenders, the Q-structure(s) Q_ξ used in the trees at limit stages ξ do not overlap $\delta(\mathcal{T}{\restriction}\xi)$ (except possibly with long extenders). They also agree with one another (in $\mathcal{T}, \mathcal{V}$ and all interpretations of $\dot{\mathcal{U}}$), and no extenders in $\mathbb{E}_+^{Q_\xi}$ are used later in the comparison (in particular for genericity iteration). This is because in P and in $P[g']$, the trees are P-suitable, and the Q-structures are produced by P-construction, and because of the agreement between $P, \mathscr{V}_1$, they are therefore identical. $\qquad\square$

5.2.2 The internal direct limit system $\mathscr{D}_1$

We adapt Definition 4.11 in the obvious manner, to which we refer the reader for details:

Definition 5.27. Work in $\mathscr{V}_1$. Define *(weak) s-iterability* for $P \in \mathscr{F}_1$ and $s \in [\mathrm{OR}]^{<\omega} \setminus \{\emptyset\}$ as in 4.11. If P is s-iterable and $s \subseteq t$ and Q is t-iterable with $P \preceq Q$, then likewise for γ_s^P, H_s^P and $\pi_{Ps,Qt} : H_s^P \to H_t^Q$. Define *strong s-iterability* as before.

Let $\mathscr{F}_1^+ = \{(P,s) \mid P \in \mathscr{F}$ and P is strongly s-iterable$\}$, and similarly let $d_1^+ = \{(P|\delta_1^P, s) \mid (P,s) \in \mathscr{F}_1^+\}$. The order $\preceq$ on d_1^+ is determined by (ug8). Define $\preceq$ on $\mathscr{F}^+$ likewise. Clearly if $(P,s) \preceq (Q,t) \preceq (R,u)$ then

$$\pi_{Ps,Ru} = \pi_{Qt,Ru} \circ \pi_{Ps,Qt}.$$

Define the system $\mathscr{D}_1 = (H_s^P, H_t^Q, \pi_{Ps,Qt} : (P,s) \preceq (Q,t) \in \mathscr{F}_1^+)$.

Given $P \in \mathscr{F}_1$ and $s \in [\mathrm{OR}]^{<\omega}$, recall that s is P-*stable* iff $\pi_{PQ}(s) = s$ for every $Q \in \mathscr{F}_1$ with $P \preceq Q$. ⊣

Remark 5.28. As in Remark 4.12, s-iterability actually implies strong s-iterability.

The following lemma yields properties (ug7), (ug8), (ug9), (ug10), (ug11), (ug12), (ug13), (ug14), (ug15), (ug16):

Lemma 5.29. *We have:*

(a) *if $P \in \mathscr{F}_1$ and $s \in [\mathrm{OR}]^{<\omega}\backslash\{\emptyset\}$ and s is P-stable, then $(P,s) \in \mathscr{F}_1^+$ and (P,s) is true (see Definition 2.1).*

(b) *$(\mathscr{F}_1^+, \preceq)$ is directed – for $(P,s), (Q,t) \in \mathscr{F}_1^+$ there is $(R,u) \in \mathscr{F}_1^+$ with $(P,s) \preceq (R,u)$ and $(Q,t) \preceq (R,u)$ (note $u = s \cup t$ suffices).*

(c) *$\mathscr{D}_1$ is lightface $\mathscr{V}_1$-definable.*

Definition 5.30. Noting that $\mathscr{I}^{\mathscr{V}_1} = \mathscr{I}^M$ is a club class of generating indiscernibles for $\mathscr{V}_1$, define $\mathscr{I}^P = i_{\mathscr{V}_1 P}``\mathscr{I}^{\mathscr{V}_1}$ whenever P is a non-dropping iterate of $\mathscr{V}_1$. ⊣

For the following, see the proof of Lemma 4.15:

Lemma 5.31. *For each $P \in \mathscr{F}_1$, P is $\{\alpha\}$-stable for every $\alpha \in \mathscr{I}^M = \mathscr{I}^P$. Therefore property (ug17) holds, as witnessed by some $s \in [\mathscr{I}^M]^{<\omega}$.*

We can now (working in $\mathscr{V}_1$) define the direct limit

$$(\mathcal{M}_{\infty 1}, \pi_{Ps,\infty} : (P,s) \in \mathscr{F}_1^+) = \mathrm{dirlim}\ \mathscr{D}_1, \tag{26}$$

and the associated $*$-map $*_1$. This notation is somewhat cumbersome, so let us also write $\mathcal{N}_\infty = \mathcal{M}_{\infty 1}$, and we will often write $*$ instead of $*_1$, where there should be no cause for confusion. By Lemmas 2.3 and 2.5, $\chi : \mathcal{N}_\infty \to \mathcal{N}_\infty^{\mathrm{ext}}$ is the identity and $\mathcal{N}_\infty = \mathcal{N}_\infty^{\mathrm{ext}}$. Property (ug20) holds as if $P \in \mathscr{F}_1$ and $\bar{Q} \in d_1^P \cap d_1$ then $\mathcal{P}_{\bar{Q}}^P = \mathscr{P}^P(\bar{Q}) = \mathscr{P}^{\mathscr{V}_1}(\bar{Q})$, because $\mathbb{E}^P$ is a translation of $\mathbb{E}^{\mathscr{V}_1}$ above δ_1^P). And (ug21) again holds if s is P-stable.

So we have established (ug1)–(ug21). For the remaining properties set $\delta = \delta_1^{\mathscr{V}_1}$ and $\mathbb{B} = \mathbb{B}_{\delta\delta_0^{\mathscr{V}_1}}^{\mathscr{V}_1}$ (for the witnesses to those properties in §2). This gives (ug22). Recall we defined τ^P in Definition 5.25. Write $\delta_{1\infty} = i_{\mathscr{V}_1 \infty}(\delta_1) = \delta_1^{\mathcal{N}_\infty}$ (replacing the notation δ_∞ of §2). As for Lemma 4.16:

Lemma 5.32. *We have:*

1. For each $\mathscr{V}_1$-stable $\alpha \in \mathrm{OR}$ and each $P \in \mathscr{F}_1$, letting and g be the P-generic filter for $\mathbb{B}^P_{\delta^P_1 \delta^P_0}$ given by $\mathscr{V}_1|\delta^P_1$, then $i_{\mathscr{V}_1 P}(\tau^{\mathscr{V}_1}{\restriction}\alpha)_g = \mathscr{V}_1|\alpha$. Moreover, $\mathscr{V}_1 \mathrel{\widehat{=}} P[g] \mathrel{\widehat{=}} P[\mathscr{V}_1|\delta^P_1]$.

2. (ug24) holds.

3. $\kappa^M_1 = \kappa^{\mathscr{V}_1}_1$ is the least measurable cardinal of $\mathcal{N}_\infty$.

4. $\kappa^{+M}_1 = \kappa^{+\mathscr{V}_1}_1 = \delta_{1\infty}$.

5.2.3 The second Varsovian model as $\mathcal{N}_\infty[*]$

Definition 5.33. Recall that $*_1$ is the $*$-map associated to the preceding construction. We define the structure

$$\mathcal{N}_\infty[*_1] = (L[\mathcal{N}_\infty, *_1], \mathcal{N}_\infty, *_1); \tag{27}$$

that is, with universe $L[\mathcal{N}_\infty, *_1]$ and predicates $\mathcal{N}_\infty$ and $*_1$. However, as mentioned above, we will often abbreviate $*_1$ with $*$, hence writing $\mathcal{N}_\infty[*]$.

Note this structure has the universe of the abstract Varsovian model of §2. Essentially by §2, we have the elementary maps

$$\pi_{\infty 1} : \mathcal{N}_\infty \to \mathcal{N}_\infty^{\mathcal{N}_\infty},$$

$$\pi^+_{\infty 1} : \mathcal{N}_\infty[*_1] \to \mathcal{N}_\infty^{\mathcal{N}_\infty}[*_1^{\mathcal{N}_\infty}],$$

where $\mathcal{N}_\infty^{\mathcal{N}_\infty}$ is computed in $\mathcal{N}_\infty$ just as $\mathcal{N}_\infty$ is computed in $\mathscr{V}_1$, and $*_1^{\mathcal{N}_\infty}$ is the $*$-map as computed in $\mathcal{N}_\infty$. Recall $*_1 \subseteq \pi_{\infty 1} \subseteq \pi^+_{\infty 1}$, and these maps are lightface definable over $\mathscr{V}_1$. $\dashv$

We next point out that $\mathcal{N}_\infty^{\mathcal{N}_\infty}$ is a $\Sigma_{\mathcal{N}_\infty}$-iterate of $\mathcal{N}_\infty$ and $\pi_{\infty 1}$ is the correct iteration map. We also want to generalize this to other iterates of $\mathscr{V}_1$.

Definition 5.34. Given a $\mathscr{V}_1$-like Vsp N, let $\mathscr{D}^N_1$ and $\mathcal{N}^N_\infty$ be defined over N just as $\mathscr{D}_1, \mathcal{N}_\infty$ are defined over $\mathscr{V}_1$, and likewise $*_1^N, \pi^N_{\infty 1}, (\pi^+_{\infty 1})^N$. If N is a correct iterate of $\mathscr{V}_1$, also define $(\mathcal{N}^{\mathrm{ext}}_\infty)_N$ (the external direct limit) relative to N, as for $\mathscr{V}_1$: given a maximal tree $\mathcal{T} \in \mathbb{U}^N_1$ (considered as a tree on N), let $b = \Sigma_N(\mathcal{T})$ and $M_\mathcal{T} = M^\mathcal{T}_b$, and let $(\mathcal{N}^{\mathrm{ext}}_\infty)_N$ be the direct limit of these models $M_\mathcal{T}$ under the iteration maps. If in fact $M_\mathcal{T} = \mathcal{P}^N_{M(\mathcal{T})}$ (the model indexed by $M(\mathcal{T})$ in the covering system $\mathscr{D}^N_1$) for each such $\mathcal{T}$, then define $\chi_N : \mathcal{N}^N_\infty \to (\mathcal{N}^{\mathrm{ext}}_\infty)_N$ as in §2. $\dashv$

Lemma 5.35. Let N be a δ^N_1-sound, non-dropping $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$.
   Then $M_\mathcal{T} = \mathscr{P}^N(M(\mathcal{T})) = \mathcal{P}^N_{M(\mathcal{T})}$[52] for each $\mathcal{T} \in \mathbb{U}^N_1$, $\mathcal{N}^N_\infty = (\mathcal{N}^{\mathrm{ext}}_\infty)_N$ and $\chi_N = \mathrm{id}$, and $\mathcal{N}^N_\infty$ is a $\delta^{\mathcal{N}^N_\infty}_1$-sound, non-dropping Σ_N-iterate of N, and hence is a $\Sigma_{\mathscr{V}_1}$-iterate of $\mathscr{V}_1$. Moreover,

$$\pi^N_{\infty 1} : \mathcal{N}^N_\infty \to \mathcal{N}^{\mathcal{N}^N_\infty}_\infty$$

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[52]Recall that the notation is $\mathscr{P}(M(\mathcal{T}))$ for P-construction over $M(\mathcal{T})$, and $\mathcal{P}_{M(\mathcal{T})}$ for the model of $\mathscr{D}_1$ indexed at $M(\mathcal{T})$.

is the iteration map according to $\Sigma_{\mathcal{N}_\infty^N}$. This holds in particular for $N = \mathscr{V}_1$ and for $N = \mathcal{N}_\infty$, so $\mathcal{N}_\infty^{\mathcal{N}_\infty}$ is a correct iterate of $\mathcal{N}_\infty$, and $\pi_{\infty 1}$ is the iteration map.

Proof. This is just Lemma 5.16 and a consequence thereof, and by standard arguments. $\qquad\square$

Like with M, working in $\mathcal{N}_\infty[*_1 {\upharpoonright} \delta_{\infty 1}]$ we can compute $\pi_{\infty 1}^+$, so $\mathcal{N}_\infty[*_1]$ has universe

$$L[\mathcal{N}_\infty, *_1 {\upharpoonright} \delta_{\infty 1}] = L[\mathcal{N}_\infty, *_1] = L[\mathcal{N}_\infty, \pi_{\infty 1}] = L[\mathcal{N}_\infty, \pi_{\infty 1}^+].$$

5.2.4 Uniform grounds of $\mathscr{V}_1$

Lemma 5.36. *Write $\varepsilon = \delta_{1\infty}$. We have:*

1. *$V_\varepsilon^{\mathcal{N}_\infty[*_1]} = V_\varepsilon^{\mathcal{N}_\infty}$.*

2. *ε is (the second) Woodin in $\mathcal{N}_\infty[*_1]$ (and $\delta_0^{\mathcal{N}_\infty}$ the first).*

3. *Property (ug23) of uniform grounds holds for $\mathcal{N}_\infty[*_1]$ at ε; that is, $\mathcal{N}_\infty[*_1] \vDash$ "ε is regular and $\mathbb{B}_\infty$ is ε-cc". Moreover, $\mathcal{N}_\infty[*_1] \vDash$ "$\mathbb{B}_\infty$ is a complete Boolean algebra".*

Proof. Part 1: As usual we have $*_1 \upharpoonright \eta \in \mathcal{N}_\infty$ for every $\eta < \varepsilon$. Now $\mathcal{N}_\infty^{\mathcal{N}_\infty}[*_1^{\mathcal{N}_\infty}]$ is a class of $\mathcal{N}_\infty$ and
$$\pi_{\infty 1}^+ : \mathcal{N}_\infty[*_1] \to \mathcal{N}_\infty^{\mathcal{N}_\infty}[*_1^{\mathcal{N}_\infty}]$$
is elementary. Let $A \in \mathcal{P}(\mathrm{OR}) \cap \mathcal{N}_\infty[*_1]$. Then $\pi_{\infty 1}^+(A) \in \mathcal{N}_\infty$. So if $A \subseteq \eta < \varepsilon$ then $\mathcal{N}_\infty$ can compute A from the set $\pi_{\infty 1}^+(A)$ and the map $\pi_{\infty 1}^+ \upharpoonright \eta = *_1 \upharpoonright \eta$. The remaining parts are now as in Lemma 4.30. $\qquad\square$

So by Theorem 2.16, $\mathcal{N}_\infty[*_1]$ is a ground of $\mathscr{V}_1$.

5.3 The second Varsovian model as the strategy mouse $\mathscr{V}_2$

Let $j : \mathcal{M}_\infty | \delta_\infty \to \mathcal{N}_\infty | \delta_0^{\mathcal{N}_\infty}$ be the restriction of the $\Sigma_{\mathcal{M}_\infty}$-iteration map. Note that for each $\nu > \kappa_1$, if $F = F^{\mathscr{V}_1 || \nu} \neq \emptyset$ is long, then $\kappa_1 < \lambda(F)$ and $\mathcal{N}_\infty | \delta_0^{\mathcal{N}_\infty}$ is definable in the codes over $\mathscr{V}_1 | \kappa_1$, and hence in $\mathcal{N}_\infty | \delta_0^{\mathcal{N}_\infty} \in \mathscr{V}_1 | \lambda$. Moreover, letting $P = \mathrm{Ult}(\mathscr{V}_1, F) \downarrow 0$, we have $\delta_0^P = \mathrm{lh}(F)$ and $P | \delta_0^P$ is an iterate of $\mathcal{N}_\infty | \delta_0^{\mathcal{N}_\infty}$. In this circumstance let

$$k_\nu : \mathcal{N}_\infty | \delta_0^{\mathcal{N}_\infty} \to P | \delta_0^P$$

be the iteration map. Now define $\mathbb{F}_{>\kappa_1}^{\mathscr{V}_1}$ as the class of all tuples $(\nu, \alpha, \beta) \in \mathrm{OR}^3$ such that $\nu > \kappa_1$, $F = F^{\mathscr{V}_1 || \nu} \neq \emptyset$ and either

  – F is short (so $\kappa_1 \leq \mathrm{crit}(F)$) and $F(\alpha) = \beta$, or

  – F is long and $k_\nu(\alpha) = \beta$.

Lemma 5.37. *$\mathbb{F}_{>\kappa_1}^{\mathscr{V}_1}$ is lightface definable over $\mathcal{N}_\infty[*_1]$.*

Proof. Write $*$ for $*_1$. Let $(\nu, \alpha, \beta) \in \mathrm{OR}^3$ with $\nu > \kappa_1$. Let $F' = F^{\mathcal{N}_\infty || \nu^*}$. We claim that $(\nu, \alpha, \beta) \in \mathbb{F}_{>\kappa_1}^{\mathscr{V}_1}$ iff either

- $F' \neq \emptyset$ is short and $F'(\alpha^*) = \beta^*$, or

- $F' \neq \emptyset$ is long and $F'(\alpha) = \beta^*$ (the argument to F' is α, not α^*!),

and moreover, if $F \neq \emptyset$ then F is short iff F' is short. This is proved like in Lemma 4.24, but the case that F is long is uses the modified P-construction. $\square$

Lemma 5.38. *Let* $\mathbb{L} = \mathbb{L}^{\mathcal{N}_\infty[*_1]}(\kappa_1)$ *(Definition 2.11, for adding a subset of κ_1). Then $\mathscr{V}_1|\kappa_1$ is $\mathbb{L}$-generic over $\mathcal{N}_\infty[*_1]$ and $\mathcal{N}_\infty[*_1][\mathscr{V}_1|\kappa_1] \mathrel{\widehat{=}} \mathscr{V}_1$.*

Proof. This follows from Lemma 4.24 almost like in the proof of Lemma 4.38, using the fact that $\mathcal{N}_\infty|\delta_0^{\mathcal{N}_\infty}$ and the iteration map j used above are definable (in the codes) over $\mathscr{V}_1|\kappa_1$, and hence available to $\mathcal{N}_\infty[*_1][\mathscr{V}_1|\kappa_1]$. $\square$

We now adapt Definition 4.39, presenting the second Varsovian model as a strategy mouse $\mathscr{V}_2$ analogous to $\mathscr{V}_1$. The sequence $\mathbb{E}^{\mathscr{V}_2}$ will have two kinds of long extenders, corresponding to $\delta_0^{\mathcal{N}_\infty}$ and $\delta_1^{\mathcal{N}_\infty}$:

Definition 5.39. Write $\gamma_0^{\mathscr{V}_2} = \gamma^{\mathcal{N}_\infty}$ and $\gamma_1^{\mathscr{V}_2} = (\kappa_1^{\mathcal{N}_\infty})^{+\mathcal{N}_\infty}$. Note that

$$\kappa_1 < \delta_0^{\mathcal{N}_\infty} < \gamma_0^{\mathscr{V}_2} < \kappa_1^{+\mathscr{V}_1} = \delta_1^{\mathcal{N}_\infty} < \gamma_1^{\mathscr{V}_2}.$$

Define the structure

$$\mathscr{V}_2 = (L[\mathbb{E}^{\mathscr{V}_2}]; \in, \mathbb{E}^{\mathscr{V}_2}),$$

with segments $\mathscr{V}_2||\nu = (\mathcal{J}_\nu[\mathbb{E}^{\mathscr{V}_2}{\upharpoonright}\nu]; \in, \mathbb{E}^{\mathscr{V}_2}{\upharpoonright}\nu, \mathbb{E}_\nu^{\mathscr{V}_2})$ and their passivizations $\mathscr{V}_2|\nu$, recursively in ν as follows:

$$\mathbb{E}_\nu^{\mathscr{V}_2} = \begin{cases} \mathbb{E}_\nu^{\mathcal{N}_\infty} & \text{if } \nu < \gamma_1^{\mathscr{V}_2} \\ \pi_{\infty 1}{\upharpoonright}(\mathcal{N}_\infty|\delta_1^{\mathcal{N}_\infty}) & \text{if } \nu = \gamma_1^{\mathscr{V}_2} \\ \mathbb{E}_\nu^{\mathscr{V}_1}{\upharpoonright}(\mathscr{V}_2|\nu) & \text{if } \nu > \gamma_1^{\mathscr{V}_2} \text{ and } \mathbb{E}_\nu^{\mathscr{V}_1} \text{ is short,} \\ k_\nu{\upharpoonright}(\mathcal{N}_\infty|\delta_0^{\mathcal{N}_\infty}) & \text{if } \nu > \gamma_1^{\mathscr{V}_2} \text{ and } \mathbb{E}_\nu^{\mathscr{V}_1} \text{ is long,} \end{cases} \qquad (28)$$

and with $\mathbb{E}^{\mathscr{V}_2} = \{(\nu, x, y) \colon \mathbb{E}_\nu^{\mathscr{V}_1} \neq \emptyset \text{ and } y = \mathbb{E}_\nu^{\mathscr{V}_1}(x)\}$ and $\mathbb{E}^{\mathscr{V}_2}{\upharpoonright}\nu$ as usual. (We verify well-definedness in Lemma 5.41.)

Definability etc over $\mathscr{V}_2$ has the predicate $\mathbb{E}^{\mathscr{V}_2}$ available by default.
Write $e_i^{\mathscr{V}_2} = \mathbb{E}_{\gamma_i^{\mathscr{V}_2}}^{\mathscr{V}_1}$ for $i = 0, 1$. $\dashv$

The fine structural concepts for segments of $\mathscr{V}_2$ are defined directly as for segments of $\mathscr{V}_1$ (Definition 4.40). The next two lemmas are direct adaptations of Lemmas 4.41, 4.42 respectively:

Lemma 5.40. *Let* $\bar{\mathscr{V}} = \mathscr{V}_2||\gamma_1^{\mathscr{V}_2}$. *Then:*

(a) $\mathbb{L} = \mathbb{L}^{\mathcal{N}_\infty[*_1]}(\kappa_1)$ *is Σ_1-definable over $\bar{\mathscr{V}}$.*

(b) $\bar{\mathscr{V}}$ *is isomorphic to a structure which is definable without parameters over $\mathscr{V}_1|\kappa_1^{+\mathscr{V}_1}$.*

(c) $\bar{\mathscr{V}}$ is sound, with $\rho_\omega^{\bar{\mathscr{V}}} = \rho_1^{\bar{\mathscr{V}}} = \delta_{1\infty}$ and $p_1^{\bar{\mathscr{V}}} = \emptyset$.

(d) $\mathrm{OR}^{\bar{\mathscr{V}}} < \xi_0$, where ξ_0 is the least $\xi > \kappa_1^{+\mathscr{V}_1}$ such that $\mathscr{V}_1|\xi$ is admissible. Therefore $\mathscr{V}_2||\nu$ is passive for every $\nu \in (\gamma_1^{\mathscr{V}_2}, \xi_0]$.

Lemma 5.41. *Let $g = g_{\mathscr{V}_1|\kappa_1}$ be the $(\mathcal{N}_\infty[*_1], \mathbb{L})$-generic determined by $\mathscr{V}_1|\kappa_1$. For every $\nu \in \mathrm{OR}$:*

1. *$\mathscr{V}_2|\nu$ and $\mathscr{V}_2||\nu$ are in $\mathcal{N}_\infty[*_1]$,*

2. *$\mathscr{V}_2|\nu$ and $\mathscr{V}_2||\nu$ are sound,*

3. *Suppose $\nu \geq \xi_0$ and let $E = F^{\mathscr{V}_2||\nu}$ and $E' = F^{\mathscr{V}_1||\nu}$.[53] Then*

    (a) *$\mathbb{L} \in \mathscr{V}_2|\nu$ and g is $(\mathscr{V}_2|\nu, \mathbb{L})$-generic,*

    (b) *$(\mathscr{V}_2|\nu)[g] =^* \mathscr{V}_1|\nu$,[54]*

    (c) *$(\mathscr{V}_2||\nu)[g] =^* \mathscr{V}_1||\nu$,*

    (d) *if $E' \neq \emptyset$ and $\mathrm{crit}(E') > \kappa_1$ then $\mathscr{V}_2||\nu$ satisfies the usual premouse axioms with respect to E (with Jensen indexing; so E is an extender over $\mathscr{V}_2|\nu$ which coheres $\mathbb{E}^{\mathscr{V}_2|\nu}$, etc),*

    (e) *if $E' \neq \emptyset$ and $\mathrm{crit}(E') = \kappa_1$ then E is a long $(\delta_{1\infty}, \nu)$-extender over $\mathcal{N}_\infty$ and*

    $$\mathrm{Ult}(\mathcal{N}_\infty|\delta_{1\infty}, E) = i_{E'}^{\mathscr{V}_1}(\mathcal{N}_\infty|\delta_{1\infty}) = \mathcal{N}_\infty^{\mathrm{Ult}(\mathscr{V}_1, E')}|i_{E'}^{\mathscr{V}_1}(\delta_{1\infty})$$

    is a lightface proper class of $\mathscr{V}_2|\nu$, uniformly in such ν, and

    (f) *if $E' \neq \emptyset$ and E' is long then E is a long $(\delta_0^{\mathcal{N}_\infty}, \nu)$-extender over $\mathcal{N}_\infty$ and*

    $$\mathrm{Ult}(\mathcal{N}_\infty|\delta_0^{\mathcal{N}_\infty}, E) = i_{E'}^{\mathscr{V}_1}(\mathscr{V}_1|\delta_0^{\mathscr{V}_1}) = \mathcal{M}_\infty^{\mathrm{Ult}(\mathscr{V}_1, E')}|i_{E'}^{\mathscr{V}_1}(\delta_0^{\mathscr{V}_1})$$

    is a lightface proper class of $\mathscr{V}_2|\nu$, uniformly in such ν.

Remark 5.42. Here the notation $=^*$ is like in Remark 4.43, except that when E' is long, we have $E' = E \circ j$, instead of $E \subseteq E'$; recall j is encoded into g. An analogous consideration applies to the proof of part 4 in the next lemma; cf. Lemmas 4.44 and 4.47 and their proofs:

Lemma 5.43.

1. *$\mathcal{N}_\infty[*_1]$ and $\mathscr{V}_2$ have the same universe.*

2. *$\mathrm{Ult}(\mathscr{V}_2, e_1^{\mathscr{V}_2}) = \mathscr{V}_2^{\mathcal{N}_\infty}$.*

3. *$\mathscr{V}_2$ is a lightface class of $\mathcal{N}_\infty[*_1]$.*

4. *$\mathcal{N}_\infty[*_1]$ is a lightface class of $\mathscr{V}_2$.*

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[53] Also, $M|\theta_0^M$ and $\mathscr{V}_1||\gamma^{\mathscr{V}_1}$ are "generically equivalent in the codes", and letting

$$f : (\theta_0^M, \xi_0) \to (\gamma^{\mathscr{V}_1}, \xi_0)$$

be the unique surjective order-preserving map, then $M|\alpha = M||\alpha$ are likewise equivalent with $\mathscr{V}_1|f(\alpha) = \mathscr{V}_1||f(\alpha)$ for all $\alpha \in \mathrm{dom}(f)$, but we will not need this.

[54] The notation is explained in 5.42.

5.4 Iterability of $\mathcal{N}_\infty | \delta_{1\infty}$ in $\mathcal{V}_1$ and $\mathcal{V}_2$

Adapting Definition 4.54:

Definition 5.44. Let $\mathcal{V}$ be a non-dropping $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$. For $P \in \mathscr{F}_1^{\mathcal{V}}$ let

$$H^P = \mathrm{Hull}_1^P(\delta_1^P \cup \mathscr{I}^{\mathcal{V}}),$$

$\bar{P}$ its transitive collapse and $\pi_{\bar{P}P} : \bar{P} \to P$ the uncollapse map. Recall here that by Lemma 5.16, $\bar{P}$ is a $\delta_1^{\bar{P}}$-sound $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$. Define $(\mathcal{N}_\infty^{\overline{\mathrm{ext}}})_{\mathcal{V}}$ as the direct limit of the iterates $\bar{P}$ such for P.

Recall that $\mathcal{V}$ is automatically $\kappa_1^{\mathcal{V}}$-sound. Let $\alpha \in \mathrm{OR}$ and $P \in \mathscr{F}_1^{\mathcal{V}}$. We say that α is $(P, \mathscr{F}_1^{\mathcal{V}})$-*stable* iff whenever $P \preceq Q \in \mathscr{F}_1^{\mathcal{V}}$, we have $\alpha \in H^Q$ and

$$\pi_{\bar{Q}Q} \circ i_{\bar{P}\bar{Q}} \circ \pi_{\bar{P}P}^{-1}(\alpha) = \alpha. \hspace{2cm} \dashv$$

Adapting Lemmas 4.21, 4.22, 4.55, 4.56 and 4.57 and their proofs (and using that non-dropping $\Sigma_{\mathcal{V}_1}$-iterates of $\mathcal{V}_1$ are always $\kappa_1^{\mathcal{V}}$-sound), we have:

Lemma 5.45. Let $\mathcal{V}$ be a non-dropping $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$ and $\bar{\mathcal{V}}$ be the $\delta_1^{\mathcal{V}}$-core of $\mathcal{V}$. Let $\mathcal{N} = \mathcal{N}_\infty^{\mathcal{V}}$. Then:

1. For each $P \preceq Q \in \mathscr{F}_1^{\mathcal{V}}$, we have $H^P \cap \mathrm{OR} \subseteq H^Q \cap \mathrm{OR}$.

2. For each $\alpha \in \mathrm{OR}$ there is $P \in \mathscr{F}_1^{\mathcal{V}}$ such that α is $(P, \mathscr{F}_1^{\mathcal{V}})$-stable.

3. $\mathscr{I}^{\mathcal{N}_\infty} = \mathscr{I}^{\mathcal{V}_1}$ and $i_{\mathcal{V}_1 \mathcal{N}_\infty} \restriction \mathscr{I}^{\mathcal{V}_1} = \mathrm{id} = *_1 \restriction \mathscr{I}^{\mathcal{V}_1}$,

4. $\mathscr{I}^{\mathcal{N}_\infty^{\mathcal{V}}} = \mathscr{I}^{\mathcal{V}}$ and $*_1^{\mathcal{V}} \restriction \mathscr{I}^{\mathcal{V}} = \mathrm{id}$,

5. $\mathcal{N} = \mathcal{N}_\infty^{\mathcal{V}} = i_{\mathcal{V}_1 \mathcal{V}}(\mathcal{N}_\infty) = (\mathcal{N}_\infty^{\overline{\mathrm{ext}}})_{\mathcal{V}}$ is a $\delta_1^{\mathcal{N}}$-sound $\Sigma_{\bar{\mathcal{V}}}$-iterate of $\bar{\mathcal{V}}$. Moreover, $\mathcal{N} = (\mathcal{N}_\infty^{\overline{\mathrm{ext}}})_{\mathcal{V}}$ iff $\bar{\mathcal{V}} = \mathcal{V}$ iff $\mathcal{N}$ is a $\Sigma_{\mathcal{V}}$-iterate of $\mathcal{V}$ iff $\mathcal{V}$ is $\delta_1^{\mathcal{V}}$-sound.

6. Let $\mathcal{V}'$ be a non-dropping $\Sigma_{\mathcal{V}}$-iterate $\mathcal{V}$ with $\mathcal{V} | \kappa_1^{\mathcal{V}} \triangleleft \mathcal{V}'$. Then

   (a) $\mathcal{N}_\infty^{\mathcal{V}'}$ is a $\Sigma_{\mathcal{N}_\infty^{\mathcal{V}}}$-iterate of $\mathcal{N}_\infty^{\mathcal{V}}$, and

   (b) $i_{\mathcal{V}\mathcal{V}'} \restriction \mathcal{N}_\infty^{\mathcal{V}}$ is just the $\Sigma_{\mathcal{N}_\infty^{\mathcal{V}}}$-iteration map $\mathcal{N}_\infty^{\mathcal{V}} \to \mathcal{N}_\infty^{\mathcal{V}'}$.

7. $\mathcal{N}_\infty^{\mathcal{N}}$ is a $\delta_1^{\mathcal{N}_\infty^{\mathcal{N}}}$-sound $\Sigma_{\mathcal{N}}$-iterate of $\mathcal{N}$ and $*_1^{\mathcal{V}} \subseteq \pi_{1\infty}^{\mathcal{V}} : \mathcal{N} \to \mathcal{N}_\infty^{\mathcal{N}}$ is the $\Sigma_{\mathcal{N}}$-iteration map.

Recall that $\Sigma_{\mathcal{V}, \mathcal{V} | \alpha}$ denotes the restriction of $\Sigma_{\mathcal{V}}$ to trees based on $\mathcal{V} | \alpha$. Write $(\mathcal{N}_\infty^{\mathcal{V}})^- = \mathcal{N}_\infty^{\mathcal{V}} | \delta_{1\infty}^{\mathcal{V}}$.

Lemma 5.46. Let $\mathcal{V}$ be a non-dropping $\Sigma_{\mathcal{V}_1}$-iterate of $\mathcal{V}_1$. Then:

(a) $\mathcal{V}$ is closed under $\Sigma_{\mathcal{N}_\infty^{\mathcal{V}}, (\mathcal{N}_\infty^{\mathcal{V}})^-}$ and $\Sigma_{\mathcal{N}_\infty^{\mathcal{V}}, (\mathcal{N}_\infty^{\mathcal{V}})^-} \restriction \mathcal{V}$ is lightface definable over $\mathcal{V}$.

(b) Let $\lambda \geq \delta_0^{\mathscr{V}}$, let $\mathbb{P} \in \mathscr{V}|\lambda$, and g be $(\mathscr{V}, \mathbb{P})$-generic (with g appearing in some generic extension of V). Then $\mathscr{V}[g]$ is closed under $\Sigma_{\mathcal{N}_\infty^{\mathscr{V}}, (\mathcal{N}_\infty^{\mathscr{V}})^-}$ and $\Sigma_{\mathcal{N}_\infty^{\mathscr{V}}, (\mathcal{N}_\infty^{\mathscr{V}})^-} \restriction \mathscr{V}$ is definable over the universe of $\mathscr{V}[g]$ from the parameter $x = \mathscr{V}|\lambda^{+\mathscr{V}}$, uniformly in λ.[55]

Moreover, the definability is uniform in $\mathscr{V}, x$.

Proof. By Lemma 5.18, we can define $\mathscr{V}$ from x in $\mathscr{V}[g]$, and uniformly so. To compute the δ_1-short tree strategy (for $\mathcal{N}_\infty^{\mathscr{V}}$) and determine δ_1-maximality, use Lemma 5.22 (recall this involves $*$-translation). The computation of branches at δ_1-maximal stages is like in the proof of Lemma 4.18, using Lemmas 5.45 and 3.11 (or arguing as in Footnote 27 in place of Lemma 3.11). $\qquad\square$

By Lemma 4.84 and [14, ***Theorem 10.2], $\Sigma_{\mathcal{N}_\infty^{\mathscr{V}}}$ has minimal inflation condensation. So like in Remark 4.19, it follows that $\mathscr{V}[g]$ can also compute the tail strategy $\Gamma_{\mathcal{N}_\infty^{\mathscr{V}}, (\mathcal{N}_\infty^{\mathscr{V}})^-}$ for stacks on $\mathcal{N}_\infty^{\mathscr{V}}$, based on $(\mathcal{N}_\infty^{\mathscr{V}})^-$ (restricted to stacks in $\mathscr{V}[g]$), as in fact

$$\Gamma_{\mathcal{N}_\infty^{\mathscr{V}}, (\mathcal{N}_\infty^{\mathscr{V}})^-} = (\Sigma_{\mathcal{N}_\infty^{\mathscr{V}}, (\Sigma_\infty^{\mathscr{V}})^-})^{\mathrm{stk}}.$$

Similarly:

Lemma 5.47. $\mathscr{V}_2$ *is closed under* $\Sigma_{\mathscr{V}_2, \mathscr{V}_2^-}$ *and* $\Sigma_{\mathscr{V}_2, \mathscr{V}_2^-} \restriction \mathscr{V}_2$ *is lightface definable over* $\mathscr{V}_2$.

Proof. To compute the δ_0-short and δ_1-short tree strategies in $\mathscr{V}_2$, proceed much as in the proof of Lemmas 5.46 and 4.45, naturally adapted to $\mathscr{V}_2$. Since $\mathscr{V}_2$ is a ground of $\mathscr{V}_1$ via $\mathbb{L}^{\mathscr{V}_2}$ and because of the correspondence between $\mathbb{E}^{\mathscr{V}_2}$, $\mathbb{E}^{\mathscr{V}_1}$ and $\mathbb{E}^M$, we can perform the relevant P-constructions above $\gamma_1^{\mathscr{V}_2}$ using $\mathbb{E}^{\mathscr{V}_2}$ in the natural way. For δ_0-maximal and δ_1-maximal trees, we use the 0-long and 1-long extenders in $\mathbb{E}^{\mathscr{V}_2}$ as usual. $\qquad\square$

5.5   2-Varsovian strategy premice

Definition 5.48. For a $\mathscr{V}_1$-like $\mathscr{V}$, we define the lightface $\mathscr{V}$-classes $\mathcal{N}_\infty^{\mathscr{V}}$, $*_1^{\mathscr{V}}$, $\mathcal{N}_\infty[*_1]^{\mathscr{V}}$ and $\mathscr{V}_2^{\mathscr{V}}$ over $\mathscr{V}$ just as the corresponding classes are defined over $\mathscr{V}_1$.

Also given a $\mathscr{V}_1$-like $\mathscr{V}$ and $\bar{\mathscr{V}} \trianglelefteq \mathscr{V}$ with $\kappa_1^{+\mathscr{V}} \leq \mathrm{OR}^{\bar{\mathscr{V}}}$, we define $\mathscr{V}_2^{\bar{\mathscr{V}}}$ by recursion on $\mathrm{OR}^{\bar{\mathscr{V}}}$ by setting $\mathscr{V}_2^{\mathscr{V}||(\kappa_1^{+\mathscr{V}}+\alpha)} = \mathscr{V}_2^{\mathscr{V}}||(\gamma+\alpha)$, where $\gamma = \gamma_1^{\mathscr{V}_2^{\mathscr{V}}}$. Noting that this definition is level-by-level, we similarly define $\mathscr{V}_2^{\bar{\mathscr{V}}}(\kappa)$ whenever $\bar{\mathscr{V}}$ is a $\mathscr{V}_1$-small Vsp such that $\gamma_0^{\bar{\mathscr{V}}}$ exists and κ is an inaccessible limit of $\delta_0^{\bar{\mathscr{V}}}$-cutpoints of $\bar{\mathscr{V}}$ and $\kappa < \mathrm{OR}^{\bar{\mathscr{V}}}$, level-by-level (starting by defining $\mathscr{V}_2^{\bar{\mathscr{V}}|\kappa^{+\bar{\mathscr{V}}}}$ as $\mathscr{V}_2||\gamma_1^{\mathscr{V}_1}$ is defined (in the codes) over $\mathscr{V}_1|\kappa_1^{+\mathscr{V}_1}$). We will often suppress the κ from the notation, writing just $\mathscr{V}_2^{\bar{N}}$. $\qquad\dashv$

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[55] Regarding trees $\notin V$, cf. Footnote 26.

We now want to axiomatize structures in the hierarchy of $\mathscr{V}_2$ to some extent, just like for $\mathscr{V}_1$, adapting Definitions 4.49, 4.51 and 4.52. These are very straightforward adaptations, and the reader could fill it in him/herself, but because they are reasonably detailed, we write them out for convenience:

Definition 5.49. A *base 2-Vsp* is an amenable transitive structure $\mathscr{V} = (P_\infty, F)$ such that in some forcing extension there is P such that:

1. P, P_∞ are 1-Vsps which model ZFC$^-$, $\gamma_0^P < \mathrm{OR}^P$ and $\gamma_0^{P_\infty} < \mathrm{OR}^{P_\infty}$, and P, P_∞ are $\mathscr{V}_1$-small (that is, P has no active segments satisfying "There are δ_1', κ_1' such that $\gamma_0^P < \delta_1' < \kappa_1'$ and δ_1' is Woodin and κ_1' is strong", and likewise for P_∞).

2. P has a unique Woodin cardinal $\delta_1^P > \gamma_0^P$ and a largest cardinal $\kappa_1^P > \delta_1^P$, and κ_1^P is inaccessible in P and a limit of δ_0^P-cutpoints of P; likewise for P_∞,

3. $\mathrm{OR}^P = \delta_1^{P_\infty}$, κ_1^P is the least measurable of P_∞ and $\mathscr{V}_2^P = \mathrm{cHull}_1^{\mathscr{V}}(\delta_1^{P_\infty})$,

4. $\mathcal{N}_\infty^{P_\infty}$ (defined over P_∞ like $\mathcal{N}_\infty|\gamma_1^{\mathscr{V}_2}$ is defined over $\mathscr{V}_1|\kappa_1^{+\mathscr{V}_1}$) is well-defined, and has least measurable $\kappa_1^{P_\infty}$ and second Woodin $\delta_1^{\mathcal{N}_\infty^{P_\infty}} = \mathrm{OR}^{P_\infty}$,

5. $\mathcal{N}_\infty^{P_\infty}|\delta_1^{\mathcal{N}_\infty^{P_\infty}}$ is obtained by iterating $P_\infty|\delta_1^{P_\infty}$, via a short-normal tree $\mathcal{T}$ of length $\delta_1^{\mathcal{N}_\infty^{P_\infty}}$,

6. F is a cofinal Σ_1-elementary (hence fully elementary) embedding

$$F : P_\infty|\delta_1^{P_\infty} \to \mathcal{N}_\infty^{P_\infty}|\delta_1^{\mathcal{N}_\infty P_\infty},$$

and there is a $\mathcal{T}$-cofinal branch b such that $F \subseteq i_b^{\mathcal{T}}$, and $i_b^{\mathcal{T}}(\delta_1^{P_\infty}) = \delta_1^{\mathcal{N}_\infty^{P_\infty}}$ (so b is intercomputable with F, and note that by amenability of $\mathscr{V}$, F is amenable to P_∞, and hence so is b),

7. $\rho_1^{\mathscr{V}} = \delta_1^{P_\infty} = \mathrm{OR}^P$ and $p_1^{\mathscr{V}} = \emptyset$ (so $\mathfrak{C}_1(\mathscr{V}) = \mathscr{V}_2^P$) and $\delta_1^{P_\infty}$ is Woodin in $\mathcal{J}(\mathfrak{C}_1(\mathscr{V}))$, as witnessed by $\mathbb{E}^{P_\infty}$.

8. P is $(\mathcal{J}(\mathfrak{C}_1(\mathscr{V})), \mathbb{L}^{\mathscr{V}})$-generic, where $\mathbb{L}^{\mathscr{V}}$ is defined over $\mathscr{V}$ as $\mathbb{L}$ above was defined over $\mathscr{V}_2||\gamma_1^{\mathscr{V}_2}$. $\dashv$

Remark 4.50 carries over directly.

Definition 5.50. A *2-Varsovian strategy premouse (2-Vsp)* is a structure

$$\mathscr{V} = (\mathcal{J}_\alpha^{\mathbb{E}}, \mathbb{E}, F)$$

for some sequence $\mathbb{E}$ of extenders, where either $\mathscr{V}$ is a premouse or a 1-Vsp, or:

1. $\alpha \leq \mathrm{OR}$ and $\mathscr{V}$ is an amenable acceptable J-structure,

2. $\mathscr{V}$ has at least two Woodin cardinals, the least two of which are $\delta_0^{\mathscr{V}} < \delta_1^{\mathscr{V}}$, and has an initial segment $\mathscr{V}||\gamma$ which is a base 2-Vsp,

3. $\delta_1^{\mathscr{V}} < \gamma$, so $\delta_1^{\mathscr{V}}$ is the second Woodin of $\mathscr{V}||\gamma$,

4. if $F \neq \emptyset$ and $\gamma < \mathrm{OR}^{\mathscr{V}}$ then either:

   (a) $\mathscr{V}$ satisfies the premouse axioms (for Jensen indexing) with respect to F, and $\gamma < \mathrm{crit}(F)$, or

   (b) $\mathscr{V}$ satisfies the 1-Vsp axioms for a long extender, i.e. clause 4b of Definition 4.51, for giving an iteration map on (the premouse) $\mathscr{V}|\delta_0^{\mathscr{V}}$, or

   (c) i. $\mathscr{V}^{\mathrm{pv}} = (\mathcal{J}_\alpha^{\mathbb{E}}, \mathbb{E}, \emptyset) \vDash \mathrm{ZFC}^-$,

       ii. $\mathscr{V}$ has largest cardinal μ, which is inaccessible in $\mathscr{V}$ and a limit of $\delta_0^{\mathscr{V}}$-cutpoints of $\mathscr{V}$ (where $\delta_0^{\mathscr{V}}$-*cutpoint* applies to both short extenders and long extenders over $\mathscr{V}|\delta_1^{\mathscr{V}}$)

       iii. $\mathcal{N} = \mathcal{N}_\infty^{\mathscr{V}^{\mathrm{pv}}}$ is a well-defined, and satisfies the axioms of a 1-Vsp with $\delta_1^{\mathcal{N}}$ existing (but $\mathcal{N}$ is possibly illfounded), and $\mathcal{N}$ is $(\mathrm{OR}^{\mathscr{V}} + 1)$-wellfounded, with $\delta_1^{\mathcal{N}} = \mathrm{OR}^{\mathscr{V}}$,

       iv. $\mathcal{N}|\delta_1^{\mathcal{N}}$ is a proper class of $\mathscr{V}^{\mathrm{pv}}$ has least measurable μ,

       v. F is a cofinal Σ_1-elementary embedding $F : \mathscr{V}|\delta_1^{\mathscr{V}} \to \mathcal{N}|\delta_1^{\mathcal{N}}$,

       vi. $\mathcal{N}|\delta_1^{\mathcal{N}}$ is pseudo-iterate of $\mathscr{V}|\delta_1^{\mathscr{V}}$, via short-normal tree $\mathcal{T}$, and there is a $\mathcal{T}$-cofinal branch b such that $F \subseteq i_b^{\mathcal{T}}$ (hence b is amenable to $\mathscr{V}$ and inter-definable with F over $\mathscr{V}^{\mathrm{pv}}$),

5. each proper segment of $\mathscr{V}$ is a sound 2-Vsp (defining *2-Vsp* recursively), where the fine structural language for active segments is just that with symbols for $\in, \mathbb{E}, F$,

6. some $p \in \mathbb{L}^{\mathscr{V}} = \mathbb{L}^{\mathscr{V}||\gamma}$ forces that the generic object is a 1-Vsp P of height $\delta_1^{\mathscr{V}}$ with $\mathscr{V}_2^P = \mathscr{V}||\gamma$, and there is an extension P to a 1-Vsp P^+ such that $\mathscr{V}_2^{P^+} = \mathscr{V}$ (so P^+ is level-by-level definable over $\mathscr{V}$, via inverse P-construction).

We write $\gamma_1^{\mathscr{V}} = \gamma$ above (if $\mathscr{V}$ is not a 1-Vsp). $\dashv$

Definition 5.51. A 2-Vsp $\mathscr{V}$ is $\mathscr{V}_2$-*like* iff it is proper class and in some set-generic extension, $\mathscr{V} = \mathscr{V}_2^N$ for some M_{swsw}-like premouse N. (Note this is first-order over $\mathscr{V}$.)

We write $\mathscr{V}_2 \downarrow 1 = \mathcal{N}_\infty$ and $\mathscr{V}_2 \downarrow 0 = \mathcal{N}_\infty \downarrow 0$. Let $\mathscr{V}$ be $\mathscr{V}_2$-like. We define $\mathscr{V} \downarrow 1$ and $\mathscr{V} \downarrow 0$ analogously (first-order over $\mathscr{V}$ as in the proof of Lemma 5.43 part 4). In fact, let us define $\mathscr{V} \downarrow i$ more generally, in the same first-order manner, but allowing $\mathscr{V}$ to be illfounded, but $\mathscr{V}_2$-like with respect to first-order properties. Also if N is a 1-Vsp, let $N \downarrow 1 = N$. We write $\mathscr{V}^-$ for (the 1-Vsp) $\mathscr{V}|\delta_1^{\mathscr{V}}$. We write $\Lambda_i^{\mathscr{V}}$ for the putative strategy for $\mathscr{V} \downarrow i$ for trees based on $\mathscr{V}|\delta_i^{\mathscr{V}}$, defined over $\mathscr{V}$ just as the corresponding restrictions of $\Sigma_{\mathscr{V}_2 \downarrow i}$ are defined over $\mathscr{V}_2$, via the proof of Lemma 5.47.

We write $\mathscr{V}_2 = \mathscr{V}(\mathcal{N}_\infty, *_1) = \mathscr{V}(\mathcal{N}_\infty, *_1 \!\restriction\! \delta_{1\infty}) = \mathscr{V}(\mathcal{N}_\infty, e_1^{\mathscr{V}_2})$. Given a pair $(N, *')$ or $(N, *' \!\restriction\! \delta)$ or (N, e) where N is $\mathscr{V}_1$-like and the pair has similar first-order properties as does $(\mathcal{N}_\infty, *_1)$ or $(\mathcal{N}_\infty, *_1 \!\restriction\! \delta_{1\infty})$ or $(\mathcal{N}_\infty, e_1^{\mathscr{V}_2})$ respectively, we

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define $\mathscr{V}(N, *')$ or $\mathscr{V}(N, *' \upharpoonright \delta)$ or $\mathscr{V}(N, e)$ analogously (via the proof of Lemma 5.43 part 3). ⊣

5.6 Iterability of $\mathscr{V}_2$

In this subsection we will define a normal iteration strategy $\Sigma_{\mathscr{V}_2}$ for $\mathscr{V}_2$ in V.

Definition 5.52. For $i \leq 1$, an *i-long* extender is a $(\delta_i^{\mathscr{V}}, \delta)$-extender over $\mathscr{V}$, for some premouse, 1-Vsp or 2-Vsp $\mathscr{V}$, and some δ. ⊣

Definition 5.53. Let $\mathscr{V}$ be a $\mathscr{V}_2$-like 2-Vsp. A 0-*maximal iteration tree* $\mathcal{T}$ on $\mathscr{V}$ *of length* $\lambda \geq 1$ is a system with the usual properties for 0-maximality, except that when $E_\alpha^{\mathcal{T}}$ is a *i*-long extender, then then $\mathrm{pred}^{\mathcal{T}}(\alpha + 1)$ is the least $\beta \leq \alpha$ such that $[0, \beta]_{\mathcal{T}}$ does not drop and $\delta_i^{M_\beta^{\mathcal{T}}} < \mathrm{lh}(E_\alpha^{\mathcal{T}})$.

Iteration strategies and *iterability* for (such trees on) $\mathscr{V}_2$ are defined in the obvious manner (one detail here is that if $[0, \alpha + 1]_{\mathcal{T}}$ does not drop then $M_{\alpha+1}^{\mathcal{T}}$ is a (putative) 2-Vsp, including when $E_\alpha^{\mathcal{T}}$ is 0-long). ⊣

Definition 5.54. A *short-normal tree* on a $\mathscr{V}_2$-like 2-Vsp $\mathscr{V}$ is a 0-maximal tree that uses no long extenders. Note that a short-normal tree is of the form $\mathcal{T}_0 \,^\frown \mathcal{T}_1 \,^\frown \mathcal{S}$, where $\mathcal{T}_0$ is based on $\mathscr{V} | \delta_0^{\mathscr{V}}$, either

  (i) [$\mathcal{T}_0$ has limit length or $b^{\mathcal{T}_0}$ drops] and $\mathcal{T}_1 = \mathcal{S} = \emptyset$, or

  (ii) $\mathcal{T}_0$ has successor length, $b^{\mathcal{T}_0}$ does not drop and $\mathcal{T}_1$ is above $\gamma_0^{M_\infty^{\mathcal{T}_0}}$ and based on $M_\infty^{\mathcal{T}_0} | \delta_1^{M_\infty^{\mathcal{T}_0}}$,

and if $\mathcal{T}_1 \neq \emptyset$ then either

  (i) [$\mathcal{T}_1$ has limit length or $b^{\mathcal{T}_1}$ drops] and $\mathcal{S} = \emptyset$, or

  (ii) $\mathcal{T}_1$ has successor length, $b^{\mathcal{T}_1}$ does not drop and $\mathcal{S}$ is above $\gamma_1^{M_\infty^{\mathcal{T}_1}}$.

Say that $\mathcal{T}_0 \,^\frown \mathcal{T}_1$ and $\mathcal{S}$ are the *lower, upper components* respectively, and $\mathcal{T}_i$ the *i-lower component*. ⊣

5.6.1 Condensation properties for full normalization

Definition 5.55. We define the notions $(m+1)$-*relevantly condensing*, $(m+1)$-*sub-condensing* and *n-standard* for 2-Vsps just as for 1-Vsps (see Definition 4.70), replacing the role of premice there with 1-Vsps, and replacing $\delta_0^{\mathscr{V}}, \gamma_0^{\mathscr{V}}$ with $\delta_1^{\mathscr{V}}, \gamma_1^{\mathscr{V}}$. ⊣

By Lemma 4.71 and its proof we have the following, and Remark 4.72 carries over directly:

Lemma 5.56. $\mathscr{V}_2$ *is* ω-*standard. (Thus, we take* $\mathscr{V}_2$-*like to include* ω-*standard.)*

5.6.2 Tree translation from $\mathscr{V}_1$ to $\mathscr{V}_2$

Definition 5.57. Let $\mathscr{V}$ be $\mathscr{V}_1$-like. We define *1-translatable* trees $\mathcal{T}$ on $\mathscr{V}$ like in Definition 4.60, with 0 replaced by 1 as appropriate, but add the demand that $\mathcal{T}$ uses no 0-long extenders.

Let $\mathcal{T}$ on $\mathscr{V}$ be 1-translatable. The *1-translation* of $\mathcal{T}$ is the tree on $\mathscr{V}_2^{\mathscr{V}}$ defined just as in Definition 4.62. $\dashv$

Remarks 4.61 and 4.63 carry over directly, replacing 0 with 1 as appropriate. Likewise Lemma 4.64 and its proof:

Lemma 5.58. *Let $\mathcal{T}$ on $\mathscr{V}$ be 1-translatable, where $\mathscr{V}$ is $\mathscr{V}_1$-like. Then:*

1. *The 1-translation $\mathcal{U}$ of $\mathcal{T}$ exists and is unique.*

2. $M_\alpha^{\mathcal{U}} = \mathscr{V}_2^{M_\alpha^{\mathcal{T}}}$ *and* $\gamma_1^{M_\alpha^{\mathcal{U}}} < \mathrm{OR}(M_\alpha^{\mathcal{U}})$ *for all* $\alpha < \mathrm{lh}(\mathcal{T})$.

3. $i_{\alpha\beta}^{\mathcal{U}} = i_{\alpha\beta}^{\mathcal{T}} \restriction M_\alpha^{\mathcal{U}}$ *for all* $\alpha <_{\mathcal{T}} \beta$ *such that* $(\alpha, \beta]_{\mathcal{T}}$ *does not drop.*

4. $M_{\alpha+1}^{*\mathcal{U}} = \mathscr{V}_2^{M_{\alpha+1}^{*\mathcal{T}}}$ *for all* $\alpha + 1 < \mathrm{lh}(\mathcal{T})$.

5. $i_{\alpha+1}^{*\mathcal{U}} = i_{\alpha+1}^{*\mathcal{T}} \restriction M_{\alpha+1}^{*\mathcal{U}}$ *for all* $\alpha + 1 < \mathrm{lh}(\mathcal{T})$.

5.6.3 Trees based on $\mathcal{N}_\infty | \delta_1^{\mathcal{N}_\infty}$

Toward defining $\Sigma_{\mathscr{V}_2}$, we first consider trees on $\mathscr{V}_2$ based on $\mathscr{V}_2^- = \mathcal{N}_\infty | \delta_1^{\mathcal{N}_\infty}$, adapting Definition 4.65:

Definition 5.59. Write $\Sigma_{\mathcal{N}_\infty, \mathscr{V}_2^-}^{\mathrm{sn}}$ for the strategy for $\mathcal{N}_\infty$ for short-normal trees based on $\mathscr{V}_2^-$, induced by $\Sigma_{\mathcal{N}_\infty}$. Let $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$ denote the putative strategy for short-normal trees on $\mathscr{V}_2$ based on $\mathscr{V}_2^-$, induced by $\Sigma_{\mathcal{N}_\infty, \mathscr{V}_2^-}^{\mathrm{sn}}$. This makes sense by Lemma 5.36. $\dashv$

Remark 4.66 adapts routinely. We now partially adapt Lemma 4.68, but omit the clause "and in fact, $\Lambda^{M_\alpha^{\mathcal{U}}} \subseteq \Sigma_{M_\alpha^{\mathcal{T}}}$", as we will prove this in more generality later, in Lemma 5.68. The proof of the rest is a direct adaptation:

Lemma 5.60. $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$ *yields wellfounded models. Moreover, let $\mathcal{T}$ be on $\mathcal{N}_\infty$, via $\Sigma_{\mathcal{N}_\infty, \mathscr{V}_2^-}^{\mathrm{sn}}$, and let $\mathcal{U}$ be the corresponding tree on $\mathscr{V}_2$ (so via $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$). Let*

$$\pi_\alpha : M_\alpha^{\mathcal{T}} \to M_\alpha^{\mathcal{U}} \downarrow 1 \subseteq M_\alpha^{\mathcal{U}}$$

be the natural copy map (where $\pi_0 = \mathrm{id}$). Then:

(i) $[0, \alpha]_{\mathcal{T}}$ *drops iff* $[0, \alpha]_{\mathcal{U}}$ *drops.*

(ii) *If $[0, \alpha]_{\mathcal{T}}$ drops then $M_\alpha^{\mathcal{T}} = M_\alpha^{\mathcal{U}} = M_\alpha^{\mathcal{U}} \downarrow 1$ (cf. Remark 4.66 adapted).*

(iii) *If $[0, \alpha]_{\mathcal{T}}$ does not drop then $M_\alpha^{\mathcal{T}} = M_\alpha^{\mathcal{U}} \downarrow 1$ and $M_\alpha^{\mathcal{U}} = \mathscr{V}(M_\alpha^{\mathcal{T}}, \ell)$ where $\ell : M_\alpha^{\mathcal{T}} \to \mathcal{N}_\infty^{M_\alpha^{\mathcal{T}}}$ is the correct iteration map,*

(iv) $\pi_\alpha = \mathrm{id}$; therefore, $i_\alpha^\mathcal{T} \subseteq i_\alpha^\mathcal{U}$.

Definition 5.61. Given a non-dropping $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$-iterate $\mathscr{V}$ of $\mathscr{V}_2$, let $\Psi_{\mathscr{V}, \mathscr{V}^-}$ be induced by $\Sigma_{\mathscr{V} \downarrow 1}$ just as $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$ is induced by $\Sigma_{\mathcal{N}_\infty}$ (this makes sense by Lemma 5.60). $\dashv$

5.6.4 Short-normal trees on $\mathscr{V}_2$

Definition 5.62. Let $\mathscr{V}$ be a (possibly dropping, putative) iterate of $\mathscr{V}_2$, via a short-normal tree $\mathcal{T} \,^\frown \mathcal{S}$ with lower and upper components $\mathcal{T}, \mathcal{S}$. We say that $\mathscr{V}$ is *good* iff $\mathcal{T}$ is via $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$, $\mathscr{V}$ is wellfounded and for every i-long $E \in \mathbb{E}_+^\mathscr{V}$, $\mathcal{M}_{i\infty}^{\mathscr{V}|\mathrm{lh}(E)} = P|\delta_i^P$ for some $\Sigma_{\mathscr{V} \downarrow i}$-iterate P of $\mathscr{V} \downarrow i$, and E is the corresponding iteration map.

Say that a (partial) iteration strategy Ψ for $\mathscr{V}_2$ is *good* iff all putative iterates via Ψ are good. $\dashv$

We now extend $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$ to a good short-normal 0-maximal strategy Ψ_{sn} for $\mathscr{V}_2$. We first deal with trees based on $\mathscr{V}||\gamma_1^\mathscr{V}$:

Definition 5.63. Write $\Psi_{\mathscr{V}_2, \gamma_1^{\mathscr{V}_2}}$ for the putative strategy Ψ for $\mathscr{V}_2$, for short-normal 0-maximal trees based on $\mathscr{V}_2||\gamma_1^{\mathscr{V}_2}$, as follows:

1. $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-} \subseteq \Psi$, and

2. given $\mathcal{T}$ via $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$, of successor length $\alpha+1$, where $[0, \alpha]_\mathcal{T}$ does not drop, and given a putative 0-maximal tree $\mathcal{U}$ on $M_\alpha^\mathcal{T}||\gamma_1^{M_\alpha^\mathcal{T}}$, which is above $\delta_1^{M_\alpha^\mathcal{T}}$, then $\mathcal{T} \,^\frown \mathcal{U}$ is (equivalent to a tree) via Ψ iff there is a tree $\mathcal{U}'$ on $M_\alpha^\mathcal{T} \downarrow 1$, via $\Sigma_{M_\alpha^\mathcal{T} \downarrow 1}$, with the same extenders and tree order as $\mathcal{U}$. $\dashv$

We adapt Lemma 4.75:

Lemma 5.64. $\Psi_{\mathscr{V}_2, \gamma_1^{\mathscr{V}_2}}$ *is a short-normal 0-maximal strategy (hence yields wellfounded models). Moreover, let $\mathcal{T} \,^\frown \mathcal{U}$ and $\mathcal{U}'$ be as in Definition 5.63, with $\mathcal{U} \neq \emptyset$. Then:*

1. $M_0^\mathcal{U} = M_\alpha^\mathcal{T}||\gamma_1^{M_\alpha^\mathcal{T}}$ *and* $\deg_0^\mathcal{U} = 0$,

2. $M_0^{\mathcal{U}'} = M_\alpha^\mathcal{T} \downarrow 1$ *and* $\deg_0^{\mathcal{U}'} = 0$, *so* $(M_0^\mathcal{U})^{\mathrm{pv}} = M_0^{\mathcal{U}'}|\kappa_1^{+M_0^{\mathcal{U}'}}$,

3. *for* $0 < \beta < \mathrm{lh}(\mathcal{U})$, $\beta \in \mathscr{D}_{\deg}^\mathcal{U} \Leftrightarrow \beta \in \mathscr{D}_{\deg}^{\mathcal{U}'}$, *and* $\deg_\beta^\mathcal{U} = \deg_\beta^{\mathcal{U}'}$,

4. *if* $0 < \beta < \mathrm{lh}(\mathcal{U})$ *and* $[0, \beta]_\mathcal{U}$ *drops then* $M_\beta^\mathcal{U} = M_{\beta'}^\mathcal{U}$,

5. *if* $0 < \beta < \mathrm{lh}(\mathcal{U})$ *and* $[0, \beta]_\mathcal{U}$ *does not drop then* $(M_\beta^\mathcal{U})^{\mathrm{pv}} = M_\beta^{\mathcal{U}'}|\kappa_1^{+M_\beta^{\mathcal{U}'}}$,

6. *if* $0 < \beta + 1 < \mathrm{lh}(\mathcal{U})$ *and* $[0, \beta + 1]_\mathcal{U}$ *drops then* $M_{\beta+1}^{*\mathcal{U}} = M_{\beta+1}^{*\mathcal{U}'}$ *and* $i_{\beta+1}^{*\mathcal{U}} = i_{\beta+1}^{*\mathcal{U}'}$,

7. if $0 < \beta + 1 < \text{lh}(\mathcal{U})$ and $[0, \beta + 1]_{\mathcal{U}'}$ does not drop then $i^{*\mathcal{U}}_{\beta+1} \subseteq i^{*\mathcal{U}'}_{\beta+1}$,

8. if $0 \leq \beta < \text{lh}(\mathcal{U})$ and $[0, \beta]_{\mathcal{U}}$ does not drop then $M^{\mathcal{U}'}_{\beta}$ is a $(\kappa^{M^{\mathcal{U}'}_{\beta}}_1$-sound)

$\Sigma_{M^{\mathcal{T}}_{\alpha} \downarrow 1}$-iterate of $M^{\mathcal{T}}_{\alpha} \downarrow 1$, $\mathcal{N}^{M^{\mathcal{U}'}_{\beta}}_{\infty}$ is a $\delta^{\mathcal{N}^{M^{\mathcal{U}'}_{\beta}}_{\infty}}_1$-sound $\Sigma_{M^{\mathcal{T}}_{\alpha} \downarrow 1}$-iterate of $M^{\mathcal{T}}_{\alpha} \downarrow 1$,

$$\text{Ult}(M^{\mathcal{T}}_{\alpha} \downarrow 1, F(M^{\mathcal{T}}_{\beta})) = \mathcal{N}^{M^{\mathcal{U}'}_{\beta}}_{\infty}$$

and $F(M^{\mathcal{T}}_{\beta})$ is the extender of the $\Sigma_{M^{\mathcal{T}}_{\alpha} \downarrow 1}$-iteration map.

Therefore if $\mathcal{T}, \mathcal{U}$ each have successor length, then $M^{\mathcal{T} \,\hat{}\, \mathcal{U}}_{\infty}$ is good with respect to extenders indexed $\leq \gamma^{M^{\mathcal{T} \,\hat{}\, \mathcal{U}}_{\infty}}_1$ (or all extenders in $\mathbb{E}_+(M^{\mathcal{T} \,\hat{}\, \mathcal{U}}_{\infty})$, if $b^{\mathcal{T} \,\hat{}\, \mathcal{U}}$ drops).

Proof. The (last) "therefore" clause is because $\Sigma_{\mathcal{V}_1}$ is good. The rest of the proof is like for Lemma 4.75 (although we did not yet prove that $\Psi_{\mathcal{V}_2, \mathcal{V}_2^-}$ is good, Lemma 5.60(iii) does give the instance of this with respect to $F^{\mathcal{V} || \gamma^{\mathcal{V}}_1}$, where $\mathcal{V} = M^{\mathcal{T}}_{\alpha}$, which is enough to prove part 8 as in Lemma 4.75). □

We now prove a couple of variants of the branch condensation lemma 3.11 for trees on $\mathcal{V}_1$:

Lemma 5.65. *Let $\mathcal{T}, \mathcal{U}$ be short-normal on $\mathcal{V}_1$, via $\Sigma_{\mathcal{V}_1}$, based on $\mathcal{V}_1 | \delta^{\mathcal{V}_1}_1$, with $\mathcal{T}$ of limit length, $\mathcal{U}$ successor length with $b^{\mathcal{U}}$ non-dropping and $\delta(\mathcal{U}) = \delta^{M^{\mathcal{U}}_{\infty}}_1$. Let G be V-generic. Let $b, k \in V[G]$ where b is a non-dropping $\mathcal{T}$-cofinal branch with $i^{\mathcal{T}}_b(\delta^{\mathcal{V}_1}_1) = \delta(\mathcal{T})$ and and*

$$k : M^{\mathcal{T}}_b | \delta^{M^{\mathcal{T}}_b}_1 \to M^{\mathcal{U}}_{\infty} | \delta^{M^{\mathcal{U}}_{\infty}}_1$$

is elementary with $k \circ i^{\mathcal{T}}_b = i^{\mathcal{U}}_{0\infty} \restriction (\mathcal{V}_1 | \delta^{\mathcal{V}_1}_1)$. Then $b = \Sigma_{\mathcal{V}_1}(\mathcal{T})$.

Proof. We may assume $b, k \in V$. Let $\mathcal{V} = M^{\mathcal{T}}_b$. Let $\alpha \in b$ be least with either $\alpha + 1 = \text{lh}(\mathcal{T})$ or $\delta^{M^{\mathcal{T}}_{\alpha}}_0 < \text{crit}(i^{\mathcal{T}}_{\alpha b})$. So $\bar{\mathcal{V}} = M^{\mathcal{T}}_{\alpha}$ is $\delta^{\bar{\mathcal{V}}}_0$-sound and $\mathcal{T} \restriction [\alpha, \infty)$ is on $\bar{\mathcal{V}}$, above $\delta^{\bar{\mathcal{V}}}_0$. Let $\beta \in b^{\mathcal{U}}$ be analogous for $\mathcal{U}$.

Define $\pi : \mathcal{V} \to M^{\mathcal{U}}_{\infty}$ in the natural way, extending k, like in Lemma 3.11. Also define $\bar{k} : \bar{\mathcal{V}} \to M^{\mathcal{U}}_{\beta}$ analogously. So $\bar{k} \restriction \delta^{\bar{\mathcal{V}}}_0 \subseteq k$.

The phalanx $\mathfrak{P} = ((\bar{\mathcal{V}}, \delta^{\bar{\mathcal{V}}}_0), \mathcal{V}, \delta(\mathcal{T}))$ is iterable, via lifting trees with $(\bar{k}, k)$.

Let $c = \Sigma_{\mathcal{V}_1}(\mathcal{T})$ and $Q_c = Q(\mathcal{T}, c)$ be the Q-structure, or $Q_c = M^{\mathcal{T}}_c$ if $i^{\mathcal{T}}_c(\delta^{\mathcal{V}_1}_1) = \delta(\mathcal{T})$. Let $\mathfrak{Q}$ be the phalanx $((\bar{\mathcal{V}}, \delta^{\bar{\mathcal{V}}}_0), Q_c, \delta(\mathcal{T}))$, which is also iterable.

Let $\bar{\mathcal{V}}^+$ be a generic expansion of $M^{\mathcal{T}}_{\alpha}$ (so $\bar{\mathcal{V}}^+$ is an M_{swsw}-like premouse and $\bar{\mathcal{V}} = \mathcal{V}_1^{\bar{\mathcal{V}}^+}$). Then $\mathcal{T} \restriction [\alpha, \infty)$ can be translated to a tree $\mathcal{T}^+$ on $\bar{\mathcal{V}}^+$, which is above $\kappa^{+\bar{\mathcal{V}}^+}_0$. Let $\mathcal{V}^+ = M^{\mathcal{T}^+}_b$ and $Q^+_c = Q(\mathcal{T}^+, c)$ or $Q^+_c = M^{\mathcal{T}^+}_c$ accordingly. Then the phalanxes

$$\mathfrak{P}^+ = ((\bar{\mathcal{V}}^+, \kappa^{\bar{\mathcal{V}}^+}_0 + 1), \mathcal{V}^+, \delta(\mathcal{T})) \text{ and } \mathfrak{Q}^+ = ((\bar{\mathcal{V}}^+, \kappa^{\bar{\mathcal{V}}^+}_0 + 1), Q^+_c, \delta(\mathcal{T}))$$

are iterable, since trees on them translate to trees on $\mathfrak{P},\mathfrak{Q}$. Note that $\bar{\mathscr{V}}^+$ is $\kappa_0^{\bar{\mathscr{V}}^+}$-sound, and $\mathscr{V}^+, Q_c^+$ are $\delta(\mathcal{T})$-sound. But then comparing $\mathfrak{P}^+$ versus $\mathfrak{Q}^+$ gives $b = c$. $\qquad\square$

Lemma 5.66. *Let* $\mathcal{T}, \mathcal{U}$ *be short-normal dsr trees on* $\mathscr{V}_1$, *via* $\Sigma_{\mathscr{V}_1}$, *based on* $\mathscr{V}_1|\delta_1^{\mathscr{V}_1}$, *with* $\mathcal{T}, \mathcal{U}$ *of limit length. Suppose there is* $\alpha < \mathrm{lh}(\mathcal{T})$ *such that* $[0, \alpha]_\mathcal{T}$ *does not drop and* $\mathcal{T}\!\restriction\![\alpha, \infty)$ *is above* $\delta_0^{M_\alpha^\mathcal{T}}$, *and there is an analogous such* $\beta < \mathrm{lh}(\mathcal{U})$; *fix the least such* α, β. *Let* $c = \Sigma_{\mathscr{V}_1}(\mathcal{U})$. *Suppose that if* c *is non-dropping then* $\delta(\mathcal{U}) < \delta_1^{M_c^\mathcal{U}}$. *Let* G *be* V-*generic. Let* $b \in V[G]$ *where* b *is a* $\mathcal{T}$-*cofinal branch such that if* b *is non-dropping then* $\delta(\mathcal{T}) < \delta_1^{M_b^\mathcal{T}}$. *Let* $k \in V[G]$ *be such that*

$$k : Q(\mathcal{T}, b) \to Q(\mathcal{U}, c)$$

is elementary and $k \circ i_b^\mathcal{T}\!\restriction\!(\mathscr{V}_1|\delta_0^{\mathscr{V}_1}) = i_c^\mathcal{U}\!\restriction\!(\mathscr{V}_1|\delta_0^{\mathscr{V}_1})$. *Then* $b = \Sigma_{\mathscr{V}_1}(\mathcal{T})$.

Proof. This is via a straightforward variant of the proof of Lemma 5.65, noting that because $\mathcal{T}$ is dsr, $Q(\mathcal{T}, b)$ can only overlap $\delta(\mathcal{T})$ with long extenders. $\qquad\square$

It turns out that the method we used to define $\Psi_{\mathscr{V}_1}^{\mathrm{sn}}$ is not so well suited to $\mathscr{V}_2$. Instead we proceed as follows:

Definition 5.67. $\Psi_{\mathscr{V}_2}^{\mathrm{sn}}$ denotes the (putative) short-normal strategy Ψ for $\mathscr{V}_2$, defined as follows. Firstly, $\Psi_{\mathscr{V}_2, \gamma_1^{\mathscr{V}_2}} \subseteq \Psi$. Secondly, let $\mathcal{T}$ be via $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$, of successor length, such that $b^\mathcal{T}$ does not drop, and $\mathscr{V} = M_\infty^\mathcal{T}$. We define the action of Ψ on above-$\gamma_1^{\mathscr{V}}$ trees on $\mathscr{V}$.

Let $\mathcal{P} = \mathscr{V} \downarrow 1 = i_{\mathscr{V}_2\mathscr{V}}(\mathcal{N}_\infty)$, so (by Lemma 5.60) $\mathcal{P}$ is a $\Sigma_{\mathcal{N}_\infty}$-iterate of $\mathcal{N}_\infty$ and $\mathcal{P} \subseteq \mathscr{V}$. Let $e = F^{\mathscr{V}||\gamma_1^{\mathscr{V}}}$. Let (and by Lemma 5.43)

$$i_e^{\mathscr{V}} : \mathscr{V} \to \mathrm{Ult}(\mathscr{V}, e) = \mathscr{V}_2^{\mathcal{P}}$$

be the ultrapower map. Let Λ be the above-$\gamma_1^{\mathscr{V}_2^{\mathcal{P}}}$ strategy for $\mathscr{V}_2^{\mathcal{P}}$ determined by translating above-$\kappa_1^{+\mathcal{P}}$ (hence 1-translatable) trees on $\mathcal{P}$ via $\Sigma_{\mathcal{P}}$. Then for above-$\gamma_1^{\mathscr{V}}$ trees $\mathcal{U}$ on $\mathscr{V}$, $\mathcal{T}\,\widehat{\ }\,\mathcal{U}$ is via Ψ iff $\mathcal{U}$ is via the minimal $i_e^{\mathscr{V}}$-pullback of Γ. $\qquad\dashv$

Lemma 5.68. $\Psi_{\mathscr{V}_2}^{\mathrm{sn}}$ *is good.*

Proof. Clearly $\Psi_{\mathscr{V}_2}^{\mathrm{sn}}$ is well-defined and yields wellfounded models. So let $\mathcal{T}\,\widehat{\ }\,\mathcal{U}$ be via $\Psi_{\mathscr{V}_2}^{\mathrm{sn}}$, as in Definition 5.67, and $\mathscr{V}, \mathcal{P}, e, i_e = i_e^{\mathscr{V}}$ be as there. Let $i_e``\mathcal{U}$ be the minimal $i_e = i_e^{\mathscr{V}}$-copy of $\mathcal{U}$ to a tree on $\mathscr{V}_2^{\mathcal{P}}$. Let $\mathscr{W} = M_\infty^{\mathcal{T}\,\widehat{\ }\,\mathcal{U}}$ and $\gamma_1^{\mathscr{W}} \leq \alpha \leq \mathrm{OR}^{\mathscr{W}}$ be such that $\mathscr{W}||\alpha$ is active with an i-long extender. Let $\mathscr{W}' = \mathrm{Ult}_0(\mathscr{W}||\alpha, e)$. By Lemma 5.56, $\mathscr{W}' \trianglelefteq M_\infty^{i_e``\mathcal{U}}$.

Now $F^{\mathscr{W}'}$ is a correct iteration extender (via $\Sigma_{\mathcal{N}_\infty^{\mathcal{P}}}$) based on $\mathcal{N}_\infty^{\mathcal{P}}|\delta_i^{\mathcal{N}_\infty^{\mathcal{P}}}$. For let $\mathcal{U}'^+$ be the translation of $i_e``\mathcal{U}$ to a tree on $\mathcal{P}$, and $\mathscr{W}'^+ = M_\infty^{\mathcal{U}'^+}||\mathrm{OR}^{\mathscr{W}'}$. If $i = 1$ (so $\mathrm{crit}(F^{\mathscr{W}'^+}) = \kappa_1^{\mathcal{P}}$) the correctness of $F^{\mathscr{W}'}$ is by Lemma 5.45 part 6

applied to $\mathcal{P}$ and $\mathcal{P}' = \mathrm{Ult}(\mathcal{P}, F^{\mathscr{W}'+})$ and $i_{\mathcal{P}\mathcal{P}'}$. If $i = 0$ (so $F^{\mathscr{W}'+}$ is long) it is because $\Sigma_{\mathscr{V}_1}$ is good (so $F^{\mathscr{W}'+}$ is correct) and how $\mathscr{V}_2^{\mathcal{P}}$ is defined.

Let $\mathcal{S}$ be the limit length tree leading from $\mathcal{P}|\delta_i^{\mathcal{P}}$ to $\mathcal{M}_{i\infty}^{\mathscr{W}||\alpha}$, and $\mathcal{S}'$ likewise, so $\mathcal{S}' = j(\mathcal{S})$ where $j = i_e^{\mathscr{W}||\alpha,0}$ is the ultrapower map. Let $\mathcal{R}$ be the successor length tree leading from $\mathcal{P}|\delta_i^{\mathcal{P}}$ to $\mathcal{N}_{\infty}^{\mathcal{P}}|\delta_i^{\mathcal{P}}$ (given by $e{\upharpoonright}(\mathcal{P}|\delta_i^{\mathcal{P}}))$. We know that $\mathcal{S}'$ is via the short tree strategy for $\mathcal{N}_{\infty}^{\mathcal{P}}|\delta_i^{\mathcal{N}_{\infty}^{\mathcal{P}}}$, and $F^{\mathscr{W}'}$ yields the branch $\Sigma_{\mathcal{N}_{\infty}^{\mathcal{P}}}(\mathcal{S}')$. We claim the same holds for $\mathcal{S}$ and $F^{\mathscr{W}||\alpha}$; that is, $\mathcal{S}$ is via the δ_i-short tree strategy for $\mathcal{P} \downarrow i$, and $F^{\mathscr{W}||\alpha}$ yields $b = \Sigma_{\mathcal{P}}(\mathcal{S})$.

For if $i = 0$, the Q-structure used in $\mathcal{S}$ for the limit stage $\mathcal{S}{\upharpoonright}\eta$ does not overlap $\delta(\mathcal{S}{\upharpoonright}\eta)$, and is embedded by j into an iterable Q-structure used in $\mathcal{S}'$. And if $i = 1$, it is likewise through the 0-lower component of $\mathcal{S}$ (until reaching $\delta_0^{M(\mathcal{S})}$), and above there, Lemma 5.66 applies to the normalizations of $(\mathcal{T}, \mathcal{S}{\upharpoonright}\eta)$ and $(\mathcal{T}, \mathcal{R}, j(\mathcal{S}{\upharpoonright}\eta))$, using a restriction of j as the map k. (Here $\mathcal{T}$ and $\mathcal{T}^{\frown}\mathcal{R}$ can be δ_1-maximal, but one should literally apply Lemma 5.66 to the short-normal trees $\mathcal{T}'$ on $\mathscr{V}_1$, iterating to $M(\mathcal{S}{\upharpoonright}\eta)$, and $\mathcal{U}'$ on $\mathscr{V}_1$, iterating to $M(j(\mathcal{S}{\upharpoonright}\eta)))$.

Finally let b be the $\mathcal{S}$-cofinal branch determined by $F^{\mathscr{W}||\alpha}$ and b' that determined by $F^{\mathscr{W}'}$. Then we can apply Lemma 5.65 to $(\mathcal{S}, b)$ and the stack $(\mathcal{R}, (\mathcal{S}', b'))$, using $k = j{\upharpoonright}(M(\mathcal{S}))$. Therefore b is correct. □

5.6.5 Normal trees on $\mathscr{V}_2$

Much like in Definition 4.82, it is now easy to see:

Lemma 5.69. *There is a unique 0-maximal strategy Σ for $\mathscr{V}_2$ such that $\Psi_{\mathscr{V}_2}^{\mathrm{sn}} \subseteq \Sigma$. We write $\Sigma_{\mathscr{V}_2} = \Sigma$. Every iterate of $\mathscr{V}_2$ via $\Sigma_{\mathscr{V}_2}$ is a short-normal iterate of $\mathscr{V}_2$ via $\Psi_{\mathscr{V}_2}^{\mathrm{sn}}$, and hence $\Sigma_{\mathscr{V}_2}$ is good.*

Remark 5.70. Consider a 0-maximal tree $\mathcal{T}$ on $\mathscr{V}_2$ and some limit $\lambda < \mathrm{lh}(\mathcal{T})$ such that $\mathcal{T}$ uses 1-long extenders cofinally below λ. Then $\delta(\mathcal{T})$ is the least measurable of $M_\lambda^{\mathcal{T}}$, and in particular $\delta(\mathcal{T}) < \delta_0^{M_\lambda^{\mathcal{T}}}$. Suppose $E_\lambda^{\mathcal{T}}$ is short with $\mathrm{crit}(E_\lambda^{\mathcal{T}}) < \delta_1^{M_\lambda^{\mathcal{T}}}$ and total over $M_\lambda^{\mathcal{T}}$, or $E_\lambda^{\mathcal{T}}$ is 0-long. Then $\mathrm{pred}^{\mathcal{T}}(\lambda + 1) = \lambda$ and $M_{\lambda+1}^{\mathcal{T}} = \mathrm{Ult}(M_\lambda^{\mathcal{T}}, E_\lambda^{\mathcal{T}})$, and note that the short-normal tree $\mathcal{U}$ via $\Psi_{\mathscr{V}_2}^{\mathrm{sn}}$ has $\delta(\mathcal{U}) > \mathrm{lh}(E_\lambda^{\mathcal{T}})$. This could be unnatural; letting $\mathcal{U}_\lambda$ be the short-normal tree with last model $M_\lambda^{\mathcal{T}}$, it might be better to define 0-maximality by taking $\beta < \mathrm{lh}(\mathcal{U}_\lambda)$ least such that $E_\lambda^{\mathcal{T}} \in \mathbb{E}_+^{M_\beta^{\mathcal{U}_\lambda}}$, and defining $M_{\lambda+1}^{\mathcal{T}}$ to be the model produce by normally extending $\mathcal{U}_\lambda{\upharpoonright}(\beta + 1)$ with $E_\lambda^{\mathcal{T}}$. However, for our purposes here, the more naive notion of 0-maximality suffices.

Lemma 5.71. *Let $\bar{\mathscr{V}}$ be a non-dropping $\Psi_{\mathscr{V}_2, \mathscr{V}_2^-}$-iterate of $\mathscr{V}_2$, and $\mathscr{V}$ a non-dropping $\Psi_{\bar{\mathscr{V}}, \bar{\mathscr{V}}^-}$-iterate of $\bar{\mathscr{V}}^-$. Let $\pi : \bar{\mathscr{V}} \to \mathscr{V}$ be the iteration map. Let $\bar{\Psi}$ be the above-$\delta_1^{\bar{\mathscr{V}}}$ short-normal strategy for $\bar{\mathscr{V}}$ given by $\Sigma_{\mathscr{V}_2}$, and Ψ likewise for $\mathscr{V}$. Then $\bar{\Psi}$ is the minimal π-pullback of Ψ (see [14, ***10.3, 10.4]).*

Proof. Let $\mathcal{T}_1$ on $\bar{\mathscr{V}}$ be via the π-pullback of Ψ; we want to see that $\mathcal{T}_1$ is via $\bar{\Psi}$. Let $\mathcal{X}_1 = \pi``\mathcal{T}_1$, which is via Ψ. Let $\pi' = \pi{\upharpoonright}(\bar{\mathscr{V}} \downarrow 1)$. So $\pi' = i_{\bar{\mathscr{V}}\downarrow 1, \mathscr{V}\downarrow 1}$, also an iteration map.

If $\mathrm{lh}(E_0^{\bar{\mathcal{T}}_1}) < \gamma_1^{\bar{\mathcal{V}}}$ then the desired conclusion follows from the fact that $\Sigma_{\mathcal{N}_\infty}$ has mic. So suppose otherwise.

Let $\bar{j} : \bar{\mathcal{V}} \to \mathrm{Ult}(\bar{\mathcal{V}}, e_1^{\bar{\mathcal{V}}})$ and $j : \mathcal{V} \to \mathrm{Ult}(\mathcal{V}, e_1^{\mathcal{V}})$ be the ultrapower maps, and recall $\mathrm{Ult}(\mathcal{V}, e^{\mathcal{V}_1}) = \mathcal{V}_2^{\mathcal{V}\downarrow 1}$ and likewise for $\bar{\mathcal{V}}$. So the minimal j-copy $j``\mathcal{X}_1$ of $\mathcal{X}_1$, on $\mathcal{V}_2^{\mathcal{V}\downarrow 1}$, translates to a tree $\mathcal{X}_1'$ on $\mathcal{V} \downarrow 1$ which is via $\Sigma_{\mathcal{V}\downarrow 1}$ (and above $\gamma_1^{\mathcal{V}} = \kappa_1^{+\mathcal{V}\downarrow 1}$). We need to see that the minimal $\bar{j}$-copy of $\mathcal{T}_1$ translates to a tree $\mathcal{T}_1'$ on $\bar{\mathcal{V}} \downarrow 1$ via $\Sigma_{\bar{\mathcal{V}}\downarrow 1}$. Since $\Sigma_{\bar{\mathcal{V}}\downarrow 1}$ has mic (Lemma 4.84 and [14, ***Theorem 10.2]) and by [14, ***10.3, 10.4], it therefore suffices to see that that $\mathcal{X}_1' = \pi'``\mathcal{T}_1'$.

Let $\sigma' = \pi'\!\restriction\!(\bar{\mathcal{V}}|\gamma_1^{\bar{\mathcal{V}}})$ and

$$\sigma = \sigma'\!\restriction\!(\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}|\delta_1^{\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}}) = \pi\!\restriction\!\mathrm{Ult}(\bar{\mathcal{V}}|\delta_1^{\bar{\mathcal{V}}}, e_1^{\bar{\mathcal{V}}}).$$

We have
$$\mathrm{Ult}(\bar{\mathcal{V}} \downarrow 1, \sigma') = \mathrm{Ult}(\bar{\mathcal{V}} \downarrow 1, \sigma'\!\restriction\!\delta_1^{\bar{\mathcal{V}}}) = \mathcal{V} \downarrow 1$$

and the associated ultrapower map is just $\pi\!\restriction\!(\bar{\mathcal{V}} \downarrow 1)$. Given the fine structural correspondence between $\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}$ and $\bar{\mathcal{V}} \downarrow 1$, therefore

$$\mathrm{Ult}(\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}, \sigma) = \mathcal{V}_2^{\mathcal{V}\downarrow 1}$$

and the σ-ultrapower map $\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1} \to \mathcal{V}_2^{\mathcal{V}\downarrow 1}$ is just $\pi\!\restriction\!\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}$. Although σ is not the restriction of an iteration map on $\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}$, it is straightforward to see we still have $\widetilde{\mathcal{X}}_1 = \sigma``\widetilde{\mathcal{T}}_1$ (that is, $\widetilde{\mathcal{X}}_1$ is the minimal σ-copy of $\widetilde{\mathcal{T}}_1$), meaning that:

- $\widetilde{\mathcal{X}}_1$ has the same tree, drop and degree structure as has $\widetilde{\mathcal{T}}_1$,

- for each $\alpha + 1 < \mathrm{lh}(\widetilde{\mathcal{T}}_1)$, we have $M_\alpha^{\widetilde{\mathcal{X}}_1}|||\mathrm{lh}(E_\alpha^{\widetilde{\mathcal{X}}_1}) = \mathrm{Ult}_0(M_\alpha^{\widetilde{\mathcal{T}}_1}|||\mathrm{lh}(E_\alpha^{\widetilde{\mathcal{T}}_1}), \sigma)$,

- for each $\alpha < \mathrm{lh}(\widetilde{\mathcal{T}}_1)$, if $d = \deg_\alpha^{\mathcal{T}_1}$ then $M_\alpha^{\widetilde{\mathcal{X}}_1} = \mathrm{Ult}_d(M_\alpha^{\widetilde{\mathcal{T}}_1}, \sigma)$, and if α is a successor then $M_\alpha^{*\widetilde{\mathcal{X}}_1} = \mathrm{Ult}_d(M_\alpha^{*\widetilde{\mathcal{T}}_1}, \sigma)$, and

- the resulting ultrapower maps $M_\alpha^{\widetilde{\mathcal{T}}_1} \to M_\alpha^{\widetilde{\mathcal{X}}_1}$ and $M_\alpha^{*\widetilde{\mathcal{T}}_1} \to M_\alpha^{*\widetilde{\mathcal{X}}_1}$ (via σ) commute with the iteration maps of $\widetilde{\mathcal{T}}_1$ and $\widetilde{\mathcal{X}}_1$.

These are just standard properties of minimal copying, so we already know the corresponding properties hold with respect to $(\pi, \mathcal{T}_1, \mathcal{X}_1)$, $(\bar{j}, \mathcal{T}_1, \widetilde{\mathcal{T}}_1)$, and $(j, \mathcal{X}_1, \widetilde{\mathcal{X}}_1)$. One can now deduce them for $(\sigma, \widetilde{\mathcal{T}}_1, \widetilde{\mathcal{X}}_1)$ with some commutativity, and in particular that

$$j \circ \pi\!\restriction\!(\bar{\mathcal{V}}|\delta_1^{\bar{\mathcal{V}}}) = \pi\!\restriction\!(\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}) \circ \bar{j}.$$

But then because $\mathcal{T}_1', \mathcal{X}_1'$ are translations of $\widetilde{\mathcal{T}}_1, \widetilde{\mathcal{X}}_1$, and given the fine structural correspondence between $\bar{\mathcal{V}} \downarrow 1$ and $\mathcal{V}_2^{\bar{\mathcal{V}}\downarrow 1}$, and likewise between $\mathcal{V} \downarrow 1$ and $\mathcal{V}_2^{\mathcal{V}\downarrow 1}$, it follows that $\mathcal{X}_1' = \pi'``\mathcal{T}_1'$, as desired. $\qquad\square$

Lemma 5.72. $\Sigma_{\mathcal{V}_2}$ *has minimal inflation condensation (mic).*

Proof. We just discuss short-normal trees. Let $\mathcal{T} = \mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1$ and $\mathcal{X} = \mathcal{X}_0 \,\widehat{}\, \mathcal{X}_1$ be as before, but with respect to $\mathscr{V}_2$ and $\Sigma_{\mathscr{V}_2}$; in particular we have

$$\mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1 \rightsquigarrow_{\min} \mathcal{X}_0 \,\widehat{}\, (\mathcal{X}_1 {\restriction} \lambda),$$

where λ is a limit ordinal and $\lambda + 1 = \mathrm{lh}(\mathcal{X}_1)$. We must show that $\mathcal{X}$ is a minimal inflation of $\mathcal{T}$. Now $\Psi_{\mathscr{V}_1, \mathscr{V}_2^-}$ has mic, since $\Sigma_{\mathscr{V}_1}$ does, by Lemma 4.84 and [14, ***Theorem 10.2]. So we may assume $\mathcal{T}_1 \neq \emptyset$, so we get $\alpha, \beta, \Pi_0, \pi$ like before, with analogous properties (with δ_1 replacing δ_0). Let $\eta < \mathrm{lh}(\mathcal{T}_1)$ be the limit ordinal and c the $\mathcal{T}_1 {\restriction} \eta$-cofinal branch and

$$\Pi : \mathcal{T}_0 \,\widehat{}\, (\mathcal{T}_1 {\restriction} \eta) \,\widehat{}\, c \hookrightarrow_{\min} \mathcal{X}$$

the minimal tree embedding determined by extending the inflation $\mathcal{T} \rightsquigarrow_{\min} \mathcal{X}_0 \,\widehat{}\, (\mathcal{X}_1 {\restriction} \lambda)$ to $\mathcal{X}$ in the unique possible way. We want $c = [0, \eta)_{\mathcal{T}_1}$.

Let $\bar{\mathscr{V}} = M_\alpha^{\mathcal{T}_0}$, $\mathscr{V} = M_\beta^{\mathcal{X}_0}$ and $\bar{j} : \bar{\mathscr{V}} \to \mathrm{Ult}(\bar{\mathscr{V}}, e_1^{\bar{\mathscr{V}}})$ and $j : \mathscr{V} \to \mathrm{Ult}(\mathscr{V}, e_1^{\mathscr{V}})$ be the ultrapower maps, and recall $\mathrm{Ult}(\mathscr{V}, e^{\mathscr{V}_1}) = \mathscr{V}_2^{\mathscr{V} \downarrow 1}$ and likewise for $\bar{\mathscr{V}}$. So the minimal j-copy $j``\mathcal{X}_1$ of $\mathcal{X}_1$, on $\mathscr{V}_2^{\mathscr{V} \downarrow 1}$, translates to a tree $\mathcal{X}_1'$ on $\mathscr{V} \downarrow 1$ which is via $\Sigma_{\mathscr{V} \downarrow 1}$ and is above $\kappa_1^{+(\mathscr{V} \downarrow 1)}$. Likewise, $\mathcal{T}_1$ translates to a tree $\mathcal{T}_1'$ on $\bar{\mathscr{V}} \downarrow 1$ via $\Sigma_{\bar{\mathscr{V}} \downarrow 1}$ which is above $\kappa_1^{+(\bar{\mathscr{V}} \downarrow 1)}$.

Let $\widehat{\mathcal{T}}_1 = \pi``\mathcal{T}_1$. By Lemma 5.71, $\mathcal{X}_0 \,\widehat{}\, \widehat{\mathcal{T}}_1$ is via $\Sigma_{\mathscr{V}_2}$. Lifting with π, it is easy to see that

$$\mathcal{X}_0 \,\widehat{}\, \widehat{\mathcal{T}}_1 \rightsquigarrow_{\min} \mathcal{X}_0 \,\widehat{}\, (\mathcal{X}_1 {\restriction} \lambda)$$

and that it suffices to see that

$$\mathcal{X}_0 \,\widehat{}\, \widehat{\mathcal{T}}_1 \rightsquigarrow_{\min} \mathcal{X}_0 \,\widehat{}\, \mathcal{X}_1$$

(see [14, ***Theorem 10.7] for details; there is a straightforward correspondence between these inflations and those for $\mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1$).

So relabelling, we may assume $\mathcal{T}_0 = \mathcal{X}_0$ and $\Pi {\restriction} \mathcal{T}_0 = \mathrm{id}$, so $\bar{\mathscr{V}} = \mathscr{V}$ and $\pi = \mathrm{id}$ and $j = \bar{j}$. Let $\widetilde{\mathcal{T}}_0 = \widetilde{\mathcal{X}}_0$ be the short-normal tree leading from $\mathscr{V}_2$ to $\mathrm{Ult}(\mathscr{V}, e_1^{\mathscr{V}})$. Then

$$\widetilde{\mathcal{T}}_0 \,\widehat{}\, \bar{j}``\mathcal{T}_1 \rightsquigarrow_{\min} \widetilde{\mathcal{X}}_0 \,\widehat{}\, (j``\mathcal{X}_1 {\restriction} \lambda),$$

as can be seen by lifting all relevant structures up by the extender $e^{\bar{\mathscr{V}}_1} = e^{\mathscr{V}_1}$ with the relevant degree ultrapowers. Letting $\mathcal{T}_0' = \mathcal{X}_0'$ be $\mathcal{T}_0 = \mathcal{X}_0$ but as a tree on $\mathcal{N}_\infty$ (and recall $\mathcal{T}_1', \mathcal{X}_1'$ were introduced above), it follows that

$$\mathcal{T}_0' \,\widehat{}\, \mathcal{T}_1' \rightsquigarrow_{\min} \mathcal{X}_0' \,\widehat{}\, (\mathcal{X}_1' {\restriction} \lambda).$$

Since $\Sigma_{\mathcal{N}_\infty}$ has mic, therefore

$$\mathcal{T}_0' \,\widehat{}\, \mathcal{T}_1' \rightsquigarrow_{\min} \mathcal{X}_0' \,\widehat{}\, \mathcal{X}_1'.$$

But the ultimate minimal tree embedding Π' determined by this inflation is induced naturally by Π above, and in particular $c = [0, \eta)_{\mathcal{T}_1'}$, so $c = [0, \eta)_{\mathcal{T}_1}$, as desired. $\qquad\square$

5.7 Self-iterability of $\mathscr{V}_2$

Lemma 5.73. $\mathscr{V}_2$ *is definable over its universe from the parameter* $\mathcal{N}_\infty | \kappa_1^{\mathcal{N}_\infty}$.

Proof. This is an essentially direct adaptation of the proof of Lemma 5.17, but using Lemma 5.17 at the point that Remark 3.2 was used there. □

Note that in Theorem 5.79, we will improve the lemma above, showing that, in fact, $\mathscr{V}_2$ is definable without any parameters over its universe. But just using Lemma 5.73 and adapting Lemma 5.18, we have:

Lemma 5.74. *Let* $\mathscr{V}$ *be a non-dropping* $\Sigma_{\mathscr{V}_2}$*-iterate of* $\mathscr{V}_2$. *Let* $\lambda \in \mathrm{OR}$ *with* $\lambda \geq \delta_1^{\mathscr{V}_2}$ *and* $\mathbb{P} \in \mathscr{V} | \lambda^{+\mathscr{V}}$ *and* g *be* $(\mathcal{V}, \mathbb{P})$*-generic. Then* $\mathscr{V}$ *is definable over the universe of* $\mathscr{V}[g]$ *from the parameter* $x = \mathscr{V} | \lambda^{+\mathscr{V}}$.

We will now state a key fact on the self-iterability of $\mathscr{V}_2$ (and more). As usual, we will give the proof in a special case which illustrates the main new features, but the full proof will be handled by [16], as it involves $*$-translation:

Theorem 5.75. *Let* $G \subseteq \lambda$ *be set generic over* $\mathscr{V}_2$, *where* $\lambda \geq \delta_1^{\mathscr{V}_2}$. *Let* $x = \mathscr{V}_2 | \lambda^{+\mathscr{V}_2}$. *Then:*

1. $\mathscr{V}_2$ *is closed under* $\Sigma_{\mathscr{V}_2}$ *and* $\Sigma_{\mathscr{V}_2} \restriction \mathscr{V}_2$ *is lightface definable over* $\mathscr{V}_2$.

2. $\mathscr{V}_2[G]$ *is closed under* $\Sigma_{\mathscr{V}_2}$ *and* $\Sigma_{\mathscr{V}_2} \restriction (\mathscr{V}_2[G])$ *is definable over the universe of* $\mathscr{V}_2[G]$ *from the parameter* x, *uniformly in* x.

3. $\mathscr{V}_2$ *is closed under* $\Sigma_{\mathcal{N}_\infty}$ *and* $\Sigma_{\mathcal{N}_\infty} \restriction \mathscr{V}_2$ *is lightface definable over* $\mathscr{V}_2$. *(Recall that* $\mathcal{N}_\infty = \mathscr{V}_2 \downarrow 1$ *is a* $\Sigma_{\mathscr{V}_1}$*-iterate of* $\mathscr{V}_1$.*)*

4. $\mathscr{V}_2[G]$ *is closed under* $\Sigma_{\mathcal{N}_\infty}$ *and* $\Sigma_{\mathcal{N}_\infty} \restriction (\mathscr{V}_2[G])$ *is definable over the universe of* $\mathscr{V}_2[G]$ *from the parameter* x, *uniformly in* x.

In order to prove the theorem, we again use modified P-constructions (in general, incorporating $*$-translation), in the context of the following notions of P-suitability. The full proof will rely on $*$-translation, and so will be given in [16]. Here we will restrict our attention to dsr (defined in this context below) trees only, for illustration purposes (but the notion of P-suitability below does not have such a restriction). We restrict to trees in $\mathscr{V}_2$ (as opposed to $\mathscr{V}_2[G]$), as this simplifies things, and we can reduce other trees to this case.

Definition 5.76. *Let* $\mathcal{T} \in \mathscr{V}_2$ *be an iteration tree on* $\mathscr{V}_2$. *Say that* $\mathcal{T}$ *is P-suitable for* $\mathscr{V}_2$ *iff there are* $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, E, F, \mathscr{V}, \eta, \delta, \iota$ *such that:*

1. $\mathcal{T} = \mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1 \,\widehat{}\, \mathcal{T}_2$ *is short-normal on* $\mathscr{V}_2$, *according to* $\Sigma_{\mathscr{V}_2}$, *with 0-lower and 1-lower components* $\mathcal{T}_0, \mathcal{T}_1$ *and upper component* $\mathcal{T}_2 \neq \emptyset$,

2. $E, F \in \mathbb{E}^{\mathscr{V}_2}$ *are* $\mathscr{V}_2$*-total and 1-long,*

3. $\mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1$ *is the successor-length tree on* $\mathscr{V}_2$ *induced by* E,

4. $\mathscr{V} = M_\infty^{\mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1} = \mathrm{Ult}(\mathscr{V}_2, E)$,

5. $\delta_1^{\mathscr{V}} = \mathrm{lh}(E) < \lambda^{++\mathscr{V}_2} < \iota = \mathrm{lgcd}(\mathscr{V}_2|\mathrm{lh}(F)) < \mathrm{lh}(F)$,

6. $\mathcal{T}_2 \in \mathscr{V}_2|\iota$; note $\mathcal{T}_2$ is on $\mathscr{V}$ and is above $\delta_1^{\mathscr{V}}$,

7. η is a strong $\{\delta_0^{\mathscr{V}_2}, \delta_1^{\mathscr{V}_2}\}$-cutpoint of $\mathscr{V}_2$,

8. $\mathcal{T}_2$ has limit length, $\lambda^{++\mathscr{V}_2} \leq \eta < \delta = \delta(\mathcal{T}_2) < \iota$, η is the largest cardinal of $\mathscr{V}_2|\delta$, $\mathcal{T}_2$ is definable from parameters over $\mathscr{V}_2|\delta$, and $\mathscr{V}_2|\delta$ is generic over $M(\mathcal{T}_2)$ for the above-ξ extender algebra of $M(\mathcal{T}_2)$ at δ, for some $\xi < \delta$.

Now let $\mathcal{T} \in \mathscr{V}_2$ be a tree on $\mathcal{N}_\infty = \mathscr{V}_2 \downarrow 1$. Say that $\mathcal{T}$ is *P-suitable for $\mathscr{V}_2$* iff the conditions above hold, except that $\mathcal{T}$ is short-normal on $\mathcal{N}_\infty$, according to $\Sigma_{\mathcal{N}_\infty}$, $\mathcal{T}_0$ is the lower component of $\mathcal{T}$, $\mathcal{T}_1$ is based on $M_\infty^{\mathcal{T}_0}|\delta_1^{M_\infty^{\mathcal{T}_0}}$ and is above $\delta_0^{M_\infty^{\mathcal{T}_0}}$, $\mathcal{T}_0 ^\frown \mathcal{T}_1$ has successor length and does not drop, and $\mathcal{T}_2$ is on $\mathcal{N} = M_\infty^{\mathcal{T}_1}$ and is above $\delta_1^{\mathcal{N}}$ (so $\mathcal{N} = \mathrm{Ult}(\mathcal{N}_\infty, E)$, and note that $\mathcal{T}_0 ^\frown \mathcal{T}_1$ can also be considered as a tree on $\mathscr{V}_2$, with properties as above). $\dashv$

The corresponding P-constructions are as follows; no proper class models show up, because we are now working up above the real Woodin cardinals. We must now restrict our attention to dsr trees (as defined immediately below).

Definition 5.77. Let $\mathbb{P} \in \mathscr{V}_2$ and $\delta_1^{\mathscr{V}_2} \leq \lambda \in \mathrm{OR}$ with $\mathbb{P} \subseteq \lambda$, and G be $(\mathscr{V}_2, \mathbb{P})$-generic.

Let $\mathcal{T} \in \mathscr{V}_2$, on either $\mathscr{V}_2$ or $\mathcal{N}_\infty$, be P-suitable for $\mathscr{V}_2$, and adopt notation as in Definition 5.76. Say that $\mathcal{T}$ is *dsr* iff $M(\mathcal{T})$ has only 2 Woodin cardinals.

Suppose that $\mathcal{T}$ is dsr, but $M(\mathcal{T})$ is not a Q-structure for itself. Then the *P-construction* $\mathscr{P}^{\mathscr{V}_2|\iota}(M(\mathcal{T}))$ of $\mathscr{V}_2|\iota$ over $M(\mathcal{T})$ (recalling that ι is the largest cardinal of $\mathscr{V}_2|\mathrm{lh}(F)$) is defined like the P-constructions used to compute the δ_1-short tree strategy for $\mathscr{V}_1$, noting that the iteration map $j : \mathscr{V}_2|\delta_0^{\mathscr{V}_2} \to M_\infty^{\mathcal{T}_0}|\delta_0^{M_\infty^{\mathcal{T}_0}}$, which is determined by E, is in $\mathscr{V}_2|\eta$ (and note that there are no 1-long extenders in $\mathbb{E}^{\mathscr{V}_2} {\upharpoonright} [\delta, \iota]$). $\dashv$

Lemma 5.78. *Let $\mathcal{T}$ be dsr P-suitable for $\mathscr{V}_2$, on either $\mathscr{V}_2$ or $\mathcal{N}_\infty$. Suppose $M(\mathcal{T})$ is not a Q-structure for itself. Then the P-construction $\mathscr{P}^{\mathscr{V}_2|\iota}(M(\mathcal{T}))$ reaches the Q-structure $Q(\mathcal{T}, b)$, where $b = \Sigma_{\mathscr{V}_2}(\mathcal{T})$ or $b = \Sigma_{\mathcal{N}_\infty}(\mathcal{T})$.*

Proof. We first consider P-suitable trees $\mathcal{T}$ on $\mathcal{N}_\infty$ in $\mathscr{V}_2$. So adopt the notation of Definition 5.76 for this, with $\mathcal{N} = \mathrm{Ult}(\mathcal{N}_\infty, E)$. Because $\mathcal{T}_2$ is above $\delta_1^{\mathcal{N}}$, the Q-structure $Q = Q(\mathcal{T}_2, b)$ exists, where $b = \Sigma_{\mathcal{N}_\infty}(\mathcal{T})$. Suppose $Q \neq M(\mathcal{T}_2)$. We want to see that the P-construction reaches Q. To verify this, we run a comparison analogous some earlier in the paper, modulo the generic at δ, and after appropriate translation of long extenders. We need to specify the phalanxes we compare.

On the P-construction side, we just have M.

On the Q-structure side, we proceed as follows. Let $\bar{\mathcal{N}}$ be the $\delta_0^{\mathcal{N}}$-core of $\mathcal{N}$ (this is not $M_\infty^{\mathcal{T}_0}$, as $\mathcal{N}_\infty$ itself is not $\delta_0^{\mathcal{N}_\infty}$-sound). Note that $\delta_0^{\bar{\mathcal{N}}} = \delta_0^{\mathcal{N}}$ and $\bar{\mathcal{N}}$ is $\delta_0^{\mathcal{N}}$-sound. Let $\bar{\mathcal{N}}^+$ be a generic expansion of $\bar{\mathcal{N}}$ (to a premouse), via a filter which is M-generic (for the same forcing $\mathbb{L}^{\mathcal{N}}$. So $\bar{\mathcal{N}} = \mathscr{V}_1^{\bar{\mathcal{N}}^+}$. Then

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$\mathcal{T}_1 \,\widehat{\,}\, \mathcal{T}_2$ translates to a tree $\mathcal{T}_1^+ \,\widehat{\,}\, \mathcal{T}_2^+$ on $\bar{\mathcal{N}}^+$ which is above $\kappa_0^{+\bar{\mathcal{N}}^+} = \delta_0^{\bar{\mathcal{N}}}$. Let $Q^+ = Q(\mathcal{T}_2^+, b)$. Define the phalanx

$$\mathfrak{Q}^+ = ((\bar{\mathcal{N}}^+, \kappa_0^{\bar{N}} + 1), Q^+, \delta).$$

Note that $\mathfrak{Q}^+$ is iterable, as it corresponds to iterating the phalanx

$$\mathfrak{Q} = ((\bar{\mathcal{N}}, \delta_0^{\bar{\mathcal{N}}}), Q, \delta).$$

(Note here that the only extenders overlapping δ in $\mathbb{E}_+^Q$ are long, since $\mathcal{T}$ is dsr.)

We now compare $\mathfrak{Q}^+$ with M, above δ, modulo the generic at δ, translating extenders with critical point $\kappa_0^{\bar{\mathcal{N}}^+}$ on the $\mathfrak{Q}^+$ side, and those with critical point κ_0^M on the M side, much as before. Much like in the proof of Lemma 5.16, and using the $\kappa_0^{\bar{\mathcal{N}}^+}$-soundness of $\bar{\mathcal{N}}^+$ (which is by Lemma 5.14), the comparison is trivial, so the P-construction reaches Q, as desired. Regarding the equivalence modulo the generic at δ, although $\mathscr{V}_2|\delta$ is extender algebra generic over Q, (for an extender algebra $\mathbb{B}$ at δ, above some ξ), it doesn't seem immediate that it is also generic for the corresponding extender algebra of Q^+ (although the extenders correspond, it seems there might still be further axioms in Q^+ which cause problems). However, this is not a problem. Note that

$$\mathcal{P}(< \delta) \cap (Q[\mathscr{V}_2|\delta]) = \mathcal{P}(< \delta) \cap (M(\mathcal{T})[\mathscr{V}_2|\delta]) = \mathcal{P}(< \delta) \cap (\mathscr{V}_2|\delta).$$

We can force over $Q[\mathscr{V}_2|\delta]$ with $\mathbb{L}' = \mathbb{L}^{\mathscr{V}_2} * \dot{\mathbb{L}}^{\mathscr{V}_1} \in \mathscr{V}_2|\delta$, adding $(\mathscr{V}_1|\kappa_1^{\mathscr{V}_1}, M|\kappa_0^M)$, which results in $Q[M|\delta]$, and similarly

$$\mathcal{P}(< \delta) \cap (Q[M|\delta]) = \mathcal{P}(< \delta) \cap (M(\mathcal{T})[M|\delta]) = \mathcal{P}(< \delta) \cap (M|\delta).$$

Since $\bar{\mathcal{N}}^+|\kappa_0^{\bar{\mathcal{N}}^+}$ was taken M-generic for $\mathbb{L}^{\bar{\mathcal{N}}} \in M(\mathcal{T})$, we can therefore force further with $\mathbb{L}^{\bar{\mathcal{N}}}$ to reach $Q[M|\delta, \bar{\mathcal{N}}^+|\kappa_0^{\bar{\mathcal{N}}^+}]$, which computes Q^+. But the product $(\mathbb{B} * \dot{\mathbb{L}}') \times \mathbb{L}^{\bar{\mathcal{N}}}$ can be reversed, and so $M|\delta$ is also Q^+-generic for $\mathbb{B} * \dot{\mathbb{L}}'$. Moreover, $\mathbb{B}$ is definable from parameters over $M(\mathcal{T}^+) = Q^+|\delta$. The same holds for all models that appear above $\mathfrak{Q}^+$ in the comparison. This gives the usual fine structural correspondence between models above $\mathfrak{Q}^+$ and their generic extensions given by adjoining $M|\delta$. On the M-side, the extension to $M[\bar{\mathcal{N}}^+|\kappa_0^{\bar{\mathcal{N}}^+}]$ is via $\mathbb{L}^{\bar{\mathcal{N}}}$, which is small relative to δ in M. Likewise for all models on the M-side of the comparison. So we also get the appropriate fine structural correspondence on the M-side.

Now consider trees $\mathcal{T}$ on $\mathscr{V}_2$; we adopt the relevant notation from Definition 5.76. If $\mathrm{lh}(E_0^{\mathcal{T}_2}) < \gamma_1^{\mathscr{V}}$, then since $\rho_1^{\mathscr{V}||\gamma_1^{\mathscr{V}}} = \delta_1^{\mathscr{V}}$, $\mathcal{T}_2$ immediately drops in model to $\mathscr{V}||\gamma_1^{\mathscr{V}}$, and this cannot be undone (since $\mathcal{T}_2$ is short-normal). Since $\mathscr{V}|\gamma_1^{\mathscr{V}} = \mathcal{N}||\kappa_1^{+\mathcal{N}}$, where $\mathcal{N}$ is as in the previous case, with corresponding iteration strategy for such trees, everything in this situation is as above. So suppose $\gamma_1^{\mathscr{V}} < \mathrm{lh}(E_0^{\mathcal{T}_2})$. Let $E^+ \in \mathbb{E}^M$ with $\mathrm{lh}(E^+) = \mathrm{lh}(E)$. Let $R = \mathrm{Ult}(M, E^+)$. Then $\mathscr{V} = \mathscr{V}_2^R$ and (since $\gamma_1^{\mathscr{V}} < \mathrm{lh}(E_0^{\mathcal{T}_2})$), $\mathcal{T}_2$ translates to a tree $\mathcal{T}_2^+$ on R, which is above κ_1^{+R}. Let $Q^+ = Q(\mathcal{T}_2^+, b)$. It is straightforward to see that $Q = Q(\mathcal{T}_2, b)$ has no 1-long extenders overlapping δ (recall that $Q \neq M(\mathcal{T}_2)$,

and use the smallness of M), so Q^+ can only have extenders overlapping δ with critical point κ_0. Define the phalanx

$$\mathfrak{Q} = ((M, \kappa_0 + 1), Q^+, \delta).$$

Clearly $\mathfrak{Q}$ is iterable. We compare $\mathfrak{Q}$ versus M, again above δ etc, like before. This time the equivalence modulo the generic is a little different, because $R|\kappa_1^{+R}$ is not M-generic, but instead is in M. However, over Q, we adjoin $\mathscr{V}_2|\delta$ with the extender algebra, then adjoin $M|\kappa_1^{+M}$, reaching $Q[M|\delta]$. Like in the previous case, this two-step forcing iteration is definable from parameters over $Q|\delta$ and the Woodinness of δ ensures genericity. But $R|\kappa_1^{+R} \in M|\delta$, so Q^+ is (simply) definable from parameters over $Q[M|\delta]$, and Q^+ (simply) defines Q from parameters. This (together with exactly how these definitions are made and the parameters used) is enough for the fine structural analysis of the comparison. $\qquad \square$

Without discussing $*$-translation, we are limited to sketching the proof of that $\mathscr{V}_2$ can iterate itself and $\mathcal{N}_\infty$:

Sketch of proof for Theorem 5.75.
Lemma 5.47 handles trees based on $\mathcal{N}_\infty|\delta_1^{\mathcal{N}_\infty} = \mathscr{V}_2|\delta_1^{\mathscr{V}_2}$.

So consider dsr trees $\mathcal{U} = \mathcal{U}_0 \,^\frown \mathcal{U}_1 \,^\frown \mathcal{U}_2$, with lower component $\mathcal{U}_0, \mathcal{U}_1$ on $M_\infty^{\mathcal{U}_0}$, based on $M_\infty^{\mathcal{U}_0}$, and above $\delta_0^{M_\infty^{\mathcal{U}_0}}$, $b^{\mathcal{U}_0 \,^\frown \mathcal{U}_1}$ non-dropping, and $\mathcal{U}_2$ on $M_\infty^{\mathcal{U}_0 \,^\frown \mathcal{U}_1}$, above $\delta_1^{M_\infty^{\mathcal{U}_0 \,^\frown \mathcal{U}_1}}$. Let $E \in \mathbb{E}^{\mathscr{V}_2}$ be such that E is 1-long and $\lambda^{+\mathscr{V}_2} < \iota = \mathrm{lgcd}(\mathscr{V}_2|\mathrm{lh}(E))$ and $\mathcal{U}_0 \,^\frown \mathcal{U}_1 \in \mathscr{V}_2|\iota$. Then letting $\mathcal{T}_0 \,^\frown \mathcal{T}_1$ on $\mathcal{N}_\infty$ result from E, $\mathcal{N} = M_\infty^{\mathcal{T}_0 \,^\frown \mathcal{T}_1} = \mathscr{V}_1^{\mathrm{Ult}(\mathscr{V}_2, E)}$ is a correct iterate of $M_\infty^{\mathcal{U}_0 \,^\frown \mathcal{U}_1}$, and $\mathscr{V}_2$ knows the iteration map j. So given that $\mathscr{V}_2$ computes the restriction of $\Sigma_{\mathcal{N}}$ to above-$\delta_1^{\mathcal{N}}$ trees correctly, it can use j to form minimal copies of trees $\mathcal{U}_2$ (of the form above) to correct trees $\mathcal{T}_2$, and then $\mathcal{U}_2$ is correct, because $\Sigma_{\mathscr{V}_1}$ has minimal inflation condensation (Lemma 4.84), and hence so does $\Sigma_{\mathcal{N}_\infty}$, by [14, ***Theorem 10.2]. (Note also that dsr-ness is preserved by the copying.) Finally, arbitrary (dsr) trees $\mathcal{T}_2$ can be reduced to P-suitable trees by the usual minimal genericity inflation technique.

By [16], the foregoing generalizes to arbitrary (not just dsr) trees, so that $\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2}$ is definable over $\mathscr{V}_2$. Since $\Sigma_{\mathcal{N}_\infty}$ has minimal inflation condensation, $\mathscr{V}_2 \vDash \text{``}\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2}$ has minimal inflation condensation".

Since the least $\mathscr{V}_2$ indiscernible is countable in V, and $\Sigma_{\mathcal{N}_\infty}$ has minimal inflation condensation in V, by [14, ***Remark 9.2], $\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2}$ extends to canonically to set-generic extensions $\mathscr{V}_2[G]$ of $\mathscr{V}_2$ (via the method in the proof of [14, ***Remark 9.2]), and letting $\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2[G]}$ be the extension, every tree via $\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2[G]}$ embeds via a minimal tree embedding arising from minimal inflation into some tree in $\mathscr{V}_2$ via $\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2}$, and therefore $\Sigma_{\mathcal{N}_\infty}^{\mathscr{V}_2[G]}$ also agrees with $\Sigma_{\mathcal{N}_\infty}^{V[G]}$ if G is V-generic. For the definability in $\mathscr{V}_2[G]$ from the parameter x, use 5.74 to recover $\mathscr{V}_2$, from which we compute(d) the strategy.

For trees on $\mathscr{V}_2$, i.e. computing $\Sigma_{\mathscr{V}_2}^{\mathscr{V}_2}$ and $\Sigma_{\mathscr{V}_2}^{\mathscr{V}_2[G]}$, it is very similar, using Lemma 5.72. $\qquad \square$

5.8 The mantle and eventual generic HOD of M

In this section we prove the main facts regarding the eventual generic HOD and the mantle:

Theorem 5.79. *Let U_2 be the universe of $\mathscr{V}_2$. Then:*

1. *$U_2 = \mathrm{HOD}^{U_2[G]}$ for all $\mathscr{V}_2$-generics $G \subseteq \mathrm{Coll}(\omega, \lambda)$, for all $\lambda \geq \delta_1^{\mathscr{V}_2} = \kappa_1^{+M}$, and likewise $U_2 = \mathrm{HOD}^{M[H]}$ for all M-generics $H \subseteq \mathrm{Coll}(\omega, \lambda)$.*

2. *U_2 has no proper ground, so U_2 is the mantle and smallest ground of M.*

3. *U_2 is the mantle of all set generic extensions of M.*

4. *$\mathscr{V}_2$ is definable without parameters over U_2, and in fact over any set-generic extension of U_2.*

Proof. Work in $\mathscr{V}_2[G]$ where $G \subseteq \mathrm{Coll}(\omega, \lambda)$ is $\mathscr{V}_2$-generic and $\lambda \geq \delta_1^{\mathscr{V}_2}$. Say that $\mathscr{V}$ is a λ-*candidate* iff $\mathscr{V}$ is a $\mathscr{V}_2$-like 2-Vsp and there is $H \subseteq \mathrm{Coll}(\omega, \lambda)$ which is $\mathscr{V}$-generic and $\mathscr{V}[H] \cong \mathscr{V}_2[G]$.

Note that $\mathscr{V}$ is determined in $\mathscr{V}_2[G]$ by $\bar{\mathscr{V}} = \mathscr{V}|\lambda^{+\mathscr{V}}$, by Lemma 5.74, and moreover, by the uniformity of its proof, $\bar{\mathscr{V}} \mapsto \mathscr{V}$ is definable over (the universe of) $\mathscr{V}_2[G]$. (We can recover the universe U of $\mathscr{V}$ from $\bar{\mathscr{V}}$, via (the proof of) Woodin-Laver, and we can recover $\mathscr{V}$ from $\bar{\mathscr{V}}$ and U via (the proof of) Lemma 5.74.) So there are only set-many λ-candidates. Note that $\mathscr{V}_2$ is a λ-candidate.

Recall here that $\mathscr{V}_2$-*like* is assumed to include whatever first-order facts satisfied by $\mathscr{V}_2$ to make our arguments work. In particular, it should include the statement/proof of Lemma 5.74, and also the statements

- $\mathscr{V}$ is fully iterable in every set generic extension $\mathscr{V}[g]$ of $\mathscr{V}$, via the strategy $\Sigma_{\mathscr{V}}^{\mathscr{V},g}$ defined as in the proof of Theorem 5.75; and

- $\mathscr{V} \downarrow 1$ is fully iterable in every set generic extension $\mathscr{V}[g]$ of $\mathscr{V}$, via the strategy $\Sigma_{\mathscr{V}\downarrow 1}^{\mathscr{V},g}$ defined as in the proof of Theorem 5.75.

Now using this iterability (which holds in $\mathscr{V}_2[G]$ with respect to each λ-candidate $\mathscr{V}$), we want to define a kind of simultaneous "comparison" of all λ-candidates. For this, we will not directly attempt to compare the λ-candidates $\mathscr{V}$ themselves by least disagreement (due to familiar problems with showing that the comparison terminates), but, as we have done elsewhere in the paper, instead compare generic expansions of the $\mathscr{V} \downarrow 1$, and then use this to infer a comparison of the λ-candidates (and it doesn't seem obvious that this comparison of λ-candidates is by least disagreement).

However, we only have iterability for the generic expansions N above their κ_0^N, which isn't enough to expect a standard comparison of these premice by least disagreement either (they need not agree below their κ_0^N, as this part is just generic). Instead, like in the proof of Lemma 4.81, we will first form a "mutual genericity iteration" at an appropriate Woodin cardinal, and after this converges, move to comparison "modulo a generic" above that point.

So, work in $\mathscr{V}_2[G,G']$, where $G' \subseteq \mathrm{Coll}(\omega, \lambda^{+\mathscr{V}_2[G]})$ is $\mathscr{V}_2[G]$-generic. Fix for each λ-candidate $\mathscr{V}$ a generic expansion $P = P_{\mathscr{V}}$ of $\mathscr{V} \downarrow 1$. So $(\mathscr{V}_1 \downarrow 1) \subseteq P$ and $(\mathscr{V}_1 \downarrow 1) = \mathscr{V}_1^P$. (Moving to $\mathscr{V}_2[G,G']$ ensures these Ps exist.) Let Σ_P be the iteration strategy for P for 0-maximal trees $\mathcal{T}$ with $\mathrm{lh}(E_0^{\mathcal{T}}) > \kappa_0^P$ given by the proof of Lemma 4.81 (translating to trees via $\Sigma_{\mathscr{V}}^{\mathscr{V}_2[G,G']}$, etc). Let $D^P \in \mathbb{E}^P$ be the least P-total extender with $\mathrm{crit}(D^P) = \kappa_0^P$. Let δ^P be the least Woodin cardinal of $P|\lambda(D^P)$ such that $\delta^P > \kappa_0^P$ (so $\delta^P > \kappa_0^{+P}$).

Recall the meas-lim extender algebra (see [17]), used in the proof of Lemma 4.81. Write $\mathbb{B}^P$ for the (meas-lim) extender algebra of $P|\mathrm{lh}(D^P)$, at δ^P, formed with extenders $E \in \mathbb{E}^P$ such that $\mathrm{crit}(E) > \kappa_0$ and $\nu(E)$ is a limit of measurables of $P|\mathrm{lh}(D^P)$, as witnessed by $\mathbb{E}^P$. We will now form a mutual genericity iteration of all P as above, for the image of $\mathbb{B}^P$, producing padded trees $\mathcal{T}_P$ on P, above κ_0^{+P}, based on $P|\delta^P$ (and hence $\mathcal{T}_P$ immediately drops in model to $P|\mathrm{lh}(D^P)$, noting that $\rho_1^{P|\mathrm{lh}(D^P)} = \kappa_0^{+P}$), inserting some linear iterations at successor measurables to space things conveniently. Let $\mathscr{P}$ be the set of all $P_{\mathscr{V}}$, for λ-candidates $\mathscr{V}$ (where "λ-candidate" is still as computed in $\mathscr{V}[G]$, but $\mathscr{P} \subseteq \mathscr{V}[G,G']$). Fix an enumeration $\langle P_{\beta} \rangle_{\beta}$ of $\mathscr{P}$, and let C be a set of ordinals coding $\langle P_{\beta}|\delta^{P_{\beta}} \rangle_{\beta}$. We define a sequence $\langle \mathcal{T}_{\alpha}^P \rangle_{\alpha \leq \iota}$ of approximations to the final trees $\mathcal{T}_P = \mathcal{T}_{\iota}^P$. We start with $\mathcal{T}_0^P$ being the trivial tree on P. Suppose we have defined $\mathcal{T}_{\alpha}^P$ for each $P \in \mathscr{P}$. This will be a 0-maximal successor-length padded tree on P, based on $P|\mathrm{lh}(D^P)$, above κ_0^{+P}. Let $\delta_{\alpha} = \sup_{P \in \mathscr{P}} \delta(\mathcal{T}_{\alpha}^P)$, where $\delta(\mathcal{T}) = \sup_{\beta+1 < \mathrm{lh}(\mathcal{T})} \mathrm{lh}(E_{\beta}^{\mathcal{T}})$. If $b^{\mathcal{T}_{\alpha}^P}$ drops below the image of $P|\mathrm{lh}(D^P)$ then let $\gamma_{\alpha}^P = \mathrm{OR}(M^{\mathcal{T}_{\alpha}^P})$, and otherwise let $\gamma_{\alpha}^P = j^{\mathcal{T}_{\alpha}^P}(\delta^P)$, where $j^{\mathcal{T}_{\alpha}^P} : P|\mathrm{lh}(D^P) \to M_{\infty}^{\mathcal{T}_{\alpha}^P}$ is the iteration map. Let $K_{\alpha}^P = M^{\mathcal{T}_{\alpha}^P}||\gamma_{\alpha}^P$. Let $D_{\alpha} = (C, C_{\alpha})$ where C_{α} codes $\left\langle (K_{\alpha}^{P_{\beta}})^{\mathrm{pv}} \right\rangle_{\beta}$ as

$$C_{\alpha} = \{(\beta, \gamma) \in \mathrm{OR}^2 \mid \gamma \in \mathbb{E}(K_{\alpha}^{P_{\beta}})\}$$

(where $\mathbb{E}(K_{\alpha}^{P_{\beta}})$ is taken as a set of ordinals in a canonical fashion). Let G_{α}^P be the least $E \in \mathbb{E}_+(K_{\alpha}^P)$ such that E is K_{α}^P-total and

1. $E = F(K_{\alpha}^P)$, or

2. $\nu(E)$ is a limit of measurable cardinals of K_{α}^P, as witnessed by $\mathbb{E}^{K_{\alpha}^P}$, and $E{\restriction}\nu(E)$ induces an extender algebra axiom false of D_{α}, or

3. $\mathrm{crit}(E) < \sup(C)$, or $\mathrm{crit}(E)$ is not a cardinal in $\mathscr{V}_2[G,G']$,

if there is such an E, and $G_{\alpha}^P = \emptyset$ otherwise. If there is $P \in \mathscr{P}$ such that $G_{\alpha}^P = \emptyset$ and $b^{\mathcal{T}_{\alpha}^P}$ does not drop below the image of $P|\mathrm{lh}(D^P)$ and $\gamma_{\alpha}^P \leq \mathrm{lh}(G_{\alpha}^{P'})$ for all P' such that $G_{\alpha}^{P'} \neq \emptyset$, then we stop the construction, and set $\iota = \alpha$, and $\mathcal{T}^P = \mathcal{T}_{\alpha}^P$ for all P. Otherwise, let $\xi_{\alpha} = \min_{P \in \mathscr{P}} \mathrm{lh}(G_{\alpha}^P)$, and set $E_{\alpha}^P = G_{\alpha}^P$ if $\mathrm{lh}(G_{\alpha}^P) = \xi_{\alpha}$, and $E_{\alpha}^P = \emptyset$ otherwise. Let γ be least such that either $\gamma + 1 = \mathrm{lh}(\mathcal{T}_{\alpha}^P)$ or $\xi_{\alpha} < \mathrm{lh}(E_{\gamma}^{\mathcal{T}_{\alpha}^P})$, and set $\mathcal{T}_{\alpha+1}^P = \mathcal{T}_{\alpha}^P{\restriction}(\gamma + 1) \frown \langle E_{\alpha}^P \rangle$ (as a 0-maximal tree, with last extender used being E_{α}^P, which might be empty).

Now suppose we have defined $\mathcal{T}_\alpha^P$ for all $\alpha < \eta$ and $P \in \mathscr{P}$, where η is a limit. Let $\xi = \liminf_{\alpha < \eta} \xi_\alpha$. Then $\mathcal{T}_\eta^P$ is the natural lim inf of the sequence $\langle \mathcal{T}_\alpha^P \rangle_{\alpha < \eta}$. That is, $E_\gamma^{\mathcal{T}_\eta^P} = E$ iff $\mathrm{lh}(E) < \xi$ and eventually all $\alpha < \eta$ have $E_\gamma^{\mathcal{T}_\alpha^P} = E$, and $\mathcal{T}_\eta^P$ is via Σ_P, and has successor length. This determines $\mathcal{T}_\eta^P$.

This determines the entire construction. The first claim is very much like in the proof of Lemma 4.81:

CLAIM 1. We have:

1. Each $\mathcal{T}_\alpha^P$ is 0-maximal on P, and if $b^{\mathcal{T}_\alpha^P}$ drops below the image of $P|\mathrm{lh}(D^P)$ then K_α^P is active, so $G_\alpha^P \neq \emptyset$.

2. $\iota < \infty$.

3. Therefore there is $P' \in \mathscr{P}$ such that $b^{\mathcal{T}_\iota^{P'}}$ does not drop below the image of $P'|\mathrm{lh}(D^{P'})$.

CLAIM 2. For every $P \in \mathscr{P}$, $b^{\mathcal{T}^P}$ does not drop below the image of $P|\mathrm{lh}(D^P)$, and $j^{\mathcal{T}^P}(\delta^P) = j^{\mathcal{T}^{P'}}(\delta^{P'})$.

Proof. Because $b^{\mathcal{T}_\iota^{P'}}$ does not drop below the image of $P'|\mathrm{lh}(D^{P'})$, and $j^{\mathcal{T}^{P'}}(\delta^{P'})$ is a limit of measurables of $K_\iota^{P'}$, and $G_\iota^{P'} = \emptyset$, $j^{\mathcal{T}^{P'}}(\delta^{P'})$ must be a limit cardinal of $\mathcal{V}_2[G, G']$. Therefore every $\mathcal{T}^P$ uses cofinally many non-empty extenders indexed below $j^{\mathcal{T}^{P'}}(\delta^{P'}) = \xi = \liminf_{\alpha < \iota} \xi_\alpha$, ξ is a limit cardinal of $M_\infty^{\mathcal{T}^P}$, and note that (C, C_ι) is generic over $M(\mathcal{T}^P)$ for its extender algebra at ξ, since $\xi \leq \mathrm{lh}(G_\iota^P)$ if $G_\iota^P \neq \emptyset$.

Now suppose that $b^{\mathcal{T}^P}$ drops below the image of $P|\mathrm{lh}(D^P)$, or $j^{\mathcal{T}^P}(\delta^P) > \xi$. Let $Q \trianglelefteq M_\infty^{\mathcal{T}^P}$ be the Q-structure for ξ. It is straightforward to see that Q does not overlap ξ, and note that we can compare Q versus $U = \mathrm{Ult}(P', F(M_\infty^{\mathcal{T}^{P'}}))$ as premice Q^+ and U^+ over $(M(\mathcal{T}^P), M(\mathcal{T}^{P'}))$, so by M_{swsw}-likeness, it follows that $Q^+ \triangleleft U^+$. Expanding U^+ to $U^+[C, C_\iota]$, where ξ is still regular, we can now argue like in the proof of Claim 1 of the proof of Lemma 4.81 for a contradiction. (Although $\mathcal{T}^P \in U^+[P|\delta^P]$, we work in $U^+[C, C_\iota]$ because the reasons for the extenders used in $\mathcal{T}^P$ are encoded into (C, C_ι), and we need this to obtain the contradiction.) $\qquad\square$

So $M_\infty^{\mathcal{T}^P}$ is active with an image E^P of D^P, and E^P is P-total with $\mathrm{crit}(E^P) = \kappa_0^P$. Let $U_P = \mathrm{Ult}(P, E^P)$. Then ξ is a strong cutpoint of $U_P, U_{P'}$, and both extend to premice over $(U_P|\xi, U_{P'}|\xi)$. So we can simultaneously compare all U_P above ξ, modulo this generic equivalence. (With a simple instance of normalization, the resulting trees can easily be rearranged as trees on the phalanxes $\Phi(\mathcal{T}^P)$.) This produces a final iterate W_P of P, with $W_P =_\xi^* W_{P'}$, ξ is Woodin in W_P and $\xi < \kappa_0^{W_P}$.

CLAIM 3. $\mathcal{V}_1^{W_P} = \mathcal{V}_1^{W_{P'}}$ for all $P, P' \in \mathscr{P}$.

Proof. By §4.5, $\mathscr{V}_1 = \mathscr{V}_1^M$ depends only on the equivalence class $\mathscr{E}$. So the corresponding fact holds for W_P. But we can take $G, G' \subseteq \mathrm{Coll}(\omega, < \kappa_0)$ which are $W_P, W_{P'}$-generic respectively with $W_P[G] \stackrel{\wedge}{=} W_{P'}[G']$, and then $\mathscr{E}^{W_P} = \mathscr{E}^{W_{P'}}$, which suffices. $\square$

Now let $\mathscr{V}$ be a λ-candidate and $P = P_{\mathscr{V}}$, so $\mathscr{V} \downarrow 1 = \mathscr{V}_1^P$. Note that $\mathcal{T}_P \,\widehat{}\, \langle E_P \rangle \,\widehat{}\, \mathcal{U}_P$, after normalization, is translatable, where $\mathcal{U}_P$ is the tree leading from U_P to W_P. Let $\mathcal{T}_{\mathscr{V}\downarrow 1}$ on $\mathscr{V} \downarrow 1 = \mathscr{V}_1^P$ be its (short-normal) translation on $\mathscr{V} \downarrow 1$. Then $M_\infty^{\mathcal{T}_{\mathscr{V}}} = \mathscr{V}_1^{W_P}$, which by the previous claim is independent of $\mathscr{V}$. So the trees $\mathcal{T}_{\mathscr{V}\downarrow 1}$ iterate the various $\mathscr{V} \downarrow 1$ to a common model $\mathscr{V}_1^*$.

Let $\mathcal{T}_{\mathscr{V}\downarrow 1} = \mathcal{T}_0 \,\widehat{}\, \mathcal{T}_1$ where $\mathcal{T}_0$ is based on $\mathscr{V}|\delta_1^{\mathscr{V}} = (\mathscr{V} \downarrow 1)|\delta_1^{\mathscr{V}\downarrow 1}$, and $\mathcal{T}_1$ is on $M_\infty^{\mathcal{T}_0}$, and is above $\delta_1^{M_\infty^{\mathcal{T}_0}} = \delta_1^{\mathscr{V}_1^*}$. Then $\mathcal{T}_0$ translates to a tree $\mathcal{U}_0$ on $\mathscr{V}$, and $M_\infty^{\mathcal{T}_0} = M_\infty^{\mathcal{U}_0} \downarrow 1$. Now $\mathcal{T}_1$ essentially translates to a tree $\mathcal{U}_1$ on $M_\infty^{\mathcal{U}_0}$. The extenders used in $\mathcal{U}_1$ are just those with indices those used in $\mathcal{T}_1$, together with 1 further 1-long extender, which is an image of $e_1(M_\infty^{\mathcal{U}_0})$. That is, let $\alpha_0 = 0$, and let α_1 be least such that $[0, \alpha_1]_{\mathcal{T}_0}$ is non-dropping and $\kappa_1^{+M_{\alpha_1}^{\mathcal{T}_0}} < \mathrm{lh}(E_{\alpha_1}^{\mathcal{T}_0})$. Then $\mathcal{U}_1 {\restriction} (\alpha_1 + 1)$ is a direct translation of $\mathcal{T}_1$, though note that if it is non-trivial, it drops in model immediately to $M_\infty^{\mathcal{U}_0} || \gamma_1^{M_\infty^{\mathcal{U}_0}}$, or some segment thereof. Then, $\mathcal{U}_1$ uses an extra extender; if $\alpha_1 = 0$ then $E_{\alpha_1}^{\mathcal{U}_1} = e_1(M_\infty^{\mathcal{U}_0})$, and otherwise $E_{\alpha_1}^{\mathcal{U}_1} = F(M_\infty^{\mathcal{U}_0})$ (which is an image of $e_1(M_\infty^{\mathcal{U}_0})$). This results in $M_{\alpha_1+1}^{\mathcal{U}_1} = \mathscr{V}_2(M_{\alpha_1}^{\mathcal{T}_1})$. After this, noting that $\mathcal{T}_1 {\restriction} [\alpha_1, \infty)$ is 1-translatable on $M_{\alpha_1}^{\mathcal{T}_1}$ (in particular, uses no 0-long extenders), we set $\mathcal{U}_1 {\restriction} [\alpha + 1, \infty)$ to be its 1-translation. Write $\mathcal{U}_{\mathscr{V}} = \mathcal{U}_0 \,\widehat{}\, \mathcal{U}_1$.

So we end up with $M_\infty^{\mathcal{U}_{\mathscr{V}}} = \mathscr{V}_2(M_\infty^{\mathcal{T}_{\mathscr{V}}})$, but $M_\infty^{\mathcal{T}_{\mathscr{V}}}$ was independent of $\mathscr{V}$. So write $\mathscr{V}_2^*$ for this common iterate of the λ-candidates $\mathscr{V}$.

Using the strategies Σ_P, we can define $\mathcal{U}_{\mathscr{V}}$ uniformly in $\mathscr{V}$. So let Γ be the proper class of all ordinals fixed by all the iteration maps $i^{\mathcal{U}_{\mathscr{V}}}$. Let $\bar{\mathscr{V}}_2 = \mathrm{cHull}_1^{\mathscr{V}_2^*}(\Gamma)$ and $\pi : \bar{\mathscr{V}}_2 \to \mathscr{V}_2^*$ the uncollapse map. Since $\mathrm{rg}(i_{\mathcal{U}_{\mathscr{V}_2}}) \subseteq \mathrm{rg}(\pi)$, this determines an elementary $\pi : \bar{\mathscr{V}}_2 \to \mathscr{V}_2$ by factoring. But $\mathscr{V}_2$ is a set-ground of $\mathscr{V}_2[G, G']$, so by [5], $\bar{\mathscr{V}}_2 = \mathscr{V}_2$ and $\pi = \mathrm{id}$.

So we have defined $\mathscr{V}_2$ over the universe of $\mathscr{V}_2[G, G']$ from the parameter λ, and so by homogeneity, in fact over the universe of $\mathscr{V}_2[G]$ from λ. The uniformity then gives that we can define $\mathscr{V}_2$ over the universe of any set-generic extension of $\mathscr{V}_2$, from no parameter.

We can now easily complete the proof of the theorem. Part 4 was just established above. Part 1: Let U_2 be the universe of $\mathscr{V}_2$ and $G \subseteq \mathrm{Coll}(\omega, \lambda)$ be $\mathscr{V}_2$-generic, where $\lambda \geq \delta_1^{\mathscr{V}_2}$. It now easily follows that $U_2 \subseteq \mathrm{HOD}^{U_2[G]}$, so actually $U_2 = \mathrm{HOD}^{U_2[G]}$. And if H is $\mathrm{Coll}(\omega, \lambda)$-generic over M, then since M is an $\mathbb{L}^{\mathscr{V}_2}$-extension of $\mathscr{V}_2$, it follows that $U_2 = \mathrm{HOD}^{M[H]}$. Part 2: If $W \subseteq U_2$ is a ground of U_2, then we get $\mathrm{HOD}^{U_2[G]} \subseteq W_2$ if λ is sufficiently large, so $U_2 \subseteq W$, so $U_2 = W$. Part 3: The fact that U_2 is the mantle of all set-generic extensions of U_2 now follows from the set-directedness of set-grounds. $\square$

Corollary 5.80. *We have:*

1. The κ_1-mantle $\mathscr{M}^M_{\kappa_1}$ of M is the universe of $\mathscr{V}_2$, so κ_1 is the least ordinal with this property.

2. If $G \subseteq \mathrm{Coll}(\omega, < \kappa_1)$ is M-generic then $\mathrm{HOD}^{M[G]}$ is the universe of $\mathscr{V}_2$.

Proof. Note that $\mathscr{V}_2 \subseteq \mathrm{HOD}^{M[G]} \subseteq \mathscr{M}^M_{\kappa_1}$, using Theorem 5.79. So we just need to see that $\mathscr{M}^M_{\kappa_1} \subseteq \mathscr{V}_2$. Let X be a set of ordinals in $\mathscr{M}^M_{\kappa_1}$. Let $E \in \mathbb{E}^M$ be M-total with $\mathrm{crit}(E) = \kappa_1$, and $U = \mathrm{Ult}(M, E)$. Then $j(X) \in \mathscr{M}^U_{j(\kappa_1)}$ and $j{\restriction}\sup X \in \mathscr{V}_2$, so it suffices to find a $< j(\kappa_1)$-ground W of U with $W \subseteq \mathscr{V}_2$. But this can be done like in the proof of Theorem 4.36, or as follows:[56] Let W be the result of the P-construction of U over $\mathscr{V}_2|\gamma_1^{\mathscr{V}_2}$ (in the style of that used to construct $\mathscr{V}_2$). Note that $(\mathscr{V}_1|\kappa_1, M|\kappa_0)$ is generic over W for the two-step forcing iteration given by $\mathbb{L}^{\mathscr{V}_2}$ followed by $\mathbb{L}^{\mathscr{V}_1}$, and $W[\mathscr{V}_1|\kappa_1, M|\kappa_0] \cong U$. So W is a $< j(\kappa_1)$-ground of U. But $W \subseteq \mathscr{V}_2$, since $W||\mathrm{lh}(E) = \mathscr{V}_2|\mathrm{lh}(E)$, and we can inductively compute the extender sequence of W above $\mathrm{lh}(E)$ using $E{\restriction}\mathscr{V}_2$ and $\mathrm{Ult}(\mathscr{V}_2, E{\restriction}\mathscr{V}_2) = \mathscr{V}_2^U$. $\qquad\square$

Acknowledgments

References

[1] Erik W. Closson. *The Solovay Sequence in Derived Models Associated to Mice.* PhD thesis, University of California, Berkeley, 2008.

[2] Gunter Fuchs, Joel David Hamkins, and Jonas Reitz. Set theoretic geology. *Annals of Pure and Applied Logic*, 2015.

[3] Gunter Fuchs and Ralf Schindler. Inner model theoretic geology. *Journal of Symbolic Logic*, 81(3):972–996, 2016.

[4] Haim Gaifman. Elementary embeddings of models of set-theory and certain subtheories. In *Axiomatic set theory (Proc. Sympos. Pure Math., Vol. XIII, Part II, Univ. California, Los Angeles, Calif., 1967)*, pages 33–102. Amer. Math. Soc., Providence, R. I., 1974.

[5] Joel David Hamkins, Greg Kirmayer, and Norman Lewis Perlmutter. Generalizations of the kunen inconsistency. *Annals of Pure and Applied Logic*, 163(12), 2012.

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[56] The method used here was actually the method used in the original proof of Theorem 4.36.

[6] Richard Laver. Certain very large cardinals are not created in small forcing extensions. *Annals of Pure and Applied Logic*, 149:1–6, 2007.

[7] G. Sargsyan. *Hod Mice and the Mouse Set Conjecture*, volume 236 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2015.

[8] Grigor Sargsyan and Ralf Schindler. Varsovian models I. *Journal of Symbolic Logic*, 83(2):496–528, 2018.

[9] Ralf Schindler. A note on the $< \kappa$ mantle. Handwritten, https://ivv5hpp.uni-muenster.de/u/rds/kappa_mantle.pdf.

[10] Ralf Schindler. Varsovian models II (parts 1–6). Handwritten, https://ivv5hpp.uni-muenster.de/u/rds/VM2.pdf, https://ivv5hpp.uni-muenster.de/u/rds/VM2_2.pdf, https://ivv5hpp.uni-muenster.de/u/rds/VM2_5.pdf, https://ivv5hpp.uni-muenster.de/u/rds/VM2_6.pdf, https://ivv5hpp.uni-muenster.de/u/rds/VM2_7.pdf, https://ivv5hpp.uni-muenster.de/u/rds/VM2_8.pdf.

[11] Ralf Schindler. The long extender algebra. *Archive for Mathematical Logic*, pages 73–82, 2018.

[12] Ralf Schindler and John Steel. The self-iterability of $L[E]$. *Journal of Symbolic Logic*, 74(3):751–779, 2009.

[13] Farmer Schlutzenberg. Background construction for λ-indexed mice. arXiv:2101.00889v2.

[14] Farmer Schlutzenberg. Full normalization for transfinite stacks. arXiv:2102.03359v2.

[15] Farmer Schlutzenberg. Local mantles of $L[x]$. arXiv:2103.12925v1.

[16] Farmer Schlutzenberg. Notes on $*$-translation. In preparation.

[17] Farmer Schlutzenberg. Ordinal definability in $L[\mathbb{E}]$. arXiv:2012.07185v1.

[18] Farmer Schlutzenberg. *Measures in mice*. PhD thesis, University of California, Berkeley, 2007. arXiv: 1301.4702v1.

[19] Farmer Schlutzenberg. "Θ is woodin in HOD" directly from mice. Available at https://www.math.uci.edu/~mzeman/CMI/talk-schlutzenberg-4.pdf, 2016.

[20] Farmer Schlutzenberg. The definability of the extender sequence $\mathbb{E}$ from $\mathbb{E} \upharpoonright \aleph_1$ in $L[\mathbb{E}]$. arXiv: 1906.00276v1, 2019.

[21] Farmer Schlutzenberg. Choice principles in local mantles. arXiv: 2006.01119v2, 2020.

[22] Farmer Schlutzenberg. Iterability for (transfinite) stacks. *Journal of Mathematical Logic*, 21(2), 2021. arXiv:1811.03880v4.

[23] John Steel. Local hod computation. Handwritten notes. Available at http://math.berkeley.edu/~steel/papers/publications.html, 2016.

[24] John R. Steel. Normalizing iteration trees and comparing iteration strategies. Forthcoming book available at http://math.berkeley.edu/~steel/papers/publications.html.

[25] Toshimichi Usuba. The downward directed grounds hypothesis and very large cardinals. *Journal of Mathematical Logic*, 2017.

[26] Toshimichi Usuba. Extendible cardinals and the mantle. *Archive for Mathematical Logic*, 58:71–75, 2019.

[27] W. Hugh Woodin. Email to the second author. Personal communication, dated February 1, 2016.

[28] W. Hugh Woodin. The continuum hypothesis, the generic-multiverse of sets, and the ω-conjecture. In *Set theory, arithmetic, and foundations of mathematics: theorems, philosophies*, volume 36 of *Lecture Notes in Logic*, pages 13–42. Cambridge University Press, 2011.