Volume and L^2-Betti numbers of aspherical manifolds

Roman Sauer

University of Chicago

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Definition of L^2-Betti numbers

L^2-Betti numbers – analytically [Atiyah]

Let $\widetilde{M} \to M$ be the universal cover of a compact Riemannian manifold, and let $\mathscr{F} \subset \widetilde{M}$ be a $\pi_1(M)$-fundamental domain. Then define

$$b_i^{(2)}(M) = \lim_{t\to\infty} \int_{\mathscr{F}} \operatorname{tr}_{\mathbb{C}} e^{-t\Delta_i}(x,x)\,dvol(x).$$

L^2-Betti numbers – homologically [Farber; Lück]

Let M be an arbitrary space and $\Gamma = \pi_1(M)$. Let

$$b_i^{(2)}(M) = \dim_{L(\Gamma)} H_i(L(\Gamma) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M})) \in [0,\infty].$$

Dimension function for finite von Neumann algebras [Lück]

$\dim_{\mathcal{A}}(M)$ is defined for any finite von Neumann algebra $\mathcal{A}$ with trace $\operatorname{tr} : \mathcal{A} \to \mathbb{C}$ and any $\mathcal{A}$-module. For example, $\dim_{\mathcal{A}}(\mathcal{A}p) = \operatorname{tr}(p)$.

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Some properties of L^2-Betti numbers

Basic properties

- $\pi_1(M)$ finite $\Rightarrow b_i^{(2)}(M) = b_i(\widetilde{M})/|\pi_1(M)|$
- $\sum_{i\geq 0}(-1)^i b_i^{(2)}(M) = \chi(M) = \sum_{i\geq 0}(-1)^i b_i(M)$.
- $\bar{M} \to M$ d-sheeted cover $\Rightarrow b_i^{(2)}(\bar{M}) = d \cdot b_i^{(2)}(M)$.

Vanishing and non-vanishing

- Amenable groups are a large class of groups including solvable ones. If M aspherical, i.e. $\widetilde{M}$ contractible, and $\pi_1(M)$ amenable, then $b_i^{(2)}(M) = 0$ [Cheeger-Gromov].
- If M is a $2n$-dimensional hyperbolic manifold then $b_i^{(2)}(M) = 0$ unless $i = n$.

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Volume theorems

- Let M be an n-dimensional, closed, aspherical manifold.

Theorem 1 [Gromov; S.]

If (M, g) has a lower Ricci curvature bound $\mathrm{Ricci}(M, g) \geq -(n-1)g$, then

$$b_i^{(2)}(M) \leq \mathrm{const}_n \mathrm{vol}(M, g) \quad \text{for every } i \geq 0.$$

- Gromov states this inequality in his book *Metric structures on Riemannian and non-Riemannian spaces* along with an idea (*randomization*) which he attributes to Connes.
- We provide the first complete proof of that inequality. The rigorous implementation of Gromov's idea uses tools and ideas from Gaboriau's theory of L^2-Betti numbers of measured equivalence relations and spaces with groupoid actions of such.

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Corollary (Minimal volume estimate)

$$b_i^{(2)}(M) \leq \mathrm{const}_n \, \mathrm{minvol}(M)$$

Definition (Minimal volume)

$$\mathrm{minvol}(M) = \inf\{\mathrm{vol}(M, g);\ -1 \leq \sec(g) \leq 1\}.$$

Vanishing theorems

- Let M be an n-dimensional, closed, aspherical manifold.

Theorem 2 [S.]

If M is covered by open, amenable sets such that every point belongs to at most n sets, then

$$b_i^{(2)}(M) = 0 \qquad \text{for every } i \geq 0.$$

- $U \subset M$ is called amenable if $\pi_1(U)$ maps to an amenable subgroup of $\pi_1(M)$.
- There is also a version of this theorem for arbitrary spaces.
- The proof uses a geometric construction, which is based on the Rokhlin theorem from ergodic theory, and the randomization idea mentioned before.

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Corollary (Gap theorem for vanishing of $b_i^{(2)}$)

There is a constant $\epsilon_n > 0$ only depending on n such that

$$\mathrm{minvol}(M) < \epsilon_n \Rightarrow b_i^{(2)}(M) = 0 \quad \textit{for every } i \geq 0.$$

- The implication “Theorem ⇒ Corollary” follows from Gromov’s work.

Proof of Theorem 2 – a first try

General technique of bounding the i-th L^2-Betti number by equivariant coverings

- Let $\Gamma = \pi_1(M)$. Construct, using the geometric assumption somehow, a Γ-equivariant open covering $\mathcal{U}$ of $\widetilde{M}$.
- Let us say that $\mathcal{U}$ is indexed by an index set $\subset \Gamma \times \mathbb{N}$.
- One obtains $f \in \operatorname{map}\big(\widetilde{M}, \Delta(\Gamma \times \mathbb{N})\big)^{\Gamma}$ (nerve map).
- Let $\mathscr{F}_i$ be a set of Γ-representatives of $\Delta(\Gamma \times \mathbb{N})^{(i)}$.
  Let $C_i(f) \in \mathbb{N}$ be the number of i-cells in $\mathscr{F}_i$ hit by $f(\widetilde{M})$.
- Since $\widetilde{M} \simeq E\Gamma$, there is an equivariant homotopy retract

$$\begin{tikzcd} \widetilde{M} \arrow[r, "f"'] & \Delta(\Gamma \times \mathbb{N}) \arrow[l, bend right] \end{tikzcd}.$$

- By rank considerations,

$$b_i^{(2)}(M) \leq C_i(f).$$

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- Let $C_i(f) \in \mathbb{N}$ be the number of equivariant i-cells hit by $f(\widetilde{M})$. In other words, ...
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Proof of Theorem 2 – Randomization

Connes-Gromov's philosophy – in probabilistic terms

Rephrasing what we tried before

- View $\Omega = \mathrm{map}(\widetilde{M}, \Delta(\Gamma \times \mathbb{N}))$ as a Borel space with a Γ-action.
- We tried to find a Γ-invariant point measure $f \in \Omega$ such that

$$b_i^{(2)}(M) \leq C_i(f) = \#(i\text{-cells in } \mathscr{F}_i \text{ hit by } f) \quad \text{is arbitrarily small.}$$

Randomized Problem

- View $C_i : \Omega \to \mathbb{Z}$ as a random variable.
- Instead of seeking a point measure, find any Γ-invariant probability measure μ on Ω such that the expected value

$$\mathbb{E}_{(\Omega,\mu)}(C_i) = \int_{\Omega} C_i(f) d\mu(f) \quad \text{is arbitrarily small.}$$

- Show that $b_i^{(2)}(M) \leq \mathbb{E}_{(\Omega,\mu)}(C_i)$.

Proof of Theorem 2 – Randomization

Connes-Gromov's philosophy – in probabilistic terms

Deterministic Problem

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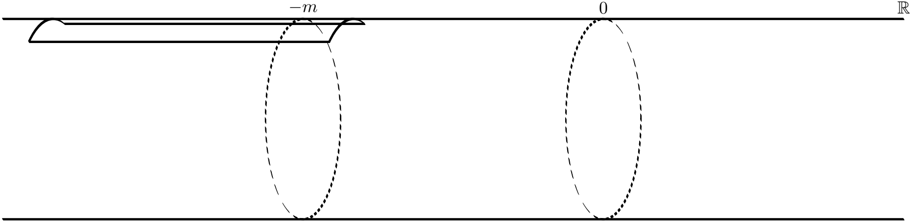
Proof of Theorem 2 – the toy case $M = S^1$

Making the expected value $\mathbb{E}_{(\Omega,\mu)}(C_i)$ arbitrarily small for suitable μ

- Let $\mathbb{Z} = \langle t \rangle$ act on $X = S^1$ by an irrational angle α.
- Consider the diagonal action $\mathbb{Z} \curvearrowright X \times \widetilde{M} = S^1 \times \mathbb{R}$.
  (thanks to C. Löh for a lot of help with the picture!)



- Consider the "tile" on top of the cylinder with arclength α and length m.
- We now look at its $\mathbb{Z}$-orbit....

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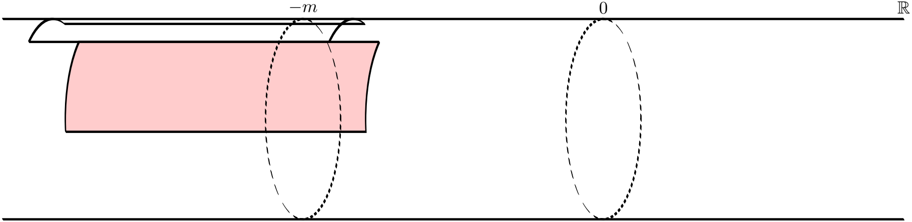

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- Here's the t-translate.

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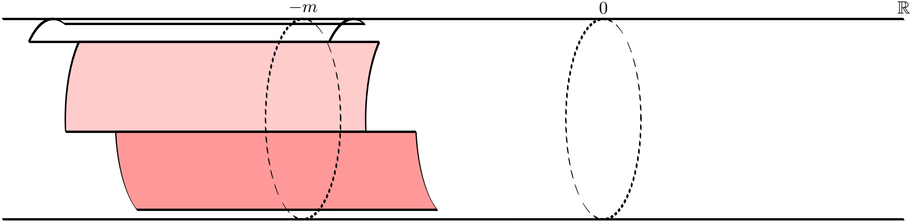

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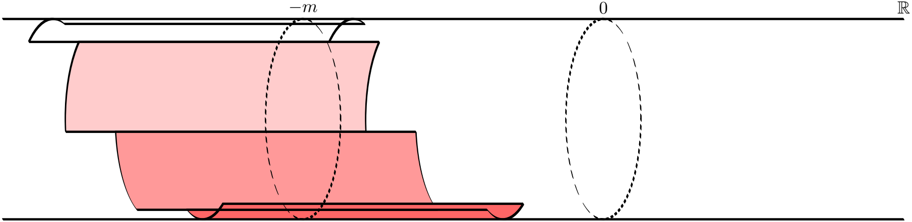

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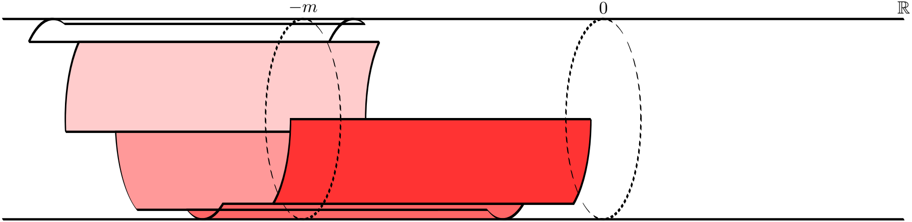

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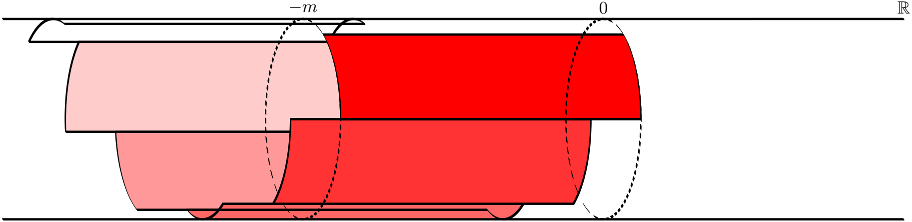

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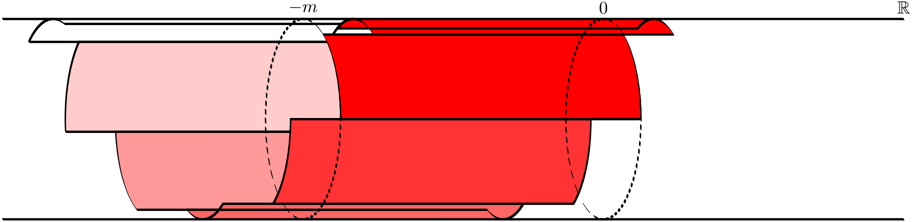

- After m steps the tiles almost close up, and the whole $\mathbb{Z}$-orbit almost tesselates $X \times \widetilde{M}$.

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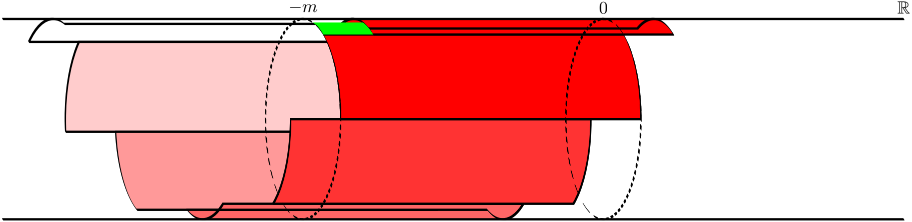

- Add the $\mathbb{Z}$-orbit of the small green tile, and one gets a tesselation of $X \times \widetilde{M}$ indexed by $\mathbb{Z} \times \{1, 2\}$.

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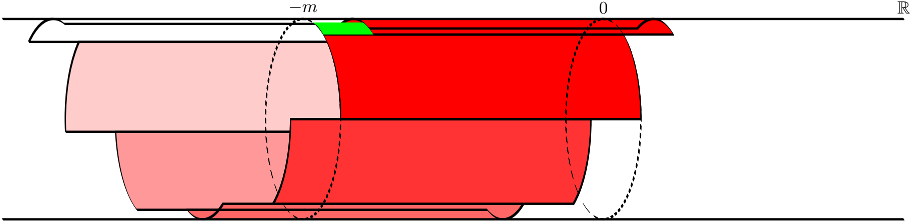

- Increase the size of the tiles in the the induced tesselation of $\{z\} \times \widetilde{M}$ a little to get a covering.
- For every $z \in S^1$ we obtain a map $\{z\} \times \widetilde{M} \to \Delta(\mathbb{Z} \times \{1,2\})$.

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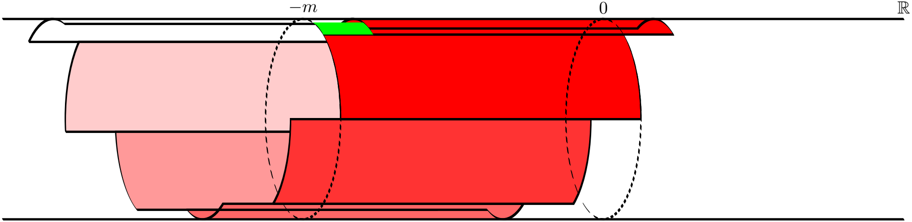

- Equivalently, we have a map

$$\phi : S^1 \to \Omega = \mathrm{map}\big(\widetilde{M}, \Delta(\mathbb{Z} \times \{1,2\})\big).$$

- The push-forward of the Haar measure on S^1 by ϕ is the desired measure μ.
- One has $\mathbb{E}_{(\Omega,\mu)}(C_i) \leq 2/m$.

L^2-Betti numbers and actions on probability spaces

What's behind $b_i^{(2)}(M) \leq \mathbb{E}_{(\Omega,\mu)}(C_i)$?

Group measure space construction [Murray-von Neumann]

The *group measure space construction*

$$L^\infty(\Omega,\mu) \hookrightarrow L^\infty(\Omega,\mu)\overline{\rtimes}\Gamma \hookleftarrow L(\Gamma)$$

is a completion of the algebraic crossed product $L^\infty(X) \rtimes \Gamma$ with respect to the trace

$$\mathrm{tr}\Big(\sum f_\gamma \gamma\Big) = \int_\Omega f_1(x) d\mu(x).$$

L^2-Betti numbers and Induction

$$b_i^{(2)}(M) = \dim_{L^\infty(\Omega,\mu)\overline{\rtimes}\Gamma} H_i\big(L^\infty(\Omega,\mu)\overline{\rtimes}\Gamma \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M})\big)$$
$$\big[= \dim_{L(\mathcal{R})} \mathrm{Tor}_i^{\mathbb{C}\mathcal{R}}\big(L(\mathcal{R}), L^\infty(\Omega)\big)\big]$$

Proof of Theorem 2 – the general case

Where are the technical difficulties?

- Let $X = \{0,1\}^\Gamma$. For each amenable $U \subset M$ construct a certain tesselation on each $X \times p^{-1}(U)$ and combine them. Crucial ingredient: the generalized *Rokhlin lemma* from ergodic theory [Ornstein-Weiss].
- $\rightsquigarrow$ Γ-invariant measure μ on $\Omega = \mathrm{map}\big(\widetilde{M}, \Delta(\Gamma \times \mathbb{N})\big)$ such that $\mathbb{E}_{(\Omega,\mu)}(C_n)$ is arbitrarily small.
- $\rightsquigarrow$ construct a fundamental class $\sum f_i \otimes \sigma_i$ in

$$H_n\big(L^\infty(\Omega) \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M})\big) \xleftarrow{\cong} H_n(\mathbb{R} \otimes_{\mathbb{Z}\Gamma} C_*(\widetilde{M})\big) = H_n(M,\mathbb{R}).$$

  such that $\sum \mu(\mathrm{supp}(f_i))$ is arbitrarily small.
- $\rightsquigarrow$ $b_i^{(2)}(M)$ arbitrarily small [Schmidt].

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